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Navier–Stokes–Fourier equations as a parabolic limit of a general hyperbolic system of rational extended thermodynamics

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ABSTRACT

This paper aims to prove that a general system of 14 balance laws for a compressible, possibly dense, gas that satisfies the universal principles of Rational Extended Thermodynamics (RET) converges to the Navier–Stokes–Fourier equations in the first step of the Maxwellian iteration. Moreover, in a theory not far from equilibrium, we show that the production terms of the hyperbolic system are uniquely determined as soon as the heat conductivity, the shear viscosity, and the bulk viscosity are assigned. The obtained results are tested on the RET theories for rarefied monatomic and polyatomic gases.

1. Introduction

In the thermodynamics of irreversible processes (TIP) for a compressible gas, the balance laws of mass, momentum, and energy

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+\frac{\partial \rho v_i}{\partial x_i}=0,\\
&\frac{\partial \rho v_j}{\partial t}+\frac{\partial}{\partial x_i}\left(\rho v_i v_j-t_{ij}\right)=0,\\
&\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}+\rho\varepsilon\right)+\frac{\partial}{\partial x_i}\left\{\left(\frac{\rho v^2}{2}+\rho\varepsilon\right)v_i-t_{ik}v_k+q_i\right\}=0,
\end{aligned}\tag{1}$$

are completed by the classical Navier–Stokes–Fourier (NSF) constitutive equations

$$\sigma_{\langle ij\rangle}=2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}},\qquad \Pi=-\nu\frac{\partial v_i}{\partial x_i},\qquad q_i=-\kappa\frac{\partial T}{\partial x_i},\tag{2}$$

where ρ, $t_{ij}=-(p+\Pi)\delta_{ij}+\sigma_{\langle ij\rangle}$, ε, v_i, T and q_i $(i,j=1,2,3)$ are, respectively, the mass density, stress tensor, specific internal energy, velocity, temperature, and heat flux; p, Π and $\sigma_{\langle ij\rangle}$, are the pressure, the dynamical pressure and the traceless shear viscosity tensor, while κ, μ and ν are the heat conductivity, shear, and bulk viscosity coefficients.[1]

The system (1)–(2) has a parabolic structure, and its validity is limited to processes that satisfy the so-called *local equilibrium* assumption.

The theory of modern Rational Extended Thermodynamics (RET) originates from Ruggeri's idea [1] to consider a system of 14 balance laws

$$\begin{aligned}
&\frac{\partial F}{\partial t}+\frac{\partial F_k}{\partial x_k}=0,\\
&\frac{\partial F_i}{\partial t}+\frac{\partial F_{ik}}{\partial x_k}=0,\\
&\frac{\partial G_{ii}}{\partial t}+\frac{\partial G_{iik}}{\partial x_k}=0,\\
&\frac{\partial H_{ij}}{\partial t}+\frac{\partial H_{ijk}}{\partial x_k}=P_{ij},\\
&\frac{\partial I_{lli}}{\partial t}+\frac{\partial I_{llik}}{\partial x_k}=Q_{lli},
\end{aligned}\tag{3}$$

for the extended 14 field components

$$\mathbf{u}\equiv\left(\rho, v_i, T, \sigma_{\langle ij\rangle}, \Pi, q_i\right).$$

The first 5 scalar equations of (3) with $F=\rho$, $F_i=\rho v_i$ and $G_{ll}=2\rho\varepsilon+\rho v^2$ correspond to the conservation laws (1) with the momentum flux F_{ij} and energy flux G_{lli}, while the remaining 9 equations replace the NSF Eqs. (2). Due to the overwhelming difficulties, the system's closure was obtained in [1] utilizing the entropy principle under some suitable assumptions, which were not motivated at the mesoscopic scale. Later on, Liu and Müller [2] proposed, in the case of a rarefied monatomic gas, to identify the system of balance laws (3) as the system obtained by taking the moments of the distribution function of the Boltzmann equation, which implies $G_{ii}=F_{ii}$, $H_{ij}=F_{ij}$, $H_{ijk}=G_{ijk}$, $P_{ll}=0$, $I_{lli}=$

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[1] The symbol $\langle ij\rangle$ indicates a traceless tensor (deviatoric tensor), italic indices run from 1, 2, 3, and repeated indexes indicate summation with respect to the indexes.

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G_{lli}. In this case, the equations are 13 as Π is identically zero. More recently, Arima, Taniguchi, Ruggeri, and Sugiyama [3] considered a rarefied polyatomic gas; in this case, two hierarchies of moments were written. These two hierarchies, with $H_{ij} = F_{ij}$ and $I_{lli} = G_{lli}$, are a particular case of (3) as well.

The general principles of RET used to close the system were based on the systematic use of universal principles like the entropy principle, the Galilean invariance, and the convexity of entropy density. In rarefied monatomic and polyatomic gases, it was proved that the parabolic limit is the NSF theory. RET is now a well-established theory, and the results are summarized in the book by Müller, and Ruggeri for monatomic gases [4] and in the books by Ruggeri and Sugiyama [5,6] for polyatomic classical and relativistic gases. In recent years, efforts have been made in order to include dense gases in the RET theory ([6], and references therein); however, a RET theory for the general structure (3) has not yet been obtained.

In this paper, we investigate the conditions under which the system (3) converges in the first step of the Maxwellian iteration to the NSF system. We will see that the convergence requires a simple compatibility condition. Under this condition, the production terms as functions of the phenomenological coefficients (heat conductivity, shear viscosity, and bulk viscosity) are determined.

In the second part of the paper, we prove that the symmetric form of the system that arises from the entropy principle automatically satisfies the compatibility condition. Therefore any system (3), compatible with the universal principles of RET, converges formally to the NSF equations in the parabolic limit.

2. Galilean invariance

The system (3) is a particular case of a general quasi-linear hyperbolic system of balance laws:

$$\frac{\partial \mathbf{F}^0(\mathbf{u})}{\partial t} + \frac{\partial \mathbf{F}^i(\mathbf{u})}{\partial x^i} = \mathbf{f}(\mathbf{u}) \tag{4}$$

with

$$\mathbf{F}^0 = \begin{pmatrix} \rho \\ \rho v_i \\ G_{ll} \\ H_{ij} \\ I_{lli} \end{pmatrix}, \qquad \mathbf{F}^k = \begin{pmatrix} \rho v_k \\ F_{ik} \\ G_{llk} \\ H_{ijk} \\ I_{llik} \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ P_{ij} \\ Q_{lli} \end{pmatrix}.$$

According to the theorem given in [7], the system (4) is Galilean invariant if it exists a matrix depending only on the velocity, $\mathbf{X}(\mathbf{v})$, such that

$$\mathbf{F}^0 = \mathbf{X}(\mathbf{v})\,\hat{\mathbf{F}}^0, \qquad \mathbf{F}^k - v^k\mathbf{F}^0 = \mathbf{X}(\mathbf{v})\,\hat{\mathbf{F}}^k, \qquad \mathbf{f} = \mathbf{X}(\mathbf{v})\,\hat{\mathbf{f}}, \tag{5}$$

where the hat indicates the quantities evaluated for zero velocity. The general expression of $\mathbf{X}(\mathbf{v})$ is given in [7]. In the present case becomes

$$\mathbf{X}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_i & 0 & 0 & 0 & 0 \\ v^2 & 2v_i & 1 & 0 & 0 \\ v_iv_j & 2\delta^{h_1}_{(i}v_{j)} & 0 & \delta^{h_1}_i\delta^{h_2}_j & 0 \\ v^2v_i & 3v_{(l}v_l\delta^{h_1}_{i)} & v_i & 2\delta^{(h_1}_i v_{h_2)} & \delta^{h_1}_i \end{pmatrix},$$

and (5) implies

$$\begin{aligned}
&F_{ij} - F_iv_j = \rho v_iv_j + \hat{F}_{ij},\\
&G_{lli} - G_{ll}v_i = \rho v^2 + 2v_l\hat{F}_{li} + \hat{G}_{lli},\\
&H_{ij} = \rho v_iv_j + \hat{H}_{ij},\\
&H_{ijk} - H_{ij}v_k = 2v_{(i}\hat{F}_{j)k} + \hat{H}_{ijk},\\
&I_{lli} = \rho v^2v_i + \hat{G}_{ll}v_i + 2\hat{H}_{il}v_l + \hat{I}_{lli},\\
&I_{llik} - I_{lli}v_k = 3v_{(l}v_l\hat{F}_{i)k} + v_i\hat{G}_{llk} + 2v_l\hat{H}_{lik} + \hat{I}_{llik},\\
&P_{ij} = \hat{P}_{ij},\\
&Q_{lli} = 2v_l\hat{P}_{li} + \hat{Q}_{lli}.
\end{aligned} \tag{6}$$

With the conventional hydrodynamic variables, we have (see (1), (3))

$$\hat{G}_{ll} = 2\rho\varepsilon, \qquad \hat{F}_{ij} = -t_{ij} = (p+\Pi)\delta_{ij} - \sigma_{\langle ij\rangle}, \qquad \hat{G}_{lli} = 2q_i.$$

The pressure p and the specific internal energy ε is given by the thermal and caloric equations of state as the function of the mass density ρ and the temperature T.

Considering, as usual, processes that are linear with respect to the non-equilibrium variables, we have the following representation for the production terms:

$$\hat{P}_{ij} = a_3\Pi\delta_{ij} + a_2\sigma_{\langle ij\rangle}, \qquad \hat{Q}_{lli} = a_1q_i,$$

and from the last two equations of (6) we obtain:

$$P_{ij} = a_3\Pi\delta_{ij} + a_2\sigma_{\langle ij\rangle}, \qquad Q_{lli} = 2a_3\Pi v_i + 2a_2\sigma_{\langle li\rangle}v_l + a_1q_i, \tag{7}$$

where the scalars a_1, a_2, a_3 are functions of ρ and T to be determined.

3. Maxwellian iteration and NSF limit

The Maxwellian iteration was introduced by Ikenberry and Truesdell [8], and it is substantially composed of (i) an identification of the relaxation times and (ii) a formal power expansion of the solution in terms of the relaxation times: a sort of Chapman–Enskog procedure at a macroscopic level. In general, the *first iterates* (i.e. the values at the first step of the iterative procedure) are obtained from the right-hand side of the balance laws by putting the "zeroth" iterates (equilibrium values) into the left-hand side. The second iterates are obtained from the right-hand side by putting the first iterates into the left-hand side, and so on. Therefore we need to consider the balance laws with production terms in (3) in which on the left-hand side we put the equilibrium values and on the left-hand side, the production terms (7) at the first step of the iterative procedure appear (we omit the apex (1) which is typically used to indicate the first iterates). Given the tensorial character of the system, we have:

$$\hat{H}^E_{ij} = a\delta_{ij}, \qquad \hat{H}^E_{ijk} = 0, \qquad \hat{I}^E_{lli} = 0, \qquad \hat{I}^E_{llik} = b\delta_{ik}, \qquad \left(\hat{F}^E_{ij} = p\delta_{ij}\right),$$

where a and b are functions of ρ and T, and the superscript E denotes values evaluated at equilibrium for which $P_{ij} = 0, Q_{lli} = 0$ and as consequence $\Pi = 0, \sigma_{\langle ij\rangle} = 0, q_i = 0$.

Taking into account the Galilean decomposition (6), we have

$$\begin{aligned}
&H^E_{ij} = \rho v_iv_j + \hat{H}^E_{ij} = \rho v_iv_j + a\delta_{ij},\\
&H^E_{ijk} = H^E_{ij}v_k + 2v_{(i}\hat{F}^E_{j)k} + \hat{H}^E_{ijk} = \rho v_iv_jv_k + a\delta_{ij}v_k + 2pv_{(i}\delta_{j)}k,\\
&I^E_{lli} = \rho v^2v_i + \hat{G}^E_{ll}v_i + 2\hat{H}^E_{il}v_l + \hat{I}^E_{lli} = \rho v^2v_i + 2\rho\varepsilon v_i + 2av_i,\\
&I^E_{llik} = I^E_{lli}v_k + 3v_{(l}v_l\hat{F}^E_{i)k} + v_i\hat{G}^E_{llk} + 2v_l\hat{H}^E_{lik} + \hat{I}^E_{llik}\\
&\quad = \rho v^2v_iv_k + 2\rho\varepsilon v_iv_k + 2av_iv_k + 3pv_{(l}v_l\delta_{i)k} + b\delta_{ik}.
\end{aligned} \tag{8}$$

It is seen that the equilibrium part of the tensors that appear in the balance laws $(3)_{4,5}$ are completely determined except for the two equilibrium functions a and b, while the production terms (7) depend on the scalar equilibrium functions a_1, a_2, a_3. All these 5 scalars are functions of (ρ, T) and are to be determined.

Theorem 1. *The necessary and sufficient condition for the system of balance laws* $(3)_{4,5}$ *in the approximation of first-order Maxwellian iteration to coincide with the NSF Eqs. (2) is that the coefficients a and b satisfy the relation:*

$$a = \rho\left(\frac{b_\rho}{2p_\rho} - \varepsilon\right). \tag{9}$$

Moreover, the following relations between the coefficients a_1, a_2, a_3 *and the phenomenological coefficients* κ, μ, ν *hold:*

$$\begin{aligned}
&a_1 = -\frac{1}{\kappa}\left\{b_T - 2\left(\varepsilon + \frac{a}{\rho}\right)p_T\right\},\\
&a_2 = \frac{p}{\mu},\\
&a_3 = \frac{1}{\nu}\left(\rho a_\rho + \frac{Tp_T}{\rho c_v}a_T - a - \frac{2}{3}p\right),
\end{aligned} \tag{10}$$

where for any function $f \equiv f(\rho,T)$, we indicate $f_\rho = (\partial f/\partial\rho)_T$ and $f_T = (\partial f/\partial T)_\rho$.

Proof. The first step of the Maxwellian iteration applied to Eq. $(3)_4$, taking into account (8), provides

$$\frac{\partial}{\partial t}\left(\rho v_i v_j + a\delta_{ij}\right) + \frac{\partial}{\partial x_k}\left(\rho v_i v_j v_k + a\delta_{ij}v_k + 2pv_{(i}\delta_{j)k}\right) = a_3\Pi\delta_{ij} + a_2\sigma_{\langle ij\rangle},$$

which can be rewritten as:

$$\dot{a}\delta_{ij} + 2\rho v_{(i}\dot{v}_{j)} + a\delta_{ij}\frac{\partial v_l}{\partial x_l} + 2p\frac{\partial v_{(i}}{\partial x_{j)}} + 2v_{(i}\frac{\partial p}{\partial x_{j)}} = a_3\Pi\delta_{ij} + a_2\sigma_{\langle ij\rangle}, \tag{11}$$

where the dot indicates the material derivative $\dot{f} = \frac{\partial f}{\partial t} + v_l\frac{\partial f}{\partial x_l}$ and parentheses around a set of indices represent the symmetrization with respect to those indices. The trace part of Eq. (11) provides:

$$\dot{a} + \frac{2}{3}\rho v_l\dot{v}_l + a\frac{\partial v_l}{\partial x_l} + \frac{2}{3}p\frac{\partial v_l}{\partial x_l} + \frac{2}{3}v_l\frac{\partial p}{\partial x_l} = a_3\Pi, \tag{12}$$

and its traceless part is:

$$2\rho v_{\langle i}\dot{v}_{j\rangle} + 2p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + 2v_{\langle i}\frac{\partial p}{\partial x_{j\rangle}} = a_2\sigma_{\langle ij\rangle}. \tag{13}$$

Since we are now considering the first iteration of the procedure, we need to insert in Eq. (12) and in Eq. (13) the expressions for $\dot{\rho}$, $\dot{v}_i$, $\dot{\varepsilon}$ obtained from Eq. (1) at equilibrium, namely:

$$\dot{\rho} = -\rho\frac{\partial v_l}{\partial x_l}, \qquad \dot{v}_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \qquad \dot{\varepsilon} = -\frac{p}{\rho}\frac{\partial v_l}{\partial x_l} \;\rightarrow\; \dot{T} = -\frac{Tp_T}{\rho c_v}\frac{\partial v_l}{\partial x_l}, \tag{14}$$

where $c_v = \varepsilon_T$ indicates the specific heat at constant volume. Since $\dot{a} = a_\rho\dot{\rho} + a_T\dot{T}$, we find:

$$\dot{a} = -\left(\rho a_\rho + \frac{Tp_T}{\rho c_v}a_T\right)\frac{\partial v_l}{\partial x_l}.$$

Eq. (12) is then written as:

$$\Pi = -\frac{1}{a_3}\left(\rho a_\rho + \frac{Tp_T}{\rho c_v}a_T - a - \frac{2}{3}p\right)\frac{\partial v_l}{\partial x_l}. \tag{15}$$

By comparing Eq. (15) to the equation for Π in the NSF theory, namely $\Pi = -\nu\partial v_l/\partial x_l$, we obtain the coefficient a_3 given in $(10)_3$.

Eq. (13) can be written as:

$$2p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = a_2\sigma_{\langle ij\rangle}. \tag{16}$$

By comparing Eq. (16) to the expression for $\sigma_{\langle ij\rangle}$ provided by the NSF theory, namely, $\sigma_{\langle ij\rangle} = 2\mu\partial v_{\langle i}/\partial x_{j\rangle}$, we find the coefficient a_2 given in $(10)_2$.

The first step of the Maxwellian iteration on the last equation of the system (3), taking into account Eq. (8), takes the explicit form:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\rho v^2 v_i + 2\rho\varepsilon v_i + 2av_i\right)\\ &\quad+\frac{\partial}{\partial x_k}\left(\rho v^2 v_i v_k + 2\rho\varepsilon v_i v_k + 2av_i v_k + 3pv_{(l}v_l\delta_{i)k} + b\delta_{ik}\right) =\\ &\qquad 2a_3\Pi v_i + 2a_2\sigma_{\langle li\rangle}v_l + a_1 q_i,\end{aligned}$$

which, after some manipulation with the use of (11), can be written as

$$\begin{aligned}&\left(-\rho v^2 + 2\rho\varepsilon + 2a\right)\dot{v}_i + 2\rho v_i\dot{\varepsilon} - 2\frac{a}{\rho}v_i\dot{\rho} + 2pv_i\frac{\partial v_l}{\partial x_l} - v^2\frac{\partial p}{\partial x_i}\\ &\quad+\frac{\partial b}{\partial x_i} - 2av_i\frac{\partial v_l}{\partial x_l} = a_1 q_i.\end{aligned}$$

Making use of Eq. (14), we write

$$-2\left(\varepsilon + \frac{a}{\rho}\right)\frac{\partial p}{\partial x_i} + \frac{\partial b}{\partial x_i} = a_1 q_i.$$

Then, we obtain:

$$q_i = \frac{1}{a_1}\left\{\left[b_\rho - 2\left(\varepsilon + \frac{a}{\rho}\right)p_\rho\right]\frac{\partial\rho}{\partial x_i} + \left[b_T - 2\left(\varepsilon + \frac{a}{\rho}\right)p_T\right]\frac{\partial T}{\partial x_i}\right\}. \tag{17}$$

By comparing Eq. (17) with the expression for q_i provided by the NSF theory, namely $q_i = -\kappa\partial T/\partial x_i$, we obtain

$$b_\rho - 2\left(\varepsilon + \frac{a}{\rho}\right)p_\rho = 0, \qquad \frac{1}{a_1}\left\{b_T - 2\left(\varepsilon + \frac{a}{\rho}\right)p_T\right\} = -\kappa.$$

The first relation above represents the compatibility condition (9), while the second one directly provides the expression of a_1 given in the $(10)_1$. □

Therefore, if we assign the thermal and caloric equations of state $p \equiv p(\rho,T)$, $\varepsilon \equiv \varepsilon(\rho,T)$ and we know the phenomenological coefficients κ, μ, ν as functions of ρ and T, then thanks to the previous theorem only one scalar function $b \equiv b(\rho,T)$ is arbitrary, being the remaining four a, a_1, a_2 and a_3 given by (9) and (10).

4. Entropy principle and symmetric form

In this section, we show that the compatibility condition (9) is automatically satisfied under the general principles of RET, i.e. the Galilean invariance of the balance laws and the entropy principle.

First, we recall that a system of balance laws (4) is compatible with an entropy principle if any classical solution of (4) also satisfies the supplementary entropy law:

$$\frac{\partial h^0}{\partial t} + \frac{\partial h^i}{\partial x_i} = \Sigma \geq 0. \tag{18}$$

Under the thermodynamic stability condition that h^0 is a convex function with respect the field of densities $\mathbf{u} \equiv \mathbf{F}^0$, Ruggeri and Strumia, starting from the papers by Godunov [9] and Boillat [10], proved the following theorem [11]:

Theorem 2 (*Main Field and Symmetric Form*). *The compatibility between the system of balance laws (4) and the supplementary balance law (18) with h^0 being a convex function of $\mathbf{u} \equiv \mathbf{F}^0$ implies the existence of the main field $\mathbf{u}'$ that satisfies*

$$dh^\alpha = \mathbf{u}'\cdot d\mathbf{F}^\alpha, \qquad \Sigma = \mathbf{u}'\cdot\mathbf{f} \geq 0, \qquad \alpha = 0,1,2,3. \tag{19}$$

If we introduce the four-potential defined by

$$h'^\alpha = \mathbf{u}'\cdot\mathbf{F}^\alpha - h^\alpha, \tag{20}$$

then $(19)_1$ can be rewritten as

$$dh'^\alpha = \mathbf{F}^\alpha\cdot d\mathbf{u}'.$$

Choosing the components of $\mathbf{u}'$ as field variables, we have

$$\mathbf{F}^\alpha = \frac{\partial h'^\alpha}{\partial\mathbf{u}'},$$

and then the original system (4) with $x^0 = t$ can be rewritten in a symmetric form with Hessian matrices:

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial h'^\alpha}{\partial\mathbf{u}'}\right) = \mathbf{f} \quad\Longleftrightarrow\quad \frac{\partial^2 h'^\alpha}{\partial\mathbf{u}'\partial\mathbf{u}'}\frac{\partial\mathbf{u}'}{\partial x^\alpha} = \mathbf{f}. \tag{21}$$

From (20) h'^0 is the Legendre transform of h^0 and therefore is a convex function of $\mathbf{u}'$ and therefore the Hessian matrix with $\alpha = 0$ in (21) is positive definite. As a consequence, the system is symmetric hyperbolic according to the Friedrichs definition.

In [7], it was proved that the Galilean invariance implies the following condition:

$$\mathbf{u}' = \hat{\mathbf{u}}'\mathbf{X}^{-1}(\mathbf{v}) = \hat{\mathbf{u}}'\mathbf{X}(-\mathbf{v}). \tag{22}$$

The expression (19) with the Galilean theorem (5) and (22) are equivalent to (see for the details [6,7]):

$$d\hat{h}^0 = \hat{\mathbf{u}}'\cdot d\hat{\mathbf{F}}^0, \qquad d\hat{h}^i = \hat{\mathbf{u}}'\cdot\, d\hat{\mathbf{F}}^i, \qquad \Sigma = \hat{\Sigma} = \hat{\mathbf{u}}'\cdot\hat{\mathbf{f}} \geq 0, \tag{23}$$

with constraints

$$\hat{\mathbf{u}}'\mathbf{A}^r\hat{\mathbf{F}}^0 = 0, \qquad \hat{\mathbf{u}}'\mathbf{A}^r\hat{\mathbf{F}}^j = -\hat{h}'^0\delta^{rj}, \tag{24}$$

where

$$\mathbf{A}^r = \left(\frac{\partial \mathbf{X}}{\partial v_r}\right)_{\mathbf{v}=0}.$$

Theorem 3. *If the general system (3) satisfies the entropy law (18) with convex entropy and it is Galilean invariant, then the compatibility condition (9) is automatically satisfied and therefore the system converges in the first step of the Maxwellian iteration to the NSF system.*

Proof. In the present case, denoting the main field as

$$\mathbf{u}' \equiv \left(\lambda, \lambda_i, \mu, \Lambda_{ij}, \zeta_i\right),$$

Eq. (23) is rewritten as

$$\begin{aligned} d\hat{h}^0 &= \hat{\lambda} d\rho + \hat{\mu} d\hat{G}_{ll} + \hat{\Lambda}_{ij} d\hat{H}_{ij} + \hat{\xi}_i d\hat{I}_{lli}, \\ d\hat{h}_k &= \hat{\lambda}_i d\hat{F}_{ik} + \hat{\mu} d\hat{G}_{llk} + \hat{\Lambda}_{ij} d\hat{H}_{ijk} + \hat{\xi}_i d\hat{I}_{llik}, \\ \Sigma &= \hat{\Lambda}_{ij}\hat{P}_{ij} + \hat{\xi}_i \hat{Q}_{lli}. \end{aligned} \tag{25}$$

From Eq. $(24)_1$, we obtain

$$\hat{\lambda}_i = -\frac{1}{\rho}\left(\hat{G}_{ll}\hat{\xi}_i + 2\hat{H}_{il}\hat{\xi}_l\right). \tag{26}$$

We map the velocity independent 11 fields $\hat{\mathbf{F}}^0 \equiv \{\rho, 0, \hat{G}_{ll}, \hat{H}_{ij}, \hat{I}_{lli}\}$ into the 11 independent components of the main field evaluated at zero velocity $\tilde{\mathbf{u}}' \equiv \{\hat{\lambda}, 0, \hat{\mu}, \hat{\Lambda}_{ij}, \hat{\xi}_i\}$, excluding $\hat{\lambda}_i$, which depends on the other components due to the constraint (26).[2]

Introducing the following 4-potentials

$$\begin{aligned} \tilde{h}'^0 &= \tilde{\mathbf{u}}' \cdot \hat{\mathbf{F}}^0 - \hat{h}^0, \\ \tilde{h}'_i &= \tilde{\mathbf{u}}' \cdot \hat{\mathbf{F}}_i - \hat{h}_i, \end{aligned}$$

Eq. (23) provides

$$d\tilde{h}'^0 = \hat{\mathbf{F}}^0 \cdot d\tilde{\mathbf{u}}', \qquad d\tilde{h}'_i = \hat{\mathbf{F}}_i \cdot d\tilde{\mathbf{u}}' + \left(\tilde{\mathbf{u}}' - \hat{\mathbf{u}}'\right) d\hat{\mathbf{F}}_i(\tilde{\mathbf{u}}').$$

Specifically, we have

$$\begin{aligned} d\tilde{h}'^0 &= \rho d\hat{\lambda} + \hat{G}_{ll} d\hat{\mu} + \hat{H}_{ij} d\hat{\Lambda}_{ij} + \hat{I}_{lli} d\hat{\xi}_i, \\ d\tilde{h}'_k &= -\hat{\lambda}_i d\hat{F}_{ik} + \hat{G}_{llk} d\hat{\mu} + \hat{H}_{ijk} d\hat{\Lambda}_{ij} + \hat{I}_{llik} d\hat{\xi}_i, \end{aligned} \tag{27}$$

where $\hat{F}_{ik}$ is a function of $\tilde{\mathbf{u}}'$.

Equilibrium is a process in which the productions P_{ij} and Q_{lli} vanish. The entropy production $(25)_3$ becomes minimum and vanishes at equilibrium, then we obtain the necessary conditions that at equilibrium the main field components relative to the balance laws with production term vanish [6]:

$$\hat{\Lambda}_{ij} = 0, \qquad \hat{\xi}_i = 0.$$

This indicates that $\hat{\Lambda}_{ij}$ and $\hat{\xi}_i$ are the fields that characterize the non-equilibrium state. The only components that are not zero are $\hat{\lambda}$ and $\hat{\mu}$. In particular, at equilibrium [6]:

$$\hat{\lambda} = -\frac{g}{T}, \qquad \hat{\mu} = \frac{1}{2T}. \tag{28}$$

Therefore, we adopt $\hat{\lambda}$ and $\hat{\mu}$ as the fields that characterize an equilibrium state.

As we are interested in the tensors at equilibrium, it is enough to take the potentials at first order in the non-equilibrium main field variables, then

$$\tilde{h}'_k = \beta\hat{\zeta}_k + \mathcal{O}(2), \tag{29}$$

where β is a function of $\hat{\lambda}$ and $\hat{\mu}$. From (26) near an equilibrium state with recalling $\hat{H}_{li} = a\delta_{li} + \mathcal{O}(1)$, we obtain

$$\hat{\lambda}_i = -2\hat{\zeta}_i\left(\varepsilon + \frac{a}{\rho}\right) + \mathcal{O}(2),$$

and from $(27)_2$ we have the following two relations:

$$\frac{\partial \tilde{h}'_k}{\partial \hat{\lambda}} = -\hat{\lambda}_i \frac{\partial \hat{F}_{ik}}{\partial \hat{\lambda}} = -\hat{\lambda}_k \frac{\partial\left(p + \mathcal{O}(1)\right)}{\partial \hat{\lambda}} = 2\left(\varepsilon + \frac{a}{\rho}\right)\frac{\partial p}{\partial \hat{\lambda}}\hat{\zeta}_k + \mathcal{O}(2), \tag{30}$$

$$\frac{\partial \tilde{h}'_k}{\partial \hat{\zeta}_i} = \hat{I}_{llik} - \hat{\lambda}_l \frac{\partial \hat{F}_{lk}}{\partial \hat{\zeta}_i} = b\delta_{ik} + \mathcal{O}(1). \tag{31}$$

By putting (29) onto the left hand side of (31), we obtain

$$\beta = b,$$

and from (30) we obtain

$$\frac{\partial b}{\partial \hat{\lambda}} = 2\left(\varepsilon + \frac{a}{\rho}\right)\frac{\partial p}{\partial \hat{\lambda}}. \tag{32}$$

By changing the equilibrium field from $\{\hat{\lambda}, \hat{\mu}\}$ given by (28) to $\{\rho, T\}$, we can identify the relation (32) with (9). □

5. Examples of RET of monatomic and polyatomic rarefied gases

As examples of the application of Theorem 1, we present the case of monatomic and polyatomic rarefied gases modeled by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} = Q(f), \tag{33}$$

where f is the distribution function, ξ_i is the microscopic velocity and $Q(f)$ is the collision term.

5.1. Rarefied monatomic gases

The thermodynamic fields are identified as the moments of the distribution function

$$F_{i_1 i_2 \cdots i_k} = \int_{\mathbb{R}^3} m\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k} f\, d\boldsymbol{\xi},$$

where m is the mass of a molecule. In this case, the state of the gas is described by the velocity distribution function $f \equiv f(\mathbf{x}, \boldsymbol{\xi}, t)$, where $f(\mathbf{x}, \boldsymbol{\xi}, t)\, d\mathbf{x}\, d\boldsymbol{\xi}$ represents the number density of molecules at time t and in the volume element $d\mathbf{x}\, d\boldsymbol{\xi}$ of the phase space ($6D$ position-velocity space) centered at $(\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^3 \times \mathbb{R}^3$. Since the dynamic pressure vanishes in rarefied monatomic gases, we identify H_{ij} with F_{ij} and the trace part of the momentum flux F_{ll} with the energy density G_{ll}. Moreover, we identify I_{lli} with the energy flux $G_{lli} = F_{lli}$. Then, we have $H_{ijk} = F_{ijk}$ and $I_{llik} = F_{llik}$. Therefore, the independent fields are now the 13 fields F, F_i, F_{ij}, and F_{lli}.

The coefficients a and b are related to the equilibrium part of the moments as follows:

$$\begin{aligned} \hat{F}^E_{ij} &= \int_{\mathbb{R}^3} mC_i C_j f^M d\mathbf{C} = a\delta_{ik}, \\ \hat{F}^E_{llik} &= \int_{\mathbb{R}^3} mC^2 C_i C_k f^M d\mathbf{C} = b\delta_{ik}, \end{aligned}$$

where $C_i = \xi_i - v_i$ is the peculiar velocity ($C^2 = C_i C_i$) and f^M is the Maxwellian distribution function:

$$f^M = \frac{\rho}{m}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mC^2}{2k_B T}\right),$$

being k_B the Boltzmann constant. Then, from the integral of the moments, we obtain

$$a = p, \qquad b = 5\left(\frac{k_B}{m}\right)^2 \rho T^2,$$

with the thermal and caloric equations of state

$$p = \frac{k_B}{m}\rho T, \qquad \varepsilon = \frac{3}{2}\frac{k_B}{m}T.$$

It is easily verified that the compatibility condition (9) is satisfied.

[2] An alternative and more sophisticated method to deal with the problem that the main field evaluated for zero velocity is actually not a field was presented by Pennisi and Ruggeri in [12].

From (35), we obtain the expressions of the expansion coefficients of the production terms (10) as follows:

$$a_1 = -5\left(\frac{k_B}{m}\right)^2 \frac{p}{\kappa}, \qquad a_2 = \frac{p}{\mu}, \qquad \nu a_3 = 0. \tag{34}$$

These expressions are consistent with the phenomenological RET theory and the molecular RET theory with BGK-type collisional terms [4]. The $(34)_3$ reflects that the monatomic case is a singular limit and there is no dynamic pressure in a rarefied monatomic gas. The singular limit from the theory of polyatomic gases to monatomic gases has been studied in [13] (see also [6]).

5.2. Rarefied polyatomic gases

For polyatomic gases, due to the existence of the molecular internal modes such as the rotational and vibrational modes of a molecule, the Borgnakke–Larsen model [14] has been proposed, and the kinetic model presented in [15] has been developed in order to include molecular relaxation processes. Based on the Borgnakke–Larsen model, the velocity distribution function f is assumed to depend also on a non-negative microscopic energy parameter I, i.e., $f \equiv f(\mathbf{x}, \boldsymbol{\xi}, t, I)$. It is also assumed that the evolution of the distribution function f is described by the Boltzmann Eq. (33) and the influence of internal degrees of freedom through the collision cross-section is taken into account through the collisional term $Q(f)$.

Since for polyatomic gases the dynamic pressure is in general non-vanishing, and therefore F_{ll} is different from G_{ll}, the fields are now F, F_i, F_{ij}, G_{ll}, G_{lli}, which are the moments of the distribution function defined as follows:

$$\begin{pmatrix} F \\ F_i \\ F_{ij} \\ G_{ll} \\ G_{lli} \end{pmatrix} = \int_{\mathbb{R}^3}\int_0^\infty m \begin{pmatrix} 1 \\ \xi_i \\ \xi_i\xi_j \\ \xi^2 + \frac{2I}{m} \\ \xi_i\left(\xi^2 + \frac{2I}{m}\right) \end{pmatrix} f\,\phi(I)\,dI\,d\boldsymbol{\xi},$$

where $\phi(I)$ is the state density corresponding to I, i.e., $\phi(I)\,dI$ represents the number of internal state between I and $I + dI$.

The coefficients a and b are related to the equilibrium part of the moments as follows:

$$\hat{F}^E_{ij} = \int_{\mathbb{R}^3}\int_0^\infty mC_iC_j f^E \phi(I) dI d\mathbf{C} = a\delta_{ik},$$

$$\hat{G}^E_{llik} = \int_{\mathbb{R}^3}\int_0^\infty mC_iC_k\left(C^2 + \frac{2I}{m}\right) f^E \phi(I)\,dI\,d\mathbf{C} = b\delta_{ik},$$

where f^E is the equilibrium distribution function [6]:

$$f^E = f^M f^I, \quad \text{with} \quad f^I = \frac{1}{A(T)}\exp\left(-\frac{I}{k_BT}\right),$$

being $A(T)$ a normalization factor (partition function):

$$A(T) = \int_0^\infty \phi(I) \mathrm{e}^{-\frac{I}{k_BT}}\,\mathrm{d}I.$$

From the integral of the moments, we obtain

$$a = p, \qquad b = 2p\left(\varepsilon + \frac{k_B}{m}T\right), \tag{35}$$

with the thermal and caloric equations of state

$$p = \frac{k_B}{m}\rho T, \qquad \varepsilon = \varepsilon(T).$$

Consequently, it is easy to find that the compatibility condition (9) is satisfied.

From (35), we obtain the expressions of the expansion coefficients of the production terms (10) as follows:

$$a_1 = -\frac{2}{\kappa}\frac{k_B}{m}p\left(c_v + \frac{k_B}{m}\right), \qquad a_2 = \frac{p}{\mu}, \qquad a_3 = -\frac{p}{\nu}\frac{2c_v - 3\frac{k_B}{m}}{3c_v}.$$

These expressions are consistent with the phenomenological RET theory [3] and the molecular RET theory with BGK-type collisional terms [16] and in the monatomic limit as $c_v = 3k_B/(2\,m)$ reduce to (34).

6. Conclusions

Hyperbolic theories of balance laws for generic – non-necessarily rarefied – gases with 14 fields in the form of Eq. (3), compatible with the universal principles of Rational Extended Thermodynamics, involve tensors at equilibrium and production terms such that the system at the first step of the Maxwellian iteration coincides with the Navier–Stokes–Fourier system.

The present study sheds light on the behavior of RET theories in their parabolic limit. It may be helpful in the construction of new causal theories valid also in cases in which the kinetic theory is not applicable as, for example, the case of dense gases.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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