Online Appendix to
Only the ugly face? A theoretical model of brand dilution

Emanuele BACCHIEGA*[1], Mariachiara COLUCCI†[2], Vincenzo DENICOLÒ‡[3], and Marco MAGNANI§[3]

[1]Dipartimento di Informatica - Scienza e Ingegneria, Alma Mater Studiorum - Università di Bologna, Italy.
[2]Dipartimento di Scienze Aziendali, Alma Mater Studiorum - Università di Bologna, Italy.
[3]Dipartimento di Scienze Economiche, Alma Mater Studiorum - Università di Bologna, Italy and Centre for Economic Policy Research, CEPR, United Kingdom.
[4]Italian Regulatory Authority for Energy, Networks and the Environment (ARERA).

May 26, 2023

In this appendix we develop in detail the extensions mentioned in Section 7 of the paper.

1 Endogenous quality benchmark

Here we assume that the willingness to pay for the core product is

$$v_C = e_C + \lambda(1-\alpha)\left[e_E - (\varepsilon + \gamma e_C)\right], \tag{1}$$

where the parameter $\gamma > 0$ captures the possibility that the reference level of quality for the extension product depends on the brand owner's investment in the brand.

1.1 In-House

The profit of the brand owner is

$$\pi_B = \mu\left\{e_C + \lambda(1-\alpha)\left[e_E - (\varepsilon + \gamma e_C)\right]\right\} + \rho\frac{\left[e_E + (1-\alpha)e_C\right]^2}{4} - \frac{1}{2}\theta\beta e_E^2 - \frac{1}{2}\beta e_C^2, \tag{2}$$

so the optimal effort levels are:

$$e_C^H = \frac{\mu\{2\theta[1-(1-\alpha)\gamma\lambda] + (1-\alpha)\lambda\rho(1-\alpha+\gamma) - \rho\}}{\theta\left[2-(1-\alpha)^2\rho\right] - \rho} \tag{3}$$

*emanuele.bacchiega@unibo.it
†Corresponding author, mariachiara.colucci@unibo.it
‡vincenzo.denicolo@unibo.it
§mmagnani@arera.it

and

$$e_E^H = \frac{(1-\alpha)\mu\left\{\lambda[2-(1-\alpha)\rho(1-\alpha+\gamma)]+\rho\right\}}{\theta\left[2-(1-\alpha)^2\rho\right]-\rho}. \tag{4}$$

The equilibrium profit then is

$$\pi_B^H = \frac{\mu^2\left\{2(1-\alpha)^2\lambda^2+2\theta[1-(1-\alpha)\gamma\lambda]^2-\rho[1-(1-\alpha)\lambda(1-\alpha+\gamma)]^2\right\}}{2\left\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\right\}} - (1-\alpha)\lambda\mu\epsilon \tag{5}$$

1.2 Licensing

The profit of the brand owner now is

$$\pi_B = \mu\left\{e_C+\lambda(1-\alpha)\left[e_E-(\varepsilon+\gamma e_C)\right]\right\} + s\rho\frac{(1-\alpha)e_C+e_E-s}{2} - \frac{1}{2}e_C^2, \tag{6}$$

whereas that of the licensee does not change. Therefore, the optimal effort chosen by the licensee is still (12). Plugging it into the expression for the brand owner's profit, it is easy to see that the optimal royalty rate is still (13). Thus, the brand owner's profit becomes

$$\pi_B = \frac{2e_C\mu\{(1-\alpha)\lambda[-\alpha\rho-2\gamma(2-\rho)+\rho]+2(2-\rho)\}-(1-\alpha)\lambda\mu[4(2-\rho)\varepsilon-(1-\alpha)\lambda\mu\rho]-e_C^2\{4-[3-(2-\alpha)\alpha]\rho\}}{4(2-\rho)}, \tag{7}$$

Maximizing this expression with respect to e_C one obtains:

$$e_C^L = \frac{\mu\{4-(1-\alpha)\lambda[2\gamma(2-\rho)-(1-\alpha)\rho]-2\rho\}}{4-[3-(2-\alpha)\alpha]\rho}. \tag{8}$$

The corresponding optimal royalty rate is

$$s^L = (1-\alpha)\mu\frac{2-\rho+\{[2-(2-\alpha)\alpha]\rho-[(1-\alpha)(2-\rho)\gamma]\}\lambda}{4-[3-(2-\alpha)\alpha]\rho}. \tag{9}$$

The effort level of the licensee then becomes

$$e_E^L = \frac{(1-\alpha)\mu\rho[1+\lambda-(1-\alpha)\gamma\lambda]}{4-[3-(2-\alpha)\alpha]\rho}, \tag{10}$$

which plugged into (7) yields:

$$\pi_B^L = \frac{\mu\Big\{\mu\rho\{(1-\alpha)\lambda\{(1-\alpha)\lambda[1-2\gamma(1+\gamma-\alpha)]+2(1+2\gamma-\alpha)\}+2\}+4\mu[1-(1-\alpha)\gamma\lambda]^2-2(1-\alpha)\lambda\epsilon\{4-[3-(2-\alpha)\alpha]\rho\}\Big\}}{2\{4-[3-(2-\alpha)\alpha]\rho\}}. \tag{11}$$

1.3 Results

Proposition 1. *Under both in-house development and licensing, the equilibrium exhibits under-investment in the quality of both products.*

Proof. The first best efforts coincide with those under in-house development when $\theta = 1$. The first part of the proposition follows from the fact that the derivatives

$$\frac{\partial e_C^H}{\partial \theta} = -\frac{(1-\alpha)^2\mu\rho\{\lambda[2-(1-\alpha)\rho(1+\gamma-\alpha)]+\rho\}}{\{\theta[2-(1-\alpha)^2\rho]-\rho\}^2}, \quad \frac{\partial e_E^H}{\partial \theta} = -\frac{(1-\alpha)\mu\left[2-(1-\alpha)^2\rho\right]\{\lambda[2-(1-\alpha)\rho(1+\gamma-\alpha)]+\rho\}}{\{\theta[2-(1-\alpha)^2\rho]-\rho\}^2}, \tag{12}$$

are both negative. As for the second part, observe that the difference between the first-best effort and the effort under licensing on the core product is positive

$$e_C^{FB} - e_C^L = \frac{(1-\alpha)^2\mu(2-\rho)\rho[1+\lambda-(1-\alpha)\gamma\lambda]}{\{2-[2-(2-\alpha)\alpha]\rho\}\{4-[3-(2-\alpha)\alpha]\rho\}} > 0. \tag{13}$$

The difference between the efforts on the extension is

$$e_E^{FB} - e_E^L = \frac{(1-\alpha)\mu\left\{\lambda\left\{\left\{\rho^2\{5-\alpha\{10-\alpha[9-(4-\alpha)\alpha]-\gamma\}+\gamma\}\right\}-2\rho[6-\alpha(6-3\alpha+\gamma)+\gamma]+8\right\}+(2-\rho)\rho\right\}}{\{2-[2-(2-\alpha)\alpha]\rho\}\{4-[3-(2-\alpha)\alpha]\rho\}} > 0. \tag{14}$$

Clearly, the denominator is positive; tedious algebra confirms that the numerator is positive as well. □

Proposition 2. *At the extensive margin, if licensing is the best organizational mode (i.e., if $\pi_B^L = \pi_B^{NE} \geq \pi_B^H$), then brand extension necessarily entails brand dilution.*

Proof. The equation $\pi_B^L - \frac{\mu^2}{2} = 0$ has two solutions in μ, namely $\mu = 0$ and

$$\mu = \frac{2\lambda\epsilon\{4-[3-(2-\alpha)\alpha]\rho\}}{2(1-\alpha)\gamma^2\lambda^2(2-\rho) - 2\gamma\lambda\left\{4-\rho\left[2-(1-\alpha)^2\lambda\right]\right\} + (1-\alpha)(\lambda+1)^2\rho} = \mu^E. \tag{15}$$

By evaluating the difference between the licensee's effort and the threshold $\varepsilon + \gamma e_C^L$ one obtains

$$\left(e_E^L - \varepsilon - \gamma e_C^L\right)\Big|_{\mu=\mu_E} = \frac{(1-\alpha)\varepsilon\left\{\lambda^2\{\rho+2\gamma[\alpha\rho+\gamma(2-\rho)-\rho]\}-\rho\right\}}{2(1-\alpha)\gamma^2\lambda^2(2-\rho) - 2\gamma\lambda\left\{4-\rho\left\{2-(1-\alpha)^2\lambda\right\}\right\} + (1-\alpha)(\lambda+1)^2\rho}. \tag{16}$$

To verify that this expression is negative we proceed as follows. First, notice that both optimal efforts are positive and decreasing in γ:

$$\frac{\partial e_E^L}{\partial \gamma} = -\frac{(1-\alpha)^2\lambda\mu\rho}{4-[3-(2-\alpha)\alpha]\rho} < 0, \quad \frac{\partial e_C^L}{\partial \gamma} = -\frac{2(1-\alpha)\lambda\mu(2-\rho)}{4-[3-(2-\alpha)\alpha]\rho}. \tag{17}$$

Next, observe that for $\gamma = 0$ the difference $e_E^L - \varepsilon - \gamma e_C^L$ reduces to $e_E^L - \varepsilon$, which is negative (see the baseline model). Re-write the condition at $\gamma = 0$ as $e_E^L < \varepsilon$. Then, letting γ increase above zero entails a decrease in e_E^L, whereas ε does not term and a further negative term appears, γe_C^L. □

1.3.1 Internal development: Extensive margin(s)

The analysis parallels that of the baseline model, though the algebra is more cumbersome.

If the extension is developed in-house, the extensive margin is defined by

$$\pi_B^H = \pi_B^{NE}. \tag{18}$$

This equation has two roots in λ, which we label λ_1^{EH} and λ_2^{EH}; the roots are real as long as the discriminant

$$(1-\alpha)^2 \frac{\rho\left\{\mu^2(1+\gamma-\alpha)^2+2\mu\varepsilon(1+\gamma-\alpha)+\epsilon^2\left[(1-\alpha)^2\theta+1\right]\right\}-2\theta(\gamma\mu+\varepsilon)^2}{\rho-\theta\left[2-(1-\alpha)^2\rho\right]} \tag{19}$$

is positive. The denominator is negative under our assumptions, and the numerator is negative too if

$$\theta > \frac{\rho[\mu(1+\gamma-\alpha)+\varepsilon]^2}{2(\gamma\mu+\varepsilon)^2-(1-\alpha)^2\rho\varepsilon^2} \tag{20}$$

As long as the roots are real, $\lambda_1^{EH} < \lambda_2^{EH}$. At the smaller root there is always brand dilution:

$$\left(e_E^H-\varepsilon-\gamma e_C^H\right)\Big|_{\lambda=\lambda_1^{EH}} = -\sqrt{\frac{\rho\left\{\mu^2(1+\gamma-\alpha)^2+2\mu\varepsilon(1+\gamma-\alpha)+\epsilon^2\left[(1-\alpha)^2\theta+1\right]\right\}-2\theta(\gamma\mu+\varepsilon)^2}{\rho-\theta\left[2-(1-\alpha)^2\rho\right]}}. \tag{21}$$

At the larger root, there is always brand enhancement:

$$\left(e_E^H-\varepsilon-\gamma e_C^H\right)\Big|_{\lambda=\lambda_2^{EH}} = \sqrt{\frac{\rho\left\{\mu^2(1+\gamma-\alpha)^2+2\mu\varepsilon(1+\gamma-\alpha)+\epsilon^2\left[(1-\alpha)^2\theta+1\right]\right\}-2\theta(\gamma\mu+\varepsilon)^2}{\rho-\theta\left[2-(1-\alpha)^2\rho\right]}}. \tag{22}$$

In the ensuing results, the cautionary note of the baseline model that some intervals may be empty still applies.

Proposition 3. *At the intensive margin, a switch from internal development to licensing always decreases the quality of the core product e_C and always increases the quality of the extension e_E; therefore, it reduces the likelihood of brand dilution.*

Proof. The solution to equation $\pi_B^H = \pi_B^L$ w.r.t. θ is

$$\theta^I = \frac{\rho^2\left\{\lambda\left[2+\gamma+\alpha^2-\alpha(\gamma+2)\right]-1\right\}^2+2\lambda\rho\{\lambda[\alpha(6-3\alpha+4\gamma)-4\gamma-5]+4\}+8\lambda^2}{\rho\{-(1-\alpha)\lambda\rho\{(1-\alpha)\lambda[1-2\gamma(1-\alpha+\gamma)]-2\alpha+4\gamma+2\}+2\lambda\{(1-\alpha)\gamma\{2-\lambda[2+(1-\alpha)\gamma]\}+\lambda+2\}-2(1-\rho)\}}, \tag{23}$$

Here we restrict our attention to the case where $\theta^I > 1$. If this condition does not hold, in-house development is never a profitable extension mode.

The difference between the efforts on the core product with under in-house and licensing, evaluated at θ^I is

$$\left(e_C^H-e_C^L\right)\Big|_{\theta=\theta^I} = \frac{(1-\alpha)^2\lambda\mu\rho\{2+\lambda\{[2-(2-\alpha)\alpha]\rho-[(1-\alpha)\gamma(2-\rho)]\}-\rho\}}{\{4-[3-(2-\alpha)\alpha]\rho\}\{\lambda[2-(1-\alpha)\rho(1-\alpha+\gamma)]+\rho\}} \tag{24}$$

The denominator is clearly positive and tedious algebra shows that the numerator is positive, too, implying that the difference is positive. In turn, this implies that the threshold for brand enhancement/dilution, $\varepsilon+\gamma e_C$ has a downward jump at the intensive margin.

The difference between the efforts on the extension:

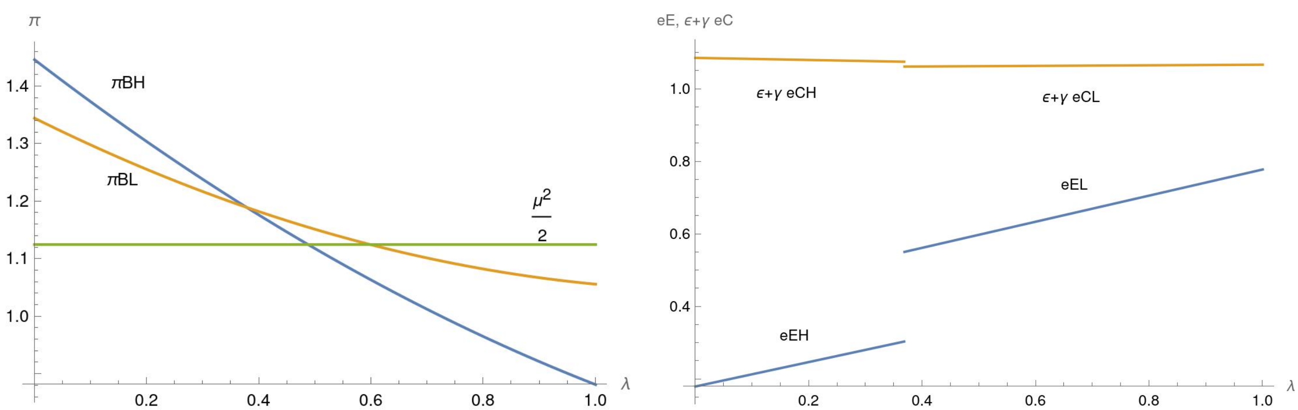
$$\left(e_E^H - e_E^L\right)\Big|_{\theta=\theta^I} = -\frac{(1-\alpha)\mu\rho[1-(1-\alpha)\gamma\lambda]\{2+\lambda\{[2-(2-\alpha)\alpha]\rho - [(1-\alpha)\gamma(2-\rho)]\} - \rho\}}{\{4-[3-(2-\alpha)\alpha]\rho\}\{\lambda[2-(1-\alpha)\rho(1-\alpha+\gamma)]+\rho\}} \tag{25}$$

by contrast, is always negative. This in turn implies that the effort on the extension jumps upward at the intensive margin.

The combination of these two observations delivers the result (see Figures 3 and 4 for a graphical representation). □

Proposition 4. *Under both in-house development and licensing, the equilibrium effort e_E is increasing in λ. Furthermore, if the technological distance θ is large enough, the brand owner develops the extension internally if λ is small, licenses the brand to a specialized licensee for intermediate values of λ, and does not engage in brand extension if λ is large.*

Figure 1 illustrates the Proposition.



(a) Profits.

(b) Efforts on extension.

Figure 1: Profits and effort levels on the extension product as a function of the size of reciprocal effect (λ). Parameter configuration: $\alpha = 0.3, \rho = 0.8, \varepsilon = 0.7, \mu = 1.5, \theta = 3.4, \gamma = 0.1$.

Proof. The derivatives of the efforts on the extension product w.r.t. λ are

$$\frac{\partial e_E^H}{\partial \lambda} = \frac{(1-\alpha)\mu[2-(1-\alpha)\rho(1+\gamma-\alpha)]}{\theta\,[2-(1-\alpha)^2\rho]-\rho} > 0, \quad \frac{\partial e_E^L}{\partial \lambda} = \frac{(1-\alpha)\mu\rho[1-(1-\alpha)\gamma]}{4-[3-(2-\alpha)\alpha]\rho} > 0, \tag{26}$$

so they clearly are both positive in the admissible parameter space.

To prove the second part of the proposition, we start by the intensive margin. The difference between in-house and licensing profits, evaluated at $\lambda = 0$, is:

$$\left(\pi_B^H - \pi_B^L\right)\Big|_{\lambda=0} = \frac{1}{2}\mu^2 \left\{ \frac{(1-\alpha)^2\rho[2\theta(1-\rho)+\rho]}{\{4-[3-(2-\alpha)\alpha]\rho\}\left(\theta\,[2-(1-\alpha)^2\rho-2)-\rho\right]} \right\} > 0. \tag{27}$$

The equation $\pi_B^H = \pi_B^L$ has two roots in λ, that are real if $\rho > \frac{2}{\theta}$. However, only one root lies in the interval $[0, 1]$, provided that the following condition holds:[1]

$$\theta > \frac{\rho^2\left\{\lambda\left[\alpha^2-\alpha(\gamma+2)+\gamma+2\right]-1\right\}^2+2\lambda\rho\{\lambda[\alpha(4\gamma-3\alpha+6)-4\gamma-5]+4\}+8\lambda^2}{\rho\{2\lambda\{(1-\alpha)\gamma\{2-\lambda[2-(1-\alpha)\gamma]\}+\lambda+2\}-2(1-\rho)-(1-\alpha)\lambda\rho\{(1-\alpha)\lambda[2\gamma(\alpha-\gamma-1)+1]-2\alpha+4\gamma+2\}\}}. \tag{28}$$

This root therefore represents the intensive margin; we label it λ^I. We then evaluate the derivative of the difference between the in-house and licensing profits at the intensive margin:

$$\left.\frac{\partial(\pi_B^H - \pi_B^L)}{\partial\lambda}\right|_{\lambda=\lambda^I} = -\mu\sqrt{\frac{(1-\alpha)^4\mu^2\rho(\theta\rho-2)}{\{4-[3-(2-\alpha)\alpha]\,\rho\}\,\{\theta\,[2-(1-\alpha)^2\rho]-\rho\}}} < 0. \tag{29}$$

These observations imply that in-house development is preferred for all $\lambda < \lambda^I$, licensing for $\lambda > \lambda^I$.

Next, consider the extensive margin. The difference $\left.\pi_B^L - \pi_B^{NE}\right|_{\lambda=1}$ is negative if

$$\varepsilon > \frac{\mu[2-(1-\alpha)\gamma][\rho(1-\alpha)-\gamma(2-\rho)]}{4-[3-(2-\alpha)\alpha]\rho}. \tag{30}$$

The equation $\pi_B^L = \pi_B^{NE}$ has two zeros in λ, which are real if the foregoing condition is met. In this case, the only root that lies in the range $[0, 1]$ is

$$\lambda^{EL} = \frac{2(1-\alpha)\gamma\mu(2-\rho)-\{4-[3-(2-\alpha)\alpha]\}(\rho A+(1-\alpha)\varepsilon-(1-\alpha)^2\mu\rho}{(1-\alpha)^2\mu\{\rho+2\gamma[\alpha\rho+\gamma(2-\rho)-\rho]\}}, \tag{31}$$

where $A = \sqrt{\frac{(1-\alpha)^2\{4(\gamma\mu+\epsilon)^2-\rho\{2\gamma\mu^2(1+\gamma-\alpha)+2\mu\varepsilon(1+2\gamma-\alpha)+[3-(2-\alpha)\alpha]\varepsilon^2\}\}}{4-[3-(2-\alpha-2)]\rho}}$. We conclude that to the left of λ_E licensing is preferred to not extending the brand, and the opposite to its right. Accordingly, the extensive margin is $\lambda^E = \min[\lambda_1^{EH}, \lambda^{EL}]$.[2]

Depending on the relative values of λ^I and λ^E, two cases may arise. If $\lambda^E > \lambda^I$, licensing is the optimal extension mode at the extensive margin. This is the case described in the statement of Proposition 4 and depicted in Figure 1. If instead $\lambda^E < \lambda^I$, then the interval where licensing is optimal vanishes. In this case, in-house development is optimal for $\lambda < \lambda^E$ and no licensing for $\lambda > \lambda^E$.

□

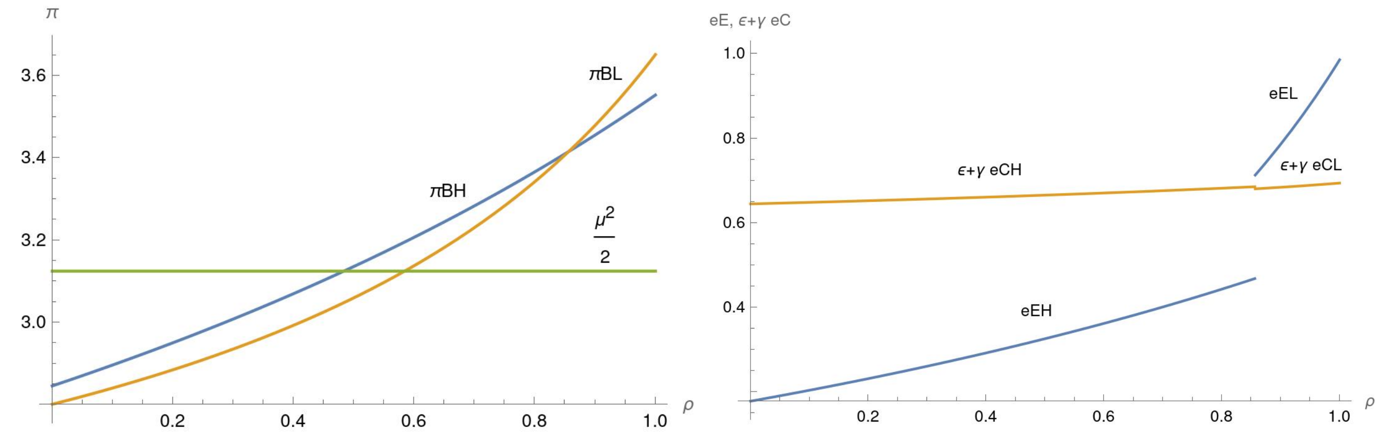
Proposition 5. *Under both in-house development and licensing, the equilibrium effort e_E is increasing in ρ. Furthermore, the brand owner does not engage in brand extension if ρ is small, develops the extension internally for intermediate values of ρ, and licenses the brand to a specialized licensee if ρ is large.*

Figure 2 illustrates the Proposition.

Proof. The first part of the proposition follows from the fact that the derivatives of the efforts on

[1] If this condition is violated, then in house extension is always preferred to licensing.

[2] Similarly to the baseline model, condition (28) guarantees that $\lambda_2^{EH} > 1$.



(a) Profits.

(b) Efforts on extension.

Figure 2: Profits and effort levels on the extension product as a function of the size of extension market (ρ). Parameter configuration: $\alpha = 0.3, \lambda = 0.4, \varepsilon = 0.7, \mu = 3, \theta = 3.4, \gamma = 0.15$.

the extension product w.r.t. ρ are

$$\frac{\partial e_E^H}{\partial \rho} = \frac{2(1-\alpha)\mu[\theta + \lambda - (1-\alpha)\gamma\theta\lambda]}{\{\theta\left[2-(1-\alpha)^2\rho\right] - \rho\}^2} > 0, \quad \frac{\partial e_E^L}{\partial \rho} = \frac{4(1-\alpha)\mu[1+\lambda-(1-\alpha)\gamma\lambda]}{\{4[3-(2-\alpha)\alpha]\rho\}^2} > 0, \tag{32}$$

are positive for all admissible values of the parameters. The proof of the second part follows the same steps as that of Proposition 4. Details are left to the reader. □

Proposition 6. *Under both in-house development and licensing, the equilibrium effort e_E is decreasing in α. Furthermore, the brand owner develops the extension internally if α is small, licenses the brand to a specialized licensee for intermediate values of α, and does not engage in brand extension if α is large.*

Figure 3 illustrates the Proposition.

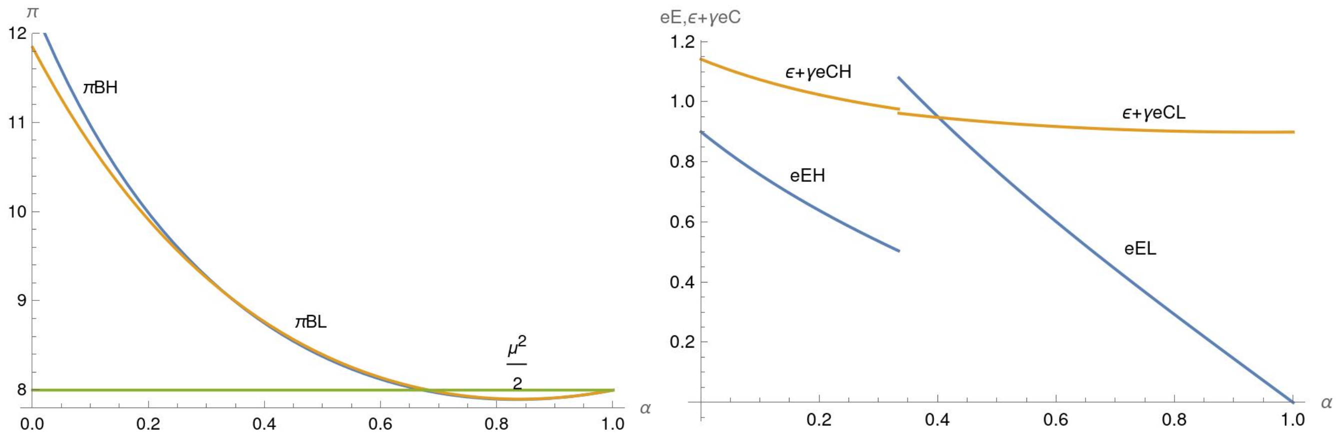
Proof. The first part of the proposition is proved by calculating the derivatives of the efforts on the extension product under the two alternative extension models:

$$\frac{\partial e_E^H}{\partial \alpha} = -\frac{\mu\left\{\left\{\rho^2\left[(1-\alpha)\lambda(3+2\gamma-3\alpha)+(1-\alpha)^4\theta\lambda+(1-\alpha)^2\theta-1\right]\right\}+2\rho\{\theta[1-2(1-\alpha)\lambda(1+\gamma-\alpha)]+\lambda\}+4\theta\lambda\right\}}{\{\theta[2-(1-\alpha)^2\rho]-\rho\}^2} < 0, \tag{33}$$

$$\frac{\partial e_E^L}{\partial \alpha} = -\frac{\mu\rho\{4+\lambda[4-4(1-\alpha)\gamma(2-\rho)-(2-\alpha)\alpha\rho-\rho]-(2-\alpha)\alpha\rho-\rho\}}{\{4-[3-(2-\alpha)\alpha]\rho\}^2} < 0. \tag{34}$$

Tedious algebra confirms that the derivatives are negative. The second part is proved like that of Proposition 4. Details are left to the reader. □

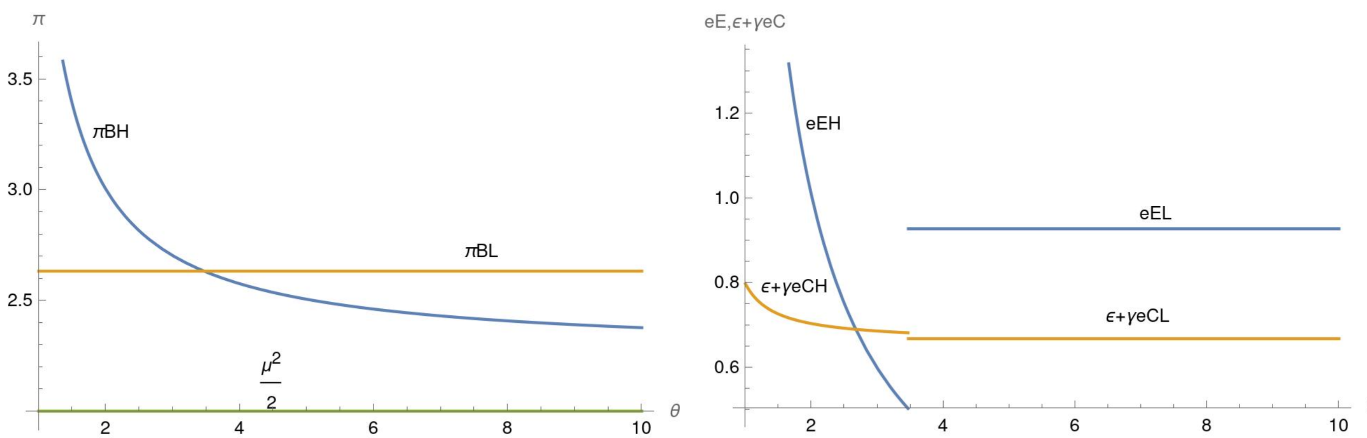
Proposition 7. *Under in-house development, the equilibrium effort e_E is decreasing in θ; under licensing, instead, it is independent of θ. Furthermore, if $\pi_B^L > \pi_B^{NE}$ then the brand owner develops*



(a) Profits. (b) Efforts on extension.

Figure 3: Profits and effort levels on the extension product as a function of the size of the perceived distance (α). Parameter configuration: $\lambda = 0.35, \rho = 0.7, \varepsilon = 0.5, \mu = 4, \theta = 4.4, \gamma = 0.1$.

the extension internally if θ is small and licenses the brand to a specialized licensee for high values of θ.



(a) Profits. (b) Efforts on extension.

Figure 4: Profits and effort levels on the extension product as a function of the technological distance (θ). Parameter configuration: $\alpha = 0.2, \rho = 0.8, \varepsilon = 0.2, \mu = 2, \lambda = 0.4, \gamma = 0.1$.

Proof. The first part follows from the fact that the derivative of the effort of the brand owner w.r.t. θ:

$$\frac{\partial e_E^H}{\partial \theta} = -\frac{(1-\alpha)\mu\left[2-(1-\alpha)^2\rho\right]\{\lambda[2-(1-\alpha)\rho(1+\gamma-\alpha)]+\rho\}}{\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\}^2} < 0, \tag{35}$$

is negative for all admissible values of the parameters. The second part is proved similarly to Proposition 4. Details are left to the reader. □

2 Licensee's outside option

In this section, we assume that there is only one potential licensee that has an outside option the value of which is Ω. Therefore, the licensee will enter into the licensing agreement only if it obtains at least Ω. To make the analysis interesting, we assume that Ω is greater than the profit that the licensee would obtain in the baseline model

$$\Omega > \tfrac{(1-\alpha)^2(1+\lambda)^2\mu^2(1-\rho)\rho}{2\{4-[3-(2-\alpha)\alpha]\rho\}^2}.$$

If this inequality was reversed, the existence of the outside option would not affect the equilibrum.

In any case, the existence of the licensee's outside option does not affect the equilibrium under in-house development. The equilibrium under licensing changes as follows.

The formulas for the profits of the licensee and the brand owner coincide with those in the baseline model, (10) and (14). The expression for the optimal effort of the licensee as a function of the royalty rate (12) does not change. By plugging it back into the licensee's profit, we obtain:

$$\pi_L(s) = \frac{\rho[(1-\alpha)e_C - s]^2}{2(2-\rho)} \tag{36}$$

The brand owner now sets s so as to solve

$$\pi_L(s) = \Omega, \tag{37}$$

which implies:

$$s = (1-\alpha)e_C - \frac{\sqrt{2}(2-\rho)\sqrt{\Omega}}{\sqrt{(2-\rho)\rho}}. \tag{38}$$

This expression replaces equation (13) in the baseline model. Substituting into (14) we obtain

$$\pi_B = \tfrac{e_C\left[\sqrt{2}(1-\alpha)\sqrt{(2-\rho)\rho}\sqrt{\Omega}+\mu(2-\rho)\right]+2\sqrt{2}\lambda\mu\sqrt{(2-\rho)\rho}\sqrt{\Omega}-(8-4\rho)\Omega-2\sqrt{2}\alpha\lambda\mu\sqrt{(2-\rho)\rho}\sqrt{\Omega}-2(1-\alpha)\lambda\mu(2-\rho)\varepsilon-e_C^2(2-\rho)}{2(2-\rho)}. \tag{39}$$

The profit-maximizing effort level therefore is:

$$e_C^L = \mu + \frac{(1-\alpha)\sqrt{\rho\Omega}}{\sqrt{1-\frac{\rho}{2}}} \tag{40}$$

By substituting the optimal effort level back into the expression for the royalty rate we get

$$s^L = (1-\alpha)\mu - \frac{\sqrt{2}\sqrt{\Omega}\{2-[2-(2-\alpha)\alpha]\rho\}}{\sqrt{(2-\rho)\rho)}}. \tag{41}$$

This is positive provided that

$$\Omega < \frac{(1-\alpha)^2\mu^2(2-\rho)\rho}{2\{2-[2-(2-\alpha)\alpha]\rho\}^2}. \tag{42}$$

Plugging back s^L and e_C^L into e_E we get

$$e_E^L = \frac{\sqrt{\rho\Omega}}{\sqrt{1-\frac{\rho}{2}}}. \tag{43}$$

Finally, the brand owner's profit in the licensing equilibrium is

$$\pi_B^L = \frac{2\sqrt{2}(1-\alpha)(\lambda+1)\mu\sqrt{(2-\rho)\rho}\sqrt{\Omega} + 2[3-(3-\alpha)\alpha]\rho\Omega - 2(1-\alpha)\lambda\mu(2-\rho)\epsilon + \mu^2(2-\rho) - 8\Omega}{2(2-\rho)}. \tag{44}$$

Obviously, the profit of the licensee is Ω.

Having derived the equilibrium for this variant of the model, we now verify that the results obtained in the main text continue to hold.

Proposition 1. *Under both in-house development and licensing, the equilibrium exhibits under-investment in the quality of both products.*

Proof. The in-house and first-best efforts are the same as in the baseline model. Thus, the proof of the first part of the proposition coincides with that of the baseline model. As for the second part, let us calculate the difference between the efforts on the core and extension products in the first best and under licensing:

$$e_C^{FB} - e_C^L = \frac{(1-\alpha)\sqrt{\rho}\left\{(1-\alpha)(\lambda+1)\mu\sqrt{(2-\rho)\rho)}\right\} - \sqrt{2\Omega}\{2-[2-(2-\alpha)\alpha]\rho\}}{\sqrt{2-\rho}\{2-[2-(2-\alpha)\alpha]\rho\}} > 0, \tag{45}$$

$$e_E^{FB} - e_E^L = \frac{(1-\alpha)\lambda\mu\sqrt{2-\rho}\left[2-(1-\alpha)^2\rho\right] + (1-\alpha)\mu\rho\sqrt{2-\rho} - \sqrt{2\rho\Omega}\{2-[2-(2-\alpha)\alpha]\rho\}}{\sqrt{2-\rho}\{2-[2-(2-\alpha)\alpha]\rho\}} > 0, \tag{46}$$

Tedious algebra shows that both differences are positivefor all admissible values of the parameters. □

Proposition 2. *If the royalty rate is positive and licensing is the best organizational mode at the extensive margin (i.e., if $\pi_B^L = \pi_B^{NE} \geq \pi_B^H$), then brand extension entails brand dilution.*

Proof. The equation $\pi_B^L - \frac{\mu^2}{2} = 0$ has two solutions in μ, namely $\mu = 0$ and

$$\mu = \frac{\Omega\{4-[3-(2-\alpha)\alpha]\rho\}}{(1-\alpha)\left[\sqrt{2}(\lambda+1)\sqrt{\Omega(2-\rho)\rho)} - \lambda(2-\rho)\epsilon\right]} = \mu^E. \tag{47}$$

By evaluating the difference between the licensee's effort and the threshold $\varepsilon + \gamma e_C^L$ one obtains

$$\left(e_E^L - \varepsilon\right)\Big|_{\mu=\mu_E} = \frac{\sqrt{\Omega\rho}}{\sqrt{1-\frac{\rho}{2}}} - \epsilon. \tag{48}$$

which is positive provided that condition (), which ensures that the royalty rate is positive, hods.

∎

In the following results, the cautionary note of the baseline model that some intervals may be empty still applies.

Proposition 3. *At the intensive margin, a switch from internal development to licensing always increases the quality of the extension e_E; therefore, it reduces the likelihood of brand dilution.*

Proof. The solution to equation $\pi_B^H = \pi_B^L$ w.r.t. θ is

$$\theta^I = \frac{2(1-\alpha)\lambda\mu\rho\left[(1-\alpha)\mu(2-\rho)+\sqrt{2\Omega(2-\rho)\rho}\right]+2(1-\alpha)\mu\rho\sqrt{2\Omega(2-\rho)\rho}-2\rho\Omega\{4-[3-(2-\alpha)\alpha]\rho\}-(1-\alpha)^2\lambda^2\mu^2(2-\rho)\left[(1-\alpha)^2\rho-2\right]}{2(1-\alpha)(\lambda+1)\mu\sqrt{2\Omega(2-\rho)\rho}[2-(1-\alpha)^2\rho]-(1-\alpha)^2\mu^2(2-\rho)\rho-2\Omega[2-(1-\alpha)^2\rho]\{4-[3-(2-\alpha)\alpha]\rho\}}, \tag{49}$$

Here we restrict our attention to the case where $\theta^I > 1$. If this condition does not hold, in-house development is never a profitable extension mode.

The difference between the efforts on the core product with under in-house and licensing, evaluated at θ^I is

$$\left(e_C^H - e_C^L\right)\Big|_{\theta=\theta^I} = \frac{\rho\left\{(1-\alpha)\lambda\mu\left[(1-\alpha)\mu(2-\rho)+2\sqrt{2\Omega(2-\rho)\rho}\right]+2(1-\alpha)\mu\sqrt{2\Omega(2-\rho)\rho}+2[3-(2-\alpha)\alpha]\rho\Omega-8\Omega\right\}}{\mu(2-\rho)\{\lambda[2-(1-\alpha)^2\rho]+\rho\}} - \frac{(1-\alpha)\sqrt{\Omega\rho}}{\sqrt{1-\frac{\rho}{2}}} \tag{50}$$

Here we have to revert to indirect methods to analyze the changes in the effort levels at the intensive margin. The r.h.s. of (50), evaluated at $\Omega = 0$ boils down to $\frac{(1-\alpha)^2\lambda\mu\rho}{\lambda[2-(1-\alpha)^2\rho]+\rho} > 0$ Furthermore, the derivative of (50) relative to the same parameter is $-\frac{(1-\alpha)\sqrt{\rho}}{\sqrt{2\Omega(2-\rho)}} < 0$, and tedious algebra shows that (50), evaluated at $\hat{\Omega}$ is always negative. As a consequence, the switch to licensing reduces the effort on the core product if Ω is small, but increases it if Ω is large.

The difference between the efforts on the extension at the intensive margin is

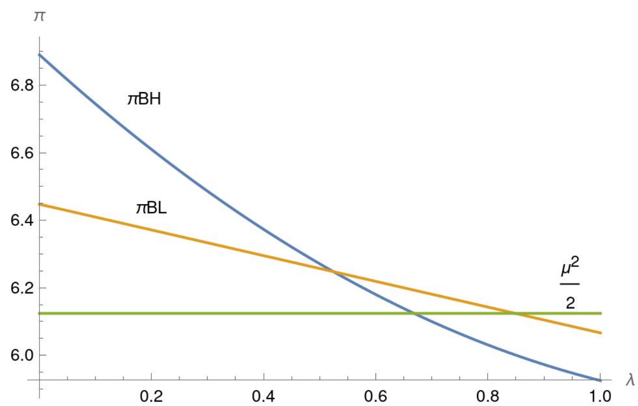
$$\left(e_E^H - e_E^L\right)\Big|_{\theta=\theta^I} = \\ \frac{(1-\alpha)\mu\left\{\lambda\left[2-(1-\alpha)^2\rho\right]+\rho\right\}}{\frac{\left[2-(1-\alpha1)^2\rho\right]\left\{(1-\alpha)^2\lambda^2\mu^2(2-\rho)\left[2-(1-\alpha)^2\rho\right]+2(1-\alpha)\lambda\mu\rho\left[(1-\alpha)\mu(2-\rho)+\sqrt{2\Omega(2-\rho)\rho}\right]+2(1-\alpha)\mu\rho\sqrt{2\Omega(2-\rho)\rho}-2\rho\Omega\{4-[3-(2-\alpha)\alpha]\rho\}\right\}}{2(1-\alpha)(1+\lambda)\mu\left[2-(1-\alpha)^2\rho\right]\sqrt{2\Omega(2-\rho)\rho}-(1-\alpha)^2\mu^2(2-\rho)\rho-2\Omega\left[2-(1-\alpha)^2\rho\right]\{4-[3-(2-\alpha)\alpha]\rho\}}+\rho} + \\ -\frac{\sqrt{\rho\Omega}}{\sqrt{1-\frac{\rho}{2}}}. \tag{51}$$

As before, we evaluate (51) at $\Omega = 0$, which returns $-\frac{(1-\alpha)\mu\rho}{\lambda[2-(1-\alpha)^2\rho]+\rho} < 0$, as well its first-order partial derivative w.r.t. Ω, $-\frac{\sqrt{\rho}}{\sqrt{2\Omega(2-\rho)}} < 0$. Therefore, we conclude that the switch to licensing always increases the effort on the extension product.

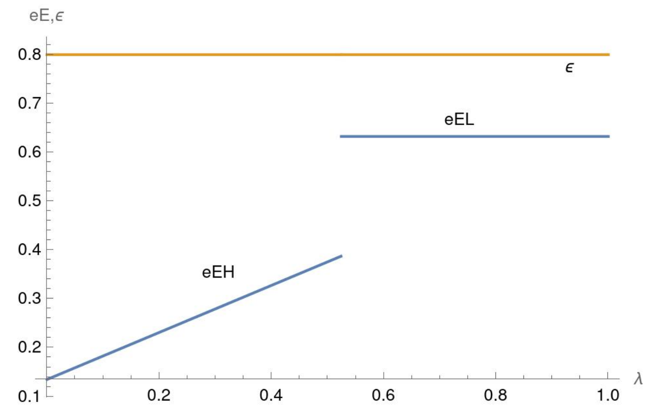
∎

Proposition 4. *Under in-house development the equilibrium effort e_E is increasing in λ, whereas under licensing, it is independent of λ. Furthermore, if the technological distance θ is sufficiently* *large, the brand owner develops the extension internally if λ is small, licenses the brand to a specialized licensee for intermediate values of λ, and does not engage in brand extension if λ is large.*

The following figure illustrates the Proposition.



(a) Profits.



(b) Efforts on extension.

Figure 5: Profits and effort levels on the extension product as a function of the size of reciprocal effect (λ). Parameter configuration: $\alpha = 0.35, \rho = 0.5, \varepsilon = 0.8, \mu = 3.5, \theta = 5, \Omega = 0.6$.

Proof. The first-order partial derivatives of the efforts on the extension product w.r.t. λ are

$$\frac{\partial e_E^H}{\partial \lambda} = \frac{(1-\alpha)\mu\left[2-(1-\alpha)^2\rho\right]}{\theta\left[2-(1-\alpha)^2\rho\right]-\rho} > 0, \quad \frac{\partial e_E^L}{\partial \lambda} = 0. \tag{52}$$

The proof of the second part follows the same logic of that of the baseline model. We start by evaluating the difference in profits at $\lambda = 0$:

$$\left(\pi_B^H - \pi_B^L\right)\Big|_{\lambda=0} = \frac{(1-\alpha)^2\theta\mu^2\rho}{2\left\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\right\}} + \frac{\Omega\{4-[3-(2-\alpha)\alpha]\rho\}}{\rho-2} - \frac{(1-\alpha)\mu\sqrt{\rho\Omega}}{\sqrt{1-\frac{\rho}{2}}}. \tag{53}$$

We observe that for $\Omega = 0$ it boils down to

$$\frac{(1-\alpha)^2\theta\mu^2\rho}{2\left\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\right\}} > 0, \tag{54}$$

which is positive. Furthermore, tedious algebra shows that its first-order derivative w.r.t. Ω:

$$\frac{\{4-[3-(2-\alpha)\alpha]\rho\}-(1-\alpha)\mu(2-\rho)\sqrt{2\rho}-2\sqrt{\Omega(2-\rho)}}{2\sqrt{\Omega(2-\rho)}(2-\rho)} > 0 \tag{55}$$

is positive as well. We then conclude that, for $\lambda = 0$ in-house is preferred to licensing. Next, we observe that the profit difference $\pi_B^H - \pi_B^L$ has only one real root in $[0,1]$, λ^I, provided that $\rho > \frac{2}{\theta}$. This defines the intensive margin, at which the derivative of the difference of the profit functions is negative:

$$\left.\frac{\partial(\pi_B^H - \pi_B^L)}{\partial\lambda}\right|_{\lambda=\lambda^I} =$$
$$= -(1-\alpha)\mu\times$$
$$\times\sqrt{8+\frac{-2\Omega\left\{\rho^2\{4-(2-\alpha)\alpha[4-(2-\alpha)\alpha+\theta]+\theta\}-2\rho[5-3(2-\alpha)\alpha+\theta]\right\}-(1-\alpha)^2\mu^2(2-\rho)\rho+2\sqrt{2}(1-\alpha)\mu\{2-[2-(2-\alpha)\alpha]\rho\}\sqrt{(2-\rho)\rho\Omega}}{(2-\rho)\{\theta[2-(1-\alpha)^2\rho-2]+\rho\}}} < 0, \tag{56}$$

meaning that π_B^L crosses π_B^H from below, which entails that to the left of λ^I in-house is preferred to licensing, and the converse to its right. Similarly to the baseline model, the intensive margin lies in the interval $[0,1]$ if

$$\theta > \frac{(1-\alpha)^2\mu^2(2-\rho)^{3/2}\{2-[1-(2-\alpha)\alpha]\rho\}+4\sqrt{2}(1-\alpha)\mu(2-\rho)\rho^{3/2}\sqrt{\Omega}-2\Omega\rho\sqrt{2-\rho}\{4-[3-(2-\alpha)\alpha]\rho\}}{2\Omega\sqrt{2-\rho}[(1-\alpha)^2\rho-2]\{4-[3-(2-\alpha)\alpha]\rho\}-(1-\alpha)^2\mu^2\rho(2-\rho)^{3/2}-4\sqrt{2}(1-\alpha)\mu(2-\rho)\sqrt{\rho}\sqrt{\Omega}((1-\alpha)^2\rho-2)}. \tag{57}$$

If this condition is not met, in-house development is always preferred to licensing.

Let us now turn to the extensive margin. The difference $\pi_B^L - \pi_B^{NE}$, evaluated at $\lambda = 0$ is positive as long as

$$\Omega < \frac{2(1-\alpha)^2\mu^2(2-\rho)\rho}{\{4-[3-(2-\alpha)\alpha]\rho\}^2}(>\hat{\Omega}). \tag{58}$$

Futhermore, the equation $\pi_B^L = \pi_B^{NE}$ has one root only in λ:

$$\lambda^{EL} = \frac{(1-\alpha)\mu\sqrt{2\Omega\rho(2-\rho)} - \Omega\{4-[3-(2-\alpha)\alpha]\rho\}}{(1-\alpha)\mu\sqrt{2\Omega\rho(2-\rho)} + (2-\rho)\epsilon}, \tag{59}$$

which lies in the $[0,1]$ interval provided that (58) holds and

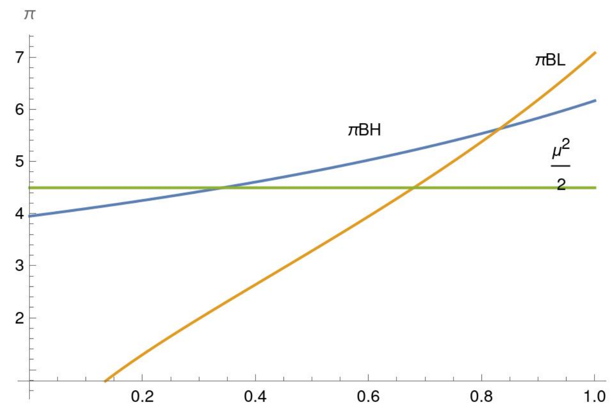
$$\varepsilon > 2\sqrt{\frac{2\Omega\rho}{2-\rho}} - \frac{\Omega\{4-[3-(2-\alpha)\alpha]\rho\}}{(1-\alpha)\mu(2-\rho)}. \tag{60}$$

It follows that licensing is preferred to no-extension to the left of λ^{EL}, and the opposite is true to the right of λ^{EL}.

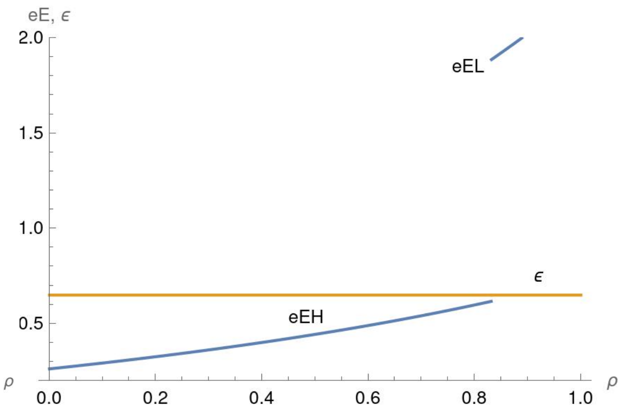
The extensive margin under in-house development coincides with that of the baseline model. The extensive margin is then $\lambda^E = \min\left[\lambda^{EL}, \lambda_1^{EH}\right]$. Depending on the relative values of the margins, one of the two possible configurations highlighted in the baseline model may emerge. □

Proposition 5. *Under both in-house development and licensing, the equilibrium effort e_E is increasing in ρ. Furthermore, the brand owner does not engage in brand extension if ρ is small, develops the extension internally for intermediate values of ρ, and licenses the brand to a specialized licensee if ρ is large.*

Figure 6 illustrates the Proposition.



(a) Profits.



(b) Efforts on extension.

Figure 6: Profits an effort levels on the extension product as a function of the size of extension market (ρ). Parameter configuration: $\alpha = 0.3, \lambda = 0.5, \varepsilon = 0.65, \mu = 3, \theta = 4, \Omega = 2.5$.

Proof. The first-order partial derivatives of the efforts on the extension product w.r.t. ρ are

$$\frac{\partial e_E^H}{\partial \rho} = \frac{2(1-\alpha)\mu(\theta+\lambda)}{\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\}^2} > 0, \quad \frac{\partial e_E^L}{\partial \rho} = \frac{\sqrt{2\Omega\rho}}{\sqrt{(2-\rho)\rho}} > 0. \tag{61}$$

The proof of the second part follows the same steps as those of Proposition 4. Details are left to the reader. $\square$

Proposition 6. *Under in-house development the equilibrium effort e_E is decreasing in α, whereas under licensing it is independent of that parameter. Furthermore, the brand owner develops the extension internally if α is small, licenses the brand to a specialized licensee for intermediate values of α, and does not engage in brand extension if α is large.*

Figure 7 illustrates the Proposition.

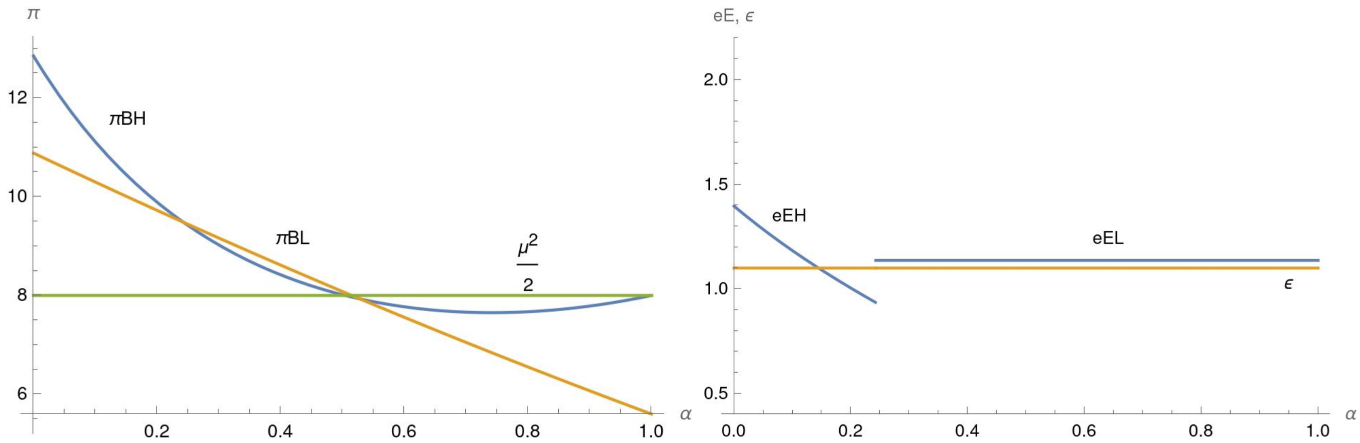
Proof. The first-order partial derivatives of the efforts on the extension product under the two alternative extension models are

$$\frac{\partial e_E^H}{\partial \alpha} = -\frac{\mu\left\{\theta\lambda\left[2-(1-\alpha)^2\rho\right]^2+\theta\rho\left[2+(1-\alpha)^2\rho\right]-\lambda\rho\left[2-3(1-\alpha)^2\rho\right]-\rho^2\right\}}{\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\}^2} < 0, \tag{62}$$

$$\frac{\partial e_E^L}{\partial \alpha} = 0. \tag{63}$$

Tedious algebra confirms that the first derivative is negative, while the second is nil. The second part of the proposition is proved following the same steps as Proposition 4. Details are left to the reader. $\square$

Proposition 7. *Under in-house development, the equilibrium effort e_E is decreasing in θ; under licensing, instead, it is independent of θ. Furthermore, if $\pi_B^L > \pi_B^{NE}$ then the brand owner develops*



(a) Profits. (b) Efforts on extension.

Figure 7: Profits and Effort levels on the extension product as a function of the perceived distance (α). Parameter configuration: $\lambda = 0.6, \rho = 0.7, \varepsilon = 1.1, \mu = 4, \theta = 3.8, \Omega = 1.2$.

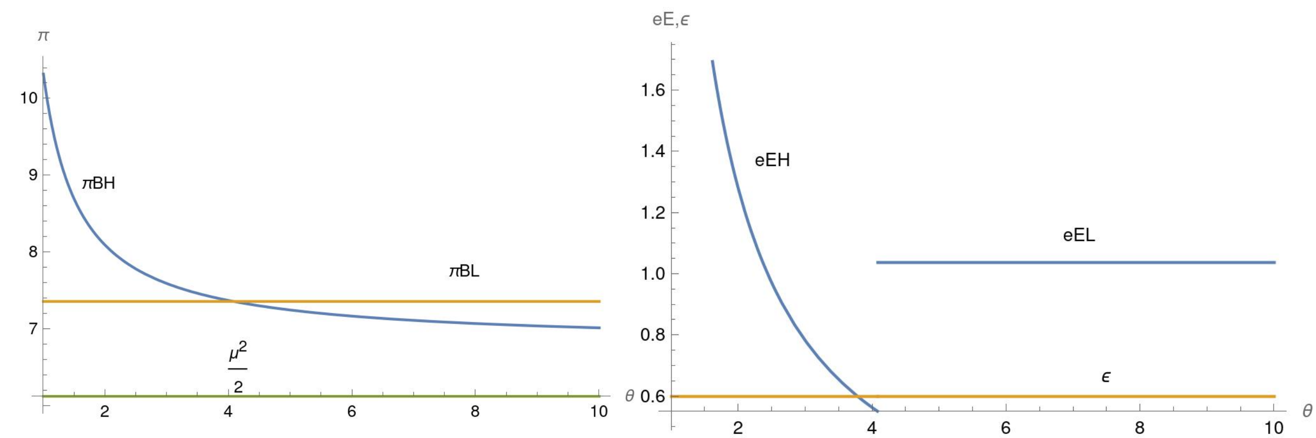
the extension internally if θ is small and licenses the brand to a specialized licensee for high values of θ.

Figure 8 illustrates the Proposition.

Proof. The partial derivative of the effort of the brand owner w.r.t. θ is

$$\frac{\partial e_E^H}{\partial \theta} = -\frac{(1-\alpha)\mu\left[2-(1-\alpha)^2\rho\right]\left\{\rho-\lambda\left[2-(1-\alpha)^2\rho\right]\right\}}{\left\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\right\}^2} < 0. \tag{64}$$

Again, a procedure like that of Proposition 4 demonstrates the second part of the Proposition. Details are left to the reader. □



(a) Profits. (b) Efforts on extension.

Figure 8: Profits an effort levels on the extension product as a function of the technological distance (θ). Parameter configuration: $\alpha = 0.3, \lambda = 0.4, \varepsilon = 0.6, \mu = 3.5, \rho = 0.7, \Omega = 1.1$.

3 Two-part tariff

Here we allow for the possibility that the licensing contract specifies a two-part tariff (s, F), where F is the fixed-fee.

In principle, a two-part tariff allows firms to solve the problem of double-marginalization. That is, the brand owner can set $s = 0$ and extract its profits from the extension only by means of the fixed fee. It could even set $s < 0$, providing a subsidy per unit of product so as to incentivize the licensee's effort.

As noted in the main text, however, such negative royalty rates are not observed in reality. The reason for this is, perhaps, that the use of fixed fees may create other types of inefficiencies. For example, suppose that the licensee is risk-averse and there are idiosyncratic shocks that affect the demand for the extension product. In this setting, the use of pure fixed-fee contracts exposes the licensee to the risk of making large payments to the brand owner even if demand turns out to be low. In these cases, the licensee will incur in losses. To reduce this risk, the brand owner may want to lower the fixed fee and increase the royalty rate.

As another example, suppose that the licensee knows the demand for the extension product but the brand owner does not. In this case, if the brand owner uses a pure fixed-fee contract, it must set the fixed fee on the basis of the expected level of demand. But when demand turns out to be low, the licensee may refuse the contract. To guarantee acceptance, the brand owner may again want to lower the fixed fee and rely more heavily on the royalty to extract its profits.

To capture these effects in a simple way, following Calzolari et al. (2020) and Condorelli and Padilla (2022) we assume that extracting rents by means of fixed fees creates deadweight losses: with a lump-sum payment of F, the brand owner earns F but the licensee loses νF, with $\nu \geq 1$. The difference $\nu - 1$ may capture various costs associated with the use of fixed fees as means of rent extraction such as those discussed above (for brevity, we shall refer to such costs as extraction costs). In the limit as $\nu \to \infty$, we re-obtain the baseline case where the brand owner uses pure royalty contracts.

Obviously, the analysis of the case of in-house development does not change. With licensing, the profit of the brand owner is still

$$\pi_B = \mu \left[e_C + \lambda(1-\alpha)\left(e_E - \varepsilon\right)\right] + s\rho\frac{(1-\alpha)e_C + e_E - s}{2} - \frac{1}{2}e_C^2 + F, \tag{65}$$

whereas that of the licensee becomes

$$\pi_L = \rho\frac{\left[(1-\alpha)e_C + e_E - s\right]^2}{4} - \frac{1}{2}e_E^2 - \nu F. \tag{66}$$

The introduction of the fixed fee does not affect the way the licensee sets its own effort, which remains:

$$e_E = \frac{\rho}{2-\rho}\left[(1-\alpha)e_C - s\right], \tag{67}$$

as in the baseline model. Substituting this effort level into (66), it appears that the profit of the licensee now is

$$\pi_L = \frac{\rho[(1-\alpha)e_C - s]^2}{2(2-\rho)} - \nu F. \tag{68}$$

Since the licensee's reservation payoff is nil, the brand owner will set

$$F = \frac{\rho[(1-\alpha)e_C - s]^2}{2\nu(2-\rho)}. \tag{69}$$

The brand owner appropriates all the licensee's profits in excess of its reservation payoff, but payoffs are not transferred from the licensee to the brand owner on a one-to-one basis. Due to the extraction costs, the rate of transformation is $\frac{1}{\nu} \leq 1$.

By plugging the optimal e_E and F back into the brand owner's profit, we can express these profits as a function of s only:

$$\begin{aligned} \pi_B \quad = \quad & \mu e_C - (1-\alpha)\lambda\mu \left\{ \frac{\rho[(1-\alpha)e_C + s]}{2-\rho} + \varepsilon \right\} + \\ & - \frac{\rho s[(1-\alpha)e_C + s]}{2-\rho} + \frac{\rho[(1-\alpha)e_C + s]^2}{2m\nu(2-\rho)} - \frac{e_C^2}{2}. \end{aligned} \tag{70}$$

Maximizing, we obtain the optimal royalty rate:

$$s = \frac{(1-\alpha)[e_C(\nu-1) - \lambda\mu\nu]}{2\nu-1}. \tag{71}$$

It appears that the optimal royalty rate is positive provided that $\nu > \frac{e_C}{e_C - \lambda\mu} > 1$. In the absence of extraction costs, ($\nu = 1$) the brand owner would set a negative royalty rate – a subsidy – to boost the incentivizing power of the contract. The assumption that $\nu > 1$ allows us to reconcile the use of two-part tariffs with the fact that negative royalty rates are not observed in reality.

By substituting the optimal royalty rate into the brand owner's profit and maximizing with respect to e_C we obtain:

$$e_C^L = \frac{1}{\mu} \frac{\nu\{4 - [3-(2-\alpha)\alpha]\rho\} - (2-\rho)}{\nu\left\{\rho\left[(1-\alpha)^2\lambda - 2\right] + 4\right\} - (2-\rho)}, \tag{72}$$

which can be plugged back into the expression for the optimal royalty rate to obtain:

$$s^L = (1-\alpha)\mu \frac{\nu\lambda\{[2-(2-\alpha)\alpha]\rho - 2\} + (\nu-1)(2-\rho)}{\nu\{4-[3-(2-\alpha)\alpha]\rho\} - (2-\rho)}. \tag{73}$$

The optimal royalty rate is increasing in ν. This is intuitive: an increase in the extractions costs lead the brand owner to rely more heavily on the royalty rate as a means of rent extraction. In terms of exogenous variables, the condition that guarantees the positivity of s^L is:

$$\nu > \frac{2-\rho}{2-\rho\lambda\{2-[2-(2-\alpha)\alpha]\rho\}}. \tag{74}$$

The equilibrium effort of the licensee is

$$e_E^L = \frac{(1-\alpha)(1+\lambda)\mu\nu\rho}{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2}. \tag{75}$$

Finally, the equilibrium profit of the brand owner under licensing is:

$$\pi_B^L = \frac{\mu\left\{\mu\left\{\nu\left\{\rho\left[(1-\alpha)^2\lambda(\lambda+2)-2\right]+4\right\}+\rho-2\right\}-2(1-\alpha)\lambda\varepsilon\left\{2-\nu\left\{4-[3-(2-\alpha)\alpha]\rho\right\}-\rho\right\}\right\}}{2\{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2\}}. \tag{76}$$

Having derived the equilibrium for this variant of the model, we now verify that the results obtained in the main text continue to hold.

Proposition 1. *Under both in-house development and licensing, the equilibrium exhibits under-investment in the quality of both products.*

Proof. The proof for the case of in-house development is the same as in the baseline model. As for the case of licensing, observe that the difference between the first-best effort and the effort under licensing on the core product is positive

$$e_C^{FB} - e_C^L = \frac{(1-\alpha)^2(\lambda+1)\mu(\nu-1)(2-\rho)\rho}{\{2-[2-(2-\alpha)\alpha]\rho\}\{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2\}} > 0. \tag{77}$$

The difference between the efforts on the extension is

$$e_E^{FB} - e_E^L = \frac{(1-\alpha)\mu\{\lambda\nu\{8-\rho\{12-5\rho-(2-\alpha)\alpha\{6-[5-(2-\alpha)\alpha]\rho\}\}\}+(\nu-1)(2-\rho)\rho\}-\lambda(2-\rho)\left[2-(1-\alpha)^2\rho\right]}{\{2-[2-(2-\alpha)\alpha]\rho\}\{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2\}}. \tag{78}$$

Tedious algebra confirms that the numerator is positive for all possible parameter values. □

Proposition 2. *At the extensive margin, if licensing is the best organizational mode (i.e., if $\pi_B^L = \pi_B^{NE} \geq \pi_B^H$), then brand extension necessarily entails brand dilution.*

Proof. The equation $\pi_B^L - \frac{\mu^2}{2} = 0$ has two solutions in μ, namely $\mu = 0$ and

$$\mu = \frac{2\lambda\varepsilon\{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2\}}{(1-\alpha)(1+\lambda)^2\nu\rho} = \mu^E > 0. \tag{79}$$

By evaluating the difference between the licensee's effort and the threshold ε one obtains

$$e_E^L - \varepsilon\Big|_{\mu=\mu_E} = -\varepsilon\frac{1-\lambda}{1+\lambda} < 0. \tag{80}$$

□

The analysis of the extensive margin(s) under internal development (Appendix E of the paper) is unchanged relative to the baseline model.

As in the baseline model, the following results require the cautionary note that some of the intervals mentioned may be empty.

Proposition 3. *Provided that the royalty rate is positive, at the intensive margin a switch from internal development to licensing always decreases the quality of the core product* e_C *and always increases the quality of the extension* e_E*; therefore, it reduces the likelihood of brand dilution.*

Proof. The solution to equation $\pi_B^H = \pi_B^L$ w.r.t. θ is

$$\theta^I = \frac{\rho^2\left\{\lambda\left[(1-\alpha)^2\lambda-2\right]-\nu\{1-[(1-\alpha)\alpha-2]\lambda\}^2\right\}+2\lambda\rho\{-\{[2-(2-\alpha)\alpha]\lambda\}+\nu\{[5-3(2-\alpha)\alpha]\lambda-4\}+2\}+4\lambda^2(1-2\nu)}{\rho\{\nu\rho[(1-\alpha)^2\lambda(\lambda+2)-2]-2[\lambda(\lambda+2)-1]\nu+\rho-2\}}, \tag{81}$$

Here we restrict our attention to the case where $\theta^I > 1$. If this condition does not hold, in-house development is never a profitable extension mode.

The difference between the efforts on the core product with under in-house and licensing, evaluated at θ^I is

$$\left(e_C^H - e_C^L\right)\Big|_{\theta=\theta^I} = \frac{(1-\alpha)^2\lambda\mu\rho\{\nu\{\lambda\{[2-(2-\alpha)\alpha]\rho\}+\rho-2\}+\rho-2\}}{\{\lambda\left[2-(1-\alpha)^2\rho\right]+\rho\}\left\{\nu\left\{4-[3-(2-\alpha)\alpha]\rho\right\}+\rho-2\right\}} \tag{82}$$

The denominator is positive. The numerator is also positive provided that the royalty rate is positive,a condition that guarantees that the term in curly brackets in the numerator is positive. Thus, the difference is positive.

The difference between the efforts on the extension is:

$$\left(e_E^L - e_E^L\right)\Big|_{\theta=\theta^I} = -\frac{(1-\alpha)\mu\rho\{\nu\{\lambda\{[2-(2-\alpha)\alpha]\rho-2\}-\rho+2\}+\rho-2\}}{\{\lambda\left[2-(1-\alpha)^2\rho\right]+\rho\}\left\{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2\right\}} \tag{83}$$

By the same reasoning, this is always negative provided that the optimal royalty rate is positive. □

Proposition 4. *Under both in-house development and licensing, the equilibrium effort* e_E *is increasing in* λ*. Furthermore, if both the technological distance* θ *and the extraction costs* ν *are sufficiently high, the brand owner develops the extension internally if* λ *is small, licenses the brand to a specialized licensee for intermediate values of* λ*, and does not engage in brand extension if* λ *is large.*

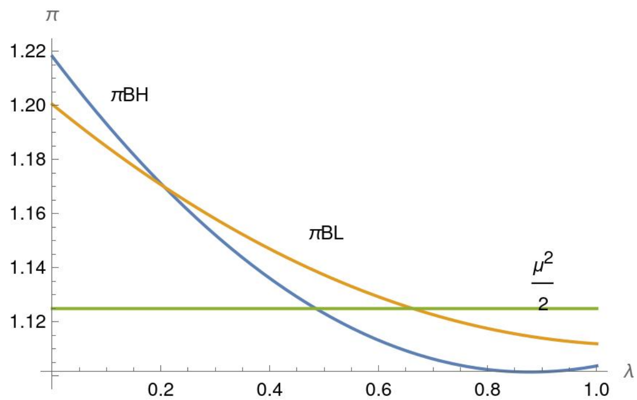
The figure 9 illustrates the Proposition.

Proof. The first part of the proposition follows from the calculation of the derivatives

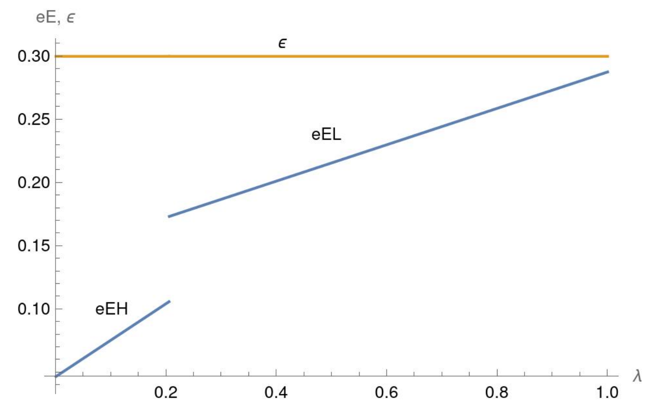
$$\frac{\partial e_E^H}{\partial\lambda} = \frac{(1-\alpha)\mu\left[2-(\alpha-1)^2\rho\right]}{\theta\left[2-(1-\alpha)^2\rho\right]-\rho} > 0, \quad \frac{\partial e_E^L}{\partial\lambda} = \frac{(1-\alpha)\mu\nu\rho}{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2} > 0. \tag{84}$$

In order to prove the second part, we start by the intensive margin. The difference between the in-house and licensing profits, $\pi_B^H - \pi_B^L t$, is quadratic in λ. Furthermore, $\left(\pi_B^H - \pi_B^L\right)\Big|_{\lambda=0}$ is positive provided that

$$\nu > \frac{\theta(2-\rho)}{2\theta(2-\rho)+\rho}. \tag{85}$$



(a) Profits.



(b) Efforts on extension.

Figure 9: Profits and effort levels on the extension product as a function of the intensity of the reciprocal effect (λ). Parameter configuration: $\alpha = 0.3, \rho = 0.3, \varepsilon = 0.3, \mu = 1.5, \theta = 3.8, \nu = 1.6$.

On the other hand, $\left(\pi_B^H - \pi_B^L\right)\Big|_{\lambda=1}$ is negative if

$$\theta > \frac{\rho\left[(1-\alpha)^4\rho + 6(2-\alpha)\alpha - 2\right] + 8}{\rho[4 - 3(2-\alpha)\alpha\rho + \rho]}. \tag{86}$$

The profit difference function is continuous in λ over $[0, 1]$, thus if the two above conditions are fulfilled the difference of the profits cuts the horizontal axis only once. This root is the intensive margin, and we label it λ^I. Finally, observe that the derivative of the difference of the profits, computed at $\lambda = \lambda^I$ is negative, which confirms our last claim:

$$\left.\frac{\partial\left(\pi_B^H - \pi_B^L\right)}{\partial\lambda}\right|_{\lambda=\lambda^I} = -\mu\sqrt{\frac{(1-\alpha)^4\mu^2\rho[2-\nu(2-\theta\rho)-\rho]}{\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\}\left\{4-\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho\right\}}} < 0. \tag{87}$$

We conclude that for all λ to the left of λ^I in house extension is preferred to licensing, and the converse to its right.

Let us now consider the extensive margin under licensing, which is defined by the locus $\pi_B^L = \pi_B^{NE}$. This equation has two roots in λ, which are real provided that

$$\varepsilon > \frac{2(1-\alpha)\mu\nu\rho}{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2}. \tag{88}$$

This said, observe that

$$\left(\pi_B^L - \pi_B^{NE}\right)\Big|_{\lambda=0} = \frac{(1-\alpha 1)^2\mu^2\nu\rho}{2\nu\{4-[3-(2-\alpha)\alpha]\rho\}+2\rho-4} > 0, \tag{89}$$

and that

$$\left(\pi_B^L - \pi_B^{NE}\right)\Big|_{\lambda=1} = (1-\alpha)\mu \left\{ \frac{2(1-\alpha)\mu\nu\rho}{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2} - \varepsilon \right\} \tag{90}$$

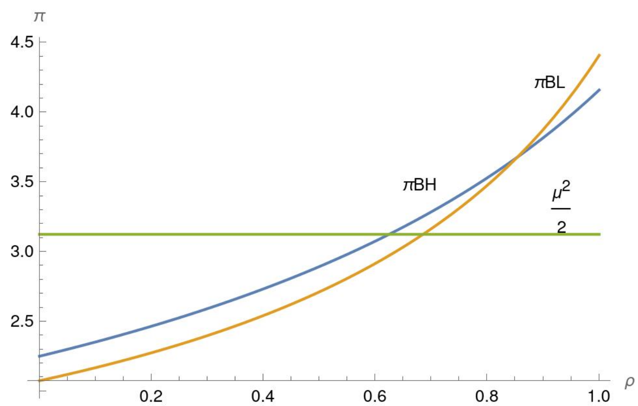
is negative as long as (88) is fulfilled. In this case there is only one root that lies in the $[0,1]$ interval, we denote it λ^{EH}. The extensive margin is thus $\lambda^E = \min[\lambda_1^{EH}, \lambda^{EL}]$.[3]

Depending on the relative values of λ^I and λ^E, two cases may arise. If $\lambda^E > \lambda^I$, licensing is the optimal extension mode at the extensive margin. This is the case described in the statement of Proposition 4 and depicted in Figure 9. If instead $\lambda^E < \lambda^I$, then the interval where licensing is optimal vanishes. In this case, in-house development is optimal for $\lambda < \lambda^E$ and no licensing for $\lambda > \lambda^E$.

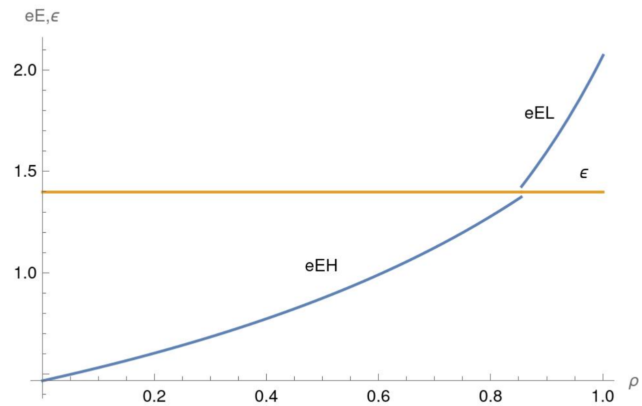
□

Proposition 5. *Under both in-house development and licensing, the equilibrium effort e_E is increasing in ρ. Furthermore, the brand owner does not engage in brand extension if ρ is small, develops the extension internally for intermediate values of ρ, and licenses the brand to a specialized licensee if ρ is large.*

The following figure illustrates the Proposition.



(a) Profits.



(b) Efforts on extension.

Figure 10: Profits and effort levels on the extension product as a function of the size of the extension market (ρ). Parameter configuration: $\alpha = 0.4, \lambda = 0.5, \varepsilon = 1.4, \mu = 2.5, \theta = 1.6, \nu = 1.8$.

Proof. The first part of the proposition follows from the calculation of the derivatives

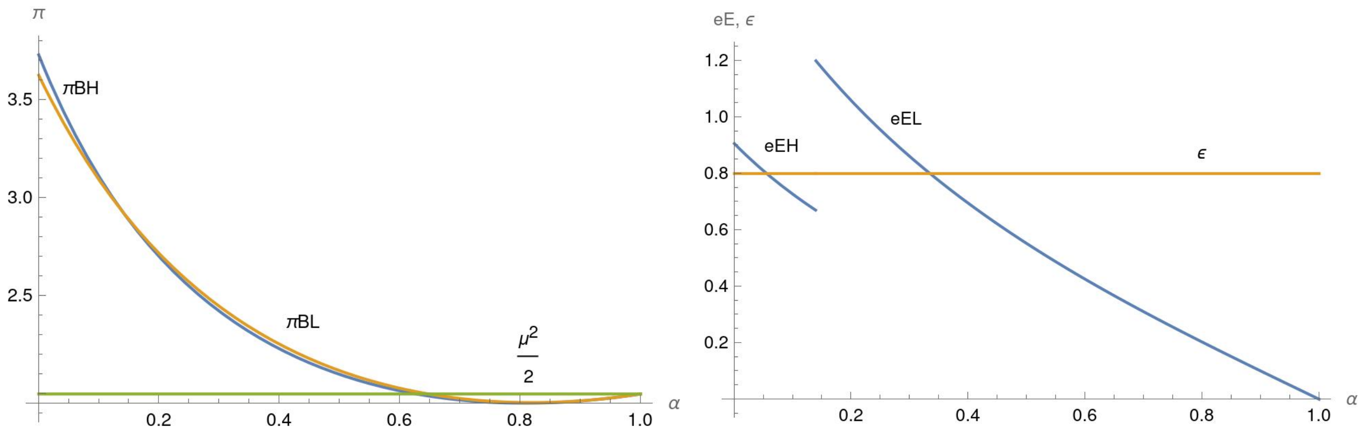
$$\frac{\partial e_E^H}{\partial \rho} = \frac{2(1-\alpha)\mu[\theta+\lambda)}{\{\theta\,[2-(1-\alpha)^2\rho]-\rho\}^2} > 0, \quad \frac{\partial e_E^L}{\partial \rho} = \frac{2(1-\alpha)(\lambda+1)\mu\nu(2\nu-1)}{\{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2\}^2} > 0. \tag{91}$$

The second part can be proved by the same reasoning as that in the proof of Proposition 4. Details are left to the reader. □

[3] In house development choices are unaffected by the licensing contract structure, thus the relevant margins coincide with those in the baseline model. Condition (86) insures that $\lambda_2^{EH} > 1$.

Proposition 6. *Under both in-house development and licensing, the equilibrium effort e_E is decreasing in α. Furthermore, the brand owner develops the extension internally if α is small, licenses the brand to a specialized licensee for intermediate values of α, and does not engage in brand extension if α is large.*

The following figure illustrates the Proposition.



(a) Profits. (b) Efforts on extension.

Figure 11: Effort levels on the extension product and profits as a function of the perceived distance (α). Parameter configuration: $\alpha = 0.3, \rho = 0.8, \varepsilon = 0.8, \mu = 2, \theta = 2.8, \nu = 3.8$.

Proof. The first-order partial derivatives of the efforts on the extension product under the two alternative extension models are

$$\frac{\partial e_E^H}{\partial \alpha} = -\frac{\mu\left\{\theta\lambda\left[(1-\alpha)^2\rho-2\right]^2+\theta\rho\left[(1-\alpha)^2\rho+2\right]+\lambda\rho\left[3(1-\alpha)^2\rho-2\right]-\rho^2\right\}}{\left\{\theta\left[2-(1-\alpha)^2\rho\right]+\rho\right\}^2} < 0, \quad (92)$$

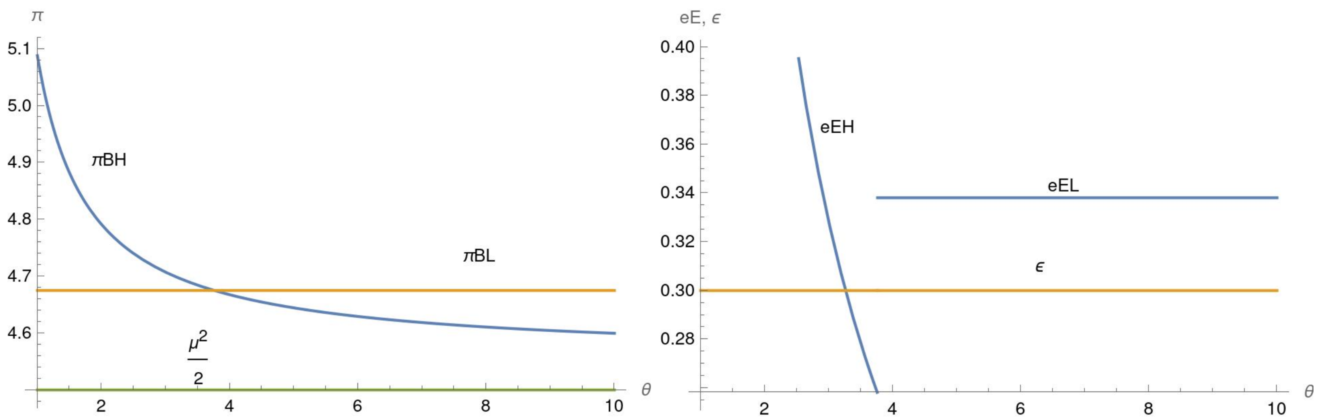
$$\frac{\partial e_E^L}{\partial \alpha} = -\frac{(1+\lambda)\mu\nu\rho\{\nu\{4-[1+(2-\alpha)\alpha]\rho\}+\rho-2\}}{\{\nu\{4-[3-(2-\alpha)\alpha]\rho\}+\rho-2\}^2} < 0. \quad (93)$$

It is clear by inspection that the second derivative is negative, tedious algebra confirms that so is the first one. The second part of the proposition is proved similarly to that of Proposition 4. Details are left to the reader. □

Proposition 7. *Under in-house development, the equilibrium effort e_E is decreasing in θ; under licensing, instead, it is independent of θ. Furthermore, if $\pi_B^L > \pi_B^{NE}$ then the brand owner develops the extension internally if θ is small and licenses the brand to a specialized licensee for high values of θ.*

Proof. The partial derivative of the effort of the brand owner w.r.t. θ is

$$\frac{\partial e_E^H}{\partial \theta} = -\frac{(1-\alpha)\mu\left[2-(1-\alpha)^2\rho\right]\left\{\rho-\lambda\left[2-(1-\alpha)^2\rho\right]\right\}}{\left\{\theta\left[2-(1-\alpha)^2\rho\right]-\rho\right\}^2} < 0. \quad (94)$$



(a) Profits.

(b) Efforts on extension.

Figure 12: Profits and effort levels on the extension product as a function of the technological distance (θ). Parameter configuration: $\alpha = 0.5, \rho = 0.4, \varepsilon = 0.3, \mu = 3, \lambda = 0.4, \nu = 2.6$.

The second part of the proposition is proved similarly to that of Proposition 4. Details are left to the reader. □

References

Calzolari, G., Denicolò, V., and Zanchettin, P. (2020). The demand-boost theory of exclusive dealing. *The RAND Journal of Economics*, 51(3):713–738.

Condorelli, D. and Padilla, J. (2022). Vertical mergers in ecosystems with consumer hold-up. Technical report, working paper.