OPTIMIZATION PROBLEMS WITH NON-LOCAL REPULSION PROBLEMI DI OTTIMIZZAZIONE CON COMPETIZIONE NON LOCALE

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Abstract. We review some optimization problems where an aggregating term is competing with a repulsive one, such as the Gamow liquid drop model, the Lord Rayleigh model for charged drops, and the ground state energy for the Hartree equation. As an original contribution, we show that for large values of the mass constraint, the ball is an unstable critical point of a functional made up as the sum of the first eigenvalue of the Dirichlet-Laplacian plus a Riesz-type repulsive energy term, in support to a recent open question raised in [MR21]

Sunto. Rivisitiamo alcuni problemi di ottimizzazione dove un termine coesivo è in competizione con uno repulsivo, come il modello della goccia liquida di Gamow, quello di Lord Rayleigh per gocce liquide cariche, a i ground state per equazioni di Hartree. Come contributo originale mostriamo che per valori grandi di massa la palla è un punto critico instabile di un funzionale costituito dalla somma del primo autovalore del Laplaciano con condizioni al bordo di tipo Dirichlet, sommato ad una energia repulsiva legata a nuclei di Riesz. Questo risultato è in supporto a una questione aperta posta in [MR21].

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1. Introduction

The present note is a follow up of a *Pini Seminar* given by the author, whose writing was invited by the organizers, whom I warmly thank. It mostly aims at offering an overview of some techniques recently developed to deal with variational models where one seeks to minimize energies where competitive repulsive terms appear. To do that, we review the

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proofs of two shape optimization problems whose variational analysis recently attracted a wide interest. In the considered models an aggregating term $\mathcal{A}$ appears coupled with a repulsive one, $\mathcal{R}$, leading to energies defined for a set $E \subset \mathbb{R}^N$ of the form

$$\mathcal{A}(E) + \mathcal{R}(E).$$

One of the by far most investigated instance in this class of shape optimization problems is the Gamow liquid drop model where $\mathcal{A}(E) = P(E)$ is the perimeter, that is the measure of the boundary of the set $E \subseteq \mathbb{R}^n$ and $\mathcal{R}$ is the Riesz-type repulsive energy

$$\mathcal{R}(E) = V(E) = \iint_{E\times E} \frac{dx\,dy}{|x-y|^{n-2}}.$$

The Gamow model seeks to study existence (or lack of existence) for the problem

$$\min\left\{P(E) + V(E) \,:\, |E| = m\right\}.$$

Remark 1.1. *Some physically relevant constants should appear in the above energy. We decided to scale them to* 1 *since they are not relevant in this note.*

Remark 1.2. On the Gamow liquid drop model
The Gamow Liquid drop model is physically unrelated to liquid drops. It is in fact an energy proposed by Gamow, and mainstreamed by its pioneering article in 1928 [Gam30], *to describe the binding energy of the nucleus of an atom, that is the energy required to detach nucleons from an atomic nucleus. The Gamow model pictures the atomic nuclei as uniformly charged* liquid *bodies where the contraposition between a surface tension, proportional to the perimeter, is compensated by the repulsive term V, the repulsion due to the charge. We refer the reader to the beautiful recent review paper* [CMT17] *about this and related models for mathematical, physical, and historical insights.*

By a simple scaling argument one can transform the above minimization problem into an equivalent one where the mass of minimizers is fixed to a constant, say the volume of the unit ball ω_n, by weighting one of the addends of the energy

$$\min\left\{P(E) + \varepsilon V(E) \,:\, |E| = \omega_n\right\}.$$

The physical prediction of the model appears more clear in this second formulation: if ε is small enough, which corresponds to small values of the mass m, the perimeter is expected to become the leading term so that existence of minimizers is expected as well. A further investigation is then to ask whether, according to the isoperimetric inequality, the ball is a rigid -that is unique, in a suitable topology- minimizer. In sharp contrast the guess when the mass -so ε- is large, is that the repulsive effect due to the presence of V scatters the mass off to infinity, leading to non-existence of minimizers.

In this note we focus mostly on existence issues (up to the original contribution given in Subsection 3.3). The rigidity of the ball for the Gamow energy was proven by Knüpfer and Muratov in [KM13] following ideas developed by Cicalese and Leonardi in [CL12] to get a (new) proof of the sharp quantitative isoperimetric inequality. Loosely speaking the strategy is to *select* a minimizing sequence (or directly a minimal set) to be regular enough. This may be done thanks to the regularizing effect ensuing from the presence of the cohesive term (for the Gamow functional, the perimeter). Later on, show that in the class of regular sets the ball is a rigid minimizer. More precisely, one seeks to show that the minimizers are $C^{1,\alpha}-$regular parametrizations on a ball, that is, sets E such that

$$\partial E = \{(1+\varphi(x))x \,:\, x \in \partial B\}\,,$$

with $\varphi : \partial B \to \mathbb{R}$ such that $\|\varphi\|_{C^{1,\alpha}}$ is small. We call such sets *nearly spherical sets.* This selection process goes often under the name of *Selection Principle.* Once a minimizing sequence of nearly spherical sets is extracted, one seeks to show by means of perturbative arguments that the ball is the only minimizer in this class of sets. This second step usually goes under the name of Fuglede expansion argument, since Fuglede developed it to show quantitative estimates for the isoperimetric inequality in the '70s for convex and nearly spherical sets [Fug89].

In fact there is a strict relation between the quest of seeking rigidity of isoperimetric problems with repulsion and that about quantitative estimates for isoperimetric inequalities. Let us hence recall what stability rigidity issue means. Given a functional $F : X \to \mathbb{R}$ which achieves its minimum on a set $\mathcal{M} \subset X$, that is such that

$$F(x) \geq F(x_{\min}) \quad \forall x_{\min} \in \mathcal{M},$$

we define a quantitative improvement of the above inequality an inequality of the form

$$F(x) - F(x_{\min}) \geq \omega(dist(x, x_{\min}))$$

where ω is a modulus of continuity and $dist$ a distance on X. The choice of ω and the distance may change depending on the considered inequality.

The most famous instance of quantitative inequalities is the isoperimetric quantitative inequality. This was proven by Fusco, Maggi, and Pratelli in [FMP08] with symmetrization techniques and by Figalli, Maggi, and Pratelli [FMP10] with mass transport ideas (this latter proof was able to deal also with anisotropic versions of the isoperimetric inequality). Eventually, the above mentioned proof by Cicalese and Leonardi was performed by means of the Selection Principle[CL12]. All these result led to the asymptotically sharp inequality

$$P(E) - P(B) \geq C(|E|)|E\Delta(B + x)|^2$$

where B is a ball with the same Lebesgue measure as E and $x \in \mathbb{R}^n$ is a point depending on E. Here $A\Delta B = (A \setminus B) \cup (B \setminus A)$ indicates the symmetric difference between sets.

The proof by Cicalese and Leonardi happens to be very flexible to deal with several other quantitative issues. In particular Brasco, De Philippis, and Velichkov in [BDPV15] exploited some of its ideas together with some regularity results for free boundary problems to get a sharp quantitative version of the Faber-Krahn inequality. Another instance of a successful application of such a strategy is the recent work by Fusco and Pratelli about stability of Riesz potential energies [FP20], stating that the Riesz energies are maximized by balls. See next subsection for details about such inequalities.

As mentioned beside being useful to get quantitative rigidity, those techniques happen to be fruitful as one considers minimization problems in presence of a repulsive term. In this note we review some results related respectively to the following two energies:

$$\begin{aligned} \mathcal{F}_R &= P(E) + Q^2 I_\alpha(E) \\ \mathcal{F}_H &= \lambda_1(E) + \varepsilon V_\alpha(E), \end{aligned} \tag{1}$$

and about related minimization problems. Here P and V_α are the perimeter and a variant of the above mentioned Riesz energy V, while λ_1 is the first eigenvalue of the Dirichlet laplacian and I_α is the inverse of a fractional capacity. See the next subsection for precise definitions.

Notations. We conclude this introductory part by collecting the notations together with some mathematical background on the functionals which play a relevant role in this note.

By E we denote a measurable subset in $\mathbb{R}^n$ and with $|E| = \mathcal{H}^n(E)$ its Lebesgue measure. $P(E)$ stands for the measure theoretic perimeter, defined by

$$P(E) = \inf \left\{ \int_E \operatorname{div} \phi \; : \; \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \; \|\phi\|_\infty \leq 1 \right\}.$$

We refer to [AFP00, Mag12] as general references about the theory of sets of finite perimeter. Here we just remark that if E has Lipschitz-regular boundary, then $P(E) = \mathcal{H}^{n-1}(\partial E)$, is the surface area of the boundary of E.

With $\lambda_1(E)$ we indicate the first eigenvalue (of the resolvent) of the Dirichlet Laplacian on E, which may be defined via the Courant variational formula

$$\lambda_1(E) = \inf_{u \in C^\infty_c(E)} \frac{\int_E |\nabla u|^2 \, dx}{\int_E u^2 \, dx}$$

With V_α we denote the Riesz energy of E

$$V_\alpha(E) = \iint_{E \times E} \frac{dx \, dy}{|x-y|^{n-\alpha}}$$

and I_α is the inverse capacitary potential defined as

$$\begin{aligned} I_\alpha(E) &= \inf_{\mu(E)=1} \iint \frac{d\mu(x) \, d\mu(y)}{|x-y|^{n-\alpha}} \\ &= \left(\inf \left\{ [u]^2_{H^{\alpha/2}} \; : \; u \in H^{\alpha/2}(\mathbb{R}^n), \; u \geq 1 \text{ on a neighbourhood of } E \; \right\} \right)^{-1}. \end{aligned} \tag{2}$$

The first minimization is done on probability measures on E (or equivalently, on Radon measures on E such that $\mu(E) = 1$), while in the second $[\cdot]_{H^{\alpha/2}}$ is the fractional seminorm defined, for a generic $s > 0$, as

$$[u]^2_{H^s} = \iint \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} \, dx \, dy.$$

Remark 1.3. About capacities *The definitions in* (2) *are two of the various definition of capacities one may find in literature. It is worth stressing that the first one comes from potential theory and is somewhat more suited to model physical problems. However these* *definitions happen to coincide if one requires some mild regularity on the sets, such as being compact. See* [LL01] *for a related discussion.*

Rearrangements and criticality of the ball. Given a measurable set E such that $|E| < +\infty$ we define by $E^\star$ the ball centered at the origin with measure $|E|$, that is

$$E^\star = B_{(|E|/\omega_n)^{1/n}}(0),$$

where $\omega_n = |B_1(0)|$. Given a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ we define its spherical rearrangement the function

$$f^\star(x) = \int_0^\infty \chi_{\{f>t\}^\star}(x)\, dt.$$

It is not difficult to show that $f^\star$ is *the* function such that $\{f^\star > t\} = \{f > t\}^\star$. This is in fact an equivalent definition of $f^\star$. By the Cavalieri formula one immediately gets that

$$\|f^\star\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)},$$

for any $p \geq 1$, while the celebrated Pólya-Szegö inequality states that the Dirichlet energy decreases through the symmetryzation process:

$$\int |\nabla f^\star|^p\, dx \leq \int |\nabla f|^p\, dx.$$

The same inequalities hold in fact for fractional Dirichlet energies

$$[f^\star]_{H^s} \leq [f]_{H^s}$$

for $s \leq 1$ see [FS08].

The last two inequalities immediately entail that the functional λ_1 is minimized by balls. This statement is referred to as Faber-Krahn inequality. Moreover I_α is maximized by balls, as it can be seen by the coarea formula (for fractional Dirichlet energies), as long as $\alpha \leq 2$.

Remark 1.4. *Also for the case $\alpha > 2$ it is conjectured that the inverse of the fractional capacity, I_α is maximized by balls, but up to our knowlodge this is still unproven.*

As the first Dirichlet-Laplacian eigenvalue, also the Riesz energy V_α has the ball as a critical point, this time as a maximum. This can again be proven by symmetrization thanks to the Riesz rearrangement inequality, see [LL01], stating that for measurable non-negative functions $f, g, h : \mathbb{R}^n \to \mathbb{R}^n$ it holds

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(y)h(x-y)\,dx\,dy \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} f^\star(x)g^\star(y)h^\star(x-y)\,dx\,dy.$$

To show this we just apply such an inequality with $h(x) = |x|^{-n+\alpha}$, $f(x) = g(x) = \chi_E(x)$.

2. The Lord Rayleigh charged drop model: a variational approach

In this section we focus on the minimization problem

$$\min\left\{\mathcal{F}_R(E) = P(E) + Q^2 I_\alpha(E) \,:\, |E| = \omega_n\right\}$$

when the charge parameter Q^2 is small. The energy, up to some physical relevant constants renormalized to 1 for expository reasons, was proposed by Lord Rayleigh in 1882 [Ray82]. In his seminal paper he proved that as long as Q surpasses an explicit threshold Q_0, the *Rayleigh limit*, then the ball is not a critical point of the energy among the subset of nearly spherical sets.

Remark 2.1. On the physical meaning of the Lord Rayleigh result *The energy proposed by Lord Rayleigh describes the behavior of an incompressible liquid drop in presence of a charge on its surface, in absence of gravity. The perimeter term is the leading order of the surface tension, while the capacitary term I_α is the electrostatic capacity of the set. The Rayleigh threshold has the following meaning: a liquid electrically charged drop cannot be spherical unless the charge is lower than a certain threshold depending on physical parameters of the liquid, of the surrounding medium, and proportional to a power of the radius. The precise value of Q_0 in 3 dimensions is given by the formula*

$$Q_0^2 = 64\pi^2\varepsilon_0\gamma r^3,$$

where γ is the surface tension, of the drop, ε_0 the permittivity of free space, and r the radius of the drop.

Later on, the Lord Rayleigh limit was observed sperimentally by Zeleny [Zel17]*. In particular it was observed that once the charge surpasses the Rayleigh limit, then the drop* *elongates and creates two singular cusps*[1] *from which the excess of the charge is carried away from the drop by little droplets ejected from the cusps. This phenomenon is at the base of the electrospray theory.*

Despite its huge interest in theoretical and applied physics, a precise variational analysis of the problem was missing until the paper by Goldman, Novaga, and the author in 2015 [GNR15]. In this they proved that, for small charges, the ball is a unique minimizer in the class of sets with uniformly bounded curvature of their boundaries.

Theorem 2.1. *Let $\alpha = 2$. Then there exists $\mathcal{Q}_0 = \mathcal{Q}_0(n, m) > 0$ such that if $Q < \mathcal{Q}_0$ then the ball is the unique minimizer in $\mathbb{R}^n$ of*

$$\min\left\{\mathcal{F}_R(E)\,:\,|E| = m\right\},$$

among sets with uniformly $C^{2,\delta}-$regular boundary, with $\delta > 0$.

This rigidity result, though the first variationally robust result in this direction, is not surprising. Let us comment on the two technical assumptions of the theorem: first, one may replace the Coulombic kernel corresponding to $n - \alpha = n - 2$ with any Riesz kernel such that $n - \alpha < n - 1$. This is going to be shown in a forthcoming paper by the same authors and we focus on that in the last part of this section. Secondly, the regularity assumption is necessary to get a well defined problem. In fact it holds the following ill-posedness result for $\mathcal{F}_R$ in the class of merely measurable sets with prescribed Lebesgue measure.

Theorem 2.2. [GNR15]*] Let $n - \alpha < n - 1$. Then there holds*

$$\inf\left\{\mathcal{F}_R(E)\,:\,|E| = m\right\} = P(B_{(m/\omega_n)^{1/n}}).$$

Hence, the problem is ill posed and does not admit minimizers.

To restore the well-posedness there are then two possible approaches. First, one may regularize the class of competitors, as in Theorem 2.1 or in [GNR18], where the authors consider the setting of convex sets. Second, one may regularize the functional itself.

[1]Those singularities were later studied by Sir Taylor in the '50s [Tay64]

This latter approach was considered by Muratov and Novaga in [MN16] In it the authors consider a Deybel-Hückel energy-type functional where the capacitary term I_α is replaced by

$$J_{\gamma,\alpha}(E) = \inf\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{d\rho(x)\,d\rho(y)}{|x-y|^{n-\alpha}} + \gamma\int_{\mathbb{R}^n}\rho^2(x)\,dx\,:\,\rho(E)=1\right\}.$$

Here the minimization is over probability measures on E which are also in $L^2(\mathbb{R}^n)$. Notice that $J_{0,\alpha} = I_\alpha$. The L^2-extra-term appearing in the definition of $J_{\gamma,\alpha}$, as long as $\gamma > 0$ entails the continuity of the functional $P+Q^2 J_{\gamma,\alpha}$ and as a consequence, its well-posedness. A fine analysis of the regularity of minimizers was later performed by De Philippis, Hirsch, and Vescovo in [DPHV] while the rigidity of the ball for small charge Q was eventually done by Mukoseeva and Vescovo in [MV].

2.1. **The cases $\alpha = 1$ and $\alpha < 1$.** All the above considerations were limited to the case $\alpha > 1$ where, thanks to Theorem 2.2, we know that minimizing under measure constraint of the energy $\mathcal{F}_R$ does not lead to a well posed problem. A natural question is to understand if ill-posedness holds for any α and n. This was partially answered in the negative by Muratov, Novaga, and the author, who showed in dimension 2 and for $\alpha = 1$ the following result.

Theorem 2.3. *[*[MNR18]*] Let $\alpha = 1$ and $n = 2$. Then there exists a (explicit) limit $\mathcal{Q}_1(m) > 0$ such that for $Q \le \mathcal{Q}_1(m)$ the ball is the only minimizer of the problem*

$$\min\left\{\mathcal{F}_R(E)\,:\,E\subset\mathbb{R}^2,\,|E|=m\right\}.$$

Moreover the problem does not admit minimizers if $Q_1 > \mathcal{Q}(m)$.

Despite the previous result offers a complete picture of the problem in dimension 2, its proof is quite difficult to generalize. In particular, to characterize the ball as only minimizer we make use of fine results in convex body theory in dimension 2. Namely we use the fact that the capacitary term I_1 satisfies (in any dimension) an inequality à-la Brunn-Minkowski [NR15] which, despite we conjecture to be true, seems difficult to adapt in higher dimension and for general values of α. The cases where $n \ge 2$ and $\alpha \le 1$ are dealt with in an upcoming paper by Goldman, Novaga and the author but by means of a different proof.

Theorem 2.4. *[Goldman, Novaga, Ruffini–Work in progress] Let* $n \geq 2$, $\alpha \leq 1$. *Then there exists an implicit threshold* $\mathcal{Q}_2(m) > 0$ *such that for* $Q < \mathcal{Q}_2(m)$ *the ball is a rigid minimizer of* $\mathcal{F}_R$ *among sets of measure* m.

We delve now into some ideas of the proof of this latter recent result. This will be at once the occasion to describe the main ideas developed for the liquid drop Gamow functional in [KM14] by Knüpfer and Muratov.

Let us first suppose that a minimizer E for $P + \varepsilon V_\alpha$ exist in the class of nearly spherical sets. In that case by a perturbative argument one can show that if φ is the function parametrizing ∂E on the boundary of a ball B, then one has

$$V_\alpha(B) - V_\alpha(E) \lesssim \|\varphi\|^2_{L^2(\partial B)}. \tag{3}$$

Remark 2.2. *We stress that a similar inequality, namely*

$$V_\alpha(F) - V_\alpha(E) \lesssim |E \Delta F|$$

holds with the mere hypothesis $\max\{|E|, |F|\} < +\infty$. *This links to the previous inequality since if* E *is a nearly spherical set parametrized by* φ *on* ∂B, *then a simple computation shows that* $|E \setminus B| \sim \|\varphi\|_{L^1(\partial B)}$. *On the other hand we shall see in a moment that the power* 2 *in the right-hand side of* (3) *is crucial.*

We now observe:

$$\begin{aligned} \int_{\partial B} |\varphi|^2 &\lesssim P(E) - P(B) && \text{by the Fuglede expansion argument} \\ &\lesssim \varepsilon(V_\alpha(B) - V_\alpha(E)) && \text{since } E \text{ is a minimizer} \\ &\lesssim \varepsilon \int_{\partial B} |\varphi|^2 && \text{by inequality (3)} . \end{aligned} \tag{5}$$

Clearly this implies that for small values of ε, forcedly $\varphi = 0$ so that E is a ball. A crucial point in this discussion is that regularity is used twice: to show the first inequality, one needs φ to be C^1-regular, while for the last one, one needs $\varphi \in C^{1,\delta}$ for some $\delta > 0$.

By this argument, the rigidity of the ball reduces to get existence and $C^{1,\delta}-$regularity of minimizers. In the case of the Gamow functional, the latter comes by the classical regularity theory for sets of finite perimeter. In fact if E is a minimizer, then by Remark 2.2 one gets that

$$P(E) \leq P(F) + Cr^{n-1+\delta}, \qquad \forall E\Delta F \subset B_r(x), \quad x \in \partial E.$$

Sets with such minimality property are called quasi-minimizers for the perimeter and they are regular up to a set of dimension lower than $n-8$. See [Mag12, AFP00].

Remark 2.3. *This idea was used in* [KM13] *only for* $\alpha > 1$*, while the whole range* $\alpha \in (0, n)$ *was later covered in* [FFM$^+$15]*, who dealt also with nonlocal versions of the perimeter.*

We delve now into some ideas of the proof of Theorem 2.4. We limit ourselves to $\alpha = 1$, which happens to be the trickiest.[2]

Remark 2.4 (A topological issue)**.** *All the below argument, together with the statement of Theorem 2.4 contain a topological hiccup, omitted in the discussion for expository reasons. Indeed one has to specify which class of minimization considers. The class of equivalence in* L^1*, where two sets are equivalent if their symmetric difference is negligible, is both physically irrelevant and mathematically incorrect. In fact for any* $\alpha > 0$ *if* F *is a set with Hausdorff dimension* $d \in (n-\alpha, n)$*, then its Lebesgue measure is null, while its capacity is not. This means that adding a suitable number of translated copies* F *to any competitor* E*, the nonlocal term* I_α *decreases, without changing the measure of the set* E*. On the other hand all the above discussion becomes formal (yet technically heavier) under the mild assumption that* $P(E) = \mathcal{H}^{n-1}(\partial E)$*.*

The proof follows the above mentioned path: show regularity of the boundary of competitors, and then apply a perturbative argument in the spirit of the Fuglede's one. The first evident difficulty comes while showing the analog of inequality (3). This since the presence of the measure μ varies as the set E varies. On the other hand for nearly spherical sets it is possible to show that

$$I_\alpha(B) - I_\alpha(E) \lesssim P(E) - P(B). \tag{6}$$

[2] This fact does not come as a complete surprise since, as α approaches 1, the term V_α is less regularizing. In this sense $\alpha = 1$ corresponds to the least regularizing case for which well-posedness is guaranteed.

We do not delve into details of such a proof, which is quite technical and long and we limit to mention that it exploits the boundary regularity of the optimal measure μ building upon recent results about the regularity of solutions of fractional elliptic equations developed by Ros-Oton and Serra [ROS14].

By (6) one can conclude reasoning as for the Gamow liquid drop model that the ball is a rigid minimizer. Let us spend now some works on the strategy to get existence and the sought regularity of minimizers.

- *Existence* By means of concentration-compactness arguments we show that a minimizer exists in a generalized sense. Namely one introduce the concept of generalized minimizers as a collection (E_i) of sets such that

$$\sum_i |E_i| = m \quad \text{and} \quad \sum_i \mathcal{F}_R(E_i) = \inf_{|E|=m} \mathcal{F}_R(E).$$

  In words, a generalized minimizer is a collection of sets with infinite mutually distance (so that the repulsive component of $\mathcal{F}_R$ does not enter into account) which minimizes the energy. To show that a generalized minimizer is in fact a classical minimizer it is then sufficient to show that some a priori mild regularity, as uniform density estimates, holds true.

- *Regularity* Regularity happens to be trickier as α increases toward 1. In fact, by comparison it is possible to show that any component E of the generalized minimizer satisfies

$$P(E) \leq P(F) + Q^2 r^{n-\alpha}, \quad E\Delta F \subset B_r.$$

  This means that a minimizer of $\mathcal{F}_R$ is a quasi-minimizer for the perimeter [Mag12]. Then, by classical regularity of quasi-minimizers it holds that as long as $\alpha < 1$, then ∂E is of class $C^{1,\alpha/2}$ and that is just enough to conclude. Unfortunately, this ceases to be true in the case $\alpha = 1$ [AFP00, Mag12].

  On the other hand the quasi-regularity satisfied by a minimizer in this case,

$$P(E) \leq P(F) + Q^2 r^{n-1}, \qquad E\Delta F \subseteq B_r,$$

implies that E is *Reifenberg flat*. We do not delve into precise definitions of Reifenberg flatness in this note but we refer to [GNR22] and the references therein for the result and the definitions. Reinfenberg flatness is a quite weak geometrical regularity for the boundary of minimizers. Roughly speaking a Reifenberg flat set is a set such that at any point of its boundary and at any scale, the distance of the boundary of the set is locally close to that of a hyperplane (possibly changing at different scales), and such that the rate of closeness is proportional to the scaling.

The idea is then that such a geometric closedness to a hyperplane can be transferred into an L^p-regularity for the harmonic measure $\mu = \mu_E$, the measure which achieves the minimum in the definition of I_α. In turn, higher summability of the harmonic measure entails the sought higher regularity of the boundary of the set.

3. A spectral optimization problem related to Hartree energies

In this section we revise a recent result by Mazzoleni and the author where it is considered the problem

$$\min\{\mathcal{F}_H(E) \,:\, |E| = \omega_n\} \tag{7}$$

where

$$\mathcal{F}_H(E) = \lambda_1(E) + \varepsilon V_\alpha(E).$$

The following result is the exact counterpart of the minimization result related to the energy $\mathcal{F}_R$.

Theorem 3.1. *Let us suppose that* $\alpha > 1$. *There exists* $\overline{\varepsilon} > 0$ *such that if* $\varepsilon \leq \overline{\varepsilon}$ *then Problem* (7) *admits the ball as a rigid minimizer.*

The idea of the proof follows the roadmap pictured in the previous section for the energy of charged drops. Namely one wishes to show that minimizers exist, and for ε small enough they are nearly spherical objects. On the other hand to apply those two steps, very different ideas are required. In particular, what is needed is a regularity theory suited for functionals as λ_1, or more in general for Dirichlet-like energies as

$$u \mapsto \int |\nabla u|^2.$$

3.1. **A glimpse on the physical background and motivation.** The energy $\mathcal{F}_H$ may be considered as a toy model for the reduced Hartree energies, who aim at describing the ground state energy of the electrons of an atom. We do not delve into the full physical and mathematical background of such energies, limiting ourselves to depict its easiest instance: the atom of Helium. In this case, denoting by u^2 the probability position of an electron, its energy is composed as the interaction of a kinetic energy, an aggregating term due to the attraction of the nucleus, and a repulsive part due to the interaction of the electrons. Namely one models in $\mathbb{R}^3$ the energy as

$$H(u) = \underbrace{\int |\nabla u|^2}_{Kinetic\ energy} - \underbrace{\int u^2(x)P(x)\,dx}_{Attraction\ nucleus-electrons} + \underbrace{\int \frac{u^2(x)\,u^2(y)}{|x-y|}\,dx\,dy}_{repulsion\ between\ electrons}.$$

A relevant physical question is to show existence and properties of ground state energies, that is to solve the problem

$$\min\left\{H(u)\ :\ \int u^2 = m\right\}.$$

Remark 3.1. *Notice that we relaxed the probability constraint $\int u^2 = 1$ to a more general constraint, $\int u^2 = m$. In* [LO14] *it was shown that for large m existence of ground states does not occur in $\mathbb{R}^n$. In fact even when the probability mass m is small, existence is not a trivial task because of possible lack of compactness phenomena. See* [Lio87].

Here $P(x) = |x|^{-1}$ is the Coulombic potential. From a mathematical point of view (but likely quite meaningless physically) it acts as a confining term and one can seek for a limiting case where P confines the probability density to stay in between a conductive insulator. Namely, given a fixed region $E \subset \mathbb{R}^n$ one replace $P(x) = |x|^{-1}$ with a capacitary measure μ_E such that

$$\mu_E(A) = \begin{cases} +\infty & \text{if } \operatorname{cap}(A \cap E^c) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2. *Despite here they are used just as a confining tool, capacitary measures play a fundamental role while seeking to show existence of optimal shape of functionals depending on elliptic operators. See* [HP05] *for insights about this topic.*

The energy we end up with is

$$H(u,E) = \int_E |\nabla u|^2 + \iint_{E\times E} \frac{u^2(x)\,u^2(y)}{|x-y|}\,dx\,dy$$

where $u \in H^1_0(E)$ is such that $\|u\|_{L^2(E)} = m$. If E has finite measure the problem of finding an optimal ground state u_E energy strongly simplifies. A relevant question is then the study of the shape optimization problem of minimizing the energy

$$E \mapsto H(E, u_E)$$

under various constraints. In this note we just make a further simplification to get eventually to the energy $\mathcal{F}_H$, which is to decouple completely the Dirichlet energy and the repulsive term in $H(\cdot, E)$. Namley, minimizing in u the first term in H leads to the functional λ_1 and with the further simplification of supposing u to be a homogeneous measure on E, the repulsive term in H reduces to V_1. With these two ansatz one recovers exactly the energy $\mathcal{F}_H$.

3.2. **Relevant ideas in the proof of Theorem 7.** We describe hereafter the main steps in the proof of Theorem 7.

- As in the case of the energy $\mathcal{F}_R$, one shows that among nearly spherical sets, the ball is a rigid minimizer by means of a perturbative argument. Then all reduces to show existence of minimizers together with their regularity.
- To show that a minimizer exists it is not as easy as for the energy $\mathcal{F}_R$. This since the equiboundedness of the map $n \mapsto \lambda_1(E_n)$ for a minimizing sequence $\{E_n\}$ does not entail compactness in L^1 of the sequence, at it is the case for sets with equibounded perimeter. What can be shown is the following.

Lemma 3.1. *Let $C > 0$ and let $R > 0$. Suppose that for $n \in \mathbb{N}$, $E_n \subset B(0, R)$ satisfies $\lambda_1(E_n) \leq C$. Then if u_n is the eigenfunction relative to λ_1, there exists $C_1(R) > 0$ such that for any $n \in \mathbb{N}$, there exists a super-level W_n of u_n with $P(W_n) \leq C_1(R)$.*

The previous lemma was developed by Bucur, building upon ideas from free boundary regularity [Buc12].

The idea is then to relax the constraint energy by means of a Lagrange multiplier[3] and, for a minimizing sequence E_n show that the sets W_n from the previous lemma constitute a minimizing sequence as well. This, together with the perimeter bound on the W_N's, implies compactness of the minimizing sequence.

A main drawback is the dependence on R of the constant C_1 of Lemma 3.1. This can be solved by showing by means of a delicate surgery argument, inspired by the proofs in [MP13], that minimizers must be equibounded.

- To get the required regularity, the main idea is to add a piecewise linear volume penalization $f(|E|)$ and to study the uncostrained problem

$$\mathcal{D} \,:\, u \mapsto \int |\nabla u|^2 + V_\alpha(\{u > 0\}) + f(|\{u > 0\}|).$$

  The point is that minimizers of $\mathcal{D}$ happen to be, for a suitable choice of f, such that $\{u > 0\}$ is a minimizer of $\mathcal{F}_H$. All the problem reduces then to adapt classical regularity tools for the free boundary $\partial\{u > 0\}$ developed in a series of paper by Alt, Caffarelli, et al. [AC81], [ACF84]. To show that it is a $C^{2,\alpha}-$ regular set up to a small negligible set.
- To get that for Q small the minimizers are actually nearly spherical sets, one exploits the quantitative Faber-Krahn inequality in sharp form, recently developed by Brasco, De Philippis, and Velichkov [BDPV15]. This states, in analogy to the quantitative isoperimetric inequality, that

$$\lambda_1(E) - \lambda_1(B) \geq C|E\Delta(B + x)|^2,$$

[3] In fact what it is needed is a piecewise linear function, since the two addends of the functional scale differently.

where again $|E| = |B|$. The idea is then to show, in analogy to the case of $\mathcal{F}_R$, that

$$|E\Delta(B+x)|^2 \lesssim P(E) - P(B) \lesssim \varepsilon(V_\alpha(B) - V_\alpha(E)) \lesssim \varepsilon|E\Delta B|^2, \tag{8}$$

and conclude as before. Here, and hereafter, by $A \lesssim B$ we mean that $A \leq CB$ where C is a dimensional constant.

Remark 3.3. *There is a technical point that is worth stressing here, linked with the hypothesis $\alpha > 1$ appearing in the statement of Theorem 3.1. What we actually are able to show is a slightly weaker version of* (8)*. Namely, that*

$$\int_{\partial B} \varphi^2 \lesssim \lambda_1(E) - \lambda_1(B) \lesssim \varepsilon(V_\alpha(B) - V_\alpha(E)) \lesssim \varepsilon \int_{\partial B} \varphi^2,$$

where $\varphi : \partial B \to \mathbb{R}$ is the function parametrizing the boundary of E. This lead of course to the conclusion, up to choose ε small enough. The right hand-side inequality is the consequence of an easy computation and the definition of V_α. On the other hand the left-hand side inequality is more involved. One need first to show[4] *(see* [BDPV15]*) that*

$$[\varphi]^2_{H^{1/2}(\partial B)} \lesssim \lambda_1(E) - \lambda_1(B)$$

and then conclude by means of a Poincaré-type inequality of the form

$$[\varphi]^2_{L^2(\partial B)} \lesssim [\varphi]^2_{H^{1/2}(\partial B)}.$$

To make this argument work one needs that $\varphi \in C^{2,\delta}(\partial B)$ with $\delta = \delta(\alpha) > 0$. Such a regularity is unfortunately achieved in [MR21] *only as $\alpha > 1$. Nonetheless we believe that this is a mere technical problem and that the minimality of the ball should hold for any $\alpha \in (0, n)$.*

[4]This requires also that barycenter of E is equal to 0, which can be supposed without losing any generality in the argument.

3.3. **Linear instability of the ball for $\mathcal{F}_H$ for large masses.** We conclude this section with some (original) remarks about non-existence. Namely, by invoking some classical result about the shape derivative of λ_1 and some more recent computations about the shape derivatives of V_α we observe that the ball is an unstable critical point for $\mathcal{F}_H$. Below some technical but classical computations are omitted.

Remark 3.4. *In* [MR21] *it was shown by simple rescaling arguments that for ε big enough, and for $\alpha \in (0,1)$, then minimizers for $\mathcal{F}_H$ cannot exist in the class of sets with uniform density estimates. The class is quite general, but the restriction on α is quite restrictive and we believe that this issue is of mere technical nature. The above claimed result goes in the direction of supporting such a conjecture.*

Remark 3.5. *Despite we limit ourselves to the case of the ball in this note, one might extend the below to some class of bounded regular sets.*

Definition 3.1. *A C^2-regular set E is a critical point of $\mathcal{F}_H$ if its first variation is null. A critical point is* stable *if its second variation is positive along any volume preserving flow Φ_t. We recall that given a free-divergence vector field $X : \mathbb{R}^n \mapsto \mathbb{R}^n$, a volume preserving flow (E_t) induced by X is a collection $E_t = \Phi_t(E)$ such that $|E_t| = |E|$, where $(t,x) \to \Phi_t(x)$ solves for $x \in \mathbb{R}^n$ the ODEs*

$$\begin{cases} \partial_t \Phi_t(x) = X(\Phi_t(x)) \\ \Phi_0(x) = X(x). \end{cases}$$

By classical results (see [Mag12]*) one can show the existence of a volume preserving flow for small time. Namely, that there exists $\varepsilon > 0$ such that for $0 \le t < \varepsilon$ a volume preserving flow exists. The first and the second variations of a functional $\mathcal{F}$ with initial velocity X, denoted by $\delta\mathcal{F}[E](X)$ and $\delta^2\mathcal{F}[E](X)$ respectively, are the first and the second derivative in 0 of the real function*

$$t \mapsto \mathcal{F}(E_t).$$

By a classical result about shape variations, one gets that for any volume preserving flow it holds

$$\delta\lambda_1[B](X) = 0,$$

and

$$\left|\delta^2\lambda_1[B](X)\right| \leq C(n,X),$$

for some $C(n,X) > 0$. See [HP05]. By the maximality of the ball for V_α one can easily compute

$$\delta V_\alpha[B](X) = 0.$$

Eventually, as a direct consequence of [FFM$^+$15, Proposition 7.2], we get that there exists $\beta = \beta(\alpha, n) > 0$ such that

$$\delta^2 V_\alpha[B](X) < -\beta\delta^2 P[B](X) = -C_1(n,X).$$

where P is the perimeter functional. These results immediately entail that for ε large enough, possibly depending on X, it holds the sought instability of the ball along X

$$\delta^2\mathcal{F}_H[B](X) < 0.$$

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