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On the Shape of the First Fractional Eigenfunction

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Abstract
We show that the first eigenfunction of the fractional Laplacian $(-\Delta)^s$, $s \in (1/2, 1)$, is superharmonic in the unitary ball up to dimension 11. To this aim, we also rely on a computer-assisted step to estimate a rather complicated constant depending on the dimension and the power s.

Keywords First eigenfunction · Qualitative properties · Concavity · Superharmonicity · Fractional Laplacian

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1 Introduction

The fractional Laplace operator is an integro-differential nonlocal operator of non-integer order. It is defined as

$$(-\Delta)^s u(x) := \frac{4^s \Gamma(n/2+s)}{\pi^{n/2}\left|\Gamma(-s)\right|}\, \text{p.v.}\!\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \qquad s \in (0,1),\ x \in \mathbb{R}^n,$$

where “p.v.” means that the integral is taken in the principal value sense. We refer to [2, 8, 10, 12] for all its basic features.

Here, we recall that it is naturally related to the fractional Sobolev space

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : [u]_s^2 := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx < \infty \right\}$$

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and, when the attention is restricted to a bounded domain $\Omega \subset \mathbb{R}^n$, to the space

$$\mathcal{H}^s_0(\Omega) = \big\{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\big\},$$

which is encoding a natural notion of homogeneous boundary conditions in $\mathbb{R}^n \setminus \Omega$: for this reason it is sometimes also known as the *restricted* fractional Laplacian. From a functional analytic perspective, $(-\Delta)^s$ is a positive self-adjoint operator on $\{u \in L^2(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n\}$ with compact inverse. It has therefore a discrete spectrum and the eigenvalues have finite multiplicity. In particular, the first eigenvalue, which we denote by $\lambda = \lambda(\Omega)$, is simple. It is known that the first eigenfunction $\phi \in \mathcal{H}^s_0(\Omega)$ is smooth inside Ω and that it can be chosen to be strictly positive.

In this paper we partially answer a conjecture raised by Bañuelos, Kulczycki, and Méndez-Hernández [6, Conjecture 1.1]:

$$\text{If } n = 1 \text{ and } \Omega = (-1, 1), \text{ then } \phi \text{ is concave in its support.} \tag{1.1}$$

This has been previously established by Bañuelos and Kulczycki [5, Theorem 4.7], for $s = \frac{1}{2}$, and by Kaßmann and Silvestre [13] and Bañuelos and DeBlassie [4, Theorem 1.1], whenever $s^{-1} \in \mathbb{N}$; moreover, in general dimension and for a general bounded Lipschitz domain, [4, Theorem 1.1] also shows that ϕ is superharmonic, again under the assumption $s^{-1} \in \mathbb{N}$. Another related result is contained in [6, Theorem 1.1], which states that ϕ is *mid-concave* (see [6, Definition 1.1]) on rectangles $\Omega = (-a_1, a_1) \times \cdots \times (-a_n, a_n) \subset \mathbb{R}^n$.

Here, we give a computer-aided proof of Eq. 1.1 for any $s \in (\frac{1}{2}, 1)$.

Theorem 1.1 *Let $s \in (\frac{1}{2}, 1)$ and $n = 1$. Let $\phi \in \mathcal{H}^s_0\big((-1, 1)\big)$ denote the first eigenfunction of $(-\Delta)^s$ on the interval $(-1, 1)$. Then*

$$\phi'' < 0 \quad \text{in } (-1, 1).$$

More generally, our approach is able to reach the following.

Theorem 1.2 *Let $s \in (\frac{1}{2}, 1)$, $2 \le n \le 11$, and $B_1 \subset \mathbb{R}^n$ denote the unitary ball. Let $\phi \in \mathcal{H}^s_0(B_1)$ denote the first eigenfunction of $(-\Delta)^s$ on B_1. Then*

$$-\Delta\phi > 0 \quad \text{in } B_1. \tag{1.2}$$

We believe the threshold $n \le 11$ to be merely technical and due to a few sub-optimal estimates involved in our analysis.

Our strategy begins with a purely analytic approach to reduce $\Delta\phi$ in an integral form. At the core of this strategy we exploit the semigroup property of $(-\Delta)^s$ in that we split

$$\begin{aligned}-\Delta\phi &= (-\Delta)^{1-s}(-\Delta)^s\phi = (-\Delta)^{1-s}\big(\lambda\phi + \mathbf{1}_{\mathbb{R}^n\setminus B_1}(-\Delta)^s\phi\big)\\ &= \lambda(-\Delta)^{1-2s}\big(\lambda\phi + \mathbf{1}_{\mathbb{R}^n\setminus B_1}(-\Delta)^s\phi\big) + (-\Delta)^{1-s}\big(\mathbf{1}_{\mathbb{R}^n\setminus B_1}(-\Delta)^s\phi\big) \qquad \text{in } B_1.\end{aligned} \tag{1.3}$$

Next, we write suitable integral representations for the different terms that appear: these use in a crucial way our standing assumption $s > \frac{1}{2}$, so that $(-\Delta)^{1-2s}$ stands for the convolution with the fundamental solution in $\mathbb{R}^n$. A central role in these formulas is played by the nonlocal Poisson kernel of B_1

$$P_s(x, y) := \frac{\gamma(n, s)}{|x-y|^n}\frac{\big(1-|x|^2\big)^s}{\big(|y|^2-1\big)^s}, \quad \gamma(n, s) := \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}\Gamma(s)\,\Gamma(1-s)}, \qquad x \in B_1,\ y \in \mathbb{R}^n\setminus B_1. \tag{1.4}$$

The splitting of $-\Delta\phi$ is performed in Section 2. A refinement of Eq. 1.3 leads to write Eq. 1.2 as an integral inequality not involving directly ϕ or λ (see Eq. 2.8 below). At this point we split our analysis in three different cases:

- For $n = 1$ and $s = \frac{3}{4}$ the argument can be concluded by hand, without too much of a hustle: this is done in Section 3.
- For $n = 1$ and $s \in (\frac{1}{2}, 1) \setminus \{\frac{3}{4}\}$ the integral quantities can be simplified a lot via estimates from below; still, the resulting inequality contains a quite complicated expression in s and we therefore plot and verify it using a computer: this is done in Section 4;
- For $n \geq 2$ a similar approach to the previous point is taken, with the important difference that in this case hypergeometric functions make their appearance in our study: these make the analysis even more complicated and we consequently need to rely even more on numerical evaluations: this is done in Section 5.

Many details will be deferred to Appendices, in the attempt of not breaking the flow of the exposition with technicalities. Nevertheless, we would like to mention that in Appendix B we derive an upper bound for the first eigenvalue λ on B_1 for general dimension n and power s, while in Appendix C we derive from a representation formula for certain s-harmonic functions some symmetry and monotonicity properties which are useful in our analysis and might be of independent interest.

1.1 Notations

We denote by B_r the n-dimensional ball of radius $r > 0$ centered at 0. We set

$$F_\tau(z) := \begin{cases} \kappa(n,\tau)|z|^{2\tau-n}, \quad \kappa(n,\tau) := \dfrac{\Gamma(\frac{n}{2}-\tau)}{4^\tau \pi^{n/2}\left|\Gamma(\tau)\right|}, & \tau \in \mathbb{R},\ \tau - \dfrac{n}{2} \notin \mathbb{N}_0, \\ (-1)^{1+\tau-n/2} \dfrac{2^{1-2\tau}\pi^{-n/2}}{\Gamma(1+\tau-\frac{n}{2})\,\Gamma(\tau)}\,|z|^{2\tau-n}\ \ln|z|, & \tau \in \mathbb{R},\ \tau - \dfrac{n}{2} \in \mathbb{N}_0, \end{cases} \quad z \in \mathbb{R}^n \setminus \{0\}. \tag{1.5}$$

Note that F_τ is the fundamental solution of $(-\Delta)^\tau$ in $\mathbb{R}^n$ if $\tau > 0$ and for $\tau \in (-1, 0)$ it is the kernel of the fractional Laplacian of order 2τ. For a measurable set $A \subset \mathbb{R}^n$, $\mathbf{1}_A$ denotes the characteristic function of A and $A^c = \mathbb{R}^n \setminus A$ the complementary set of A.

2 Set-up of the proof of Theorems 1.1 and 1.2

2.1 Representation of the Laplacian of the eigenfunction

In this paragraph we perform the splitting of $-\Delta\phi$ announced in Eq. 1.3.

Lemma 2.1 *For $s \in (0, 1)$ it holds $(-\Delta)^s\phi \in L^1(\mathbb{R}^n)$ and we have*

$$(-\Delta)^s\phi = \lambda\phi - \mathbf{1}_{B_1^c}(F_{-s} * \phi) \qquad in\ \mathbb{R}^n \setminus \partial B_1. \tag{2.1}$$

Proof Consider $\psi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \phi\,(-\Delta)^s\psi = \int_{B_1} \phi\,(-\Delta)^s\mathbb{G}_s[(-\Delta)^s\psi] = \lambda\int_{B_1} \phi\,\mathbb{G}_s[(-\Delta)^s\psi]$$

Above, we have denoted by $\mathbb{G}_s : L^2(B_1) \to \mathcal{H}^s_0(B_1)$ the solution map to the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } B_1,\ \ f \in L^2(B_1), \\ \qquad u \in \mathcal{H}^s_0(B_1), \end{cases}$$

which admits a representation in terms of the Green function

$$\begin{aligned} \mathbb{G}_s[f](x) &= \int_{B_1} G_s(x,y)\, f(y)\, dy && \text{for } x \in B_1, \\ G_s(x,y) &= F_s(x-y) - \int_{B_1^c} P_s(x,z)\, F_s(z-y)\, dz && \text{for } x, y \in B_1, \\ P_s(x,y) &= \int_{B_1} F_{-s}(z-y)\, G_s(x,z)\, dz && \text{for } x \in B_1,\ \ y \in B_1^c, \end{aligned} \tag{2.2}$$

see [7, Definition 1.9] and [1, equation (25) and Theorem 1.2]. We have then

$$\begin{aligned} \frac{1}{\lambda}\int_{\mathbb{R}^n} \phi\, (-\Delta)^s \psi &= \int_{B_1} \phi(x) \int_{B_1} F_s(x-y)\, (-\Delta)^s \psi(y)\, dy\, dx \\ &\quad - \int_{B_1} \phi(x) \int_{B_1^c} P_s(x,z) \int_{B_1} F_s(z-y)\, (-\Delta)^s \psi(y)\, dy\, dz\, dx \\ &= \int_{B_1} \phi\psi - \int_{B_1} \phi(x) \int_{B_1^c} F_s(x-y)\, (-\Delta)^s \psi(y)\, dy\, dx \\ &\quad - \int_{B_1} \phi(x) \int_{B_1^c} P_s(x,z)\, \psi(z)\, dz\, dx \\ &\quad + \int_{B_1} \phi(x) \int_{B_1^c} P_s(x,z) \int_{B_1^c} F_s(z-y)\, (-\Delta)^s \psi(y)\, dy\, dz\, dx \\ &= \int_{B_1} \phi\psi - \int_{B_1} \phi(x) \int_{B_1^c} P_s(x,z)\, \psi(z)\, dz\, dx \\ &\quad + \int_{B_1} \phi(x) \int_{B_1^c} \left(\int_{B_1^c} P_s(x,z)\, F_s(z-y)\, dz - F_s(x-y) \right) (-\Delta)^s \psi(y)\, dy\, dx. \end{aligned}$$

As it holds (see [15, equation (1.6.12')])

$$\int_{B_1^c} P_s(x,z) F_s(z-y)\, dz = F_s(x-y) \qquad \text{for } x \in B_1,\ \ y \in B_1^c,$$

then

$$\begin{aligned} \int_{\mathbb{R}^n} \phi\, (-\Delta)^s \psi &= \lambda \int_{B_1} \phi\, \psi - \lambda \int_{B_1} \phi(x) \int_{B_1^c} P_s(x,z)\, \psi(z)\, dz\, dx \\ &= \lambda \int_{B_1} \phi\, \psi - \int_{B_1^c} \psi(z) \int_{B_1} P_s(x,z)\, \lambda \phi(x)\, dx\, dz \\ &= \lambda \int_{B_1} \phi\, \psi - \int_{B_1^c} \psi\, (F_{-s} * \mathbb{G}_s[\lambda\phi]) = \lambda \int_{B_1} \phi\, \psi - \int_{B_1^c} \psi\, (F_{-s} * \phi) \end{aligned} \tag{2.3}$$

where we have used Eq. 2.2. The stated equality Eq. 2.1 holds also pointwisely in view of [18, Proposition 2.4].

Proposition 2.2 *For $s \in (\frac{1}{2}, 1)$ it holds*

$$-\Delta\phi = \lambda^2 F_{2s-1} * \phi + \big(F_{s-1} - \lambda F_{2s-1}\big) * \big[\mathbf{1}_{B_1^c}(F_{-s} * \phi)\big] \tag{2.4}$$

$$= \lambda^2 F_{2s-1} * \phi + \lambda\big(F_{s-1} - \lambda F_{2s-1}\big) * \mathbf{P}_s[\phi] \qquad \text{in } B_1, \tag{2.5}$$

where

$$\mathbf{P}_s[\phi](y) = \mathbf{1}_{B_1^c}(y) \int_{B_1} P_s(x, y)\, \phi(x)\, dx \qquad \text{for } y \in \mathbb{R}^n.$$

Proof Starting from the last lemma, we compute (all equalities hold only in B_1)

$$-\Delta\phi = (-\Delta)^{1-s}(-\Delta)^s\phi = (-\Delta)^{1-s}\big[\lambda\phi - \mathbf{1}_{B_1^c}(F_{-s} * \phi)\big] = \lambda(-\Delta)^{1-s}\phi + F_{s-1} * [\mathbf{1}_{B_1^c}(F_{-s} * \phi)].$$

Note that

$$(-\Delta)^{1-s}\phi = F_{2s-1} * (-\Delta)^s\phi = F_{2s-1} * \big[\lambda\phi - \mathbf{1}_{B_1^c}(F_{-s} * \phi)\big]$$

thus Eq. 2.4 follows. In expanded form, Eq. 2.4 reads, for $x \in B_1$,

$$\begin{aligned}
-\Delta\phi(x) &= \lambda^2 \int_{B_1} F_{2s-1}(x - y)\, \phi(y)\, dy \\
&\quad + \int_{B_1^c} \big(F_{s-1}(x - z) - \lambda F_{2s-1}(x - z)\big) \int_{B_1} F_{-s}(z - y)\, \phi(y)\, dy\, dz \\
&= \int_{B_1} \phi(y) \Big[\lambda^2 F_{2s-1}(x - y) + \int_{B_1^c} \big(F_{s-1}(x - z) - \lambda F_{2s-1}(x - z)\big) F_{-s}(z - y)\, dz\Big]\, dy.
\end{aligned}$$

Also, identity

$$(F_{-s} * \phi)(z) = \lambda \int_{B_1} P_s(y, z)\, \phi(y)\, dy \qquad \text{for } z \in B_1^c,$$

holds (we have already used this one in Eq. 2.3 exploiting Eq. 2.2). In expanded form, Eq. 2.5 reads

$$\begin{aligned}
-\Delta\phi(x) &= \lambda^2 \int_{B_1} F_{2s-1}(x - y)\, \phi(y)\, dy \\
&\quad + \lambda \int_{B_1^c} \big(F_{s-1}(x - z) - \lambda F_{2s-1}(x - z)\big) \int_{B_1} P_s(y, z)\, \phi(y)\, dy\, dz \\
&= \lambda \int_{B_1} \phi(y) \Big[\lambda F_{2s-1}(x - y) + \int_{B_1^c} P_s(y, z)\big(F_{s-1}(x - z) - \lambda F_{2s-1}(x - z)\big)\, dz\Big]\, dy.
\end{aligned}$$

We know that, by uniqueness, ϕ and $-\Delta\phi$ are radial, so that for any $x \in B_1$ fixed

$$-\Delta\phi(x) = -\frac{1}{|\partial B_{|x|}|} \int_{\partial B_{|x|}} \Delta\phi(\theta)\, d\theta.$$

Keeping this in mind, we define

$$J_\tau(x; y) := \begin{cases} \dfrac{1}{|\partial B_{|x|}|} \displaystyle\int_{\partial B_{|x|}} F_\tau(\theta - y)\, d\theta & \text{for } n \geq 2,\ x, y \in \mathbb{R}^n,\ x \neq y \\ \dfrac{F_\tau(x - y) + F_\tau(x + y)}{2} & \text{for } n = 1,\ x, y \in \mathbb{R},\ x \neq y. \end{cases} \tag{2.6}$$

Using Eq. 2.5, we then write

$$-\Delta\phi(x) = \lambda \int_{B_1} \phi(y)\left[\lambda J_{2s-1}(x;y) + \int_{B_1^c} P_s(y,z)\big(J_{s-1}(x;z) - \lambda J_{2s-1}(x;z)\big)\,dz\right] dy, \quad x \in B_1.$$

Since

$$\begin{aligned}\lambda J_{2s-1}(x;y) &+ \int_{B_1^c} P_s(y,z)\big(J_{s-1}(x;z) - \lambda J_{2s-1}(x;z)\big)\,dz = \\ &= \lambda\left(J_{2s-1}(x;y) - \int_{B_1^c} P_s(y,z)\,J_{2s-1}(x;z)\,dz\right) + \int_{B_1^c} P_s(y,z)\,J_{s-1}(x;z)\,dz,\end{aligned}$$

the positivity of $-\Delta\phi$ follows once we show

$$\lambda\left(J_{2s-1}(x;y) - \int_{B_1^c} P_s(y,z)\,J_{2s-1}(x;z)\,dz\right) + \int_{B_1^c} P_s(y,z)\,J_{s-1}(x;z)\,dz \geq 0, \quad x, y \in B_1.$$

As the second addend is clearly positive in the above inequality, we may replace λ with a larger constant (see Appendix B, equation Eq. B.1)

$$\lambda \leq \Lambda(n,s) := \frac{4^s\,\Gamma(1+s)^2\,\Gamma(1+2s+\frac{n}{2})}{(s+\frac{n}{2})\,\Gamma(\frac{n}{2})\,\Gamma(1+2s)} \tag{2.7}$$

and it is then enough to show

$$J_{2s-1}(x;y) - \int_{B_1^c} P_s(y,z)\,J_{2s-1}(x;z)\,dz + \frac{1}{\Lambda(n,s)}\int_{B_1^c} P_s(y,z)\,J_{s-1}(x;z)\,dz \geq 0, \quad s \in \left(\frac{1}{2}, 1\right),\ x, y \in B_1. \tag{2.8}$$

Note that $s - 1 - \frac{n}{2} < 0$ for all $n \in \mathbb{N}$ and $s \in (\frac{1}{2}, 1)$, but $2s - 1 - \frac{n}{2} \in \mathbb{N}_0$ if and only if $s = \frac{3}{4}$ and $n = 1$. Hence, this case differs strongly from the other cases as an effect of definitions Eqs. 1.5 and 2.6. We begin with some general estimates to simplify Eq. 2.8.

2.2 Reformulation of Eq. 2.8

The first step is to note that the left-hand side of Eq. 2.8 actually depends only on $|x|$ and $|y|$.

Lemma 2.3 *For any $n \in \mathbb{N}$, $\tau \in \mathbb{R}$, and $s \in (0,1)$ we have*

$$\begin{aligned} J_\tau(x;y) &= J_\tau(|x|e_1, |y|e_1) && x, y \in \mathbb{R}^n,\ x \neq y,\\ \int_{B_1^c} P_s(y,z) J_\tau(x;z)\,dz &= \int_{B_1^c} P_s(|y|e_1, z)\,J_\tau(|x|e_1; z)\,dz && x, y \in B_1,\ x \neq y.\end{aligned}$$

Proof First note that $F_\tau(x) = F_\tau(|x|e_1)$ for any $x \in \mathbb{R}^n \setminus \{0\}$ by definition Eq. 1.5. To see the statement for J_τ, note that this is obviously true for $n = 1$ from definition Eq. 2.6 and for $n > 1$ we have by a rotation for $x \in B_1$, $y \in \mathbb{R}^n \setminus \{x\}$

$$\begin{aligned} J_\tau(x;y) &= \frac{1}{|\partial B_{|x|}|}\int_{\partial B_{|x|}} F_\tau\big(|\theta - y|e_1\big)\,d\theta = \frac{1}{|\partial B_{|x|}|}\int_{\partial B_{|x|}} F_\tau\left(\sqrt{1 - 2\theta\cdot y + |y|^2}\,e_1\right) d\theta \\ &= \frac{1}{|\partial B_{|x|}|}\int_{\partial B_{|x|}} F_\tau\left(\sqrt{1 - 2|y|\nu_1 + |y|^2}e_1\right) d\nu = \frac{1}{|\partial B_{|x|}|}\int_{\partial B_{|x|}} F_\tau\big(\nu - |y|e_1\big)\,d\nu = J_\tau(|x|e_1; |y|e_1).\end{aligned}$$

Similarly, we have with a rotation

$$\begin{aligned}\int_{B_1^c} P_s(y,z)\, J_\tau(x;z)\,dz &= \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(s)\Gamma(1-s)} \int_{B_1^c} \frac{(1-|y|^2)^s}{(|z|^2-1)^s|y-z|^n}\, J_\tau(|x|e_1;|z|e_1)\,dz \\ &= \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(s)\Gamma(1-s)} \int_{B_1^c} \frac{(1-|y|^2)^s}{(|z|^2-1)^s\big(|y|^2-2y\cdot z+|z|^2\big)^{\frac{n}{2}}}\, J_\tau(|x|e_1;|z|e_1)\,dz \\ &= \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(s)\Gamma(1-s)} \int_{B_1^c} \frac{(1-|y|^2)^s}{(|v|^2-1)^s\big(|y|^2-2|y|v_1+|v|^2\big)^{\frac{n}{2}}}\, J_\tau(|x|e_1;|v|e_1)\,dv \\ &= \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(s)\Gamma(1-s)} \int_{B_1^c} \frac{(1-|y|^2)^s}{(|v|^2-1)^s\big||y|e_1-v\big|^n}\, J_\tau(|x|e_1;|v|e_1)\,dv \\ &= \int_{B_1^c} P_s(|y|e_1,v)\, J_\tau(|x|e_1;v)\,dv.\end{aligned}$$

In view of the last lemma Eq. 2.8 reduces to

$$J_{2s-1}(|x|e_1;|y|e_1)-\int_{B_1^c} P_s(|y|e_1,z) J_{2s-1}(|x|e_1;z)\,dz+\frac{1}{\Lambda(n,s)}\int_{B_1^c} P_s(|y|e_1,z)\, J_{s-1}(|x|e_1;z)\,dz\geq 0,$$

$$s\in\Big(\frac{1}{2},1\Big),\quad x,y\in B_1.$$

Lemma 2.4 *Let $n\geq 2$. For $\varepsilon, r>0$ and $\alpha\in\mathbb{R}$ it holds*

$$\int_{\partial B_r} \frac{dx}{\big|x-\varepsilon e_1\big|^{2\alpha}} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\, r^{n-1}\int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{\big(r^2+\varepsilon^2-2\varepsilon r t\big)^\alpha}\,dt$$

In case $r=\varepsilon$, we additionally require $2\alpha<n-1$. In particular, we have for $\tau-\frac{n}{2}\notin\mathbb{N}_0$

$$J_\tau(x;y) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\kappa(n,\tau)|x|^{n-1}\int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{\big(|x|^2+|y|^2-2|x||y|t\big)^{\frac{n}{2}-\tau}}\,dt \qquad x,y\in\mathbb{R}^n,$$

where we require additionally $|x|\neq|y|$ if $\tau>\frac{1}{2}$.

Proof Via an explicit calculation

$$\begin{aligned}\int_{\partial B_r} \frac{dx}{\big|x-\varepsilon e_1\big|^{2\alpha}} = r^{n-1}\int_{\partial B_1} \frac{dy}{\big|ry-\varepsilon e_1\big|^{2\alpha}} = r^{n-1}\int_{\partial B_1} \frac{dy}{\big(r^2-2r\varepsilon y_1+\varepsilon^2\big)^\alpha} = \\ = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\, r^{n-1}\int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{\big(r^2+\varepsilon^2-2\varepsilon r t\big)^\alpha}\,dt.\end{aligned}$$

The last part follows by setting $r=|x|$, $\varepsilon=|y|$, and $\alpha=\frac{n}{2}-\tau$.

Lemma 2.5 *For $x\in[0,1)$ and $s\in(\frac{1}{2},1)$ let*

$$\begin{aligned} f_1:[0,\infty)\setminus\{x\} &\longrightarrow \mathbb{R} & \qquad f_2:(1,\infty) &\longrightarrow \mathbb{R} \\ y &\longmapsto J_{2s-1}(xe_1;ye_1) & z &\longmapsto J_{s-1}(xe_1;ze_1).\end{aligned}$$

The following holds.

(1) If $n=1$, then f_1 is (cf. Figure 1):

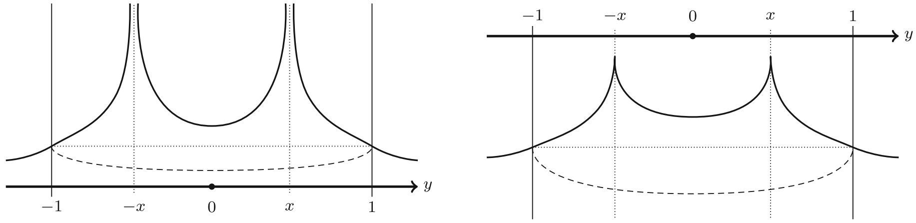

Fig. 1 On the left, a qualitative graph of f_1 and of its s-harmonic extension as entailed by the Poisson integral in Eq. 2.8 in $(-1, 1)$ (dashed line) for $\frac{1}{2} < s < \frac{3}{4}$; on the right, the analogue picture for $\frac{3}{4} < s < 1$

(a) positive for $\frac{1}{2} < s < \frac{3}{4}$,
(b) negative for $\frac{3}{4} < s < 1$,
(c) decreasing in (x, ∞),
(d) increasing and convex in $(0, x)$.

(2) If $n = 1$, $s = \frac{3}{4}$, *then* $f_1(y) = -\frac{1}{2\pi} \ln |x^2 - y^2|$, *so* f_1 *satisfies (c) and (d) in 1. Moreover,* $f_1 > 0$ *in* $[0, \sqrt{1+x^2}) \setminus \{x\}$ *and* $f_1 < 0$ *in* $(\sqrt{1+x^2}, \infty)$.
(3) f_2 *is positive and decreasing.*
(4) For $n \geq 2$ *the function* f_1 *is positive and satisfies (c) and (d) in (1).*

Proof For *(3)* note that by definition and Lemma 2.4 we have

$$f_2(z) = \begin{cases} \dfrac{2^{1-2s}\Gamma(\frac{3}{2}-s)(1-s)}{\sqrt{\pi}\,\Gamma(s)}\Big(|x-z|^{2s-3} + |x+z|^{2s-3}\Big) & n = 1; \\ \dfrac{(1-s)\Gamma(\frac{n}{2}-s+1)}{2^{2s-3}\sqrt{\pi}\Gamma(s)\Gamma(\frac{n-1}{2})}\, x^{n-1} \displaystyle\int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(x^2+z^2-2xzt)^{\frac{n}{2}-s+1}}\, dt & n > 1, \end{cases}$$

from where it easily follows that we have $f_2'(z) < 0 < f_2(z)$ for $z \geq 1$.
For *(4)* note that by Lemma 2.4 we have

$$f_1(y) = \frac{\Gamma(\frac{n}{2}-2s+1)}{2^{4s-3}\sqrt{\pi}\Gamma(2s-1)\Gamma(\frac{n-1}{2})} \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(x^2+y^2-2xyt)^{\frac{2+n}{2}-2s}}\, dt$$

and from here the statement follows again.

Lemma 2.6 *For any* $s \in (\frac{1}{2}, 1)$, $n \in \mathbb{N}$, *and* $x, y \in [0, 1)$ *it holds*

$$J_{2s-1}(xe_1; ye_1) - \int_{B_1^c} P_s(ye_1, z)\, J_{2s-1}(xe_1; z)\, dz \geq 0 \qquad \text{if } x \leq y, \tag{2.9}$$

$$J_{2s-1}(xe_1; ye_1) - \int_{B_1^c} P_s(ye_1, z)\, J_{2s-1}(xe_1; z)\, dz \geq J_{2s-1}(xe_1; 0) - J_{2s-1}(xe_1; e_1) \quad \text{if } y < x, \tag{2.10}$$

$$\int_{B_1^c} P_s(ye_1, z)\, J_{s-1}(xe_1; z)\, dz \geq \int_{B_1^c} P_s(0, z)\, J_{s-1}(xe_1; z)\, dz. \tag{2.11}$$

Proof Note first that $J_{2s-1}(xe_1; z)$ for $x \in [0, 1)$ and $z \in B_1^c$ is maximized by $J_{2s-1}(xe_1; e_1)$ using Lemma 2.5. Thus, by the positivity of P_s we have

$$\int_{B_1^c} P_s(ye_1, z)\, J_{2s-1}(xe_1; z)\, dz \le J_{2s-1}(xe_1; e_1) \int_{B_1^c} P_s(ye_1, z)\, dz = J_{2s-1}(xe_1; e_1),$$

because the integral of the Poisson kernel is normalized, see *e.g.* [7]. For $x \le y \le 1$, by Lemma 2.5, point *(1.c)* resp. *(4)*, we have $J_{2s-1}(xe_1; ye_1) \ge J_{2s-1}(xe_1; e_1)$, so that Eq. 2.9 holds. For $y < x \le 1$, by Lemma 2.5, point *(1.d)* resp. *(4)*, we have $J_{2s-1}(xe_1; ye_1) \ge J_{2s-1}(xe_1; 0)$ and thus Eq. 2.10 holds.

The last inequality Eq. 2.11 finally follows from Lemma 2.5, point *(3)*, and Proposition 5.□

In view of Eq. 2.9, it follows that Eq. 2.8 is satisfied for $|x| \le |y| < 1$. So, we can reduce our analysis of Eq. 2.8 to the range $0 \le |y| < |x| < 1$: on this we use Eqs. 2.10 and 2.11. In this way, Eq. 2.8 will be completely verified once we show the nonnegativity of the function

$$\begin{aligned} &[0, 1) \longrightarrow \mathbb{R} \\ &x \longmapsto J_{2s-1}(xe_1; 0) - J_{2s-1}(xe_1; e_1) + \frac{1}{\Lambda(n, s)} \int_{B_1^c} P_s(0, z)\, J_{s-1}(xe_1; z)\, dz. \end{aligned} \tag{2.12}$$

Noting that the last addend is positive, the positivity of the above immediately follows for those $x \in [0, 1)$ for which one has $J_{2s-1}(xe_1; 0) \ge J_{2s-1}(xe_1; e_1)$, which is what we study next.

Lemma 2.7 *Let $s \in (\frac{1}{2}, 1)$. Then there exists $x_*(n, s) \in (\frac{1}{2}, 1)$ such that*

$$J_{2s-1}(xe_1; 0) \ge J_{2s-1}(xe_1; e_1) \qquad \textit{for all } x \in [0, x_*(n, s)].$$

More precisely, one can take (Fig. 2)

$$x_*(n, s) = \begin{cases} \dfrac{2}{3} & \textit{for } n = 1 \textit{ and } s = \dfrac{3}{4}; \\[2ex] \dfrac{1}{1 + \left(2 - 2^{4s-2-n}\right)^{\frac{1}{4s-2-n}}} & \textit{for } n = 1 \textit{ and } s \ne \dfrac{3}{4}, \textit{ or } n \ge 2. \end{cases}$$

Remark 2.8

(1) Note here that $x_*(1, s) \ge \frac{3}{5}$. The statement of Lemma 2.7 hence gives an alternative proof to the mid-concavity as shown in [6, Theorem 1.1], although just in dimension $n = 1$.
(2) Similarly to the previous point, $x_*(n, s) > \frac{1}{2}$ for any $n \in \mathbb{N}$. The statement of Lemma 2.7 gives therefore super-harmonicity in the ball $B_{1/2}$ in any dimension.
(3) To show Theorems 1.1 and 1.2, it is in view of Lemma 2.7 enough to show the positivity of Eq. 2.12 for $x \in (x_*(n, s), 1)$ and $s \in (\frac{1}{2}, 1)$.

Proof of Lemma 2.7 If $n = 1$, $s = \frac{3}{4}$, then

$$J_{2s-1}(x; y) = J_{\frac{1}{2}}(x; y) = -\frac{1}{2\pi} \ln(|x^2 - y^2|) \quad \text{for } x \in [0, 1),\ y \ge 0, \text{ with } x \ne y.$$

Hence,

$$0 \le J_{2s-1}(x; 0) - J_{2s-1}(x; 1) = -\frac{1}{2\pi} \ln(x^2) + \frac{1}{2\pi} \ln(1 - x^2) = \frac{1}{2\pi} \ln\left(\frac{1}{x^2} - 1\right)$$

if and only if $x \le \frac{\sqrt{2}}{2}$ and clearly $x_*(1, \frac{3}{4}) = \frac{2}{3} < \frac{\sqrt{2}}{2}$.
If $n = 1$ and $s \neq \frac{3}{4}$, it follows that $J_{2s-1}(x; 0) \ge J_{2s-1}(x; 1)$ holds for those x's, where we have

$$2\Gamma(3/2-2s)x^{4s-3} \ge \Gamma(3/2-2s)\Big((1-x)^{4s-3} + (1+x)^{4s-3}\Big).$$

Noting that $\Gamma(3/2-2s)$ changes its sign at $s = \frac{3}{4}$, the claim amounts to checking

$$\begin{aligned} 2x^{4s-3} &\ge (1-x)^{4s-3} + (1+x)^{4s-3} \quad \text{for } s \in \Big(\frac{1}{2}, \frac{3}{4}\Big),\\ 2x^{4s-3} &\le (1-x)^{4s-3} + (1+x)^{4s-3} \quad \text{for } s \in \Big(\frac{3}{4}, 1\Big). \end{aligned}$$

A sufficient condition, is then given by

$$\begin{cases} \big(2-2^{4s-3}\big)x^{4s-3} \ge (1-x)^{4s-3} \\ 2^{4s-3}x^{4s-3} \ge (1+x)^{4s-3} \\ s \in \Big(\frac{1}{2}, \frac{3}{4}\Big) \end{cases} \quad \text{or} \quad \begin{cases} \big(2-2^{4s-3}\big)x^{4s-3} \le (1-x)^{4s-3} \\ 2^{4s-3}x^{4s-3} \le (1+x)^{4s-3} \\ s \in \Big(\frac{3}{4}, 1\Big) \end{cases}$$

which are both equivalent to

$$\begin{cases} \big(2-2^{4s-3}\big)^{\frac{1}{4s-3}} x \le 1-x, \\ 2x \le 1+x. \end{cases}$$

This gives $J_{2s-1}(x, 0) \ge J_{2s-1}(x, 1)$ for all x such that

$$x \le \frac{1}{1+\big(2-2^{4s-3}\big)^{\frac{1}{4s-3}}} =: x_*(1, s), \qquad s \in \Big(\frac{1}{2}, 1\Big) \setminus \Big\{\frac{3}{4}\Big\}.$$

It can be easily verified that $x_*(1, \cdot) : (\frac{1}{2}, 1) \setminus \{\frac{3}{4}\} \to \mathbb{R}$ satisfies

$$\begin{aligned} &\lim_{s\downarrow 1/2} x_*(1, s) = \frac{3}{5}, &&\lim_{s\uparrow 1} x_*(1, s) = 1, &&\lim_{s\to 3/4} x_*(1, s) = \frac{2}{3},\\ &\frac{d}{ds}x_*(1, s) > 0, &&\lim_{s\uparrow 1} \frac{d}{ds}x_*(1, s) = 8\ln 2, &&x_*(1, s) \in \Big(\frac{3}{5}, 1\Big). \end{aligned}$$

Finally, let $n \ge 2$. By Lemma 2.4 we have for $x, y \in [0, 1)$

$$J_{2s-1}(xe_1; ye_1) = \frac{\Gamma(\frac{n}{2}+1-2s)x^{n-1}}{4^{2s-1}\sqrt{\pi}\,\Gamma(2s-1)\Gamma(\frac{n-1}{2})} j(x, y), \tag{2.13}$$

where we put

$$j : [0, 1] \times [0, 1] \to [0, \infty], \quad j(a, b) = \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(a^2+b^2-2abt)^{\frac{n}{2}+1-2s}} dt.$$

Note here, that for $a > 0$

$$j(a,a) = (2a)^{4s-2-n} \int_{-1}^{1} (1+t)^{\frac{n-3}{2}} (1-t)^{2s-\frac{5}{2}} \, dt \begin{cases} < \infty & \text{for } s > \dfrac{3}{4}; \\ = \infty & \text{for } s \le \dfrac{3}{4}, \end{cases}$$

$$j(a,0) = a^{4s-2-n} \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \, dt = a^{4s-2-n} \frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}}{\Gamma(\frac{n}{2})}, \quad \text{and}$$

$$j(a,0) - j(a,1) = \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \Big(a^{4s-2-n} - \frac{1}{(a^2+1-2at)^{\frac{n}{2}+1-2s}} \Big) dt.$$

Since

$$a^{4s-2-n} \ge \frac{1}{(a^2+1-2at)^{\frac{n}{2}+1-2s}} \quad \Leftrightarrow \quad a^2+1-2at > a^2 \quad \Leftrightarrow \quad 1 > 2at$$

it follows that we have $j(a,0) - j(a,1) \ge 0$ for $a \in (0, \frac{1}{2}]$. To be more precise on the estimate, next note that it also follows for $a \in (\frac{1}{2}, 1)$

$$\begin{aligned} j(a,0) - j(a,1) &= \int_{-1}^{\frac{1}{2a}} (1-t^2)^{\frac{n-3}{2}} \Big(a^{4s-2-n} - \frac{1}{(a^2+1-2at)^{\frac{n}{2}+1-2s}} \Big) dt \\ &\qquad - \int_{\frac{1}{2a}}^{1} (1-t^2)^{\frac{n-3}{2}} \Big(\frac{1}{(a^2+1-2at)^{\frac{n}{2}+1-2s}} - a^{4s-2-n} \Big) dt \\ &\ge \Big(a^{4s-2-n} - \frac{1}{(a^2+1+2a)^{\frac{n}{2}+1-2s}} \Big) \int_0^1 (1-t^2)^{\frac{n-3}{2}} \, dt \\ &\qquad - \Big(\frac{1}{(a^2+1-2a)^{\frac{n}{2}+1-2s}} - a^{4s-2-n} \Big) \int_0^1 (1-t^2)^{\frac{n-3}{2}} \, dt \\ &= \frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}}{2\Gamma(\frac{n}{2})} \Big(2a^{4s-2-n} - (a+1)^{4s-2-n} - (1-a)^{4s-2-n} \Big). \end{aligned}$$

Arguing as in the case $n = 1$, $s \neq \frac{3}{4}$, we see that $j(a,0) - j(a,1) \ge 0$ holds for all $a \in (0, x_*(n,s)]$ with the claimed $x_*(n,s)$. With Eq. 2.13, this concludes the proof.

The analysis carried out so far allows us to reduce the condition in Eq. 2.8 to the following stronger one:

$$J_{2s-1}(xe_1; 0) - J_{2s-1}(xe_1; 1) + \frac{1}{\Lambda(n,s)} \int_{B_1^c} P_s(0,z) \, J_{s-1}(xe_1; z) \, dz \ge 0,$$

$$s \in \Big(\frac{1}{2}, 1\Big), \; x_*(n,s) < x < 1. \quad (2.14)$$

Before we turn to the study of Eq. 2.14, let us prove a technical lemma which will be needed in the following.

Lemma 2.9 *For $x \in (0,1)$ and $s \in (\frac{1}{2}, 1)$ it holds*

$$\int_1^{\infty} \frac{(z-x)^{2s-3} + (z+x)^{2s-3}}{z\,(z^2-1)^s} \, dz \ge \frac{\Gamma(1-s)\sqrt{\pi}}{\Gamma(\frac{3}{2}-s)} \big(1-x^2\big)^{s-2} \big(2x-1\big).$$

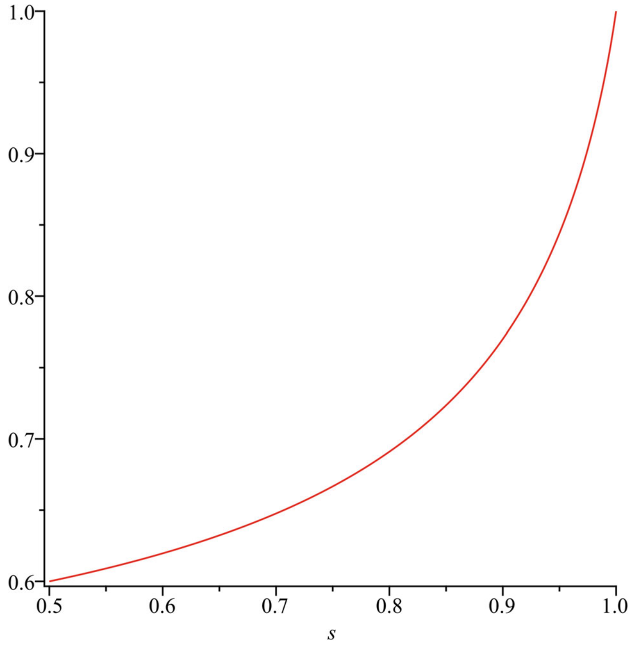

Fig. 2 A graph of $x_*(1,s)$, $\frac{1}{2} < s < 1$

Proof With the change of variables $z = \frac{1}{t}$, we write

$$\int_1^\infty \frac{(z-x)^{2s-3}+(z+x)^{2s-3}}{z\left(z^2-1\right)^s}\,dz = \int_0^1 \frac{(1-xt)^{2s-3}+(1+xt)^{2s-3}}{\left(1-t^2\right)^s}\,t^2\,dt = \int_{-1}^1 \frac{(1-xt)^{2s-3}}{\left(1-t^2\right)^s}\,t^2\,dt$$
$$= \int_{-1}^1 \frac{(1-xt)^{2s-3}}{\left(1-t^2\right)^s}\,dt - \int_{-1}^1 \frac{(1-xt)^{2s-3}}{\left(1-t^2\right)^{s-1}}\,dt.$$

We integrate the second integral in the above expression by parts, obtaining

$$\int_{-1}^1 (1-xt)^{2s-3}\left(1-t^2\right)^{1-s}\,dt = \frac{1}{x}\int_{-1}^1 (1-xt)^{2s-2}\left(1-t^2\right)^{-s} t\,dt.$$

Via another change of variable, namely $t = 2\tau - 1$, we obtain

$$\int_{-1}^1 \frac{(1-xt)^{2s-3}}{\left(1-t^2\right)^s}\,dt = 2^{1-2s}\int_0^1 \frac{(1+x-2x\tau)^{2s-3}}{\tau^s\,(1-\tau)^s}\,d\tau$$
$$= \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}(1+x)^{2s-3}\,{}_2F_1\left(3-2s,1-s;2-2s\,\middle|\,\frac{2x}{1+x}\right)$$
$$= \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}(1+x)^{2s-3}\left(1-\frac{2x}{1+x}\right)^{s-2}\left(1-\frac{x}{1+x}\right)$$
$$= \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}\left(1-x^2\right)^{s-2}$$

(where we have used Eqs. A.3 and A.9) and, similarly,

$$\begin{aligned}
\int_{-1}^{1} \frac{(1-xt)^{2s-2}}{\left(1-t^2\right)^s}\, t\, dt &= \int_{-1}^{1} \frac{(1-xt)^{2s-2}}{\left(1-t^2\right)^s}\, dt - \int_{-1}^{1} \frac{(1-xt)^{2s-2}}{(1+t)^s}\,(1-t)^{1-s}\, dt \\
&= 2^{1-2s} \int_0^1 \frac{(1+x-2x\tau)^{2s-2}}{\tau^s\,(1-\tau)^s}\, d\tau - 2^{2-2s} \int_0^1 \frac{(1+x-2x\tau)^{2s-2}}{\tau^s}\,(1-\tau)^{1-s}\, d\tau \\
&\le \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}(1+x)^{2s-2}\, {}_2F_1\Big(2-2s, 1-s; 2-2s \Big| \frac{2x}{1+x}\Big) \\
&\quad - 2^{2-2s}(1+x)^{2s-2}\Big(1-\frac{2x}{1+x}\Big) \int_0^1 \frac{(1-\frac{2x}{1+x}\tau)^{2s-3}}{\tau^s}\,(1-\tau)^{1-s}\, d\tau \\
&= \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}(1+x)^{2s-2}\, {}_2F_1\Big(2-2s, 1-s; 2-2s \Big| \frac{2x}{1+x}\Big) \\
&\quad - \frac{2^{2-2s}\Gamma(1-s)\Gamma(2-s)}{\Gamma(3-2s)}(1+x)^{2s-3}(1-x)\, {}_2F_1\Big(3-2s, 1-s; 3-2s \Big| \frac{2x}{1+x}\Big) \\
&= \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}(1+x)^{2s-2}\Big(1-\frac{2x}{1+x}\Big)^{s-1} \\
&\quad - \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}(1+x)^{2s-3}(1-x)\Big(1-\frac{2x}{1+x}\Big)^{s-1} \\
&= \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}\Big[\left(1-x^2\right)^{s-1} - (1+x)^{s-2}(1-x)^s\Big] \\
&= \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}\left(1-x^2\right)^{s-2}(1-x)\cdot 2x.
\end{aligned}$$

We then deduce

$$\int_1^{\infty} \frac{(z-x)^{2s-3}+(z+x)^{2s-3}}{z\left(z^2-1\right)^s}\, dz \ge \frac{2^{1-2s}\Gamma(1-s)^2}{\Gamma(2-2s)}\left(1-x^2\right)^{s-2}\big[1-2(1-x)\big].$$

The constant in front the above expression can be transformed using the identities on the Gamma function, namely the Legendre duplication formula

$$\Gamma(1-s)\Gamma\Big(\frac{3}{2}-s\Big) = 2^{1-2(1-s)}\sqrt{\pi}\,\Gamma(2-2s),$$

concluding the proof.

3 The one-dimensional case: $s = 3/4$

Note first, that in this case it follows from Eqs. 1.5 and 2.6 that

$$J_{2s-1}(x;y)\Big|_{s=\frac{3}{4},n=1} = J_{\frac{1}{2}}(x;y) = -\frac{1}{2\pi}\ln\left|x^2-y^2\right| \qquad \text{for } x, y \in \mathbb{R},\ x \ne y.$$

Moreover (*cf*. Eqs. B.2 and 1.4)

$$\Lambda\Big(1, \frac{3}{4}\Big) = \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{9}{4})},$$

$$P_{\frac{3}{4}}(x, z) = \frac{\sqrt{2}}{2\pi} \frac{\big(1 - x^2\big)^{3/4}}{|x - z|\big(z^2 - 1\big)^{3/4}}, \quad \text{for } x \in (-1, 1), |z| > 1,$$

$$J_{s-1}(x; y)\Big|_{s=\frac{3}{4}, n=1} = J_{-\frac{1}{4}}(x; z) = \frac{1}{4\sqrt{2\pi}}\Big(|x-y|^{-3/2} + |x+y|^{-3/2}\Big) \quad \text{for } x, y \in \mathbb{R},\ x \neq z.$$

As our goal is to verify Eq. 2.14, we have to prove

$$-2\ln x + \ln(1-x^2) + \frac{1}{2\Lambda(1, \frac{3}{4})\sqrt{\pi}} \int_1^\infty \frac{(z-x)^{-3/2} + (z+x)^{-3/2}}{z\big(z^2-1\big)^{3/4}}\, dz \geq 0 \quad \text{for } x_*\Big(1, \frac{3}{4}\Big) = \frac{2}{3} < x < 1.$$

To this aim, we estimate the integral above using Lemma 2.9, which gives us

$$\int_1^\infty \frac{(z - x)^{-3/2} + (z + x)^{-3/2}}{z\big(z^2 - 1\big)^{3/4}}\, dz \geq \frac{\Gamma(\frac{1}{4})\sqrt{\pi}}{\Gamma(\frac{3}{4})}\big(1 - x^2\big)^{-5/4}\big(2x - 1\big).$$

In this way, we are left with verifying

$$- 2\ln x + \ln(1 - x^2) + \frac{\Gamma(\frac{9}{4})}{4\Gamma(\frac{7}{4})} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\big(1 - x^2\big)^{-5/4}\big(2x - 1\big) =$$

$$= -2\ln x + \ln(1 - x^2) + \frac{5\,\Gamma(\frac{1}{4})^2}{48\,\Gamma(\frac{3}{4})^2}\big(1 - x^2\big)^{-5/4}\big(2x - 1\big) \geq 0 \qquad \text{for } \frac{2}{3} < x < 1.$$

Using the fact that

$$2x - 2 \geq -\frac{3}{2}(1 - x^2) \quad \text{and} \quad \big(1 - x^2\big)^{-5/4} \geq \big(1 - x^2\big)^{-1} \qquad \text{for } \frac{2}{3} < x < 1$$

in the following we rather show

$$-2\ln x + \ln(1-x^2) + c\big(1-x^2\big)^{-1}\Big(1 - \frac{3}{2}(1 - x^2)\Big) \geq 0 \quad \text{for } \frac{2}{3} < x < 1,\ c = \frac{5\,\Gamma(\frac{1}{4})^2}{48\,\Gamma(\frac{3}{4})^2} = 0.91...$$

We do so by re-labeling $t = 1 - x^2$ and by checking

$$- \ln(1 - t) + \ln t + ct^{-1}\Big(1 - \frac{3}{2}t\Big) \geq 0 \qquad \text{for } 0 < t < \frac{5}{9}. \tag{3.1}$$

The above inequality follows by computing the minimum of the left-hand side in the given range for t. Indeed,

$$\frac{d}{dt}\Big[- \ln(1 - t) + \ln t + ct^{-1}\Big(1 - \frac{3}{2}t\Big)\Big] = \frac{1}{1 - t} + \frac{1}{t} - \frac{c}{t^2} \geq 0 \qquad \text{for } 0 < t < \frac{5}{9}$$

if and only

$$0.4769... = \frac{c}{c + 1} \leq t < \frac{5}{9}$$

so that the left-hand side of Eq. 3.1 attains its minimum at $c/(c+1)$ where it equals

$$-\ln(1-\frac{c}{c+1})+\ln\frac{c}{c+1}+(c+1)\Big(1-\frac{3}{2}\frac{c}{c+1}\Big)=\ln c+1-\frac{c}{2}=0.45...>0.$$

Hence, Eq. 2.14 holds for $n=1$ and $s=\frac{3}{4}$.

4 The general one-dimensional case

In the following, we analyze Eq. 2.14 with $s\neq\frac{3}{4}$ and $n=1$. Recall the definition of $x_*(1,s)$ in Lemma 2.7. As an application of Lemma 2.9 and of definitions Eqs. 1.5 and 1.4 we rather check that

$$\frac{2x^{4s-3}-(1-x)^{4s-3}-(1+x)^{4s-3}}{3-4s}+\mu(1-x^2)^{s-2}\big(2x-1\big)\geq 0,$$
$$\text{for } s\in\Big(\frac{1}{2},1\Big)\setminus\Big\{\frac{3}{4}\Big\},\ x_*(1,s)<x<1,$$

where (recall Eqs. 2.7 and B.2)

$$\mu=\frac{1}{2^{2s-1}\Lambda\Gamma(s)|\Gamma(s-1)|}\frac{2^{4s-2}\sqrt{\pi}\,\Gamma(2s-1)}{\Gamma(\frac{5}{2}-2s)}=\frac{2^{2s-1}(1-s)\sqrt{\pi}}{s}\frac{\Gamma(\frac{3}{2}+s)\Gamma(2s-1)}{\Gamma(s)^3\,\Gamma(\frac{3}{2}+2s)\Gamma(\frac{5}{2}-2s)}.$$

Note that the function

$$(0,1]\ni x\longmapsto\frac{1}{3-4s}\Big(2x^{4s-3}-(1+x)^{4s-3}\Big)$$

is decreasing. Indeed, this follows by differentiation:

$$\frac{1}{3-4s}\frac{d}{dx}\Big(2x^{4s-3}-(1+x)^{4s-3}\Big)=-\Big(2x^{4(s-1)}-(1+x)^{4(s-1)}\Big)<0\qquad\text{for } x\in(0,1].$$

Then, fixing $a,b\in[\frac{3}{5},1]$ with $a<b$ we find with this for $x\in(a,b)$

$$\begin{aligned}&\frac{2x^{4s-3}-(1-x)^{4s-3}-(1+x)^{4s-3}}{3-4s}+\mu(1-x^2)^{s-2}(2x-1)\\&\quad\geq\frac{2b^{4s-3}-(1+b)^{4s-3}}{3-4s}-\frac{(1-x)^{4s-3}}{3-4s}+(1+b)^{s-2}(2a-1)\mu(1-x)^{s-2}=:q_{a,b}(s,x).\end{aligned}$$

A direct computation gives that the function $(a,b)\ni x\mapsto q_{a,b}(s,x)$ is controlled from below in (a,b) by the value

$$q_{a,b}\big(s,x_{a,b}(s)\big),\qquad\text{where } x_{a,b}(s):=1-\Big((1+b)^{s-2}(2a-1)(2-s)\mu\Big)^{\frac{1}{3s-1}}.$$

Keeping this in mind, we split

$$\Big(\frac{3}{5},1\Big)=\big(a_1,b_1\big]\cup\big(a_2,b_2\big]\cup\big(a_3,b_3\big]\cup\big(a_4,b_4\big)$$
$$a_1=\frac{3}{5},\ b_1=a_2=\frac{7}{10},\ b_2=a_3=\frac{4}{5},\ b_3=a_4=\frac{9}{10},\ b_4=1.$$

In each of these subintervals it holds that

$$q_{a_i,b_i}(s,x) \geq q_{a_i,b_i}\big(s, x_{a_i,b_i}(s)\big) > 0 \qquad \text{for } x \in (a_i, b_i),\ s \in \Big(\frac{1}{2}, 1\Big)\setminus\Big\{\frac{3}{4}\Big\},\ i \in \{1,2,3,4\},$$

see Figure 3. From this it follows that Eq. 2.14 holds for $n = 1$.

5 The higher-dimensional case

In the following we test the validity of Theorem 1.1 for the dimension $2 \leq n \leq 12$. In view of Lemma 2.7, it remains to show that, for $x \in (x_*(n,s), 1)$ and x_* as in Lemma 2.7, it holds

$$\int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \left(x^{4s-2-n} - \frac{1}{(x^2+1-2xt)^{\frac{n}{2}+1-2s}} \right) dt \\ + M(n,s) \int_1^\infty \frac{1}{\left(r^2-1\right)^s r} \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(x^2+r^2-2xrt)^{\frac{n}{2}-s+1}}\, dt\, dr \geq 0 \quad (5.1)$$

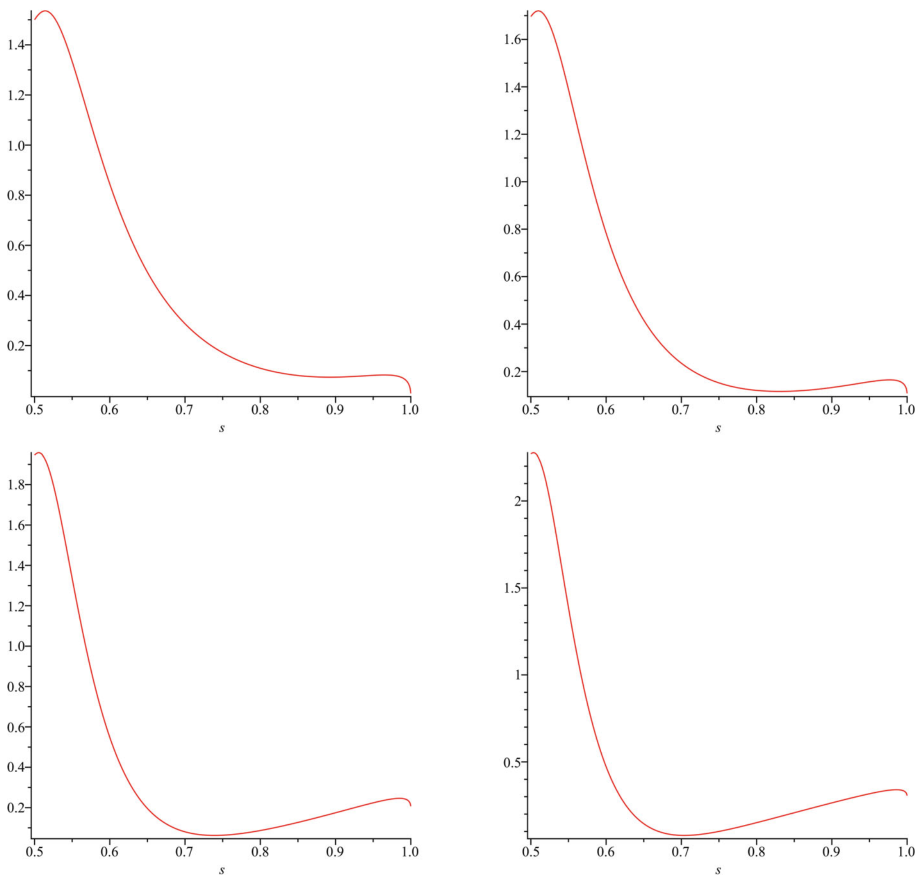

Fig. 3 A graph of $q_{a,b}(s, x_{a,b}(s))$ for $\frac{1}{2} < s < 1$ with $(a,b) = (0.9, 1)$—top left—, $(a,b) = (0.8, 0.9)$—top right—, $(a,b) = (0.7, 8)$—bottom left—, and $(a,b) = (0.6, 0.7)$—bottom right

with (recall Eqs. 1.4, 1.5, and 2.7)

$$
\begin{aligned}
M(n,s) &:= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{\gamma(n,s)\,\kappa(n,s-1)}{\Lambda(n,s)\,\kappa(n,2s-1)} \\
&= \frac{2}{\Gamma(s)\,\Gamma(1-s)} \frac{(\frac{n}{2}+s)\,\Gamma(\frac{n}{2})\,\Gamma(1+2s)}{2^{2s}\Gamma(1+s)^2\,\Gamma(\frac{n}{2}+1+2s)} \frac{\Gamma(\frac{n}{2}+1-s)}{2^{2s-3}\big|\Gamma(s-1)\big|} \frac{2^{4s-3}\Gamma(2s-1)}{\Gamma(\frac{n}{2}+1-2s)} \\
&= \frac{(1-s)\Gamma(1+2s)\Gamma(2s-1)}{s^2\Gamma(s)^4\Gamma(1-s)} \cdot \frac{(n+2s)\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1-s)}{\Gamma(\frac{n}{2}+1+2s)\Gamma(\frac{n}{2}+1-2s)}
\end{aligned}
$$

where we have used Lemma 2.4, the transformation into polar coordinates, and some reformulations of the constant using properties of the Gamma function. Note here, that

$$
\int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}}\,dt = \int_0^1 \tau^{-\frac{1}{2}}(1-\tau)^{\frac{n-3}{2}}\,d\tau = \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} = \frac{2^{n-2}\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)}
$$

and with the transformation $t+1=2\tau$ we have

$$
\begin{aligned}
\int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{\left(x^2+1-2xt\right)^{\frac{n}{2}+1-2s}}\,dt &= 2\int_0^1 \frac{(2-2\tau)^{\frac{n-3}{2}}(2\tau)^{\frac{n-3}{2}}}{\left(x^2+1-2x(2\tau-1)\right)^{\frac{n}{2}+1-2s}}\,dt \\
&= 2^{n-2}\int_0^1 \frac{(1-\tau)^{\frac{n-3}{2}}\tau^{\frac{n-3}{2}}}{\left((x+1)^2-4x\tau\right)^{\frac{n}{2}+1-2s}}\,dt \\
&= \frac{2^{n-2}}{(x+1)^{n+2-4s}} \int_0^1 \tau^{\frac{n-3}{2}}(1-\tau)^{\frac{n-3}{2}}\left(1-\frac{4x}{(x+1)^2}\tau\right)^{2s-\frac{n}{2}-1}\,dt \\
&= \frac{2^{n-2}}{(x+1)^{n+2-4s}} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)}\, {}_2F_1\!\left(\frac{n}{2}+1-2s,\,\frac{n-1}{2};\,n-1\,\Big|\,\frac{4x}{(x+1)^2}\right),
\end{aligned}
\tag{5.2}
$$

where we have used Eq. A.3. Similarly, we also have

$$
\begin{aligned}
\int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{\left(x^2+r^2+2xrt\right)^{\frac{n}{2}-s+1}}\,dt &= \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{\left(x^2+r^2-2xrt\right)^{\frac{n}{2}-s+1}}\,dt \\
&= \frac{2^{n-2}}{(x+r)^{n+2-2s}} \int_0^1 \tau^{\frac{n-3}{2}}(1-\tau)^{\frac{n-3}{2}}\left(1-\frac{4xr}{(x+r)^2}\tau\right)^{s-\frac{n}{2}-1}\,dt \\
&= \frac{2^{n-2}}{(x+r)^{n+2-2s}} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)}\, {}_2F_1\!\left(\frac{n}{2}+1-s,\,\frac{n-1}{2};\,n-1\,\Big|\,\frac{4xr}{(x+r)^2}\right).
\end{aligned}
\tag{5.3}
$$

We now perform some transformations on the hypergeometric functions appearing respectively in Eqs. 5.2 and 5.3. For this, let $\sigma = 1-2s$ or $\sigma = 1-s$ and let $t=1$ or $t=r$ (so that $t \geq 1$). Then note that

$$
\frac{n}{2}+\sigma-\frac{n-1}{2} = \frac{n}{2}+\sigma+\frac{n-1}{2}-(n-1) \quad \text{and} \quad \sqrt{1-\frac{4xt}{(x+t)^2}} = \frac{t-x}{t+x},
$$

so that, by Eq. A.10, we have

$$
\begin{aligned}
&{}_2F_1\left(\frac{n}{2}+\sigma, \frac{n-1}{2}; n-1\middle|\frac{4xt}{(x+t)^2}\right) = \\
&= 2^{2\sigma+n}\left(1+\frac{t-x}{t+x}\right)^{-2\sigma-n} {}_2F_1\left(\frac{n}{2}+\sigma, \frac{n}{2}+\sigma-\frac{n-1}{2}+\frac{1}{2}; \frac{n-1}{2}+\frac{1}{2}\middle|\left(\frac{1-\frac{t-x}{t+x}}{1+\frac{t-x}{t+x}}\right)^2\right) \\
&= \left(\frac{t}{t+x}\right)^{-2\sigma-n} {}_2F_1\left(\frac{n}{2}+\sigma, 1+\sigma; \frac{n}{2}\middle|\frac{x^2}{t^2}\right).
\end{aligned}
$$

With this, Eq. 5.1 translates to

$$
\begin{aligned}
x^{4s-2-n} &- {}_2F_1\left(\frac{n}{2}+1-2s, 2-2s; \frac{n}{2}\middle|x^2\right) \\
&+ M(n,s)\int_1^\infty \frac{r^{2s-n-3}}{(r^2-1)^s}\, {}_2F_1\left(\frac{n}{2}+1-s, 2-s; \frac{n}{2}\middle|\frac{x^2}{r^2}\right) dr \geq 0. \qquad (5.4)
\end{aligned}
$$

We exploit next the series expansion of the hypergeometric function, see Eq. A.4. We have, due to the absolute convergence of the integral and the involved infinite sum,

$$
\begin{aligned}
&\int_1^\infty \frac{r^{2s-n-3}}{(r^2-1)^s}\, {}_2F_1\left(\frac{n}{2}+1-s, 2-s; \frac{n}{2}\middle|\frac{x^2}{r^2}\right) dr = \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1-s\right)\Gamma(2-s)} \sum_{k=0}^\infty \frac{\Gamma\left(\frac{n}{2}+1-s+k\right)\Gamma(2-s+k)}{\Gamma\left(\frac{n}{2}+k\right)} \frac{x^{2k}}{k!} \int_1^\infty \frac{r^{2s-n-3-2k}}{(r^2-1)^s}\, dr \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n}{2}+1-s\right)\Gamma(2-s)} \sum_{k=0}^\infty \frac{\Gamma\left(\frac{n}{2}+1-s+k\right)\Gamma(2-s+k)}{\Gamma\left(\frac{n}{2}+k\right)} \frac{x^{2k}}{k!} \int_0^1 \frac{\rho^{n/2+k}}{(1-\rho)^s}\, d\rho \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n}{2}+1-s\right)\Gamma(2-s)} \sum_{k=0}^\infty \frac{\Gamma\left(\frac{n}{2}+1-s+k\right)\Gamma(2-s+k)}{\Gamma\left(\frac{n}{2}+k\right)} \frac{x^{2k}}{k!} \frac{\Gamma\left(\frac{n}{2}+k+1\right)\Gamma(1-s)}{\Gamma\left(\frac{n}{2}+k+2-s\right)} \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n}{2}+1-s\right)(1-s)} \sum_{k=0}^\infty \frac{\left(\frac{n}{2}+k\right)\Gamma(2-s+k)}{\frac{n}{2}+k+1-s} \frac{x^{2k}}{k!}
\end{aligned}
$$

where we have used a change of variables with $\rho = r^{-2}$ and Eq. A.2. As it holds

$$
\frac{n}{n+2-2s} \leq \frac{\frac{n}{2}+k}{\frac{n}{2}+k+1-s} \leq 1 \qquad \text{for any } n\in\mathbb{N},\ k\in\mathbb{N}\cup\{0\},\ s\in\left(\frac{1}{2},1\right), \qquad (5.5)
$$

we deduce from the above calculation that

$$
\frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma(1-s)}{2\Gamma\left(\frac{n}{2}+2-s\right)}\left(1-x^2\right)^{s-2} \leq \int_1^\infty \frac{r^{2s-n-3}}{(r^2-1)^s}\, {}_2F_1\left(\frac{n}{2}+1-s, 2-s; \frac{n}{2}\middle|\frac{x^2}{r^2}\right) dr \leq \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1-s)}{2\Gamma\left(\frac{n}{2}+1-s\right)}\left(1-x^2\right)^{s-2}
$$

Indeed, it holds

$$
\sum_{k=0}^\infty \frac{\Gamma(2-s+k)}{\Gamma(2-s)} \frac{x^{2k}}{k!} = {}_2F_1\left(2-s, 1; 1\middle|x^2\right) = \left(1-x^2\right)^{s-2},
$$

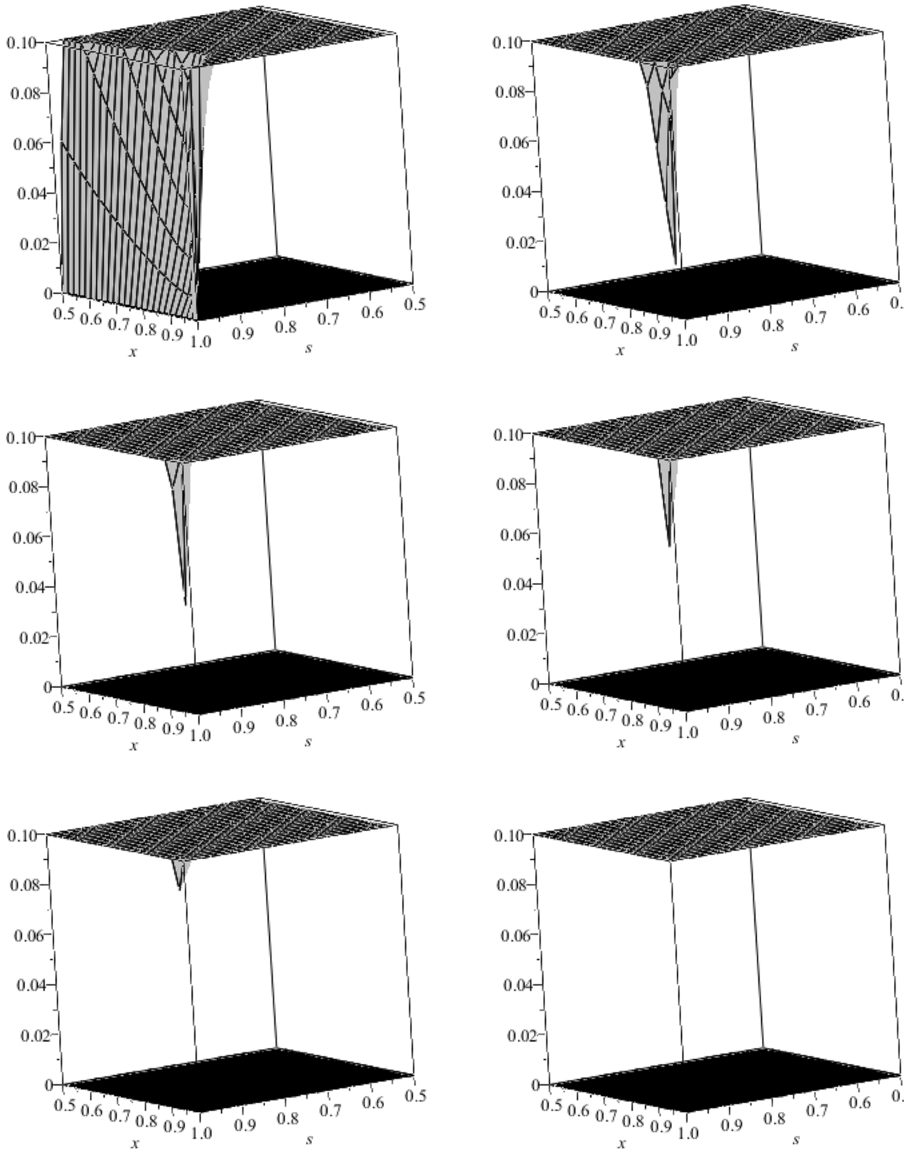

Fig. 4 The plots of

$$\begin{aligned}\left(\tfrac{1}{2},1\right)\times\left(\tfrac{1}{2},1\right) &\longrightarrow \mathbb{R}\\ (s,x) &\longmapsto \min\left\{\frac{1}{10},\, x^{4s-2-n} - {}_2F_1\left(\frac{n}{2}+1-2s,\, 2-2s;\, \frac{n}{2}\middle| x^2\right)\right.\\ &\qquad\left. + \frac{M(n,s)\Gamma\left(\frac{n}{2}+1\right)\Gamma(1-s)}{2\Gamma\left(\frac{n}{2}+2-s\right)}\left(1-x^2\right)^{s-2}\right\}\end{aligned}$$

for $n = 2$ (top left), $n = 3$ (top right), $n = 4$ (center left), $n = 5$ (center right), $n = 6$ (bottom left), and $n = 7, \ldots, 11$ (bottom right) see Eq. A.8. Hence, Eq. 5.4 is satisfied once we have

$$x^{4s-2-n} - {}_2F_1\left(\frac{n}{2}+1-2s, 2-2s; \frac{n}{2}\middle|x^2\right) + \frac{M(n,s)\Gamma\left(\frac{n}{2}+1\right)\Gamma(1-s)}{2\Gamma\left(\frac{n}{2}+2-s\right)}\left(1-x^2\right)^{s-2} \geq 0. \tag{5.6}$$

In Fig. 4 we present the plots of the left-hand side of Eq. 5.6 for $n = 2, \ldots, 11$, where this is indeed positive.

Remark 5.1 By avoiding estimate Eq. 5.5 and keeping the series expansion of the hypergeometric function, see Eq. A.4, it is possible to see that also the case $n = 12$ is actually covered by this approach. However, for larger n this keeps failing, although there always are some ranges of s where the left-hand side of Eq. 5.4 stays positive. Finally, let us mention that for $n = 127$, the left-hand side of Eq. 5.6 seems to be positive again. Indeed, again with the series expansion of the hypergeometric function, see Eq. A.4, it holds for $n \geq 4$

$${}_2F_1\left(\frac{n}{2}+1-2s, 2-2s; \frac{n}{2}\middle|x^2\right) \leq \sum_{k=0}^{\infty} \frac{\Gamma(2-2s+k)}{\Gamma(2-2s)} \frac{x^{2k}}{k!} = \left(1-x^2\right)^{2s-2},$$

see Eq. A.8. So that it remains to check

$$x^{4s-2-n} - \left(1-x^2\right)^{2s-2} + \frac{M(n,s)\Gamma\left(\frac{n}{2}+1\right)\Gamma(1-s)}{2\Gamma\left(\frac{n}{2}+2-s\right)}\left(1-x^2\right)^{s-2} \geq 0, \tag{5.7}$$

which is numerically positive for $n = 127$, see Fig. 5. Let us remark that in numerical experiments it also remained positive for any other choice of $n \geq 127$ we made. This could be driven by the fact that the left-hand side of Eq. 5.7 is greater than

$$x^{4s-2-n} - \left(1-x^2\right)^{2s-2},$$

which pointwisely diverges to $+\infty$ as $n \uparrow \infty$ for $s, x \in (\frac{1}{2}, 1)$, although not uniformly. For these reasons, we conjecture that inequality Eq. 5.7 holds true for all $n \geq 127$.

Let us mention here that our strategy strongly relies also on estimate Eq. B.1 and that a more precise estimate here could improve a lot the number of dimensions covered in our analysis.

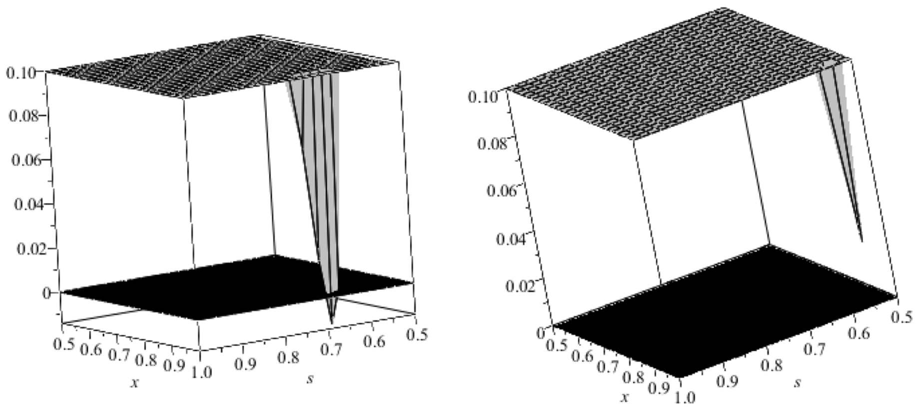

Fig. 5 On the left, Eq. 5.6 fails for $n = 12$. On the right, Eq. 5.6 is recovered for $n = 127$ via the weaker condition Eq. 5.7

Appendix A: Special Function

For the reader's convenience we list here the definitions and some properties about the special functions that we use.

Appendix A.1: The Gamma Function

As usual, the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx, \qquad \text{for } z > 0.$$

As it satisfies the recursive formula

$$\Gamma(z+1) = z\,\Gamma(z)$$

its definition can be extended using this formula to $z \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. The Gamma function satisfies in particular the duplication formula (see, *e.g.*, [3, equation 6.1.18]) s

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma\Big(z + \frac{1}{2}\Big) \qquad \text{for } z > 0. \tag{A.1}$$

Moreover, it holds (*e.g.*, [3, equation 6.1.17])

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad \text{for } z \in \mathbb{R} \setminus \mathbb{Z}.$$

Furthermore (*e.g.*, [3, equation 6.2.1]),

$$\int_0^1 t^{z-1}(1-t)^{w-1}\, dt = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}}\, dt = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}, \qquad z, w > 0. \tag{A.2}$$

Appendix A.2: The Hypergeometric Function

We collect here some facts about the hypergeometric function ${}_2F_1$. We suppose in all the following that $a, b, c, z \in \mathbb{R}$ with $c > b > 0$ and $z \in [0, 1)$, although some formulas might hold in broader generality (we refer to [3, Chapter 15]).

Recall first the integral representation

$${}_2F_1(a, b; c|z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt, \tag{A.3}$$

see [3, equation 15.3.1], and the series expansion

$${}_2F_1(a, b; c|z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^\infty \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}, \tag{A.4}$$

see [3, equation 15.1.1].

In particular one can consider $-a \in \mathbb{N} \cup \{0\}$, in which case one has that ${}_2F_1$ reduces to a polynomial of degree $-a$. For example:

$${}_2F_1(0, b; c|z) = 1, \tag{A.5}$$

$${}_2F_1(-1, b; c|z) = 1 - \frac{b}{c}\, z, \tag{A.6}$$

see [3, equation 15.4.1].

Among the many possible transformations, the following one is important to our purposes:

$$ {}_2F_1(a,b;c|z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c|z). \tag{A.7} $$

Indeed, Eq. A.7 alongside Eqs. A.5 and A.6 respectively, bears the following identities (corresponding to the particular cases $c=a$ and $c=a-1$ respectively):

$$ {}_2F_1(a,b;a|z) = (1-z)^{-b} \qquad \text{if } a>b>0, \tag{A.8} $$

$$ {}_2F_1(a,b;a-1|z) = (1-z)^{-b-1}\Big(1-\frac{a-b-1}{a-1}\,z\Big) \qquad \text{if } a-1>b>0. \tag{A.9} $$

Finally, according to [3, formula 15.3.17],

$$ {}_2F_1(a,b;2b|z) = 2^{2a}\big(1+\sqrt{1-z}\big)^{-2a}\,{}_2F_1\left(a,a-b+\frac{1}{2};b+\frac{1}{2}\Big|\Big(\frac{1-\sqrt{1-z}}{1+\sqrt{1+z}}\Big)^2\right). \tag{A.10} $$

Appendix B: A Bound On The First Eigenvalue

Let λ be the first eigenvalue of $(-\Delta)^s$ in B_1. A direct bound in terms the first eigenvalue λ_1 of the classical Dirichlet Laplacian $-\Delta$ on the same ball is given by

$$ \lambda \le (\lambda_1)^s, $$

see [17, Theorem 1.1] or, also, [9, 16]. To have a more explicit estimate—which turns out to be a better one for s away from 1 and $n=1$—recall that the function $u_1 \in C^s(\mathbb{R}^n)$, $u_1(x) = \kappa_{n,s}(1-|x|^2)_+^s$, where

$$ \kappa_{n,s} = \frac{\Gamma(n/2)4^{-s}}{\Gamma(1+s)\Gamma(s+\frac{n}{2})}, $$

satisfies $(-\Delta)^s u_1 = 1$ in B_1. In particular, we have

$$ \lambda = \min_{\substack{u\in\mathcal{H}^s_0(B_1)\\ u\neq 0}} \frac{[u]_s^2}{\|u\|^2_{L^2(B_1)}} \le \frac{[u_1]_s^2}{\|u_1\|^2_{L^2(B_1)}}. $$

Here,

$$ \begin{aligned} [u_1]_s^2 &= \int_{B_1} u_1(x)\,dx = \kappa_{n,s}\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_0^1 (1-r^2)^s\,r^{n-1}\,dr \\ &= \kappa_{n,s}\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_0^1 (1-t)^s\,t^{\frac{n}{2}-1}\,dt = \kappa_{n,s}\pi^{n/2}\frac{\Gamma(1+s)}{\Gamma(1+s+\frac{n}{2})}, \\ \|\tau\|^2_{L^2(B_1)} &= \kappa_{n,s}^2\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_0^1 (1-r^2)^{2s}\,r^{n-1}\,dr = \kappa_{n,s}^2\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_0^1 (1-t)^{2s}\,t^{\frac{n}{2}-1}\,dt \\ &= \kappa_{n,s}^2\pi^{n/2}\frac{\Gamma(1+2s)}{\Gamma(1+2s+\frac{n}{2})}, \end{aligned} $$

where we used Eq. A.2 twice. Thus,

$$ \lambda \le \frac{1}{\kappa_{n,s}}\frac{\Gamma(1+s)\Gamma(1+2s+\frac{n}{2})}{\Gamma(1+2s)\Gamma(1+s+\frac{n}{2})} = \frac{4^s\Gamma(1+s)^2\,\Gamma(1+2s+\frac{n}{2})}{(s+\frac{n}{2})\,\Gamma(\frac{n}{2})\,\Gamma(1+2s)} = \Lambda(n,s). \tag{B.1} $$

In the particular case $n = 1$, we have with the properties of the Gamma function (see Appendix A, in particular Eq. A.1)

$$\lambda \le \frac{4^s\Gamma(1+s)^2\,\Gamma(\frac{3}{2}+2s)}{(\frac{1}{2}+s)\,\Gamma(\frac{1}{2})\,\Gamma(1+2s)} = \frac{s4^s\Gamma(s)^2\,\Gamma(\frac{3}{2}+2s)}{\sqrt{\pi}(1+2s)\,\Gamma(2s)} = \frac{2s\Gamma(s)\,\Gamma(\frac{3}{2}+2s)}{(1+2s)\,\Gamma(\frac{1}{2}+s)} = \frac{\Gamma(1+s)\,\Gamma(\frac{3}{2}+2s)}{\Gamma(\frac{3}{2}+s)}. \tag{B.2}$$

Related results in this direction are contained in Dyda, Kuznetsov, and Kwaśnicki [11].

Appendix C: On the shape of some *s*-harmonic functions

We discuss here some features of s-harmonic functions in B_1 associated with particular data in B_1^c. Specifically, we assume

$$g : (1,\infty) \to \mathbb{R} \quad \text{is a non-increasing function,} \tag{C.1}$$

$$\int_{\mathbb{R}^n\setminus B_1} \frac{|g(y)|}{1+|y|^{n+2s}}\,dy < \infty. \tag{C.2}$$

We denote by

$$b_t = \sup\{y \in (1,\infty) : g(y) > t\} \quad \text{for } t \in (-\infty, \overline{g}), \qquad \overline{g} = \lim_{y\downarrow 1} g(y). \tag{C.3}$$

Let $h : \mathbb{R}^n \to \mathbb{R}$ be the s-harmonic extension of $y \to g(|y|)$ in B_1, namely

$$h(x) = \int_{B_1^c} P_s(x,y)\,g(|y|)\,dy = \gamma_{n,s}\int_{B_1^c}\left(\frac{1-|x|^2}{|y|^2-1}\right)^s \frac{g(|y|)}{|y-x|^n}\,dy, \qquad \text{for } x \in B_1. \tag{C.4}$$

Proposition C.1 *Assume Eqs. C.1, C.2, and that g is non-negative. The function h defined as in Eq. C.4 is radial, radially increasing, and subharmonic in B_1.*

Proof Starting from the representation formula Eq. C.4, we write

$$h(x) = \gamma_{n,s}\big(1-|x|^2\big)^s \int_1^\infty \frac{\rho^{n-1}\,g(\rho)}{\big(\rho^2-1\big)^s}\int_{\partial B_1}\frac{d\theta}{\big|\rho\theta - x\big|^n}\,d\rho$$

and, in view of Lemma 2.4, it holds for $x \in B_1$ and $\rho > 1$

$$\begin{aligned}
\int_{\partial B_1}\frac{d\theta}{\big|\rho\theta-x\big|^n} &= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{(\rho^2+|x|^2-2\rho|x|t)^{\frac{n}{2}}}\,dt\\
&= \frac{2^{n-1}\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_0^1 \frac{\tau^{\frac{n-3}{2}}(1-\tau)^{\frac{n-3}{2}}}{\big((\rho+|x|)^2-4\rho|x|\tau\big)^{\frac{n}{2}}}\,d\tau\\
&= \frac{2^{n-1}\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(n-1)}\big(\rho+|x|\big)^{-n}\,{}_2F_1\!\left(\frac{n}{2},\frac{n-1}{2};n-1\middle|\frac{4\rho|x|}{(\rho+|x|)^2}\right)\\
&= \frac{2^{n-1}\pi^{\frac{n-1}{2}}}{}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(n-1)}\rho^{-n}\,{}_2F_1\!\left(\frac{n}{2},1;\frac{n}{2}\middle|\frac{|x|^2}{\rho^2}\right)\\
&= 2^{n-1}\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(n-1)}\frac{\rho^{2-n}}{\rho^2-|x|^2}
\end{aligned}$$

where we have used Eqs. A.3, A.10, and A.8 in this order. Using the *layer-cake representation* for g

$$g(\rho) = \int_0^{\overline{g}} \mathbf{1}_{(1,b_t)}(\rho)\, dt, \qquad \rho > 1,$$

with b_t and $\overline{g}$ defined as in Eq. C.3, we write for $x \in B_1$

$$\begin{aligned} h(x) &= \gamma_{1,s}\big(1-|x|^2\big)^s \int_1^\infty \frac{2\rho\, g(\rho)}{\big(\rho^2-1\big)^s(\rho^2-|x|^2)}\, d\rho \\ &= \gamma_{1,s}\big(1-|x|^2\big)^s \int_0^{\overline{g}} \int_1^{b_t} \frac{2\rho}{\big(\rho^2-1\big)^s(\rho^2-|x|^2)}\, dy\, dt \end{aligned}$$

where, for any $b \geq 1$,

$$\begin{aligned} \int_1^b \frac{2\rho}{\big(\rho^2-1\big)^s(\rho^2-|x|^2)}\, d\rho &= \int_1^{b^2} \frac{dz}{(z-1)^s(z-|x|^2)} = \int_0^{b^2-1} \frac{dz}{z^s(z+1-|x|^2)} = \\ &= \big(1-|x|^2\big)^{-s} \int_0^{\frac{b^2-1}{1-|x|^2}} \frac{dw}{w^s(w+1)} = \big(1-|x|^2\big)^{-s} \int_{\frac{1-|x|^2}{b^2-1}}^\infty \frac{v^{s-1}}{v+1}\, dv. \end{aligned}$$

Therefore

$$h(x) = \gamma_{1,s} \int_0^{\overline{g}} \int_{\frac{1-|x|^2}{b_t^2-1}}^\infty \frac{v^{s-1}}{v+1}\, dv\, dt \qquad \text{for } x \in B_1.$$

As a consequence, for any $x \in B_1$,

$$\nabla h(x) = -\gamma_{1,s} \int_0^{\overline{g}} \left(\frac{1-|x|^2}{b_t^2-1}\right)^{s-1} \frac{1}{\frac{1-|x|^2}{b_t^2-1}+1} \frac{-2x}{b_t^2-1}\, dt = 2\gamma_{1,s}\, x\big(1-|x|^2\big)^{s-1} \int_0^{\overline{g}} \frac{\big(b_t^2-1\big)^{1-s}}{b_t^2-|x|^2}\, dt.$$

This proves the radial monotonicity. Moreover, for any $x \in B_1$,

$$\begin{aligned} -\Delta h(x) &= -2\gamma_{1,s}\, \mathrm{div}\left(x\big(1-|x|^2\big)^{s-1} \int_0^{\overline{g}} \frac{\big(b_t^2-1\big)^{1-s}}{b_t^2-|x|^2}\, dt \right) \\ &= -2n\gamma_{1,s}\, \big(1-|x|^2\big)^{s-1} \int_0^{\overline{g}} \frac{\big(b_t^2-1\big)^{1-s}}{b_t^2-|x|^2}\, dt - 4(1-s)\gamma_{1,s}\, |x|^2\big(1-|x|^2\big)^{s-2} \int_0^{\overline{g}} \frac{\big(b_t^2-1\big)^{1-s}}{b_t^2-|x|^2}\, dt \\ &\quad - 4\gamma_{1,s}\, |x|^2\big(1-|x|^2\big)^{s-1} \int_0^{\overline{g}} \frac{\big(b_t^2-1\big)^{1-s}}{\big(b_t^2-|x|^2\big)^2}\, dt \end{aligned}$$

is (strictly) negative for $x \in B_1$.

Remark C.2 Analogue calculations can be performed when g is non-negative and non-decreasing instead. This would lead to a radial, radially decreasing, and super-harmonic s-harmonic extension.

Remark C.3 The non-negativity assumption in Proposition 5 can be dropped: if we split $g = g^+ - g^-$ we can directly apply Proposition 5 to g^+ and Remark 5 to g^-.

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Declarations

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