



# New Properties of Holomorphic Sobolev–Hardy Spaces

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## Abstract

We give new characterizations of the optimal data space for the  $L^p(bD, \sigma)$ -Neumann boundary value problem for the  $\bar{\partial}$  operator associated to a bounded, Lipschitz domain  $D \subset \mathbb{C}$ . We show that the solution space is embedded (as a Banach space) in the Dirichlet space and that for  $p = 2$ , the solution space is a reproducing kernel Hilbert space.

**Keywords** Lipschitz domain · Sobolev–Hardy space · Neumann problem ·  $\bar{\partial}$  · Reproducing kernel Hilbert space

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## 1 Introduction

Let  $D$  be a bounded Lipschitz domain in  $\mathbb{C}$  whose boundary  $bD$  is endowed with the induced Lebesgue measure  $\sigma$ . Let  $\mathcal{H}^p(D)$  be the **holomorphic Hardy space**:

$$\mathcal{H}^p(D) := \{F \in \vartheta(D) : F^* \in L^p(bD, \sigma)\}, \quad 0 < p \leq \infty$$

with  $\vartheta(D)$  denoting the set of holomorphic functions on  $D$  and  $F^*$  the non-tangential maximal function of  $F$ . It is well-known that if  $D$  is simply connected, every element  $F$  of  $\mathcal{H}^p(D)$  admits a nontangential limit  $\dot{F}$  that lies in  $L^p(bD, \sigma)$  (see [5, Theorem 10.3]). On the other hand, since Lipschitz domains are local epigraphs, any bounded Lipschitz domain must be finitely connected. Hence, an elementary localization argument shows that any  $F \in \mathcal{H}^p(D)$  has a nontangential limit  $\dot{F}$  defined  $\sigma$ -a.e. on  $bD$ .

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We will call the set of all such nontangential limits  $h^p(bD)$ . That is,

$$h^p(bD) := \{ \dot{F} : F \in \mathcal{H}^p(D) \} \subsetneq L^p(bD, \sigma).$$

Let  $\mathcal{H}^{1,p}(D)$  be the **holomorphic Sobolev–Hardy space**

$$\mathcal{H}^{1,p}(D) := \{ G \in \vartheta(D) : G' \in \mathcal{H}^p(D) \}, \quad p > 0.$$

It is shown in [6] that, given  $g \in L^p(bD, \sigma)$  subject to the compatibility condition:

$$\int_{bD} g \, d\sigma = 0, \text{ the **Neumann problem for the } \bar{\partial} \text{ operator}**$$

$$\begin{cases} \bar{\partial} G = 0 & \text{in } D; \\ \frac{\partial G}{\partial n}(\zeta) = g(\zeta) & \text{for } \sigma\text{-a.e. } \zeta \in bD; \\ (G')^* \in L^p(bD, \sigma) \end{cases} \quad (1)$$

is solvable if and only if the data  $g$  belongs to

$$\mathfrak{n}^p(bD) := \left\{ -iT(\zeta)(\dot{G}')(\zeta) : G \in \mathcal{H}^{1,p}(D) \right\}, \quad 1 \leq p \leq \infty, \quad (2)$$

where  $\zeta \mapsto T(\zeta)$  is the unit tangent vector field for  $bD$ . Moreover, if  $g \in \mathfrak{n}^p(bD)$  then all solutions of (1) belong to  $\mathcal{H}^{1,p}(D)$ . Any two solutions of (1) differ by an additive constant, hence for any fixed  $\alpha \in D$  the space

$$\mathcal{H}_\alpha^{1,p}(D) := \{ F \in \mathcal{H}^{1,p}(D) : F(\alpha) = 0 \}$$

contains precisely one solution of (1). In the case when  $p = 2$  and  $D$  is simply-connected,  $\mathcal{H}_\alpha^{1,2}(D)$  is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{H}_\alpha^{1,2}(D)} := \int_{bD} (\dot{F}')(\zeta) \overline{(\dot{G}')(\zeta)} \, d\sigma(\zeta).$$

In this paper we explore properties of  $\mathcal{H}_\alpha^{1,2}(bD)$  and of  $\mathfrak{n}^p(bD)$ . Specifically, after recalling a few well-known basic properties of Lipschitz domains (Sect. 2), we show that the solution space  $\mathcal{H}_\alpha^{1,2}(D)$  is a reproducing kernel Hilbert space (Theorem 3.1) and for  $D = \mathbb{D}$  (the unit disc) we compute its reproducing kernel. Next we show that for  $1 < p < \infty$  there is a Banach space embedding of  $\mathcal{H}_\alpha^{1,p}(D)$  in the Dirichlet space  $\mathcal{D}_\alpha^p(D)$  (Theorem 3.3). In Sect. 4 we give various characterizations of  $\mathfrak{n}^p(bD)$  for simply connected  $D$ : in terms of  $L^p(bD, \sigma)$ -functions whose moments all vanish on  $bD$ ; or in terms of the vanishing of the Cauchy integral over  $\overline{D^c}$ , the complement of the closure of  $D$ ; as well as in terms of its conformal map (Theorems 4.1 and 4.3).

Finally, in Sect. 5 we provide a characterization of  $n^p(bD)$  for multiply connected  $D$ : in this case the aforementioned vanishing moment condition takes a more restrictive form, see Theorem 5.3.

## 2 Preliminaries

### 2.1 Lipschitz Domains

Throughout this paper the domains under consideration will be Lipschitz domains on  $\mathbb{C}$ , as defined below.

**Definition 2.1** A bounded domain  $D \subset \mathbb{C}$  with boundary  $bD$  is called a **Lipschitz domain** if there are finitely many rectangles  $\{R_j\}_{j=1}^m$  with sides parallel to the coordinate axes, angles  $\{\theta_j\}_{j=1}^m$ , and Lipschitz functions  $\phi_j : \mathbb{R} \rightarrow \mathbb{R}$  such that the collection  $\{e^{-i\theta_j} R_j\}_{j=1}^m$  covers  $bD$  and  $(e^{i\theta_j} D) \cap R_j = \{x + iy : y > \phi_j(x), x \in (a_j, b_j)\}$  for some  $a_j < b_j < \infty$ . We refer to such  $R_j$ 's as *coordinate rectangles*.

**Definition 2.2** Let  $D$  be a bounded Lipschitz domain. For any  $\zeta \in bD$ , let  $\{\Gamma(\zeta), \zeta \in D\}$  be a family of truncated (one-sided) open cones  $\Gamma(\zeta)$  with vertex at  $\zeta$  satisfying the following property: for each rectangle  $R_j$  in Definition 2.1, there exists two cones  $\Delta_1$  and  $\Delta_2$ , each with vertex at the origin and axis along the  $y$  axis such that for  $\zeta \in bD \cap e^{-i\theta_j} R_j$ ,

$$e^{-i\theta_j} \Delta_1 + \zeta \subset \Gamma(\zeta) \subset \overline{\Gamma(\zeta)} \setminus \{\zeta\} \subset e^{-i\theta_j} \Delta_2 + \zeta \subset D \cap e^{-i\theta_j} R_j.$$

It is well known that for Lipschitz  $D$ ,  $\Gamma(\zeta) \neq \emptyset$  for any  $\zeta \in bD$ ; see e.g., [4] or [12, Section 0.4]. We will sometimes refer to  $\Gamma(\zeta)$  as a *regular cone*, or a *coordinate cone*. For a function  $F$  on  $D$  and  $\zeta \in bD$ , we define the **nontangential maximal function**  $F^*(\zeta)$  and the **nontangential limit**  $\dot{F}(\zeta)$  as

$$F^*(\zeta) := \sup_{z \in \Gamma(\zeta)} |F(z)|, \quad \text{and} \quad \dot{F}(\zeta) = \lim_{\substack{z \rightarrow \zeta \\ z \in \Gamma(\zeta)}} F(z) \text{ if such limit exists.}$$

We will need an approximation scheme of  $D$  by smooth subdomains constructed by Nečas in [10], which we refer to as a **Nečas exhaustion of  $D$** . See also [8] and [12, Theorem 1.12]. (Recall that Lipschitz functions are differentiable almost everywhere; thus if  $D$  is Lipschitz and simply connected its boundary  $bD$  is a rectifiable Jordan curve that admits a (positively oriented) unit tangent vector  $T(\zeta)$  as  $\sigma$ -a.e.  $\zeta \in bD$ .)

**Lemma 2.3** [10, p. 5][12, Theorem 1.12] *Let  $D$  be a bounded Lipschitz domain. There exists a family  $\{D_k\}_{k=1}^\infty$  of smooth domains with  $D_k$  compactly contained in  $D$  that satisfy the following:*

- (a) *For each  $k$  there exists a Lipschitz diffeomorphism  $\Lambda_k$  that takes  $D$  to  $D_k$  and extends to the boundaries:  $\Lambda_k : bD \rightarrow bD_k$  with the property that*

$$\sup\{|\Lambda_k(\zeta) - \zeta| : \zeta \in bD\} \leq C/k$$

for some fixed constant  $C$ . Moreover  $\Lambda_k(\zeta) \in \Gamma(\zeta)$ .

- (b). There is a covering of  $bD$  by finitely many coordinate rectangles which also form a family of coordinate rectangles for  $bD_k$  for each  $k$ . Furthermore for every such rectangle  $R$ , if  $\phi$  and  $\phi_k$  denote the Lipschitz functions whose graphs describe the boundaries of  $D$  and  $D_k$ , respectively, in  $R$ , then  $\|(\phi_k)'\|_\infty \leq \|\phi'\|_\infty$  for any  $k$ ;  $\phi_k \rightarrow \phi$  uniformly as  $k \rightarrow \infty$ , and  $(\phi_k)' \rightarrow \phi'$  a.e. and in every  $L^p((a, b))$  with  $(a, b) \subset \mathbb{R}$  as in Definition 2.1.
- (c). There exist constants  $0 < m < M < \infty$  and positive functions (Jacobians)  $w_k : bD \rightarrow [m, M]$  for any  $k \in \mathbb{N}$ , such that for any measurable set  $F \subseteq bD$  and for any measurable function  $f_k$  on  $\Lambda_k(F)$  the following change-of-variables formula holds:

$$\int_F f_k(\Lambda_k(\eta)) w_k(\zeta) d\sigma(\eta) = \int_{\Lambda_k(F)} f_k(\eta_k) d\sigma_k(\eta_k).$$

where  $d\sigma_k$  denotes arc-length measure on  $bD_k$ . Furthermore we have

$$w_k \rightarrow 1 \quad \sigma\text{-a.e. } bD \text{ and in every } L^p(bD, \sigma) \text{ for any } 1 \leq p < \infty.$$

- (d). Let  $T_k$  denote the unit tangent vector for  $bD_k$  and  $T$  denote the unit tangent vector of  $bD$ . We have that

$$T_k \rightarrow T \quad \sigma\text{-a.e. } bD \text{ and in every } L^p(bD, \sigma) \text{ for any } 1 \leq p < \infty.$$

Note that in conclusions (b) through (d) the exponent  $p = \infty$  cannot be allowed unless  $D$  is of class  $C^1$ . Nečas exhaustions can be used to transfer well-known results for holomorphic functions over domains with smooth boundaries to Hardy space functions on Lipschitz domains. In particular, one can use it to prove Cauchy’s Theorem. See also [6, Lemma 2.7] for the proof.

**Lemma 2.4** *Let  $D$  be a bounded Lipschitz domain. Then any  $f \in h^1(bD)$  satisfies Cauchy’s Theorem. That is*

$$\int_{bD} f(\zeta) d\zeta = 0 \quad \text{for any } f \in h^1(bD).$$

Next we state some definitions and results involving Cauchy integrals and the Cauchy transform, which we first define:

**Definition 2.5** Let  $f : bD \rightarrow \mathbb{C}$ . The **Cauchy integral**  $C_D f$  of  $f$  is

$$C_D f(z) := \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in D.$$

Similarly

$$\mathbf{C}_{\overline{D}^c} f(z) := \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \overline{D}^c.$$

Finally, the **Cauchy transform**  $\mathcal{C}_D f$  of  $f$  is denoted by

$$\mathcal{C}_D f(\zeta) := (\mathbf{C}_D \dot{f})(\zeta), \quad \zeta \in bD.$$

In both integrals  $bD$  is oriented counterclockwise (that is, in the positive direction for  $D$ ).

In this paper we will use the fact that a function  $f$  in  $L^p(bD, \sigma)$  lies in  $h^p(bD, \sigma)$  if and only if the Cauchy integral of  $f$  vanishes on  $\overline{D}^c$ . This latter fact is well-known for domains with smooth boundaries; here we prove it for Lipschitz domains, see Lemma 2.6 below. We first recall the Plemelj formulas for  $f \in L^p(bD, \sigma)$ ,  $1 < p < \infty$ :

$$\mathcal{C}_D f(\zeta) = \frac{1}{2} f(\zeta) + \frac{1}{2} \mathcal{H} \mathcal{C}_{bD} f(\zeta), \quad \text{for } \sigma\text{-a.e. } \zeta \in bD, \tag{3}$$

and

$$\lim_{\substack{z \rightarrow \zeta \\ z \in \Gamma(\zeta, \overline{D}^c)}} \mathbf{C}_{\overline{D}^c} f(z) = -\frac{1}{2} f(\zeta) + \frac{1}{2} \mathcal{H} \mathcal{C}_{bD} f(\zeta) \text{ for } \sigma\text{-a.e. } \zeta \in bD. \tag{4}$$

Here

$$\mathcal{H} \mathcal{C}_{bD} f(\zeta) := \frac{1}{2\pi i} \text{P.V.} \int_{bD} \frac{f(w)}{w - \zeta} dw, \quad \text{for } \sigma\text{-a.e. } \zeta \in bD,$$

with  $bD$  oriented counterclockwise, ‘‘P.V.’’ stands for ‘‘principal value,’’ and  $\Gamma(\zeta, \overline{D}^c)$  is defined as in Definition 2.2, with  $D$  in there replaced by  $\overline{D}^c$ . Note that a Lipschitz domain  $D$  satisfies the exterior cone condition (see [7]) so the limit in (4) is well-defined. A deep result of Coifman, McIntosh, and Meyer [3] states that on bounded Lipschitz domains,  $\mathcal{H} \mathcal{C}_{bD}$  is indeed well-defined (i.e. the principal value integral exists  $\sigma$ -a.e.) and is bounded on  $L^p(bD, \sigma)$ ,  $1 < p < \infty$ . Thus, by the result of [2], the Plemelj formulas (3) and (4) hold (for more on Plemelj formulas, also see [9]).

**Lemma 2.6** *Let  $D$  be a bounded simply connected Lipschitz domain and  $1 < p < \infty$ . Assume  $f \in L^p(bD, \sigma)$ . Then  $f \in h^p(bD, \sigma)$  if and only if  $\mathbf{C}_{\overline{D}^c} f(z) = 0$  for all  $z \in \overline{D}^c$ .*

**Proof** First assume that  $\mathbf{C}_{\overline{D}^c} f(z) = 0$  for all  $z \in \overline{D}^c$ . By Eq. (4), we have

$$0 = \lim_{\substack{z \rightarrow \omega \\ z \in \Gamma(\zeta, \overline{D}^c)}} \mathbf{C}_{\overline{D}^c} f(z) = \frac{1}{2\pi i} \text{P.V.} \int_{bD} \frac{f(\zeta)}{\zeta - \omega} d\zeta - \frac{1}{2} f(\omega).$$

That is,  $\frac{1}{2}f(\omega) = \frac{1}{2\pi i} \text{P.V.} \int_{bD} \frac{f(\zeta)}{\zeta - \omega}$  for  $\sigma$ -a.e.  $\omega \in bD$ . Now, using Eq. (3), we have for  $\sigma$ -a.e.  $\omega \in bD$ ,

$$\mathcal{C}_D f(\omega) = \frac{1}{2\pi i} \text{P.V.} \int_{bD} \frac{f(\zeta)}{\zeta - \omega} d\zeta + \frac{1}{2}f(\omega) = \frac{1}{2}f(\omega) + \frac{1}{2}f(\omega) = f(\omega).$$

Thus,  $f$  is in the range of the Cauchy transform. Since the range of the Cauchy transform equals  $h^p(bD, \sigma)$  when  $D$  is bounded and simply connected and  $1 < p < \infty$  (see [8]), the backward direction is proven. For the forward direction suppose  $f \in h^p(bD, \sigma)$ . Then there exists  $F \in \mathcal{H}^p(D)$  such that  $\dot{F} = f$ . Let  $z \in \overline{D}^c$  be arbitrary and consider the function  $G_z(w) := (w - z)^{-1}$ . Then  $G_z$  is holomorphic on  $D$  and is continuous on  $\overline{D}$ . Moreover,  $\|(FG_z)^*\|_{L^p(bD, \sigma)} \leq \|F^*\|_{L^p(bD, \sigma)} \|G_z^*\|_{L^\infty(bD, \sigma)} < \infty$ . Thus  $FG_z \in \mathcal{H}^p(D)$  and by Cauchy's Theorem (Lemma 2.4) we have

$$0 = \int_{bD} (F\dot{G}_z)(\zeta) d\zeta = \mathbf{C}_{\overline{D}^c} f(z),$$

as desired. □

### 3 Properties of $\mathcal{H}_\alpha^{1,2}(D)$ for Simply Connected $D$

In this section we show that  $\mathcal{H}_\alpha^{1,2}(D)$  is a reproducing kernel Hilbert space and that it is a subset of the Dirichlet space.

#### 3.1 $\mathcal{H}_\alpha^{1,2}(D)$ is a Reproducing Kernel Hilbert Space

**Theorem 3.1** *Let  $D$  be a bounded simply connected Lipschitz domain. Then for any base point  $\alpha \in D$ :*

(a)  $\mathcal{H}_\alpha^{1,2}(D)$  is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{H}_\alpha^{1,2}(D)} := \langle (\dot{F}'), (\dot{G}') \rangle_{L^2(bD, \sigma)}.$$

(b) For any  $z \in D$ , the pointwise evaluation:  $G \mapsto E_z(G) := G(z)$  is a bounded linear functional on  $\mathcal{H}_\alpha^{1,2}(D)$ . Hence  $\mathcal{H}_\alpha^{1,2}(D)$  is a reproducing kernel Hilbert space (RKHS) with reproducing kernel  $K_\alpha^z(\cdot) = K_\alpha(\cdot, z)$ . Namely, for any  $z \in D$ , we have that

$$(K_\alpha^z)'(\zeta) \equiv \lim_{\substack{w \rightarrow \zeta \\ w \in \Gamma(\zeta)}} (K_\alpha^z)'(w)$$

exists for almost all  $\zeta \in bD$  and for  $F \in \mathcal{H}_\alpha^{1,2}(D)$  we have

$$F(z) = \int_{bD} (\dot{F}')(\zeta) \overline{(K_\alpha^z)'(\zeta)} d\sigma(\zeta), \quad z \in D. \tag{5}$$

(c) Let  $p \geq 2$  and  $g \in \mathfrak{n}^p(bD)$ . Then for any  $\alpha \in D$  the solution of the holomorphic Neumann problem (1) with boundary data  $g$  has the representation

$$G_\alpha(z) = i \int_{\zeta \in bD} g(\zeta) \overline{T(\zeta) (\dot{K}_\alpha^z)'(\zeta)} d\sigma(\zeta), \quad z \in D.$$

**Proof** To verify (3.1), note that  $\langle \cdot, \cdot \rangle_{\mathcal{H}_\alpha^{1,2}(D)}$  is a sesquilinear form and  $\langle F, F \rangle_{\mathcal{H}_\alpha^{1,2}(D)} = \|F\|_{\mathcal{H}_\alpha^{1,2}(D)}^2$ . A straightforward argument (whose details can be found in [6, Lemma 3.4]) shows that for  $1 \leq p \leq \infty$  the set  $\mathcal{H}_\alpha^{1,p}(D)$  is a Banach space with the norm defined as

$$\|F\|_{\mathcal{H}_\alpha^{1,p}(D)} = \|(\dot{F}')\|_{L^p(bD,\sigma)}.$$

Thus  $\mathcal{H}_\alpha^{1,2}(D)$  is complete under the norm  $\|\cdot\|_{\mathcal{H}_\alpha^{1,2}(D)}$ , and so  $\mathcal{H}_\alpha^{1,2}(D)$  is a Hilbert space.

Next we prove (3.1). Fix  $z \in D$  and consider the pointwise evaluation operator  $E_z$ . For any  $\alpha \in D$  and a smooth path  $\gamma_\alpha^z \subset D$  that connects  $\alpha$  to  $z$  we have

$$|E_z(G)| = |G(z)| = |G(z) - G(\alpha)| = \left| \int_{\gamma_\alpha^z} G'(w)dw \right| \leq |\gamma_\alpha^z| \sup_{w \in \gamma_\alpha^z} |G'(w)|.$$

Furthermore, for any  $w \in \gamma_\alpha^z$ , Cauchy formula and Hölder inequality give

$$|G'(w)| = \frac{1}{2\pi} \left| \int_{bD} \frac{(\dot{G}')(\zeta)}{w - \zeta} d\zeta \right| \leq \frac{|bD|^{\frac{1}{2}}}{2\pi k_z} \|(\dot{G}')\|_{L^2(bD,\sigma)} = \frac{|bD|^{\frac{1}{2}}}{2\pi k_z} \|G\|_{\mathcal{H}_\alpha^{1,2}(D)},$$

where  $k_z := \text{dist}(\gamma_\alpha^z, bD) > 0$ . Combining all of the above we see that for any  $z \in D$ ,  $E_z$  is a bounded linear functional on  $\mathcal{H}_\alpha^{1,2}(D)$ ; Hilbert space theory now grants the existence of the reproducing kernel function

$$K_\alpha^z \in \mathcal{H}_\alpha^{1,2}(D) \quad \text{with} \quad G(z) = \langle G, K_\alpha^z \rangle_{\mathcal{H}_\alpha^{1,2}(D)}.$$

Finally we verify (3.1). Let  $p \geq 2$  and  $g \in \mathfrak{n}^p(bD)$ . Suppose  $G_\alpha \in \mathcal{H}_\alpha^{1,p}(D)$  is the solution to the Neumann problem (1) with datum  $g$ . Thus  $(\dot{G}'_\alpha) = i\bar{T}g$  and  $G_\alpha \in \mathcal{H}_\alpha^{1,2}(D)$ . Hence for any  $z \in D$  we have

$$G_\alpha(z) = \langle G_\alpha, K_\alpha^z \rangle_{\mathcal{H}_\alpha^{1,2}(D)} = \int_{bD} (\dot{G}'_\alpha)(\zeta) \overline{(\dot{K}_\alpha^z)'(\zeta)} d\sigma(\zeta) = i \int_{bD} g(\zeta) \overline{T(\zeta) (\dot{K}_\alpha^z)'(\zeta)} d\sigma(\zeta),$$

as desired. □

In the case of the unit disc  $\mathbb{D}$  we obtain explicit formulas and recover the full range of  $1 \leq p \leq \infty$ :

**Theorem 3.2** 1. The reproducing kernel associated to  $\mathcal{H}_\alpha^{1,2}(\mathbb{D})$  is given by

$$K_\alpha^z(w) = \sum_{k=1}^\infty \frac{(w^k - \alpha^k)\overline{(z^k - \alpha^k)}}{2\pi k^2}, \quad z, w \in \mathbb{D}. \tag{6}$$

2. Given  $g \in n^p(b\mathbb{D})$ ,  $1 < p < \infty$  and  $\alpha := 0$ , the unique solution  $G \in \mathcal{H}_0^{1,p}(\mathbb{D})$  to the holomorphic Neumann problem (1) admits the following representation

$$G(z) = \frac{1}{2\pi} \int_{b\mathbb{D}} g(\zeta) \text{Log} \frac{1}{1 - z\bar{\zeta}} d\sigma(\zeta), \tag{7}$$

where  $\text{Log}$  denotes the principal branch of the complex logarithm.

**Proof** To prove part 1., note that since  $\mathbb{D}$  is simply connected every holomorphic function on  $\mathbb{D}$  has an antiderivative. Thus the mapping  $G \mapsto G'$  is an isometric isomorphism from  $\mathcal{H}_\alpha^{1,2}(\mathbb{D})$  onto  $\mathcal{H}^2(\mathbb{D})$ . Since  $\{\frac{1}{\sqrt{2\pi}}z^{k-1}\}_{k \in \mathbb{N}}$  is an orthonormal basis of  $\mathcal{H}^2(\mathbb{D})$ , the set of antiderivatives  $\{\frac{z^k - \alpha^k}{\sqrt{2\pi}k}\}_{k \in \mathbb{N}}$  is an orthonormal basis of  $\mathcal{H}_\alpha^{1,2}(\mathbb{D})$ . Thus, by the theory of reproducing kernel Hilbert spaces,  $K_\alpha$  as given in Eq. (6) is the reproducing kernel for  $\mathcal{H}_\alpha^{1,2}(\mathbb{D})$ .

For the proof of part 2., note that the reproducing kernel for  $\mathbb{D}$  satisfies

$$\overline{(K_0^z)'(w)} = \frac{\partial}{\partial \bar{w}} \sum_{k=1}^\infty \frac{z^k \bar{w}^k}{2\pi k^2} = \sum_{k=1}^\infty \frac{z^k \bar{w}^{k-1}}{2\pi k} = \frac{1}{2\pi w} \text{Log} \frac{1}{1 - z\bar{w}}, \quad w \in \mathbb{D}.$$

Hence for every  $\zeta \in b\mathbb{D}$ , and since  $T(\zeta) = i\zeta$ , we have

$$\overline{T(\zeta)(K_0^z)'(\zeta)} = \frac{1}{2\pi i} \text{Log} \frac{1}{1 - z\bar{\zeta}}, \quad \zeta \in b\mathbb{D}.$$

So for  $g \in n^2(b\mathbb{D})$  we have that Eq. (7) follows from the above and Theorem 3.1 part (3.1).

For  $g \in n^p(b\mathbb{D})$ ,  $1 < p < \infty$ , define  $G$  as in (7). Then  $G \in \vartheta(\mathbb{D})$  and

$$\begin{aligned} G'(z) &= \frac{1}{2\pi} \int_{b\mathbb{D}} \frac{g(\zeta)\bar{\zeta}}{1 - z\bar{\zeta}} d\sigma(\zeta) \\ &= \frac{1}{2\pi} \int_{b\mathbb{D}} \frac{g(\zeta)}{\zeta - z} d\sigma(\zeta) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{i\overline{T(\zeta)}g(\zeta)}{\zeta - z} d\zeta = \mathbf{C}_{\mathbb{D}}(i\bar{T}g)(z), \quad z \in \mathbb{D}. \end{aligned}$$

Here we used the facts that  $\zeta\bar{\zeta} = 1$  and  $d\sigma(\zeta) = \overline{T(\zeta)}d\zeta$  on  $\mathbb{D}$ . Consequently,  $(G')^* \in L^p(b\mathbb{D}, \sigma)$  by a mapping property of the Cauchy integral  $\mathbf{C}_{\mathbb{D}}$  and Cauchy

transform  $\mathcal{C}_{\mathbb{D}}$  (see [8, Lemma 2.12], noting that the assumption that  $p \neq 1, \infty$  is needed here). Moreover, from the above we also have that

$$(\dot{G}')(\zeta) = \mathcal{C}_{\mathbb{D}}(i\overline{T}g)(\zeta), \quad \text{a.e. } \zeta \in b\mathbb{D}.$$

But  $\overline{T}g \in h^p(b\mathbb{D})$  because  $g \in n^p(b\mathbb{D})$ , and  $\mathcal{C}_{\mathbb{D}}$  is the identity on  $h^p(b\mathbb{D})$ , thus

$$\frac{\partial G}{\partial n}(\zeta) = -iT(\zeta)(\dot{G}')(\zeta) = -iT(\zeta)\mathcal{C}_{\mathbb{D}}(i\overline{T}g)(\zeta) = g(\zeta), \quad \zeta \in b\mathbb{D} \quad \sigma - a.e.$$

That is,  $G$  solves (1) for  $1 < p < \infty$ . (Uniqueness was proved in [6].)

### 3.2 $\mathcal{H}_{\alpha}^{1,p}(D)$ is Embedded in the Dirichlet Space

In [1], Axler and Shields introduced the **Dirichlet space**  $\mathcal{D}_{\alpha}^2(D)$  for a general domain  $D$ , namely

$$\mathcal{D}_{\alpha}^2(D) := \left\{ F \in \vartheta(D) : F(\alpha) = 0, \int_D |F'(z)|^2 dV(z) < \infty \right\}, \quad \alpha \in D,$$

which is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{D}_{\alpha}^2(D)} := \int_D F'(z) \overline{G'(z)} dV(z).$$

(Here  $dV$  is the Lebesgue measure for  $\mathbb{C}$ .) The analogous definition of  $\mathcal{D}_{\alpha}^p(D)$  with  $1 \leq p \leq \infty$  yields a Banach space with norm

$$\|F\|_{\mathcal{D}_{\alpha}^p(D)} := \int_D |F'(z)|^p dV(z).$$

**Theorem 3.3** *Let  $D$  be a bounded simply connected Lipschitz domain and  $1 < p < \infty$ . Suppose that  $F \in \mathcal{H}_{\alpha}^{1,p}(D)$ . Then  $F \in \mathcal{D}_{\alpha}^p(D)$  and*

$$\|F\|_{\mathcal{D}_{\alpha}^p(D)} \lesssim \|F\|_{\mathcal{H}_{\alpha}^{1,p}(D)}.$$

*That is, the holomorphic Sobolev–Hardy space is embedded in the Dirichlet space.*

To prove Theorem 3.3 we need the following result:

**Lemma 3.4** *Let  $D$  be a bounded simply connected Lipschitz domain and  $1 < p < \infty$ . Suppose that  $F \in \mathcal{H}^p(D)$ . Then  $F \in \vartheta(D) \cap L^p(D)$  and*

$$\|F\|_{L^p(D)} \lesssim \|\dot{F}\|_{L^p(bD,\sigma)}.$$

That is, the holomorphic Hardy space is embedded in the Bergman space.

**Proof** In [6, Lemma 2.8] it is shown that if  $1 < p < \infty$  and  $D$  is a simply connected and bounded Lipschitz domain, then for  $F \in \mathcal{H}^p(D)$  quantities  $\|F^*\|_{L^p(bD,\sigma)}$  and  $\|\tilde{F}\|_{L^p(bD,\sigma)}$  are comparable. Thus it suffices to show that  $\|F\|_{L^p(D)} \lesssim \|F^*\|_{L^p(bD,\sigma)}$ .

Consider a Nečas exhaustion  $\{D_k\}$  of  $D$ . Then there are finitely many coordinate rectangles  $R_j := [a_j, b_j] \times (c_j, d_j)$  with Lipschitz functions  $\phi_k^j$  and  $\phi^j$  whose graphs determine  $D_k$  and  $D$ , respectively, on  $R_j$  and  $\phi_k^j$  converges uniformly to  $\phi^j$ . For any  $k \in \mathbb{N}$ ,  $x \in [a_j, b_j]$  and  $y \in (\phi_k^j(x), \phi^j(x)]$ , the point  $x + iy$  lies directly above  $x + i\phi^j(x)$  and thus  $x + iy \in \Delta_1 + (x + i\phi^j(x))$ , where  $\Delta_1$  is the cone in Definition 2.2. And so  $e^{-i\theta_j}(x + iy)$  lies in  $e^{-i\theta_j}(\Delta_1 + (x + i\phi^j(x))) \subseteq \Gamma(e^{-i\theta_j}(x + i\phi^j(x)))$ . Fix  $k \in \mathbb{N}$  so that for each  $j$  we have  $\|\phi_k^j - \phi^j\|_\infty < 1$ . Then we have for  $F \in \mathcal{H}^p(D)$

$$\begin{aligned} \iint_{D-D_k} |F(z)|^p dA(z) &\leq \sum_j \iint_{e^{-i\theta_j} R_j \cap (D-D_k)} |F(z)|^p dA(z) \\ &= \sum_j \int_{a_j}^{b_j} \int_{\phi_k^j(x)}^{\phi^j(x)} |F(e^{-i\theta_j}(x + iy))|^p dy dx \\ &\leq \sum_j \int_{a_j}^{b_j} \int_{\phi^j(x)}^{\phi_k^j(x)} |F^*(e^{-i\theta_j}(x + i\phi_j(x)))|^p dy dx \\ &\leq \sum_j \int_{a_j}^{b_j} |F^*(e^{-i\theta_j}(x + i\phi_j(x)))|^p dx \\ &\leq \sum_j \int_{a_j}^{b_j} |F^*(e^{-i\theta_j}(x + i\phi_j(x)))|^p |e^{i\theta_j}(1 + i\phi_j'(x))| dx \\ &= \sum_j \int_{bD \cap e^{-i\theta_j} R_j} |F^*(\zeta)|^p d\sigma(\zeta) \lesssim \|F^*\|_{L^p(bD,\sigma)}^p. \end{aligned}$$

Since  $D_k$  is compactly contained in  $D$  and  $k$  is fixed,  $\text{dist}(D_k, bD) > d$  for some constant  $d$  depending on  $D$ . So, similar to the argument of the proof of part (3.1) of Theorem 3.1, by the Cauchy integral formula we have

$$\iint_{D_k} |F(z)|^p dA(z) \lesssim \|F^*\|_{L^p(bD,\sigma)}^p,$$

completing the proof to  $\|F\|_{L^p(D)} \lesssim \|F^*\|_{L^p(bD,\sigma)}$ . □

**Proof of Theorem 3.3** Let  $F \in \mathcal{H}_\alpha^{1,p}(D)$ . Then  $F(\alpha) = 0$  and  $F' \in \mathcal{H}^p(D)$ . By Lemma 3.4, we also have  $F' \in \vartheta(D) \cap L^p(D)$  giving that  $F \in \mathcal{D}_\alpha^p(D)$ , as desired.  $\square$

### 4 Characterizations of $\mathfrak{n}^p(bD)$ for Simply Connected $D$

**Theorem 4.1** *Let  $D$  be a bounded simply connected Lipschitz domain and  $1 \leq p \leq \infty$ . Then  $\mathfrak{n}^p(bD)$  defined as in (2) is closed in the  $L^p(bD, \sigma)$ -norm. Moreover, for*

$$\begin{aligned} \mathfrak{n}_1 &:= \{Tg : g \in h^p(bD)\}, \\ \mathfrak{n}_2 &:= \left\{ f \in L^p(bD, \sigma) : \int_{bD} \zeta^k f(\zeta) d\sigma(\zeta) = 0 \text{ for all } k = 0, 1, 2, \dots \right\}, \\ \mathfrak{n}_3 &:= \{f \in L^p(bD, \sigma) : \mathbf{C}_{\overline{bD}}(\overline{T}f) = 0\} \end{aligned}$$

we have that  $\mathfrak{n}^p(bD) = \mathfrak{n}_1 = \mathfrak{n}_2$ . If  $1 < p < \infty$ , then we also have  $\mathfrak{n}^p(bD) = \mathfrak{n}_3$ .

**Proof** The inclusion  $\mathfrak{n}^p(bD) \subseteq \mathfrak{n}_1$  is immediate from (2). The reverse inclusion holds because  $D$  is simply connected and thus all holomorphic functions on  $D$  have antiderivatives. As  $h^p(bD)$  is closed in the  $L^p(bD, \sigma)$ -norm, we see that  $\mathfrak{n}_1$ , and thus  $\mathfrak{n}^p(bD)$  is also closed. Next, the identity  $\mathfrak{n}^p(bD) = \mathfrak{n}_2$  follows from the fact that  $T(\zeta)d\sigma(\zeta) = d\zeta$  and the well-known result of Smirnov that  $g \in L^p(bD, \sigma)$  lies in  $h^p(bD)$  if and only if

$$\int_{bD} \zeta^k g(\zeta) d\zeta = 0 \quad \text{for } k = 0, 1, 2, \dots$$

See, for example, [5, Theorem 10.4]. Finally, the identity  $\mathfrak{n}^p(bD) = \mathfrak{n}_3$  for  $1 < p < \infty$  follows from Lemma 2.6.  $\square$

We may also characterize the elements of  $\mathfrak{n}^p(bD)$  for a bounded simply connected Lipschitz domain  $D$  via its Riemann maps. We shall need the following description of the tangent vector.

**Lemma 4.2** *Let  $D$  be a bounded simply connected Lipschitz domain and  $\psi : D \rightarrow \mathbb{D}$  be a conformal map. Then the tangent vector  $T$  of  $bD$  (which is defined a.e.) can be written as*

$$T = i \frac{|\dot{\psi}'|}{\dot{\psi}'} \dot{\psi}, \quad \sigma\text{-a.e. on } bD.$$

**Proof** Let  $\phi : \mathbb{D} \rightarrow D$  be defined as  $\phi = \psi^{-1}$ . Since  $bD$  is Lipschitz, it is a Jordan curve so by Carathéodory’s theorem  $\phi$  extends to a homeomorphism of  $\overline{\mathbb{D}}$  onto  $\overline{D}$ . Thus  $\psi$  extends to a homeomorphism of  $\overline{D}$  onto  $\overline{\mathbb{D}}$  and that extension restricted to  $bD$  must equal  $\dot{\psi}$ . By [5, Theorem 3.13], we have that  $\phi' \in \mathcal{H}^1(\mathbb{D})$  so that  $(\dot{\phi}')$  exists  $\sigma$ -a.e.,  $\phi$  is absolutely continuous on  $b\mathbb{D}$ , and

$$\frac{d}{dt} \phi(e^{it}) = i e^{it} (\dot{\phi}')(e^{it}). \tag{8}$$

Thus we can write the unit tangent vector  $T$  via  $(\dot{\phi}')$  for almost all  $\zeta \in \partial D$ . To do so, first note that for  $r < 1$

$$\psi'(\phi(re^{it})) = \frac{1}{\phi'(re^{it})}.$$

Since  $\phi$  is conformal and  $(\dot{\phi}')$  exists and is nonzero a.e., we see that the nontangential limit  $(\dot{\psi}')$  exists a.e. and satisfies

$$(\dot{\psi}')(z) = \frac{1}{(\dot{\phi}')(\dot{\psi}(z))}. \tag{9}$$

Choose  $t_0$  so that  $\zeta = \phi(e^{it_0})$ . Then by Eqs. (8) and (9) we have

$$\begin{aligned} T(\zeta) &= \frac{\frac{d}{dt}\phi(e^{it})}{\left|\frac{d}{dt}\phi(e^{it})\right|} \Big|_{t=t_0} \\ &= \frac{(\dot{\phi}')(e^{it_0})ie^{it_0}}{|(\dot{\phi}')(e^{it_0})|} = i \frac{(\dot{\phi}')(\psi(\zeta))\dot{\psi}(\zeta)}{|(\dot{\phi}')(\psi(\zeta))|} = i \frac{|(\dot{\psi}')(z)|\dot{\psi}(\zeta)}{(\dot{\psi}')(z)} \sigma\text{-a.e.}, \end{aligned}$$

as desired. □

**Theorem 4.3** *Let  $D$  is a bounded simply connected Lipschitz domain, and  $1 \leq p \leq \infty$ . Let  $\psi : D \rightarrow \mathbb{D}$  be a conformal map with  $\alpha := \psi^{-1}(0) \in D$ . Then*

$$n^p(bD) = \left\{ \frac{(\dot{\psi}')}{|(\dot{\psi}')|} \dot{F} : F \in \mathcal{H}^p(D), F(\alpha) = 0 \right\}.$$

**Proof** First by Proposition 4.1, one has

$$n^p(bD) = \{T\dot{G} : G \in \mathcal{H}^p(D)\}.$$

Making use of Lemma 4.2, we further obtain

$$n^p(bD) = \left\{ \frac{(\dot{\psi}')}{|(\dot{\psi}')|} \dot{\psi}\dot{G} : G \in \mathcal{H}^p(D) \right\}.$$

Note that  $\psi$  is conformal on  $D$  and continuous on  $\bar{D}$ . In particular,  $\psi$  has only one zero at  $\alpha$  and that zero is simple. Letting  $F := \psi G$ , then

$$G \in \mathcal{H}^p(D) \text{ if and only if } F \in \mathcal{H}^p(D), F(\alpha) = 0.$$

The proof is complete. □

Note that for  $D = \mathbb{D}$  we can choose  $\psi(z) = z$ , in which case Theorem 4.3 takes an especially simple form, namely

$$n^p(b\mathbb{D}) = \{\dot{F} : F \in \mathcal{H}^p(\mathbb{D}), F(0) = 0\}.$$

### 5 A Characterization of $\mathcal{H}^p(bD)$ for Multiply Connected $D$

Let  $D$  be a bounded Lipschitz domain. Then there exists  $N \geq 1$ , such that the boundary  $bD$  consists of  $N$  closed rectifiable curves. Here and throughout we denote by  $\gamma_1, \gamma_2, \dots, \gamma_N$  those closed curves of  $bD$  endowed with the positive orientation, with  $\gamma_N$  denoting the outer curve of  $bD$  (that is,  $D$  lies in the set of points inside of  $\gamma_N$ ).

In order to characterize  $\mathcal{H}^p(bD)$  we need to understand which elements of  $\mathcal{H}^p(D)$  admit holomorphic antiderivatives. According to classical complex analysis theory, a continuous complex-valued function has an antiderivative in a domain  $D$  (which may be simply or multiply-connected) if and only if the line integral of the function along every closed contour (i.e. piecewise  $C^1$  path) in  $D$  is zero. See, for instance, [11, Theorem 6.44]. This leads us to the following:

**Proposition 5.1** *Let  $D$  be a bounded Lipschitz domain and let the boundary of  $D$  be denoted as above. For  $1 \leq p \leq \infty$  and  $F \in \mathcal{H}^p(D)$  we have that  $F$  is the complex derivative of a holomorphic function on  $D$  if and only if*

$$\int_{\gamma_j} \dot{F}(\zeta)d\zeta = 0 \quad \text{for all } j = 1, \dots, N. \tag{10}$$

**Proof** Let  $\{D_k\}$  be Nečas exhaustion of  $D$  as defined in Lemma 2.3. We will use the notation of Lemma 2.3 throughout this proof. For each  $k$  and  $1 \leq j \leq N$ , let  $\gamma_j^k$  denote portion of  $bD_k$  such that  $\Lambda_k(\gamma_j^k) = \gamma_j$ .

First, assume  $F$  is a derivative of a holomorphic function on  $D$ . For each  $k$  the curve  $\gamma_j^k$  is a closed contour in  $D$ . Thus, by the Fundamental Theorem of Calculus, we have

$$\int_{\gamma_j^k} F(\zeta)d\zeta = 0, \quad j = 1, \dots, N.$$

Thus

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \int_{\gamma_j^k} F(\zeta)d\zeta = \lim_{k \rightarrow \infty} \int_{\gamma_j^k} F(\eta_k)T_k(\eta_k)d\sigma_k(\eta_k) \\ &= \lim_{k \rightarrow \infty} \int_{\gamma_j} F(\Lambda_k(\eta))T_k(\Lambda_k(\eta))w_k(\eta)d\sigma(\eta) = \int_{\gamma_j} \dot{F}(\eta)T(\eta)d\sigma(\eta) = \int_{\gamma_j} \dot{F}(\zeta)d\zeta, \end{aligned} \tag{11}$$

where we used the Dominated Convergence Theorem with the dominating function  $M|F^*|$  (here we are using the fact that  $F \in \mathcal{H}^p(D)$  so that  $F^* \in L^1(bD, \sigma)$ ), obtaining (10).

Conversely, assume (10) holds. Fixing a point  $a \in D$ , we shall show that for any  $z \in D$ , and any contour  $\eta$  in  $D$  connecting  $a$  and  $z$ , the following line integral

$$\int_{\eta} F(\zeta)d\zeta$$

is independent of the choice of the path.

Indeed, let  $\eta_1$  and  $\eta_2$  be two contours joining  $a$  and  $z$  and let  $\beta = \eta_1 \cup (-\eta_2)$  be the closed contour starting and ending at  $a$  (here  $-\eta_2$  is  $\eta_2$  oriented in the opposite direction). Without loss of generality, suppose  $\beta$  is oriented counterclockwise and has no self-intersections. If the domain bounded by  $\beta$  is a subset of  $D$ , then  $\int_{\beta} F(\zeta)d\zeta = 0$  by Cauchy’s theorem. Else, for some  $m$  between 1 and  $N$  there are  $m$  components of  $bD$ , say,  $\gamma_1, \dots, \gamma_m$ , that lie inside the domain bounded by  $\beta$ , while the remaining components  $\gamma_{m+1}, \dots, \gamma_N$  lie outside of such domain. With same notation as before, for a Nečas exhaustion  $\{D_k\}$ , we choose  $k$  large enough so that  $D_k$  contains  $\beta$ ,  $\gamma_1^k, \dots, \gamma_m^k$  lie inside of  $\beta$ , and  $\gamma_{m+1}^k, \dots, \gamma_N^k$  lie outside of  $\beta$ . By a generalized version of Cauchy’s theorem (see, for example, [11, Theorem 8.9]),

$$\int_{\beta} F(\zeta)d\zeta = \sum_{\ell=1}^m \int_{\gamma_{\ell}^k} F(\zeta)d\zeta \quad \text{for any large } k.$$

By an argument similar to the proof of Eq.(11) we have

$$\int_{\beta} F(\zeta)d\zeta = \lim_{k \rightarrow \infty} \sum_{\ell=1}^m \int_{\gamma_{\ell}^k} F(\zeta)d\zeta = \sum_{\ell=1}^m \int_{\gamma_{\ell}} \dot{F}(\zeta)d\zeta = 0,$$

where we used (10) in the last equality. Equivalently,

$$\int_{\eta_1} F(\zeta)d\zeta = \int_{\eta_2} F(\zeta)d\zeta,$$

thus

$$H(z) := \int_{\eta} F(\zeta)d\zeta$$

is well defined and is a holomorphic antiderivative of  $F$  on  $D$ . □

**Remark 5.2** By Cauchy’s theorem (in Lemma 2.4), we have

$$\sum_{j=1}^N \int_{\gamma_j} \dot{F}(\zeta)d\zeta = \int_{bD} \dot{F}(\zeta)d\zeta = 0,$$

for any  $F \in \mathcal{H}^p(D)$ . Then we can refine the statement of Proposition 5.1 by requiring that only  $(N - 1)$ -many terms in Eq. (10) vanish. Without loss of generality, we choose

the first  $(N - 1)$  terms. Hence, Eq. (10) is equivalent to

$$\int_{\gamma_j} \dot{F}(\zeta) d\zeta = 0 \quad \text{for all } j = 1, \dots, N - 1. \tag{12}$$

**Theorem 5.3** *Let  $D$  be a bounded Lipschitz domain and  $1 \leq p \leq \infty$ . Then with  $\mathfrak{n}^p(bD)$  as in (2) we have*

$$\mathfrak{n}^p(bD) = \left\{ Tf : f \in h^p(bD), \int_{\gamma_j} f(\zeta) d\zeta = 0, \quad 1 \leq j \leq N - 1 \right\}. \tag{13}$$

If  $D$  is simply connected then the above identity reads  $\mathfrak{n}^p(bD) = \mathfrak{n}_1$ , see Theorem 4.1 (we should perhaps point out that the congruence of  $\mathfrak{n}^p(bD)$  with the two spaces  $\mathfrak{n}_2$  and  $\mathfrak{n}_3$  proved therein relies upon results that are classically stated for simply connected  $D$ ).

**Proof** Let

$$L_0^p(bD, \sigma) := \left\{ g \in L^p(bD, \sigma) : \int_{bD} g(\zeta) d\sigma(\zeta) = 0 \right\}$$

and

$$L_{00}^p(bD, \sigma) := \left\{ g \in L^p(bD, \sigma) : \int_{\gamma_j} g(\zeta) d\sigma(\zeta) = 0, \quad 1 \leq j \leq N \right\}.$$

Obviously  $L_{00}^p(bD, \sigma) \subset L_0^p(bD, \sigma)$ . We claim that

$$\mathfrak{n}^p(bD) = \{g \in L_{00}^p(bD, \sigma) : \overline{T}g \in h^p(bD)\}. \tag{14}$$

Indeed, if  $g \in \mathfrak{n}^p(bD)$  there exists a  $G \in \vartheta(D)$  with  $G' \in \mathcal{H}^p(D)$  such that  $g = -iT(\dot{G}')$ , see (2); hence  $\overline{T}g = -i(\dot{G}') \in h^p(bD)$ . Moreover Proposition 5.1 gives that

$$\int_{\gamma_j} g(\zeta) d\sigma(\zeta) = -i \int_{\gamma_j} (\dot{G}')(\zeta) d\zeta = 0, \quad j = 1, \dots, N - 1$$

proving that  $g \in L_{00}^p(bD, \sigma)$  and concluding the proof of the forward inclusion. For the reverse inclusion, suppose  $g \in L_{00}^p(bD, \sigma)$  and  $g = T\dot{F}$  for some  $F \in \mathcal{H}^p(D)$ . Then

$$\int_{\gamma_j} \dot{F}(\zeta) d\zeta = \int_{\gamma_j} g(\zeta) d\sigma(\zeta) = 0, \quad j = 1, \dots, N - 1$$

and it follows from Proposition 5.1 and Eq. (12) that  $F$  has an antiderivative  $G \in \vartheta(D)$ . Note that  $iG \in \mathcal{H}^{1,p}(D)$  by definition. Thus,  $g = T\dot{F} = -iT(i\dot{G}') \in \mathfrak{n}^p(bD)$ . The

proof of (14) is concluded. Equation (13) now follows since for  $g$  as above we have  $g = Tf$  with  $f := \overline{T}g$ .  $\square$

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## Declarations

**Conflict of interest** The authors declare no competing interests.

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## References

1. Axler, S., Shields, A.L.: Univalent multipliers of the Dirichlet space. *Michigan Math. J.* **32**(1), 65–80 (1985)
2. Bagemihl, F.: The Plemelj formulas with unrestricted approach and the continuity of Cauchy-type integrals. *Proc. Nat. Acad. Sci.* **55**(5), 1046–1048 (1966)
3. Coifman, R., McIntosh, A., Meyer, Y.: L'intégrale de Cauchy définit un opérateur borné sur  $L^2$  pour les courbes Lipschitziennes. *Ann. Math.* **116**, 361–387 (1982)
4. Dahlberg, B.: Weighted norm inequalities for the Lusin area integral and the non-tangential maximal functions for functions harmonic in a Lipschitz domain. *Stud. Math.* **67**(3), 297–314 (1980)
5. Duren, P.: *Theory of  $H^p$  spaces*. Academic Press, New York (1970)
6. Gryc, W., Lanzani, L., Xiong, J., Zhang, Y.: Boundary value problems for holomorphic functions on Lipschitz planar domains. Preprint. [arXiv: 2402.06747](https://arxiv.org/abs/2402.06747)
7. Kenig, C.: *Harmonic analysis techniques for second order elliptic boundary value problems*. CBMS Regional Conference Series in Mathematics, no. 83, American Mathematical Society, Providence, RI, pp. xii+146 (1994)
8. Lanzani, L.: Cauchy transform and Hardy spaces for rough planar domains. *Analysis, geometry, number theory: the mathematics of Leon Ehrenpreis* (Philadelphia, PA, 1998), vol. 251, pp. 409–428, *Contemporary Mathematics*, American Mathematical Society, Providence, RI (2000)
9. Muskhelishvili, N.: *Singular integral equations*. Translated by Radok, J., Wolters-Noordhoff Publishing, Groningen, pp. xiv+447 (1958)
10. Nečas, J.: *Direct methods in the theory of elliptic equations*. Springer-Verlag, Berlin Heidelberg (2012)
11. Stewart, I., Tall, D.: *Complex analysis*. Cambridge University Press, Cambridge (2018)
12. Verchota, G.: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.* **59**(3), 572–611 (1984)

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