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ON KOSTANT ROOT SYSTEMS FOR LIE SUPERALGEBRAS

IVAN DIMITROV AND RITA FIORESI

ABSTRACT. We study the eigenspace decomposition of a basic classical Lie superalgebra under the adjoint action of a toral subalgebra, thus extending results of Kostant. In recognition of Kostant's contribution we refer to the eigenspaces appearing in the decomposition as Kostant roots. We then prove that Kostant root systems inherit the main properties of classical root systems. Our approach is combinatorial in nature and utilizes certain graphs naturally associated with Kostant root systems. In particular, we reprove Kostant's results without making use of the Killing form.

Keywords: Kostant root systems, Simple roots, Parabolic subalgebras, Hermitian symmetric pairs.

1. INTRODUCTION

1.1. Background. Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{t}$ be a toral subalgebra of $\mathfrak{g}$, i.e., $\mathfrak{t}$ is an abelian subalgebra consisting of semisimple elements. Under the adjoint action of $\mathfrak{t}$, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{m} \oplus \left(\oplus_{\nu\in R}\, \mathfrak{g}_\nu\right), \tag{1.1}$$

where

$$\mathfrak{m} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}) = \{x \in \mathfrak{g} \,|\, [t, x] = 0 \text{ for every } t \in \mathfrak{t}\}$$

is a reductive subalgebra of $\mathfrak{g}$,

$$\mathfrak{g}_\nu = \{x \in \mathfrak{g} \,|\, [t, x] = \nu(t)x \text{ for every } t \in \mathfrak{t}\}$$

and

$$R = \{\nu \in \mathfrak{t}^*\backslash\{0\} \,|\, \mathfrak{g}_\nu \neq 0\}.$$

The elements of R are called *Kostant roots* (or $\mathfrak{t}$-*roots* for short) of $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ with corresponding root decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\, \mathfrak{g}_\alpha), \tag{1.2}$$

where Δ and $\mathfrak{g}_\alpha$ are defined analogously to R and $\mathfrak{g}_\nu$ above.

Roots systems and the corresponding decompositions (1.2) provide the cornerstone for developing the classification and structure theory of simple complex Lie algebras. The

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decompositions (1.1) and (1.2) are closely related as follows. The inclusion $\mathfrak{t} \subset \mathfrak{h}$ gives rise to a natural projection $\pi : \mathfrak{h}^* \longrightarrow \mathfrak{t}^*$ that relates (1.1) to (1.2) as follows:

$$\mathfrak{m} = \mathfrak{h} \oplus (\oplus_{\alpha \in \ker \pi} \mathfrak{g}_\alpha), \qquad R = \pi(\Delta) \backslash \{0\}, \qquad \mathfrak{g}_\nu = \oplus_{\alpha \in \pi^{-1}(\nu)} \mathfrak{g}_\alpha.$$

It is thus natural to expect that (1.1) and R would inherit (at least some) properties from (1.2) and Δ.

In [Ko] Kostant studied the decomposition (1.1) and the set R in the case when $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$. (See Remark 5.5 for a discussion of the condition $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$.) In particular he proved that each $\mathfrak{g}_\nu$ is an irreducible $\mathfrak{m}$-module, that $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \mathfrak{g}_{\mu+\nu}$ whenever $\mu, \nu, \mu + \nu \in R$, that R admits bases of simple roots, etc. In recognition of Kostant's work, we call R the *Kostant root system* (or the $\mathfrak{t}$-*root system* for short) associated with the pair $(\mathfrak{g}, \mathfrak{t})$ and the elements of R – *Kostant roots* (or $\mathfrak{t}$-*roots*).

In [G] Greenstein studied Kostant roots systems for affine Lie algebras. He showed that $\mathfrak{g}_\nu$ is not necessarily an irreducible $\mathfrak{m}$-module and proved that a sufficient condition of its irreducibility is $\dim \mathfrak{g}_\alpha = 1$ for every $\alpha \in \pi^{-1}(\nu)$.

1.2. Reductive Lie superalgebras. The goal of this paper is to extend Kostant's results to Lie superalgebras. The simple finite dimensional Lie superalgebras fall into three subclasses: basic classical, strange, and Cartan-type superalgebras. In terms of roots and root decompositions, the basic classical superalgebras are the closest analogs of simple finite dimensional Lie algebras. The basic classical superalgebras are the following:

$$\mathfrak{sl}(m|n), m \neq n, \quad \mathfrak{psl}(m|m), \quad \mathfrak{osp}(m|2n), \quad \mathrm{D}(2,1;a), \quad \mathrm{G}(3), \quad \mathrm{F}(4).$$

Among the basic classical superalgebras, $\mathfrak{psl}(m|m)$ are the only ones that are not Kac-Moody superalgebras in the sense of Serganova, [Se2]. A manifestation of this feature is the fact that the $\mathfrak{psl}(2|2)$ has root spaces that are two dimensional. Moreover, the projection $\mathfrak{sl}(2|2) \longrightarrow \mathfrak{psl}(2|2)$ identifies the roots and the corresponding root spaces of $\mathfrak{sl}(2|2)$ and $\mathfrak{psl}(2|2)$ and, hence, $\mathfrak{sl}(2|2)$ also admits two dimensional root spaces. This problem is resolved only by enlarging $\mathfrak{sl}(2|2)$ to $\mathfrak{gl}(2|2)$. Even though the root spaces of $\mathfrak{sl}(m|m)$ and $\mathfrak{psl}(m|m)$ are one dimensional for $m \geqslant 3$, the $\mathfrak{t}$-root spaces $\mathfrak{g}_\nu$ are not necessarily irreducible $\mathfrak{m}$-modules when $\mathfrak{g} = \mathfrak{psl}(m|m)$ even if $m > 2$, see Remark 5.6. Thus, in our work, we study $\mathfrak{gl}(m|m)$ instead of $\mathfrak{psl}(m|m)$.

Below we identify the class of Lie superalgebras for which we develop the theory of Kostant roots.

Definition 1.1. A finite dimensional Lie superalgebra $\mathfrak{g}$ is called a *reductive superalgebra* if

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l, \tag{1.3}$$

where $\mathfrak{a}$ is an abelian Lie algebra and each $\mathfrak{g}_j$ for $1 \leqslant j \leqslant l$ is a simple Lie algebra or a superalgebra isomorphic to one of the following:

$$\mathfrak{sl}(m|n), m \neq n, \quad \mathfrak{gl}(m|m), m \geqslant 1, \quad \mathfrak{osp}(m|2n), \quad \mathrm{D}(2,1;a), \quad \mathrm{G}(3), \quad \mathrm{F}(4).$$

If $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a reductive superalgebra, $\mathfrak{t}$ is a *toral subalgebra* of $\mathfrak{g}$ if $\mathfrak{t} \subset \mathfrak{g}_{\bar{0}}$ is an abelian algebra which acts semisimply on $\mathfrak{g}$.

Note that, unlike the roots systems of reductive Lie algebras, the roots systems of reductive superalgebras admit bases which are not conjugate under the action of the Weyl group; this is a feature they share with the Kostant root systems of simple Lie algebras.

1.3. Main results. Theorems 1.2 – 1.4 below are the main results of this paper concerning the decomposition (1.1) of a reductive Lie superalgebra $\mathfrak{g}$ with respect to a toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. First we prove that the centralizer $\mathfrak{m}$ of $\mathfrak{t}$ itself is a reductive Lie superalgebra.

Theorem 1.2. *Let $\mathfrak{g}$ be a reductive Lie superalgebra, let $\mathfrak{t}$ be a toral subalgebra of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{m} \oplus (\oplus_{\nu \in R} \mathfrak{g}_\nu)$ be the decomposition* (1.1)*. Then $\mathfrak{m}$ is a reductive Lie superalgebra.*

The remaining two theorems establish properties of the set R of $\mathfrak{t}$-roots. For these two theorems we work under the assumption that $\mathfrak{t} = \mathfrak{z}(\mathfrak{g})$.

Theorem 1.3. *In the notation of Theorem 1.2 above, assume that $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$. If Σ is a base of Δ, then $\pi(\Sigma)\backslash\{0\}$ is a base of R if and only if Σ contains a base of $\Delta_{\mathfrak{m}}$, the root system of $\mathfrak{m}$. Moreover, every base S of R is of the form $\pi(\Sigma)\backslash\{0\}$ for some base Σ of Δ containing a base of $\Delta_{\mathfrak{m}}$. In particular, the Kostant root system R admits a base.*

Theorem 1.4. *Under the assumptions of Theorem 1.3 above we have*

(i) *for any $\nu \in R$, the $\mathfrak{m}$-module $\mathfrak{g}_\nu$ is irreducible;*
(ii) *if $\mu, \nu \in R$ and $\mu + \nu \in R$, then $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \mathfrak{g}_{\mu+\nu}$;*
(iii) *if $\mu, \nu \in R$, and $\mu + k\nu \in R$, where $k \in \mathbb{Z}_{>0}$, then $\mu + j\nu \in R$ for every $0 \leqslant j \leqslant k$.*

1.4. Approach. Theorems 1.2 – 1.4 above are the analogs of results in [Ko].

Theorem 1.2 is relatively straightforward but one needs to take special care to ensure that in $\mathfrak{m}$ every component of $[\mathfrak{m}, \mathfrak{m}]$ isomorphic to $\mathfrak{sl}(m|m)$ can be completed to a component isomorphic to $\mathfrak{gl}(m|m)$.

Theorem 1.3 relies on the results of [DFG] to characterize the bases of R.

Theorem 1.4 is the most difficult of the three. Kostant's approach to studying the properties of the set R and the decomposition (1.1) parallels the classical approach to root systems and root decompositions and makes a heavy use of a form on $\mathfrak{t}$ inherited from the restriction on $\mathfrak{h}$ of the Killing form of $\mathfrak{g}$. This approach does not extent to the case when $\mathfrak{g}$ is a superalgebra because a non-degenerate invariant form on $\mathfrak{g}$ is, in general, not positive-definite. In order to overcome this problem we introduce a graph $\Gamma_{\Delta,\Sigma}$ associated with a root system Δ and a base Σ of Δ and translate properties (i) – (iii) into properties of $\Gamma_{\Delta,\Sigma}$. We then prove that the graphs $\Gamma_{\Delta,\Sigma}$ behave "functorially" with respect to projections like π. In the case when $\mathfrak{g}$ is a non-exceptional basic classical Lie superalgebra, part (i) of Theorem 1.4 then follows from the observation that the corresponding graphs $\Gamma_{\Delta,\Sigma}$ are the same as the graphs associated with appropriate Kostant root systems of Lie algebras. In the case when $\mathfrak{g}$ is an exceptional Lie superalgebra, part (i) of Theorem 1.4 is a direct calculation. In order to provide a proof of Theorem 1.4 independent of [Ko] we also prove part (i) in the case when $\mathfrak{m} \subset \mathfrak{g}_{\bar{0}}$ using a result of Stembridge, [St]. Once we prove part (i) of Theorem 1.4, the remaining statements are established using the functorial properties of the graphs $\Gamma_{\Delta,\Sigma}$.

1.5. Structure and contents. The paper is organized as follows. In Section 2 we discuss the general properties of reductive Lie superalgebras and prove Theorem 1.2. In Section 3 we prove Theorem 1.3. In Section 4 we introduce and study the graphs $\Gamma_{\Delta,\Sigma}$. In Section 5 we prove Theorem 1.4. Section 6 applies Theorem 1.3 to studying Hermitian symmetric pairs of Lie superalgebras. Explicit calculations that complete the proof of part (i) of Theorem 1.4 for $\mathfrak{g} = \mathrm{D}(2,1;a), \mathrm{G}(3)$, and $\mathrm{F}(4)$ are provided in the Appendix.

1.6. Notation and conventions.

- The base field is $\mathbb{C}$ and, unless explicitly stated otherwise, all vector spaces, Lie algebras and superalgebras, etc. are defined over $\mathbb{C}$.
- The sets of positive (non-negative, etc.) integers are denoted by $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geqslant 0}$, etc.).
- For a subset X of a vector space V, the span of X is denoted by $\langle X \rangle$.
- The elements of $\mathbb{Z}/2\mathbb{Z}$ are denoted by $\bar{0}$ and $\bar{1}$; if V is vector superspace (i.e, a $\mathbb{Z}/2\mathbb{Z}$-graded vector space), its graded components are denoted by $V_{\bar{0}}$ and $V_{\bar{1}}$.
- If Γ is a graph, $v(\Gamma)$ and $e(\Gamma)$ denote the vertices and the edges of Γ.
- If $\mathfrak{l}$ is a Lie superalgebra, $\mathfrak{z}(\mathfrak{l})$ denotes the center of $\mathfrak{l}$; if $\mathfrak{k}$ is a subalgebra of $\mathfrak{l}$, $\mathfrak{c}(\mathfrak{k})$ denotes the centralizer of $\mathfrak{k}$ in $\mathfrak{l}$.

2. Reductive superalgebras

Throughout the paper $\mathfrak{g}$ denotes a reductive Lie superalgebra with a decomposition (1.3). Note that, unlike the case when $\mathfrak{g}$ is a reductive Lie algebra, it is not true that $\mathfrak{a}$ is the center of $\mathfrak{g}$. Instead, we have the following relationship.

Lemma 2.1. *Let $\mathfrak{g}$ be a reductive Lie superalgebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then*

$$\dim \mathfrak{h} = \mathrm{rk}\, \mathfrak{g} + \dim \mathfrak{z}(\mathfrak{g}),$$

where $\mathrm{rk}\, \mathfrak{g}$ denotes the dimension of $\langle \Delta \rangle \subset \mathfrak{h}^*$, the vector space spanned by the roots of $\mathfrak{g}$.

Proof. Notice that $\mathfrak{h}$ decomposes as $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_l$, where $\mathfrak{h}_j$ is a Cartan subalgebra of $\mathfrak{g}_l$. Consider $\mathfrak{g}_j$. If $\mathfrak{g}_j \cong \mathfrak{gl}(m|m)$ for some m, then $\dim \mathfrak{z}(\mathfrak{g}) = 1$ and $\mathrm{rk}\, \mathfrak{g}_j = \dim \mathfrak{h}_j - 1$; if, on the other hand, $\mathfrak{g}_j$ is not isomorphic to $\mathfrak{gl}(m|m)$, then $\mathfrak{z}(\mathfrak{g}) = 0$ and $\mathrm{rk}\, \mathfrak{g}_j = \dim \mathfrak{h}_j$. Taking into account the above, the decomposition

$$\mathfrak{z}(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{z}(\mathfrak{g}_1) \oplus \ldots \mathfrak{z}(\mathfrak{g}_l)$$

completes the proof. □

Let $\mathfrak{t}$ be a toral subalgebra of $\mathfrak{g}$. Fixing a Cartan subalgebra $\mathfrak{h}$ with $\mathfrak{t} \subset \mathfrak{h}$, we denote by π the natural projection $\pi : \mathfrak{h}^* \longrightarrow \mathfrak{t}^*$. It is convenient to introduce the sets $\bar{\Delta} := \Delta \cup \{0\}$ and $\bar{R} := R \cup \{0\}$, where Δ and R are respectively the $\mathfrak{h}$-roots and the $\mathfrak{t}$-roots of $\mathfrak{g}$. Note that $\pi(\Delta) = \bar{R}$.

Our first goal is to prove Theorem 1.2. We start by establishing a result which is of independent interest. Recall that a subset $\Sigma \subset \Delta$ is a *base* of Δ if Σ is linearly independent and any element of Δ is an integral linear combination of elements in Σ with all coefficients in $\mathbb{Z}_{\geqslant 0}$ or $\mathbb{Z}_{\leqslant 0}$.

Lemma 2.2. *Let $\mathfrak{g}$ be a reductive Lie superalgebra with roots Δ. If W is a subspace of $\mathfrak{h}^*$, then there exists a base Σ of Δ such that every element of $\Delta \cap W$ is a combination of elements of the set $\Sigma_W := \Sigma \cap W$.*

Proof. It suffices to prove the statement in the case when $\mathfrak{g}$ is a Kac-Moody Lie superalgebra, i.e., one of the superalgebras $\mathfrak{sl}(m|n), m \neq n, \mathfrak{gl}(m|m), m \geqslant 1, \mathfrak{osp}(m|2n)$, $\mathrm{D}(2,1;a)$, $\mathrm{G}(3)$, or $\mathrm{F}(4)$. Fix a linear function $f \in (\mathfrak{h}^*)^*$ which takes real values on Δ and such that, for every $\alpha \in \Delta$, we have $\alpha \in W$ if and only if $\alpha \in \ker f$. The set $P := \{\alpha \in \Delta \mid f(\alpha) \geqslant 0\}$ is a parabolic subset of Δ, see [DFG]. Proposition 2.10 in [DFG] implies that there exists a base Σ of Δ such that every positive root with respect to Σ belongs to P. In particular, f takes only non-negative values on the elements of Σ. Now let $\alpha \in \Delta \cap W$. Then

$$\alpha = \sum_j c_j \beta_j + \sum_k d_k \gamma_k,$$

where $\beta_j \in \Sigma_W$, $\gamma_k \in \Sigma \backslash \Sigma_W$ and all coefficients c_j, d_k are non-negative or non-positive. Applying f we obtain

$$0 = \sum_k d_k f(\gamma_k),$$

which shows that, unless $d_k = 0$ for every k, there exist k_1 and k_2 such that one of $f(\gamma_{k_1})$ and $f(\gamma_{k_2})$ positive and the other one is negative. This contradicts the fact that f takes only non-negative values on the elements of Σ. Hence $d_k = 0$ for every k, i.e., α is a linear combination of elements of Σ_W. □

A base Σ with the above property is called a *W-adapted base of* Δ.

Proof of Theorem 1.2. As in the proof of Lemma 2.2 above it suffices to prove the statement in the case when $\mathfrak{g}$ is a Kac-Moody superalgebra. Since $\mathfrak{m} = \mathfrak{h} \oplus (\oplus_{\alpha \in \ker \pi} \mathfrak{g}_\alpha)$, Lemma 2.2 applies. Fix a $\ker \pi$-adapted base Σ of Δ and set $\Sigma_{\mathfrak{m}} := \Sigma \cap \ker \pi$. Consider the Dynkin diagram associated with Σ and let

$$\Sigma_{\mathfrak{m}} = \Sigma_1 \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_l$$

be the decomposition of $\Sigma_{\mathfrak{m}}$ into connected components. Denote the superalgebra generated by $\oplus_{\alpha \in \Sigma_k} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ by $\mathfrak{g}'_k$. Note that $\mathfrak{g}'_k$ is either a Kac-Moody superalgebra or is isomorphic to $\mathfrak{sl}(m_k|m_k)$ with $m_k \geqslant 1$.

We need to show that we can "complete" the subalgebras $\mathfrak{g}'_k$ to Kac-Moody superalgebras $\mathfrak{g}_k$, i.e., that there exist Kac-Moody superalgebras $\mathfrak{g}_k$ such that $\mathfrak{g}'_k \subset \mathfrak{g}_k$ and the sum $\mathfrak{g}_1 + \mathfrak{g}_2 + \cdots + \mathfrak{g}_l$ is direct. First we set $\mathfrak{g}_k := \mathfrak{g}'_k$ unless $\mathfrak{g}'_k$ is isomorphic to $\mathfrak{sl}(m_k|m_k)$. In order to deal with the subalgebras $\mathfrak{g}'_k$ isomorphic to $\mathfrak{sl}(m_k|m_k)$, we consider two cases for $\mathfrak{g}$.

If $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(m|n), m \neq n, \mathfrak{gl}(m|m), m \geqslant 1$, or $\mathfrak{osp}(m|2n)$, the roots of $\mathfrak{g}$ are expressed in terms of linear functions $\varepsilon_i, \delta_j \in \mathfrak{h}^*$ with dual elements $h_i \in \mathfrak{h}$, see [Ka]. For each k we then define

$$\mathfrak{g}_k := \mathfrak{g}'_k + (\oplus_i \mathbb{C} h_i),$$

where the sum is over all indices i that appear in the expression in terms of ε's and δ's of a root in Σ_k. This definition agrees with the previous definition of $\mathfrak{g}_k$ when $\mathfrak{g}'_k$ is not isomorphic to $\mathfrak{sl}(m_k|m_k)$ and $\mathfrak{g}_k \cong \mathfrak{gl}(m_k|m_k)$ when $\mathfrak{g}_k \cong \mathfrak{sl}(m_k|m_k)$. Moreover, the sum $\mathfrak{g}_1 + \mathfrak{g}_2 + \cdots + \mathfrak{g}_l$ is direct.

If $\mathfrak{g}$ is isomorphic to $\mathrm{D}(2,1;a)$, $\mathrm{G}(3)$, or $\mathrm{F}(4)$, we notice, by inspecting the list of possible Dynkin diagrams, that $\mathfrak{g}'_k$ is isomorphic to $\mathfrak{sl}(m_k|m_k)$ for at most one index k and, for this k, $m_k = 1$, see the Appendix. If there is no such k we are done. Assume now that $\mathfrak{g}'_1 \cong \mathfrak{sl}(1|1)$. Let $h \in \mathfrak{h}$ be such that $[h, \mathfrak{g}_k] = 0$ for $k > 1$ but $[h, \mathfrak{g}'_1] \neq 0$. Then $\mathfrak{g}_1 := \mathbb{C}h \oplus \mathfrak{g}'_1 \cong \mathfrak{gl}(1|1)$ and the sum $\mathfrak{g}_1 + \mathfrak{g}_2 + \cdots + \mathfrak{g}_l$ is direct.

To complete the proof, we set $\mathfrak{a} := \mathfrak{c}_{\mathfrak{m}}(\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l)$. The decomposition

$$\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$$

shows that $\mathfrak{m}$ is a reductive Lie superalgebra. □

We complete this section by recording some properties of the roots of reductive superalgebras that will be used in the rest of the paper.

Proposition 2.3. *Let $\mathfrak{g}$ be a reductive Lie superalgebra with roots Δ. Then*

(i) $-\Delta = \Delta$.
(ii) *If $\alpha, \beta \in \Delta$, then $\alpha + \beta \in \Delta \cup \{0\}$ if and only if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$.*
(iii) *If Σ is a base of Δ and γ is a positive root with respect to Σ, then there exist positive roots $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_N, \gamma_{N+1} = 0 \in \bar{\Delta}$ such that $\gamma_i - \gamma_{i+1} \in \Sigma$ for every $1 \leqslant i \leqslant N$.*

Proof. Statements (i) and (ii) are just rephrasings of the analogous statements of Proposition 2.5.5 in [Ka]. Statement (iii) follows from the fact that, if $\gamma \notin \Sigma$ is a positive root, then there exists $\alpha \in \Sigma$ such that $\gamma - \alpha$ is a (positive) root. To prove this fact, assume to the contrary that $\gamma - \alpha \notin \Delta$ for any $\alpha \in \Sigma$. This implies that $\mathfrak{g}_\gamma$ is a lowest weight space of a proper submodule of the component of the adjoint representation of $\mathfrak{g}$ containing $\mathfrak{g}_\gamma$. This contradiction completes the proof. □

3. Bases

In this section we prove Theorem 1.3 and establish some result about bases of root systems that will be needed in the rest of the paper. The definition of a base of Δ extends in a natural way to subsets S of R.

Definition 3.1. A subset $S \subset R$ is a *base* of R if S is linearly independent and any element of R is an integral linear combination of elements of S with all coefficients in $\mathbb{Z}_{\geqslant 0}$ or $\mathbb{Z}_{\leqslant 0}$. The cardinality of S is called *rank of R*.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We start with the observation that $\ker \pi = \langle \Delta_{\mathfrak{m}} \rangle$. Indeed, for any $\alpha \in \Delta_{\mathfrak{m}}$ and every $t \in \mathfrak{t}$, we have $\alpha(t) = 0$ because $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$, proving $\langle \Delta_{\mathfrak{m}} \rangle \subset \ker \pi$. Moreover, Lemma 2.1 applied to $\mathfrak{m}$ implies that

$$\dim \ker \pi = \dim \mathfrak{h}^* - \dim \mathfrak{t}^* = \dim \mathfrak{h} - \dim \mathfrak{t} = \mathrm{rk}\, \mathfrak{m} = \dim \langle \Delta_{\mathfrak{m}} \rangle,$$

proving that $\ker \pi = \langle \Delta_{\mathfrak{m}} \rangle$.

Let $\Sigma = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l\}$ be a base of Δ, where $\alpha_1, \ldots, \alpha_k \in \Delta_{\mathfrak{m}}$ and $\beta_1, \ldots, \beta_l \in \Delta \backslash \Delta_{\mathfrak{m}}$. Then $\pi(\Sigma) \backslash \{0\} = \{\pi(\beta_1), \ldots, \pi(\beta_l)\}$. It is clear that every element of R is an integral linear combination of elements of $\pi(\Sigma) \backslash \{0\}$ with all coefficients in $\mathbb{Z}_{\geqslant 0}$ or $\mathbb{Z}_{\leqslant 0}$. Hence

$\pi(\Sigma)\backslash\{0\}$ is a base of R if and only if $\pi(\beta_1), \ldots, \pi(\beta_l)$ are linearly independent. The latter itself is equivalent to $\langle\beta_1, \ldots, \beta_l\rangle \cap \ker \pi = 0$. Since $\ker \pi = \langle\Delta_{\mathfrak{m}}\rangle$, we decude that $\pi(\Sigma)\backslash\{0\}$ is a base of R if and only if $\langle\beta_1, \ldots, \beta_l\rangle \cap \langle\Delta_{\mathfrak{m}}\rangle = 0$ and the last condition is equivalent to $\langle\alpha_1, \ldots, \alpha_k\rangle = \langle\Delta_{\mathfrak{m}}\rangle$. Since $\alpha_1, \ldots, \alpha_k$ are linearly independent, $\langle\alpha_1, \ldots, \alpha_k\rangle = \langle\Delta_{\mathfrak{m}}\rangle$ is equivalent to $\{\alpha_1, \ldots, \alpha_k\}$ being a base of $\Delta_{\mathfrak{m}}$.

Next we prove that every base of R is of the form $\pi(\Sigma)\backslash\{0\}$. Let S be a base of R and consider the set

(3.1) $P := \{\alpha \in \Delta \,|\, \pi(\alpha)$ is a combination of elements of S with coefficients in $\mathbb{Z}_{\geqslant 0}\}$.

It is a parabolic subset of Δ. Lemma 2.2 and Proposition 2.10 in [DFG] imply that there exists a base $\Sigma = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l\}$ such that $\{\alpha_1, \ldots, \alpha_k\}$ is a basis of $\Delta_{\mathfrak{m}}$ and $\{\beta_1, \ldots, \beta_l\}$ is a subset of P. Then both S and $\{\pi(\beta_1), \ldots, \pi(\beta_l)\}$ are bases of R which generate the same cone in the $\mathbb{Q}$-vector space spanned by R. Hence $S = \{\pi(\beta_1), \ldots, \pi(\beta_l)\} = \pi(\Sigma)\backslash\{0\}$.

Finally, the existence of bases of R follows immediately from the existence of bases of Δ containing bases of $\Delta_{\mathfrak{m}}$ which follows from Lemma 2.2. □

Bases of R are closely related to positive systems in R. A subset $R^+ \subset R$ is called *a positive system* if

(i) $R = R^+ \cup R^-$, (ii) $R^+ \cap R^- = \emptyset$, (iii) $\nu_1, \nu_2 \in R^+, \nu_1 + \nu_2 \in R \implies \nu_1 + \nu_2 \in R^+$.

An element $\nu \in R^+$ is called *indecomposable* if it cannot be written as $\nu = \nu_1 + \nu_2$ with $\nu_1, \nu_2 \in R^+$.

Proposition 3.2. *If S is a base of R, then the subset of R consisting of $\mathbb{Z}_{\geqslant 0}$-combinations of S is a positive system whose indecomposable elements are the elements of S. Conversely, if R^+ is a positive system in R, then the set of indecomposable elements is a base of R.*

Proof. The first statement is obvious. To prove the converse, notice that

$$P := \{\alpha \in \Delta \,|\, \pi(\alpha) \in R^+\}$$

is a parabolic subset of Δ and then proceed as in the paragraph following (3.1) above. □

Corollary 3.3. *Let $\nu \in R$ be a primitive root, i.e., $k\nu \in R$ implies $|k| \geqslant 1$. Then there exists a base S of R containing ν.*

Proof. Consider a hyperplane $\mathcal{H}$ in the real vector space spanned by R which contains no elements of R and such that $\pm\nu$ are the elements of R closest to $\mathcal{H}$. Let R^+ be the set of elements of R on the same side of $\mathcal{H}$ as ν. Then R^+ is a positive system in R and $\nu \in R$ is indecomposable. By Proposition 3.2, ν belongs to the base of R^+. □

It is convenient to introduce some additional notation in which a functorial property of the projection $\Delta \longrightarrow \bar{R}$ can be expressed naturally. Let I be a base of $\Delta_{\mathfrak{m}}$ and assume that $I \neq \emptyset$, i.e., that $\mathfrak{t} \neq \mathfrak{h}$. Then $\mathfrak{t}^* \cong \mathfrak{h}^*/\langle I\rangle$ and $\bar{R} = \pi(\Delta)$. Slightly abusing notation we denote $\bar{R}$ by $\Delta/\langle I\rangle$. Notice that the notation $\Delta/\langle I\rangle$ carries more information than just the Kostant root system R, namely the base I of $\Delta_{\mathfrak{m}}$. We will use the notation $\Delta/\langle I\rangle$ without an explicit reference to R or $\mathfrak{m}$. We also denote the natural projection $\Delta \longrightarrow \Delta/\langle I\rangle$ by π_I. In fact, $\Delta/\langle I\rangle$ and π_I make sense as long as I is a subset of a base of Δ. In what

follows, whenever using the notation $\Delta/\langle I\rangle$ (or π_I) we will implicitly assume that I is contained in a base of Δ. Finally, we will also use the self-explanatory notation $R/\langle I\rangle$ and $\pi_I : R \longrightarrow R/\langle I\rangle$.

As a direct consequence of Theorem 1.3 we have the following statement.

Proposition 3.4. *The projection $\mathfrak{h}^* \mapsto \mathfrak{h}^*/\langle I\rangle$ induces a bijection*

$$\{\text{bases of } \Delta \text{ containing } I\} \quad \longleftrightarrow \quad \{\text{bases of } \Delta/\langle I\rangle\}. \qquad \square$$

Next we establish a functorial property of projections of root systems which will play a central role in the rest of the paper.

Proposition 3.5. *Consider Kostant root systems R_1 and R_2 with corresponding projections $\pi_1 : \Delta \longrightarrow \bar{R}_1$ and $\pi_2 : \Delta \longrightarrow \bar{R}_2$. Then the following are equivalent:*

(i) *There exists a surjection $\pi_{12} : \bar{R}_1 \longrightarrow \bar{R}_2$ such that $\pi_2 = \pi_{12} \circ \pi_1$;*
(ii) *there exist a base Σ of Δ and nested subsets I_1 and I_2 of Σ, i.e. $I_1 \subset I_2 \subset \Sigma$ such that $\bar{R}_1 = \Delta/\langle I_1\rangle$ and $\bar{R}_2 = \Delta/\langle I_2\rangle$.*

Assuming that (i) *and* (ii) *hold, $S_2 \subset R_2$ is a base of R_2 if and only if $S_2 = \pi_{12}(S_1) \setminus \{0\}$ for some base S_1 of R_1 containing $\pi_1(I_2 \setminus I_1)$.*

Proof. The fact that (ii) implies (i) is clear. To prove that (i) implies (ii), assume that π_{12} exists. Then $\Delta_{\mathfrak{m}_1} \subset \Delta_{\mathfrak{m}_2}$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the reductive subalgebras of $\mathfrak{g}$ corresponding to R_1 and R_2 respectively. We can then choose $I_1 \subset I_2$ such that $\Delta_{\mathfrak{m}_j} = \langle I_j\rangle$ for $j = 1, 2$. The existence a base Σ containing I_2 (and hence $I_1 \subset I_2 \subset \Sigma$) follows from Theorem 1.3.

If S_2 is a base of R_2, then $S_2 = \pi_2(\Sigma)\setminus\{0\}$ for some base Σ of Δ containing I_2. Since $I_1 \subset I_2$, then $S_1 := \pi_1(\Sigma) \setminus \{0\}$ is a base of R_1 containing $\pi_1(I_2 \setminus I_1)$. Conversely, if S_1 is a base of R_1 containing $\pi_1(I_2 \setminus I_1)$ and $S_2 = \pi_{12}(S_1) \setminus \{0\}$, then $S_1 = \pi_1(\Sigma) \setminus \{0\}$ for some base Σ containing I_2 (hence, $\Sigma \supset I_2 \supset I_1$). Then, applying Theorem 1.3 and the equivalence of (i) and (ii), we conclude that $\pi_2(\Sigma) \setminus \{0\} = \pi_{12}(\pi_1(\Sigma)) \setminus \{0\} = \pi_{12}(S_1) \setminus \{0\} =: S_2$ is a base of R_2. $\square$

Note that Proposition 3.5 can be generalized by replacing Δ with any Kostant root system R. However we will not need a statement of this generality, so we omit this discussion. Moreover, the constructions and results so far can be carried over to infinite dimensional Kac-Moody Lie superalgebras. Since the modifications are minimal, we leave the details to the reader.

4. Graphs of Kostant root systems

In this section we want to associate various graphs to a Kostant root system R and relate their properties to the properties of the decomposition (1.1).

Definition 4.1. Let R be a Kostant root system with a base S. The graph $\Gamma_{R,S}$ is defined as follows: the vertices of $\Gamma_{R,S}$ are the elements of R and there is an edge between two vertices μ and ν if and only if $\mu - \nu \in \pm S$. We say that the edge between μ and ν is *labelled* by the corresponding simple root $\mu - \nu$ or $\nu - \mu$.

Definition 4.2. In the notation above, if $I \subset S$, $\pi_I : R \longrightarrow R/\langle I\rangle$, and $\tau \in R/\langle I\rangle$, the graph $\Gamma^{\tau}_{R,S,I} \subset \Gamma_{R,S}$ is defined as follows: $v(\Gamma^{\tau}_{R,S,I}) = \pi^{-1}(\tau)$ and there is an edge between two vertices μ and ν if and only if $\mu - \nu \in \pm I$.

Clearly the graph $\Gamma^{\nu}_{\Delta,\Sigma,I}$, where $I \subset \Sigma$ are bases of $\Delta_{\mathfrak{m}}$ and Δ respectively, is related to the Kostant root space $\mathfrak{g}_{\nu}$. More precisely we have the following statement.

Proposition 4.3. *Let $\mathfrak{g}$ be a reductive Lie superalgebra with decomposition* (1.1)*. Assume that Σ is a base of Δ containing a base I of $\Delta_{\mathfrak{m}}$. If $\nu \in R$, then $\mathfrak{g}_{\nu}$ is an irreducible $\mathfrak{m}$-module if and only if the graph $\Gamma^{\nu}_{\Delta,\Sigma,I}$ is connected.*

Proof. Note that $\Gamma^{\nu}_{\Delta,\Sigma,I}$ is the set of weights of the $\mathfrak{m}$-module $\mathfrak{g}_{\nu}$. If $\Gamma^{\nu}_{\Delta,\Sigma,I}$ is connected then every weight space of $\mathfrak{g}_{\nu}$ generates all of its weight spaces and since any submodule of $\mathfrak{g}_{\nu}$ is a weight submodule, we conclude that $\mathfrak{g}_{\nu}$ is irreducible. Conversely, if $\Gamma^{\nu}_{\Delta,\Sigma,I}$ is not connected, every connected component contains a highest weight and hence $\mathfrak{g}_{\nu}$ is not irreducible. □

Definition 4.4. Let R be a Kostant root system with a base S and let I be a subset of S with corresponding projection π_I. Set $S/\langle I\rangle := \pi_I(S)\backslash\{0\}$. If Γ is a subgraph of $\Gamma_{R,S}$ such that $v(\Gamma) \cap \langle I\rangle = \emptyset$, we define the subgraph $\pi(\Gamma)$ of $\Gamma_{R/\langle I\rangle,S/\langle I\rangle}$ as follows: $v(\pi(\Gamma)) = \pi(v(\Gamma))$ and $e(\pi(\Gamma)) = \pi(e(\Gamma))\backslash\{\text{loops}\}$, where a loop is an edge whose vertices coincide.

The following statement is obvious.

Proposition 4.5. *In the notation of Definition 4.4, if Γ is connected, so is $\pi(\Gamma)$.* □

We now want to make few observations on graphs that will be instrumental for proving Theorem 1.3 and the irreducibility of the $\mathfrak{m}$-module $\mathfrak{g}_{\nu}$. Let Σ be a base of Δ and assume that $I \subset J \subset \Sigma$. By Proposition 3.5 we have the commutative diagram

(4.1)

$$\begin{tikzcd} & \Delta \arrow[dl, "\pi_I"'] \arrow[dr, "\pi_J"] & \\ \Delta/\langle I\rangle \arrow[rr, "\pi_{I,J}"] & & \Delta/\langle J\rangle \end{tikzcd}$$

where π_I, π_J, and $\pi_{I,J}$ are the projections described by Proposition 3.5.

For a nonzero element $\nu \in \Delta/\langle J\rangle$ consider the graphs $\Gamma^{\nu}_{\Delta,\Sigma,J} \subset \Gamma_{\Delta,\Sigma}$ and $\Gamma^{\nu}_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle} \subset \Gamma_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle}$. The lemma below compares $\pi_I(\Gamma^{\nu}_{\Delta,\Sigma,J})$ and $\Gamma^{\nu}_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}$ as subgraphs of $\Gamma_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle}$.

Lemma 4.6. *In the notation above we have*

(i) $v(\pi_I(\Gamma^{\nu}_{\Delta,\Sigma,J})) = v(\Gamma^{\nu}_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle})$;
(ii) $e(\pi_I(\Gamma^{\nu}_{\Delta,\Sigma,J})) \subset e(\Gamma^{\nu}_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle})$;
(iii) *if $\Gamma^{\tau}_{\Delta,\Sigma,K}$ is connected for every $K \subset \Sigma$ and every nonzero $\tau \in \Delta/\langle K\rangle$, then $e(\pi_I(\Gamma^{\nu}_{\Delta,\Sigma,J})) = e(\Gamma^{\nu}_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle})$.*

Proof. Statements (i) and (ii) follow from the commutativity of (4.1).

To prove (iii) we first consider the case when the cardinality $J \setminus I$ equals 1. Let $J = I \cup \{\alpha\}$. Set $\lambda := \pi_I(\alpha) \in \Sigma/\langle I\rangle$ and fix $\mu \in v(\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle})$. Note that $J/\langle I\rangle = \{\lambda\}$. Then $v(\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}) \subset \mu + \mathbb{Z}\lambda$, say

$$v(\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}) = \{\mu + a\lambda \mid a \in A\}$$

for some finite set $A \subset \mathbb{Z}$. Moreover, two vertices $\mu + a'\lambda$ and $\mu + a''\lambda$ are connected if and only if $|a' - a''| = 1$. Furthermore,

$$v(\Gamma^\nu_{\Delta,\Sigma,J}) = \sqcup_{a\in A}\, v(\Gamma^{\mu+a\lambda}_{\Delta,\Sigma,I}). \tag{4.2}$$

The edges of $\Gamma^\nu_{\Delta,\Sigma,J}$ are labeled by elements of $J = I \cup \{\alpha\}$; more precisely, elements of I label edges within components $\Gamma^{\mu+a\lambda}_{\Delta,\Sigma,I}$ and α labels edges between different components in (4.2). Since $\Gamma^\nu_{\Delta,\Sigma,J}$ is connected, A is an interval and any two consecutive components in (4.2) are connected by α. This proves that every edge of $\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}$ lifts to an edge of $\Gamma^\nu_{\Delta,\Sigma,J}$, i.e., that $e(\pi_I(\Gamma^\nu_{\Delta,\Sigma,J})) = e(\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle})$.

To complete the proof for general $I \subset J \subset \Sigma$, consider an edge e of $\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}$ between the vertices μ' and μ'' labeled by $\lambda \in J/\langle I\rangle$. Let $\lambda = \pi_I(\alpha)$ with $\alpha \in J$. Set $K := I \cup \{\alpha\}$ and $\tau := \pi_{I,K}(\mu') = \pi_{I,K}(\mu'')$. We have already proved that $e(\pi_I(\Gamma^\tau_{\Delta,\Sigma,K})) = e(\Gamma^\tau_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,K/\langle I\rangle})$. In particular e lifts to an edge of $\Gamma^\tau_{\Delta,\Sigma,K}$ labeled by α. Since $v(\Gamma^\tau_{\Delta,\Sigma,K}) \subset v(\Gamma^\nu_{\Delta,\Sigma,J})$ and $\alpha \in J$, we conclude that e lifts to an edge of $\Gamma^\nu_{\Delta,\Sigma,J}$. □

We complete this section by recording a result which is crucial for the proof of Theorem 1.4 (i).

Proposition 4.7. *In the notation of Lemma 4.6, if $\Gamma^\nu_{\Delta,\Sigma,J}$ is connected, so is $\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}$.*

Proof. Assume that $\Gamma^\nu_{\Delta,\Sigma,J}$ is connected. Proposition 4.5 implies that $\pi_I(\Gamma^\nu_{\Delta,\Sigma,J})$ is connected. Since, by Lemma 4.6 (i) and (ii), $\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}$ has the same vertices as $\pi_I(\Gamma^\nu_{\Delta,\Sigma,J})$ and, possibly, more edges, we conclude that $\Gamma^\nu_{\Delta/\langle I\rangle,\Sigma/\langle I\rangle,J/\langle I\rangle}$ is connected as well. □

5. Proof of Theorem 1.4

We start with proving part (i) of Theorem 1.4 in the case when $\mathfrak{m} \subset \mathfrak{g}_{\bar{0}}$. The proof is based on a result of Stembridge, see [St], Section 1. Let Λ denote the weight lattice of a semisimple Lie algebra $\mathfrak{s}$ and let $Q \subset \Lambda$ denote the root lattice of $\mathfrak{s}$. According to Corollary 1.13 in [St], each nontrivial coset of Λ/Q contains exactly one minuscule weight. Further, Remark 1.11 and Proposition 1.12 in [St] combine into the following statement.

Proposition 5.1. *Let $\Lambda/Q = \{\omega_1 = [0], \omega_2, \dots, \omega_l\}$. Then there exist $\lambda_1, \dots, \lambda_l$, with $\lambda_i \in \omega_i$, such that any simple module W of $\mathfrak{s}$, whose support is in ω_i, has λ_i as one of its weights.* □

Proposition 5.2. *Theorem 1.4 (i) holds under the additional assumption that $\mathfrak{m} \subset \mathfrak{g}_{\bar{0}}$.*

Proof. Let $\mathfrak{s} = [\mathfrak{m}, \mathfrak{m}]$. Then $\mathfrak{s}$ is a semisimple Lie algebra, $\mathfrak{g}_\nu$ is an $\mathfrak{s}$-module which is irreducible if and only if $\mathfrak{g}_\nu$ is an irreducible $\mathfrak{m}$-module, since $\mathfrak{t}$ acts semisimply on it and $\mathfrak{m} = \mathfrak{s} \oplus \mathfrak{t}$. Fix a Cartan subalgebra $\mathfrak{h}_\mathfrak{s}$ of $\mathfrak{s}$. Notice that all the weights of $\mathfrak{g}_\nu$, with respect to $\mathfrak{h}_\mathfrak{s}$, belong to the same coset of Λ/Q and that the weight spaces of $\mathfrak{g}_\nu$ are all one dimensional. Moreover, $\mathfrak{g}_\nu$ is a semisimple $\mathfrak{s}$-module. Assuming that it is not irreducible,

we would conclude that it contains (at least) two simple submodules W' and W''. Then, by Proposition 5.1, there would be λ_i which is a weight space of both W' and W''. This contradicts the fact that all the weight spaces of $\mathfrak{g}_\nu$ are one dimensional. $\square$

Remark 5.3. In the case when $\mathfrak{g}$ is a Lie algebra, Proposition 5.2 provides a proof of Theorem 1.4 (i) which is different from Kostant's proof. Moreover, the proof above also establishes the analogous result of Greenstein, [G].

Remark 5.4. Notice that the spaces $\mathfrak{g}_\nu$ are pairwise non-isomorphic as $\mathfrak{m}$-modules, but not necessarily as $\mathfrak{s}$-modules. The simplest example is provided by the decomposition of the adjoint representation of $\mathfrak{sl}_3$ as a module over a subalgebra of $\mathfrak{sl}_3$ isomorphic to $\mathfrak{sl}_2$ and generated by root spaces.

Remark 5.5. The hypothesis $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$ is essential for the irreducibility of the $\mathfrak{m}$-modules $\mathfrak{g}_\nu$. A simple example when $\mathfrak{g}_\nu$ fail to be irreducible if $\mathfrak{t} \neq \mathfrak{z}(\mathfrak{m})$ is provided by taking $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{t} = \mathbb{C}\,t$, where $t = (E_{11} - E_{33})$. Then $\mathfrak{m} = \mathfrak{h}$, the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices and $\mathfrak{t} \neq \mathfrak{z}(\mathfrak{m}) = \mathfrak{m}$. Moreover, decomposition (1.1) in this case becomes $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where both $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are two-dimensional and hence reducible $\mathfrak{m}$-modules.

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. It is sufficient to prove the theorem when $\mathfrak{g}$ is a simple Lie algebra or is isomorphic to one of the superalgebras $\mathfrak{sl}(m|n)$ with $m \neq n$, $\mathfrak{gl}(m|m)$ for $m \geqslant 1$, $\mathfrak{osp}(m|2n)$, $\mathrm{D}(2,1;a)$, $\mathrm{G}(3)$, or $\mathrm{F}(4)$ and throughout the proof we will work under this assumption.

(i) Proposition 5.2 deals with the case when $\mathfrak{g}$ is a simple Lie algebra.

Next we consider the case when $\mathfrak{g}$ is isomorphic to one of the superalgebras $\mathfrak{sl}(m|n)$ with $m \neq n$, $\mathfrak{gl}(m|m)$ for $m \geqslant 1$, or $\mathfrak{osp}(m|2n)$. The proof in this case follows from the observation that $\bar{\Delta}$ is isomorphic to the Kostant root system $\tilde{\Delta}/\langle I \rangle$ for an appropriate simple Lie algebra $\tilde{\mathfrak{g}}$ and the funtoriality of the graphs associated with root systems. Here are the details.

We say that two Kostant root systems R' and R'' are isomorphic if there is an isomorphism $\varphi : V' \longrightarrow V''$, where V' and V'' are the vector spaces spanned by R' and R'' respectively, which restricts to a bijection between R' and R''. Comparing the roots of the Lie superalgebras, [Ka], with the Kostant roots of classical simple Lie algebras, [DR], we see that $\bar{\Delta}$ is isomorphic to $\tilde{\Delta}/\langle I \rangle$, where

(a) if $\mathfrak{g} \cong \mathfrak{sl}(m|n)$ or $\mathfrak{g} \cong \mathfrak{gl}(m|m)$, then Δ is isomorphic to $\tilde{\Delta}$ with $\tilde{\mathfrak{g}} = \mathfrak{sl}_{m+n}$ or $\tilde{\mathfrak{g}} = \mathfrak{sl}_{2m}$ respectively;
(b) if $\mathfrak{g} \cong \mathfrak{osp}(2m|2n)$, then Δ is isomorphic to $\tilde{\Delta}/\langle I \rangle$ with $\tilde{\mathfrak{g}} = \mathfrak{so}_{2N}$ for appropriate N and I (this case is referred as "type D, case I" in [DR]);
(c) if $\mathfrak{g} \cong \mathfrak{osp}(2m+1|2n)$, then Δ is isomorphic to $\tilde{\Delta}/\langle I \rangle$ with $\tilde{\mathfrak{g}} = \mathfrak{so}_{2N+1}$ for appropriate N and I (this case is referred as "type B, case I" in [DR]).

Note that the choices for $\tilde{\Delta}$ and I above are not unique.

Let now that R be a Kostant root system of $\mathfrak{g}$ and assume that $\bar{\Delta} \cong \tilde{\Delta}/\langle I \rangle$ as above. Then Proposition 3.5 implies that there exist a base $\tilde{\Sigma}$ of $\tilde{\Delta}$ and subsets $I \subset J \subset \tilde{\Sigma}$ for which diagram (4.1) takes the form

$$\begin{tikzcd} & \tilde{\Delta} \arrow[dl, "\pi_I"'] \arrow[dr, "\pi_J"] & \\ \bar{\Delta} = \Delta/\langle I \rangle \arrow[rr, "\pi_{I,J}"] & & \Delta/\langle J \rangle = \bar{R}\,. \end{tikzcd}$$

Let $\nu \in R$. Then $\tilde{\mathfrak{g}}_\nu$ is irreducible and, by Proposition 4.3, the graph $\Gamma^\nu_{\tilde{\Delta},\tilde{\Sigma},J}$ is connected. Hence, by Proposition 4.7, the graph $\Gamma^\nu_{\Delta,\tilde{\Sigma}/\langle I \rangle,J/I}$ is connected too. Finally, applying Proposition 4.3 again we conclude that $\mathfrak{g}_\nu$ is irreducible thus completing the proof in this case.

When $\mathfrak{g}$ is isomorphic to one of the superalgebras $\mathrm{D}(2,1;a)$, $\mathrm{G}(3)$, or $\mathrm{F}(4)$, the irreducibility of $\mathfrak{g}_\nu$ is a direct verification which, with a little bit of care, can be carried out by hand. We provide the details of this calculation in the Appendix.

(ii) Let μ, ν and $\mu + \nu \in R$. Since $\mathfrak{g}_{\mu+\nu}$ is an irreducible $\mathfrak{m}$-module, it suffices to prove that $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \neq 0$.

First we consider the case when μ is primitive, i.e., $k\mu \in R$ implies $|k| \geqslant 1$. By Corollary 3.3 we may choose a base S of R such that $\mu \in S$. Fix a base Σ of Δ and $I \subset \Sigma$ such that $\bar{R} = \Delta/\langle I \rangle$ and $S = \Sigma/\langle I \rangle$. Let $\alpha \in \Delta$ be such that $\pi_I(\alpha) = \mu$ and let $J := I \cup \{\alpha\}$. Lemma 4.6 applies and we conclude that any edge of $\Gamma^\tau_{R,S,I}$ lifts to an edge of $\Gamma^\tau_{\Delta,\Sigma,J}$, where $\tau := \pi_{IJ}(\nu) = \pi_{IJ}(\mu + \nu)$. The assumption that μ, ν, and $\mu + \nu \in R$ implies that ν and $\mu + \nu$ are vertices of $\Gamma^\tau_{R,S,I}$ connected by an edge labeled by μ. This edge lifts to an edge labeled by α connecting two vertices of $\Gamma^\tau_{\Delta,\Sigma,J}$ – one in $\Gamma^\nu_{\Delta,\Sigma,I}$ and another one in $\Gamma^{\mu+\nu}_{\Delta,\Sigma,I}$. Say, $\beta \in \Gamma^\nu_{\Delta,\Sigma,I}$ and $\gamma \in \Gamma^{\mu+\nu}_{\Delta,\Sigma,I}$. In other words, we have $\alpha, \beta, \gamma \in \Delta$ such that $\alpha + \beta = \gamma$ and $\pi_I(\alpha) = \mu, \pi_I(\beta) = \nu$, and $\pi_I(\gamma) = \mu + \nu$. Since, by Proposition 2.3 (ii), $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_\gamma \neq 0$, we conclude that

$$[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \supset [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_\gamma \neq 0,$$

which completes the proof in this case.

The case when ν is primitive is treated in the same way. Before we proceed to the case when neither μ nor ν is primitive, we need to introduce some notation.

Let Σ be a base of Δ and let $\alpha \in \Sigma$. For $s \in \mathbb{Z}$, consider the set

$$\Delta[\alpha, s] := \{\beta \in \Delta \mid \text{ the coefficient of } \alpha \text{ in the } \Sigma - \text{decomposition of } \beta \text{ is divisible by } s\}.$$

The axiomatic description of the root systems of Lie superalgebras in [Se1] implies that $\Delta[\alpha, s]$ is a root system itself. If $\bar{R} = \Delta/\langle I \rangle$ is a Kostant root system, set $R[\alpha, s] := R \cap \Delta[\alpha, s]$. Notice that, if $\alpha \notin I$, then I is contained in a base of $\Delta[\alpha, s]$ because every element of Δ which is in the span of I belongs to $\Delta[\alpha, s]$. Moreover, $\bar{R}[\alpha, s] = \Delta[\alpha, s]/\langle I \rangle$.

Assume now that $\mu = s_1\mu'$ and $\nu = s_2\nu'$, where μ' and ν' are primitive. Fix a base Σ of Δ and $I \subset \Sigma$ such that $\bar{R} = \Delta/\langle I \rangle$ and $\alpha, \beta, \gamma \in \Delta$ with $\pi_I(\alpha) = \mu, \pi_I(\beta) = \nu, \pi_I(\gamma) = \mu + \nu$. We may assume that both μ and ν are positive with respect to $\Sigma/\langle I \rangle$ and hence $s_1, s_2 \in \mathbb{Z}_{>0}$. The assumption $\mu = s_1\mu'$ and $\nu = s_2\nu'$ implies that, for every $\theta \in \Sigma \setminus I$, s_1

and s_2 divide the respective coefficients c_1^θ and c_2^θ of θ in the decompositions of α and β with respect to the base Σ. Assume first that, for some $\theta \in \Sigma \setminus I$, the coefficients c_1^θ and c_2^θ have a common divisor $d > 0$ and consider $\Delta[\theta, d]$. Its cardinality is smaller than that of Δ and $\mu, \nu, \mu+\nu \in R[\theta, d]$. Hence an induction on the cardinality of Δ, using the result for $\mu, \nu, \mu+\nu \in R[\theta, d]$, implies that $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \neq 0$. Assume now that, for every $\theta \in \Sigma \setminus I$, c_1^θ and c_2^θ are coprime. This implies in particular that $c_1^\theta \geqslant s_1$, $c_2^\theta \geqslant s_2$, and s_1 and s_2 are coprime. Then, for every $\theta \in \Sigma \setminus I$, we have $c_3^\theta = c_1^\theta + c_2^\theta \geqslant s_1 + s_2 \geqslant 5$, where c_3^θ is the coefficient of θ in the decompositions of γ with respect to the base Σ.

A quick inspection of the highest roots of root systems shows that only the root system of E_8 allows coefficients greater than 4. In the labeling of [B], the highest root of E_8 is

$$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$$

and, moreover, there is no root in which both α_4 and α_5 enter with coefficient 5. Hence, the only cases we have not reduced to a smaller root system yet is when $I = \Sigma \setminus \{\alpha_4\}$ or $I = \Sigma \setminus \{\alpha_5\}$ and $\mu = 2\mu'$ and $\nu = 3\mu'$. In both cases it is easy to find roots $\alpha \in \pi_I^{-1}(\mu)$, $\beta \in \pi_I^{-1}(\nu)$ such that $\alpha + \beta$ is a root. This completes the proof.

(iii) Without lost of generality we may assume that ν is primitive. Fix a base Σ of Δ, $I \subset \Sigma$ such that $\bar{R} = \Delta/\langle I \rangle$, and $\alpha \in \Sigma \setminus I$ with $\pi_I(\alpha) = \nu$. We have to consider separately two cases for μ.

Assume first that μ is not proportional to ν. Set $J := I \cup \{\alpha\}$ and $\tau := \pi_{IJ}(\mu) = \pi_{IJ}(\mu + j\nu)$, see diagram (4.1). By part (i) above, $\Gamma^\tau_{\Delta,\Sigma,J}$ is connected and so is $\Gamma^\tau_{R,\Sigma/\langle I\rangle,\Sigma/\langle I\rangle}$ by Proposition 4.7. Since the vertices of the latter graph are all elements of R of the form $\mu + j\nu$ and the edges are labeled by ν, we conclude that $\mu, \mu + k\nu \in R$ implies that $\mu + j\nu$ for every j between 0 and k.

If μ is proportional to ν, it is sufficient to prove that $k\nu \in R$ with $k \in \mathbb{Z}_{>0}$ implies $j\nu \in R$ for every $0 \leqslant j \leqslant k$. Let $\gamma \in \pi_I^{-1}(k\nu)$. Then γ is in the span of J and the coefficient of α in the decomposition of γ with respect to $J \subset \Sigma$ equals k. Consider a sequence $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_N$ as in Proposition 2.3 (iii). Then the projection under π_I of the set $\{\gamma_j \mid 1 \leqslant j \leqslant N+1\}$ is exactly the set $\{j\nu \mid 0 \leqslant j \leqslant k\}$. □

We complete this section with a remark justifying why we consider $\mathfrak{gl}(m|m)$ instead of $\mathfrak{psl}(m|m)$.

Remark 5.6. If $\mathfrak{g} = \mathfrak{sl}(m|m)$, then $\mathfrak{g}_\nu$ may be a reducible $\mathfrak{m}$-module. Here is an example for $m \geqslant 3$ in the usual presentation of $\mathfrak{sl}(m|m)$ as $2m \times 2m$-matrices. Let $\mathfrak{t}$ be the span of $H_1 := E_{11} - E_{22}, H_2 := E_{33} + \ldots + E_{2m-3,2m-3}$, and $H_3 := E_{2m-1,2m-1} - E_{2m,2m}$. Then $\mathfrak{m} = \mathfrak{h} + \mathfrak{g}'$, where $\mathfrak{g}' \cong \mathfrak{sl}(m-2|m-2)$ is the subalgebra of traceless $(m-2) \times (m-2)$-matrices indexed by $3 \leqslant i, j \leqslant 2m-3$. It is immediate to verify that $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$. Define $\nu \in \mathfrak{t}^*$ by $\nu(H_1) = 1$, $\nu(H_2) = 0$, and $\nu(H_3) = -1$. Then $\mathfrak{g}_\nu$ is spanned by $E_{1,2m-1}$ and $E_{2m,2}$ and each of these vectors spans an $\mathfrak{m}$-submodule of $\mathfrak{g}_\nu$. We leave it to the reader to check that, after projecting, we obtain an example for $\mathfrak{g} = \mathfrak{psl}(m|m)$ as well.

6. Hermitian symmetric pairs and admissible systems

In this section we provide an application of the results of section 3 to Hermitian symmetric pairs. If $\mathfrak{g}$ is a simple Lie algebra and $(\mathfrak{g}, \mathfrak{k})$ is a non-compact Hermitian symmetric pair, then $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-, \tag{6.1}$$

where $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{g}$ with one dimensional center $\mathfrak{t}$, $\mathfrak{p}^\pm$ are abelian subalgebras of $\mathfrak{g}$ which are also irreducible $\mathfrak{k}$-modules. (For a detailed discussion on Hermitian symmetric pairs, see [H-C].) It is clear that $\mathfrak{k} = \mathfrak{c}(\mathfrak{t})$ and $\mathfrak{t} = \mathfrak{z}(\mathfrak{k})$. Moreover, the $\mathfrak{t}$-roots of $\mathfrak{g}$ are $R = \{\pm\nu\}$ with $\mathfrak{g}^{\pm\nu} = \mathfrak{p}^\pm$. In other words, the decomposition (6.1) is just a particular case of (1.1). By Theorem 1.3, R is the projection of a base S of Δ and $\nu = \pi(\alpha)$ for some $\alpha \in S$. The fact that $R = \{\pm\nu\}$ implies that the coefficient of α in the decomposition of any positive root of $\mathfrak{g}$ into a combination of simple roots must equal either 0 or 1. This condition is equivalent to the condition that the coefficient of α in the decomposition the highest root of $\mathfrak{g}$ equals 1, providing the list of all non-compact Hermitian symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ in the case when $\mathfrak{g}$ is a simple Lie algebra. The compact Hermitian symmetric pairs are just the pairs $(\mathfrak{g}, \mathfrak{g})$.

In [CFV] Carmeli, Fioresi, and Varadarajan introduced a generalization of Hermitian symmetric pairs for basic classical Lie superalgebras $\mathfrak{g} \ncong \mathfrak{psl}(m|m)$ (see also [CF]). More precisely, $(\mathfrak{g}, \mathfrak{k})$ is a *Hermitian symmetric pair* corresponding to the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{6.2}$$

if the following conditions are satisfied:

(i) $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \subset \mathfrak{g}_{\bar{0}}$ contains a Cartan subalgebra of $\mathfrak{g}$;
(ii) for every simple ideal $\mathfrak{s}$ of $\mathfrak{g}_{\bar{0}}$, $(\mathfrak{s}, \mathfrak{k} \cap \mathfrak{s})$ is a Hermitian symmetric pair;
(iii) $\mathfrak{p}_{\bar{1}} = \mathfrak{g}_{\bar{1}}$.

The roots $\Delta_\mathfrak{k}$ of $\mathfrak{k}$ and $\Delta_\mathfrak{p}$ of $\mathfrak{p}$ are called respectively *compact* and *non-compact*. A positive system $\Delta^+ \subset \Delta$ is *admissible* if

$$(\Delta_\mathfrak{k} + \Delta_\mathfrak{p}^+) \cap \Delta \subset \Delta_\mathfrak{p}^+ \quad \text{and} \quad (\Delta_\mathfrak{p}^+ + \Delta_\mathfrak{p}^+) \cap \Delta \subset \Delta_\mathfrak{p}^+.$$

The following proposition appears as Theorem 3.3 in [CFV]; we provide a proof based on Theorem 1.3.

Proposition 6.1. *Every admissible positive system $\Delta_{\bar{0}}^+$ in $\Delta_{\bar{0}}$ can be extended to an admissible positive system $\Delta^+ \subset \Delta$.*

Proof. The crucial observation is that $\langle \Delta_\mathfrak{k} \rangle \cap \Delta = \Delta_\mathfrak{k}$. Note that, since $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{g}_{\bar{0}}$, $\langle \Delta_\mathfrak{k} \rangle \cap \Delta_{\bar{0}} = \Delta_\mathfrak{k}$ is automatic. However, the fact that $\langle \Delta_\mathfrak{k} \rangle$ contains no odd roots follows from the properties (i) and (ii) of the decomposition (6.2) and is not true for a general reductive subalgebra $\mathfrak{k}$ of $\mathfrak{g}_{\bar{0}}$. The proof of this property is a straightforward verification for each $\mathfrak{k}$, see [CFV] for the list of all possible $\mathfrak{k}$.

Let $S_{\bar{0}}$ be the base of $\Delta_{\bar{0}}^+$. Since $\Delta_{\bar{0}}^+$ is admissible, $I := S_{\bar{0}} \cap \Delta_\mathfrak{k}$ is a base of $\Delta_\mathfrak{k}$ and hence $\langle \Delta_\mathfrak{k} \rangle = \langle I \rangle$. Let S_R be a base of the Kostant system $\bar{R} := \Delta / \langle I \rangle$. Then S_R is a projection

of a base S of Δ containing I and the positive system Δ^+ with base S is admissible and contains $\Delta_{\bar{0}}^+$. $\square$

The root system R above (and all of its bases) can be described explicitly for each Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{k})$. As a consequence, it is easy to obtain quickly all admissible positive systems $\Delta^+ \subset \Delta$ extending a given admissible system $\Delta_{\bar{0}}^+ \subset \Delta_{\bar{0}}$. Indeed, it is sufficient to provide a list of all positive systems $R^+ \subset R$ extending $R_{\bar{0}}^+ \subset R_{\bar{0}}$, where $R_{\bar{0}} := \pi(\Delta_{\bar{0}})\backslash\{0\}$ and $R_{\bar{1}} := \pi(\Delta_{\bar{1}})$. Below we provide the complete list of Kostant root systems R and their positive systems R^+ corresponding to Hermitian symmetric pairs $(\mathfrak{g}, \mathfrak{k})$, cf. the proof of Theorem 3.3 in [CFV]. In most cases we use natural parametrizations of the positive systems R^+ and $R_{\bar{0}}^+$ by the elements of symmetric or dihedral groups of small order; depending on the particular context, we denote a group of order 2 by $\mathcal{S}_2$ or $\mathcal{Z}_2$.

6.1. If $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m \neq n$ and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ admits a real form isomorphic to $\mathfrak{su}(p,q) \oplus \mathfrak{su}(r,s) \oplus \mathfrak{u}(1)$ where $p+q = m, r+s = n$ and we allow p, q, r, or s to take value zero. We may (and will) assume that $0 \leqslant p \leqslant q \leqslant m, 0 \leqslant r \leqslant s \leqslant n$.

If both p and q are nonzero, then $\mathfrak{k} = (\mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C}) \oplus (\mathfrak{sl}_r \oplus \mathfrak{sl}_s \oplus \mathbb{C}) \oplus \mathbb{C}$ and

$$R_{\bar{0}} = \{\pm(\bar{\varepsilon}_1 - \bar{\varepsilon}_2), \pm(\bar{\delta}_1 - \bar{\delta}_2)\}, \quad R_{\bar{1}} = \{\pm(\bar{\varepsilon}_i - \bar{\delta}_j) \,|\, 1 \leqslant i, j \leqslant 2\}.$$

The positive systems R^+ are indexed by elements of the symmetric group $\mathcal{S}_4$ permuting $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\delta}_1, \bar{\delta}_2$ and the positive systems $R_{\bar{0}}^+$ are indexed by elements of $\mathcal{S}_2 \times \mathcal{S}_2 \subset \mathcal{S}_4$. Hence the positive systems R^+ extending a given $R_{\bar{0}}^+$ are indexed by the cosets in $\mathcal{S}_4/\mathcal{S}_2 \times \mathcal{S}_2$. In particular, there are 6 such extensions.

If $p = 0$ but $q \neq 0$, then $\mathfrak{k} = \mathfrak{sl}_m \oplus (\mathfrak{sl}_r \oplus \mathfrak{sl}_s \oplus \mathbb{C}) \oplus \mathbb{C}$ and

$$R_{\bar{0}} = \{\pm(\bar{\delta}_1 - \bar{\delta}_2)\}, \quad R_{\bar{1}} = \{\pm(\bar{\varepsilon} - \bar{\delta}_j) \,|\, j = 1, 2\}.$$

The positive systems R^+ are indexed by elements of the symmetric group $\mathcal{S}_3$ and the positive systems $R_{\bar{0}}^+$ are indexed by elements of $\mathcal{S}_2 \subset \mathcal{S}_3$. Hence the positive systems R^+ extending a given $R_{\bar{0}}^+$ are indexed by the cosets in $\mathcal{S}_3/\mathcal{S}_2$. In particular, there are 3 such extensions.

The case when $p \neq 0$ but $q = 0$ is analogous to the case when $p = 0$ but $q \neq 0$.

If $p = q = 0$, then $\mathfrak{k} = \mathfrak{sl}_m \oplus \mathfrak{sl}_n \oplus \mathbb{C}$ and

$$R_{\bar{0}} = \emptyset, \quad R_{\bar{1}} = \{\pm(\bar{\varepsilon} - \bar{\delta})\}.$$

The positive systems R^+, and, as a tautology, the positive systems R^+ extending a given $R_{\bar{0}}^+$, are indexed by elements of the symmetric group $\mathcal{S}_2$. In particular, there are 2 such extensions.

6.2. If $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$, $m > 0$, and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ admits a real form isomorphic to $\mathfrak{so}_{\mathbb{R}}(2, 2m-1) \oplus \mathfrak{sp}_n(\mathbb{R})$ and hence $\mathfrak{k} \simeq (\mathfrak{sl}_2 \oplus \mathfrak{so}_{2m-1} \oplus \mathbb{C}) \oplus (\mathfrak{sl}_n \oplus \mathbb{C})$. Thus

$$R_{\bar{0}} = \{\pm\bar{\varepsilon}, \pm 2\bar{\delta}\}, \quad R_{\bar{1}} = \{\pm\bar{\varepsilon} \pm \bar{\delta}, \pm\bar{\delta}\}.$$

The positive systems R^+ are indexed by elements of the dihedral group $\mathcal{D}_4$ acting on the square with vertices $\{\pm\bar{\varepsilon} \pm \bar{\delta}\}$ (or, equivalently, the square with vertices $\{\pm\bar{\varepsilon}, \pm\bar{\delta}\}$). The positive systems $R_{\bar{0}}^+$ are indexed by elements of $\mathcal{Z}_2 \times \mathcal{Z}_2 \subset \mathcal{D}_4$. Hence the positive systems

R^+ extending a given $R^+_{\bar{0}}$ are indexed by the cosets in $\mathcal{D}_4/\mathcal{Z}_2 \times \mathcal{Z}_2$. In particular, there are 2 such extensions.

6.3. If $\mathfrak{g} = \mathfrak{osp}(1|2n)$ and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ admits a real form isomorphic to $\mathfrak{sp}_n(\mathbb{R})$ and hence $\mathfrak{k} \simeq \mathfrak{sl}_n \oplus \mathbb{C}$. Thus

$$R_{\bar{0}} = \{\pm 2\bar{\delta}\}, \quad R_{\bar{1}} = \{\pm\bar{\delta}\}.$$

The positive systems R^+ are indexed by elements of the group $\mathcal{Z}_2$ as are the positive systems $R^+_{\bar{0}}$. Hence there is a unique positive system R^+ extending a given $R^+_{\bar{0}}$.

6.4. If $\mathfrak{g} = \mathfrak{osp}(2|2n)$ and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ admits a real form isomorphic to $\mathfrak{so}_{\mathbb{R}}(2) \oplus \mathfrak{sp}_n(\mathbb{R})$ or a real form isomorphic to $\mathfrak{so}_{\mathbb{R}}(2) \oplus \mathfrak{sp}(n)$. In the former case $\mathfrak{k} \simeq \mathbb{C} \oplus (\mathfrak{sl}_n \oplus \mathbb{C})$ and in the latter $\mathfrak{k} \simeq \mathbb{C} \oplus \mathfrak{sp}_n$.

When $\mathfrak{k} \simeq \mathbb{C} \oplus (\mathfrak{sl}_n \oplus \mathbb{C})$, we have

$$R_{\bar{0}} = \{\pm 2\bar{\delta}\}, \quad R_{\bar{1}} = \{\pm\bar{\varepsilon} \pm \bar{\delta}\}.$$

There are 6 positive systems R^+ and 2 positive systems $R^+_{\bar{0}}$. Hence there are 3 positive systems R^+ extending a given $R^+_{\bar{0}}$.

When $\mathfrak{k} \simeq \mathbb{C} \oplus \mathfrak{sp}_n$, we have

$$R_{\bar{0}} = \emptyset, \quad R_{\bar{1}} = \{\pm\bar{\varepsilon}\}.$$

The positive systems R^+, and, as a tautology, the positive systems R^+ extending a given $R^+_{\bar{0}}$, are indexed by elements of the group $\mathcal{Z}_2$. In particular, there are 2 such extensions.

6.5. If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$, $m > 1$, and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ admits a real form isomorphic to $\mathfrak{so}_{\mathbb{R}}(2, 2m-2) \oplus \mathfrak{sp}_n(\mathbb{R})$ or a real form isomorphic to $\mathfrak{so}^*(2m) \oplus \mathfrak{sp}(n)$. In the former case $\mathfrak{k} \simeq (\mathfrak{so}_{2m-2} \oplus \mathbb{C}) \oplus (\mathfrak{sl}_n \oplus \mathbb{C})$ and in the latter $\mathfrak{k} \simeq (\mathfrak{sl}_m \oplus \mathbb{C}) \oplus (\mathfrak{sl}_n \oplus \mathbb{C})$.

When $\mathfrak{k} \simeq (\mathfrak{so}_{2m-2} \oplus \mathbb{C}) \oplus (\mathfrak{sl}_n \oplus \mathbb{C})$, we have

$$R_{\bar{0}} = \{\pm\bar{\varepsilon}, \pm 2\bar{\delta}\}, \quad R_{\bar{1}} = \{\pm\bar{\varepsilon} \pm \bar{\delta}, \pm\bar{\delta}\}$$

and when $\mathfrak{k} \simeq (\mathfrak{sl}_m \oplus \mathbb{C}) \oplus (\mathfrak{sl}_n \oplus \mathbb{C})$, we have

$$R_{\bar{0}} = \{\pm 2\bar{\varepsilon}, \pm 2\bar{\delta}\}, \quad R_{\bar{1}} = \{\pm\bar{\varepsilon} \pm \bar{\delta}\}.$$

In both cases the positive systems R^+ are indexed by elements of the dihedral group $\mathcal{D}_4$ and the positive systems $R^+_{\bar{0}}$ are indexed by elements of $\mathcal{Z}_2 \times \mathcal{Z}_2 \subset \mathcal{D}_4$. Hence the positive systems R^+ extending a given $R^+_{\bar{0}}$ are indexed by the cosets in $\mathcal{D}_4/\mathcal{Z}_2 \times \mathcal{Z}_2$. In particular, in each case there are 2 such extensions.

6.6. If $\mathfrak{g} = \mathrm{D}(2, 1; \alpha)$ and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ admits a real form isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ or a real form isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In the former case $\mathfrak{k} \simeq \mathbb{C}^3$ is a Cartan subalgebra of $\mathfrak{g}$ and in the latter $\mathfrak{k} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}$.

When $\mathfrak{k} \simeq \mathbb{C}^3$ is a Cartan subalgebra, the Kostant roots are just the usual roots

$$R_{\bar{0}} = \{\pm 2\bar{\varepsilon}_1, \pm 2\bar{\varepsilon}_2, \pm 2\bar{\varepsilon}_3\}, \quad R_{\bar{1}} = \{\pm\bar{\varepsilon}_1 \pm \bar{\varepsilon}_2 \pm \bar{\varepsilon}_3\}.$$

There are 32 positive systems R^+ and 8 positive systems $R^+_{\bar{0}}$. Hence there are 4 positive systems R^+ extending a given $R^+_{\bar{0}}$.

When $\mathfrak{k} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}$, we have

$$R_{\bar{0}} = \{\pm 2\bar{\varepsilon}\}, \quad R_{\bar{1}} = \{\pm\bar{\varepsilon}\}.$$

Both the positive systems R^+ and $R_{\bar{0}}^+$ are indexed by $\mathcal{Z}_2$ and every positive system $R_{\bar{0}}^+$ extends to a unique positive system R^+.

6.7. If $\mathfrak{g} = \mathrm{F}(4)$ and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ admits a real form isomorphic to $\mathfrak{su}_2 \oplus \mathfrak{so}_{\mathbb{R}}(2,5)$ or a real form isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_{\mathbb{R}}(7)$. In the former case $\mathfrak{k} \simeq \mathfrak{sl}_2 \oplus (\mathfrak{so}_5 \oplus \mathbb{C})$ and in the latter $\mathfrak{k} \simeq \mathbb{C} \oplus \mathfrak{so}_7$.

When $\mathfrak{k} \simeq \mathfrak{sl}_2 \oplus (\mathfrak{so}_5 \oplus \mathbb{C})$, we have

$$R_{\bar{0}} = \{\pm\bar{\varepsilon}\}, \quad R_{\bar{1}} = \{\pm\frac{1}{2}\bar{\varepsilon}\}$$

and when $\mathfrak{k} \simeq \mathbb{C} \oplus \mathfrak{so}_7$, we have

$$R_{\bar{0}} = \{\pm\bar{\delta}\}, \quad R_{\bar{1}} = \{\pm\frac{1}{2}\bar{\delta}\}.$$

In both cases both the positive systems R^+ and $R_{\bar{0}}^+$ are indexed by $\mathcal{Z}_2$ and every positive system $R_{\bar{0}}^+$ extends to a unique positive system R^+.

6.8. If $\mathfrak{g} = \mathrm{G}(3)$ and $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, then $\mathfrak{k}$ is isomorphic to $\mathfrak{k} \simeq \mathrm{G}_2 \oplus \mathbb{C}$. We have

$$R_{\bar{0}} = \{\pm 2\bar{\delta}\}, \quad R_{\bar{1}} = \{\pm\bar{\delta}\}.$$

Both the positive systems R^+ and $R_{\bar{0}}^+$ are indexed by $\mathcal{Z}_2$ and every positive system $R_{\bar{0}}^+$ extends to a unique positive system R^+.

Note finally that the discussion above in combination with Theorem 1.4 recovers Lemma 3.4 of [CFV].

Appendix. Irreducibility of $\mathfrak{g}_\nu$ for $\mathfrak{g} \cong \mathrm{D}(2,1;\alpha)$, $\mathrm{F}(4)$, and $\mathrm{G}(3)$

The irreducibility of $\mathfrak{g}_\nu$ for the exceptional Lie superalgebras amounts to a direct check. For completeness we list the bases of $\mathrm{D}(2,1;\alpha)$, $\mathrm{F}(4)$, $\mathrm{G}(3)$ up to Weyl group equivalence (W-equivalence for short) and we list the routine checks that we leave to the reader. Taking into account that the case of $\mathfrak{m}$ even is taken care by Proposition 5.2, our strategy is as follows:

Step 1: Pick $I_1 = \{\alpha\}$, α odd, in a base of Δ and consider $\pi_{I_1} : \Delta \longrightarrow R_1 = \Delta/\langle I_1 \rangle$. Check that $\mathfrak{g}_\nu$ is irreducible for all $\nu \in R_1$. Repeat this step for all possible choices of I_1.

Step 2: Enlarge I_1 to $I_2 = \{\alpha, \beta\}$, where β is chosen in any possible way in the same base of Δ as α. Consider $\pi_{I_2} : \Delta \longrightarrow R_2 = \Delta/\langle I_2 \rangle$. Check that $\mathfrak{g}_\nu$ is irreducible for all $\nu \in R_2$. Repeat this step for all possible choices of I_2.

Step 3: If $\mathfrak{g} = \mathrm{F}(4)$, enlarge I_2 to $I_3 = \{\alpha, \beta, \gamma\}$, where γ is chosen in any possible way in the same base of Δ as I_2. Consider $\pi_{I_3} : \Delta \longrightarrow R_3 = \Delta/\langle I_3 \rangle$. Check that $\mathfrak{g}_\nu$ is irreducible for all $\nu \in R_3$. Repeat this step for all possible choices of I_3.

Since the rank of exceptional Lie superalgebras is at most 4 this procedure will terminate after at most 3 steps. Notice that our choices can always be made up to W-equivalence.

We now provide the list of all possible sets I_1, I_2 and I_3 as above, leaving the necessary checks to the reader.

7.1. $\underline{\mathfrak{g} = \mathrm{D}(2,1;\alpha)}$. In this case $\Delta_{\bar{0}} = \{\pm 2\delta_i\}$, $\Delta_{\bar{1}} = \{\pm\delta_1 \pm \delta_2 \pm \delta_3\}$, $i, j = 1, 2, 3$, and $\mathfrak{g}_{\bar{0}} = A_1 \oplus A_1 \oplus A_1$. Because of the Weyl group equivalence, signs are immaterial. Notice

the symmetry in the root expressions with respect to a permutation of the indices (which is not a Weyl equivalence). With this in mind, there is only one base to consider:

$$S = \{\delta_1 - \delta_2 - \delta_3,\ 2\delta_2,\ 2\delta_3\}.$$

The only choice for I_1 is $I_1 = \{\delta_1 - \delta_2 - \delta_3\}$. After renaming δ_i, the only choice for I_2 is $I_2 = \{\delta_1 - \delta_2 - \delta_3, 2\delta_1\}$.

7.2. $\underline{\mathfrak{g} = \mathrm{G}(3)}$. In this case $\Delta_{\bar{0}} = \{\epsilon_i - \epsilon_j,\ \pm\epsilon_i,\ \pm 2\delta\}$, $\Delta_{\bar{1}} = \{\pm\epsilon_i \pm \delta,\ \pm\delta)\}$, $i, j = 1, 2, 3$, $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, and $\mathfrak{g}_{\bar{0}} = G_2 \oplus A_1$. There are four inequivalent bases:

$$\begin{aligned}
S_1 &= \{\epsilon_2,\ \epsilon_3 - \epsilon_2,\ \delta + \epsilon_1\}\,;\\
S_2 &= \{\delta - \epsilon_3,\ \epsilon_3 - \epsilon_2,\ -\delta - \epsilon_1\}\,;\\
S_3 &= \{-\delta + \epsilon_3,\ \delta - \epsilon_2,\ \epsilon_2\}\,;\\
S_4 &= \{\epsilon_3 - \epsilon_2,\ -\delta + \epsilon_2,\ \delta\}\,.
\end{aligned}$$

It is sufficient to consider $I_1 = \{\delta + \epsilon_1\}$. As for I_2, there are the following possibilities (with some redundancies):

$$\{\delta + \epsilon_1,\ \delta \pm \epsilon_2\}, \quad \{\delta + \epsilon_1,\ \alpha\},$$

where $\alpha = \delta$, ϵ_1, ϵ_2, $\epsilon_1 \pm \epsilon_2$, $\epsilon_2 \pm \epsilon_3$.

7.3. $\underline{\mathfrak{g} = \mathrm{F}(4)}$. In this case $\Delta_{\bar{0}} = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, \pm\delta\}$, $\Delta_{\bar{1}} = \{(1/2)(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)\}$, $i, j = 1, 2, 3$, and $\mathfrak{g}_{\bar{0}} = B_3 \oplus A_1$. There are six inequivalent bases:

$$\begin{aligned}
S_1 &= \left\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, -\epsilon_1\right\};\\
S_2 &= \left\{-\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)\right\};\\
S_3 &= \left\{-\epsilon_1, \tfrac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \delta), \epsilon_2 - \epsilon_3, -\tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)\right\};\\
S_4 &= \left\{\tfrac{1}{2}(-\epsilon_1 - \epsilon_2 + \epsilon_3 + \delta), -\tfrac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \delta), \tfrac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \delta), \epsilon_1 - \epsilon_2\right\};\\
S_5 &= \left\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \delta), -\epsilon_1, \delta, \epsilon_1 - \epsilon_2\right\};\\
S_6 &= \left\{\delta, \epsilon_2 - \epsilon_3, -\tfrac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \delta), \epsilon_1 - \epsilon_2\right\}.
\end{aligned}$$

Up to W-equivalence, there is only one choice for I_1:

$$I_1 = \{\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)\}.$$

Examining which bases contain I_1 we have the following choices for I_2 consisting of two odd roots (with some redundancies):

$$\begin{aligned}
I_2^1 &= \{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta),\ \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)\};\\
I_2^2 &= \{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta),\ \tfrac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \delta)\};\\
I_2^3 &= \{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta),\ \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \delta)\};\\
I_2^4 &= \{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta),\ \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta)\};\\
I_2^5 &= \{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta),\ \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta)\}.
\end{aligned}$$

Since W permutes the indices, I_2^1 is equivalent to I_2^3 and I_2^2 is equivalent to I_2^5.

The choices for I_2 consisting of an even and an odd root are:

$$\{\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta),\ \alpha\}, \qquad \alpha \in \{\delta, \epsilon_1, \epsilon_1 \pm \epsilon_2\}.$$

Notice that the case when $\alpha = \epsilon_1$ is the same as I_2^1, the case when $\alpha = \epsilon_1 + \epsilon_2$ is the same as I_2^2, and the case when $\alpha = \delta$ is the same as I_2^2, so the only new case to verify is when $\alpha = \epsilon_1 - \epsilon_2$.

Finally, we determine the possible choices for I_3. There is only one set I_3 containing 3 odd roots – the odd roots in S_4. The remaining possibilities for I_3 are:

$$I_2^i \cup \{\alpha\} \qquad \text{and} \qquad \{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\delta), \alpha, \beta\},$$

where $\alpha, \beta \in \{\delta, \epsilon_1, \epsilon_1 \pm \epsilon_2\}$.

Verifying that the spaces $\mathfrak{g}_\nu$ are irreducible $\mathfrak{m}$-modules in the cases listed above is a tedious but straightforward calculation which we omit.

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Ivan Dimitrov: Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6, Canada

E-mail address: dimitrov@queensu.ca

Rita Fioresi: Departamento di Matematica, Universitá di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy

E-mail address: rita.fioresi@unibo.it