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A direct proof of the Sharp Gårding inequality for symbols with limited smoothness

Massimo Cicognani[1]

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Abstract
We give a proof of the (possibly optimal) Sharp Gårding inequality for system operators with symbol of limited smoothness directly from the original symmetrization arguments by Friedrichs and Kumano-Go. The fact that only a few derivatives of the regularized symbol are really important was already there.

Keywords Sharp Gårding · Limited smoothness

1 Introduction and main results

The Sharp Gårding inequality is a powerful tool in the study of systems of PDE. Let $P = p(x, D_x) = (P_{jk})$ be an $\ell \times \ell$ matrix of operators $P_{jk} = p_{jk}(x, D_x)$ with matrix symbol $p(x,\xi) = (p_{jk}(x,\xi)) \in S^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, that is satisfying

$$|\partial_\xi^\alpha D_x^\beta p(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad \langle\xi\rangle = \sqrt{1+|\xi|^2}. \tag{1.1}$$

Assume that the Hermitian part $p' = (p + p^*)/2$ of $p(x,\xi)$ is positive semidefinite. Then, there exists $C > 0$ such that

$$\Re(Pu, u) \ge -C\|u\|^2_{H^{(m-\mu)/2}}, \quad \mu = \rho - \delta, \tag{1.2}$$

for every $u \in \mathcal{S}$. In particular, for $p(x,\xi) \in S^m_{1,0}$ we have

$$\Re(Pu, u) \ge -C\|u\|^2_{H^{(m-1)/2}}. \tag{1.3}$$

✉ Massimo Cicognani
massimo.cicognani@unibo.it

1 Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato, 5, 40126 Bologna, Italy

Hörmander [4] proved inequality (1.3) for scalar operators and Lax-Nirenberg [7] extended this result to systems. Friedrichs [3], Kumano-Go [5] and others improved it and simplified the proof.

For scalar operators, there is the great strengthening $\mu = 2(\rho - \delta)$ in (1.2) due to Fefferman and Phong [2] but for matrix operators with smooth symbol the bound for μ remains $\mu = \rho - \delta$.

In many applications operators with symbol of limited smoothness are involved. Let us consider $p(x, \xi)$ in the class $C^s S^m_{1,0}$ of symbols with C^s regularity in the space variable x defined by

$$\|\partial_\xi^\alpha p(x, \xi)\|_{C^s} \le C_\alpha \langle \xi \rangle^{m-|\alpha|}. \tag{1.4}$$

For any fixed $\delta \in]0, 1[$ one can regularize the symbol obtaining a splitting

$$p(x, \xi) = p^\sharp(x, \xi) + p^b(x, \xi), \quad p^\sharp(x, \xi) \in S^m_{1,\delta}, \; p^b(x, \xi) \in C^s S^{m-s\delta}_{1,\delta}, \tag{1.5}$$

e.g. Taylor [9]. If p' is positive semidefinite, then the Hermitian part of $p^\sharp(x, \xi)$ is positive semidefinite as well. Applying (1.2) to $P^\sharp(x, D_x)$ and using the boundedness

$$P^b(x, D_x) : H^m \to H^{m-s\delta},$$

the sharp Gårding inequality (1.2) for $P(x, D_x)$ holds true with a order

$$\mu \le 1 - \delta, \; \mu \le s\delta.$$

Negotiating on δ as done in [9], one obtains (1.2) for $p(x, \xi) \in C^s S^m_{1,0}$ with

$$\mu = \frac{s}{s+1}. \tag{1.6}$$

Taylor's bound (1.6) gives $\mu \to 1$ for $s \to \infty$ but it is not optimal. By means of the paradifferential calculus, Bony [1] proved that the best possible bound $\mu = 1$ is achieved already for $s = 2$. For $0 < s < 2$ Bony obtained the bound $\mu = s/2$ which is better than Taylor's one for $1 < s < 2$ but it is worse for $0 < s < 1$.

Conjugating the operator with the FBI transform, Taturu [8] proved a generalization of the Sharp Gårding inequality for regular symbols from which he obtained also inequality (1.2) for symbols $p(x, \xi) \in C^s S^m_{1,0}$ with

$$\mu = \mu^*(s) = \begin{cases} 1, \; s \ge 2, \\ 2s/(s+2), \; 0 < s < 2. \end{cases} \tag{1.7}$$

We believe this one the optimal estimate for C^s symbols, agreeing with Tataru.

Our aim is to show that a generalization of the Sharp Gårding inequality for regular symbols, sufficient to get $\mu = \mu^*(s)$ in the case of C^s limited smoothness, can be proved directly from Friedrichs symmetrization, that is going back to the original proofs of (1.2) in [3, 5, 6].

As in Tataru's result, what is really important is the order of $\partial_x^\beta p(x,\xi)$ with $|\beta| = 2$, let us denote $m + m_2$ this order. From $p(x,\xi) \in S^m_{\rho,\delta}$ clearly we have $m_2 \le 2\delta$. In case of equality one can not obtain better than $\mu = \rho - \delta$ in (1.2) but we can improve this bound in the case $m_2 < 2\delta$. As we will see later on, this is exactly what happens for $p^\sharp(x,\xi) \in S^m_{1,\delta}$ in the splitting (1.5) of $p(x,\xi) \in C^s S^m_{1,0}$.

For sake of simplicity, from now on we take $\rho = 1$ which is the case of our interest. Here we prove the following generalization of inequality (1.2) for regular symbols.

Theorem 1.1 *Let $P = p(x, D_x) = (P_{jk})$ be an $\ell \times \ell$ matrix of operators $P_{jk} = p_{jk}(x, D_x)$ with matrix symbol $p(x,\xi) = (p_{jk}(x,\xi)) \in S^m_{1,\delta}$, $0 \le \delta < 1$, and such that*

$$\partial_x^\beta p(x,\xi) \in S^{m+m_1}_{1,\delta},\ |\beta| = 1; \quad \partial_x^\beta p(x,\xi) \in S^{m+m_2}_{1,\delta},\ |\beta| = 2. \tag{1.8}$$

Assume that the Hermitian part $p' = (p + p^*)/2$ of $p(x,\xi)$ is positive semidefinite.

Then, there exists $C > 0$ such that

$$\Re(Pu, u) \ge -C\|u\|^2_{(m-\mu^\sharp)/2} \tag{1.9}$$

for every $u \in \mathcal{S}$, with

$$\mu^\sharp = \begin{cases} \min\{1 - m_1, 1 - m_2/2\}, & 2\delta - 1 \le m_2/2, \\ \min\{1 - m_1, 2(1-\delta)\}, & 2\delta - 1 > m_2/2. \end{cases} \tag{1.10}$$

For the largest possible $m_2 = 2\delta$ of course we have $2\delta - 1 < m_2/2$ hence the general bound $\mu^\sharp = 1 - \delta$. The same we have with $m_1 = \delta$ and any $m_2 \le 2\delta$.

With $m_2 < 2\delta$ and $m_1 < \delta$ there is a gain. For instance, for $m_1 = m_2 = 0$ we have $\mu^\sharp = 1$ for $0 \le \delta \le 1/2$ and $\mu^\sharp = 2(1-\delta)$ for $1/2 < \delta < 1$. Spending such a gain we can prove the result for symbols of limited smoothness.

Theorem 1.2 *Let $P = p(x, D_x) = (P_{jk})$ be an $\ell \times \ell$ matrix of operators with symbol $p(x,\xi) = (p_{jk}(x,\xi)) \in C^s S^m_{1,0}$. Assume that the Hermitian part $p' = (p + p^*)/2$ of $p(x,\xi)$ is positive semidefinite.*

Then, there exists $C > 0$ such that

$$\Re(Pu, u) \ge -C\|u\|^2_{(m-\mu^*(s))/2} \tag{1.11}$$

for every $u \in \mathcal{S}$, with

$$\mu^*(s) = \begin{cases} 1, & s \ge 2, \\ 2s/(s+2), & 0 < s < 2. \end{cases} \tag{1.12}$$

2 Proof of Theorem 1.1

We follow the proof of Friedrichs [3] and Kumano-go [5, 6].

Let $p(x, \xi) \in S^m_{1,\delta}$ and for $\delta' \geq 2\delta - 1$, $\tau = (1 + \delta')/2$ $(\geq \delta)$ let us consider

$$p_0(x, \xi) = \int p(x, \xi + \sigma \langle \xi \rangle^\tau) q(\sigma)^2 d\sigma \tag{2.1}$$

where $q(\sigma) \geq 0$ is a smooth function of $\sigma \in \mathbb{R}^n$ with support for $|\sigma| < 1$, $q(\sigma) = q(-\sigma)$, $\int q(\sigma)^2 d\sigma = 1$.

In the original proof $\tau = (1+\delta)/2$ that is $\delta' = \delta$ from the beginning. We take some advantage by fixing $\delta' \in [2\delta - 1, 1[$ related to m_2 later on.

Performing a change of variable in the integral (2.1) we have

$$p_0(x, \xi) = \int p(x, \zeta) F(\xi, \zeta)^2 d\zeta \tag{2.2}$$

with

$$F(\xi, \zeta) = q((\zeta - \xi)\langle \xi \rangle^{-\tau}) \langle \xi \rangle^{-\tau n/2}. \tag{2.3}$$

To obtain a symmetric operator, we introduce the double symbol $p_F(\xi, x', \xi')$, such that $p_F(\xi, x, \xi) = p_0(x, \xi)$, defined by

$$p_F(\xi, x', \xi') = \int F(\xi, \zeta) p(x', \zeta) F(\xi', \zeta) d\zeta. \tag{2.4}$$

We denote again $p_F(x, \xi)$ the simplified symbol of the operator $P_F(x, D_x)$.

If the matrix is $p(x, \xi)$ is positive semidefinite, then P_F is a positive operator:

$$(P_F u, u) \geq 0, \ \ u \in \mathcal{S},$$

see Theorem 4.3 in [6].

Taking $\tau = (1 + \delta')/2 > \delta$ (this is the case with the original choice $\delta' = \delta$ of [6]), from the proof of Theorem 4.2 in [6] we have that the simplified symbol $p_F(x, \xi)$ of the operator P_F belongs to the class $S^m_{1,\delta}$ and has an asymptotic expansion

$$\begin{aligned} p_F(x, \xi) \sim p(x, \xi) + \sum_{|\beta|=1} \psi_\beta(\xi) p_{(\beta)}(x, \xi) + \sum_{|\alpha+\beta|\geq 2} \psi_{\alpha,\beta}(\xi) p^{(\alpha)}_{(\beta)}(x, \xi), \\ \psi_\beta \in S^{-1}, \ \ \psi_{\alpha,\beta} \in S^{\tau(|\alpha|-|\beta|)}. \end{aligned} \tag{2.5}$$

Looking at the orders of ψ_β and $\psi_{\alpha,\beta}$ and at the orders of $\partial_x^\beta p(x, \xi)$ for $|\beta| \leq 2$ in (1.8), from the above expansion we get

$$\begin{aligned} & p(x, \xi) = p_F(x, \xi) + p_1(x, \xi) + p_2(x, \xi), \\ & p_1(x, \xi) \in S^{m-(1-m_1)}_{1,\delta}, \ \ p_2(x, \xi) \in S^{m-\mu_2}_{1,\delta}, \\ & \mu_2 = \mu_2(\delta') = \min\{1 - \delta', 1 + \delta' - m_2\}. \end{aligned} \tag{2.6}$$

In the limit case $\tau = (1+\delta')/2 = \delta$ the proof of Thoerem 4.2 in [6] still gives (2.6) with the difference $p_2(x,\xi) \in S^{m-\mu_2}_{\delta,\delta}$ instead of $S^{m-\mu_2}_{1,\delta}$ and what we loose in this case is the complete asymptotic expansion (2.5) which is not essential for our aims.

The positivity of the operator P_F and the orders of P_1, P_2 in the splitting (2.6) yield inequality (1.2) for $P = P_F + P_1 + P_2$ with

$$\mu \le \min\{1 - m_1, \mu_2\}.$$

The order of P_1 gives the bound $\mu^\sharp \le 1 - m_1$ for $\mu^\sharp$ in (1.10). Then, we have to maximize $\mu_2 = \mu_2(\delta')$ in (2.6) for $2\delta - 1 \le \delta' < 1$ in order to get the best possible second bound. Since

$$\max_{2\delta-1\le\delta'<1} \mu_2(\delta') = \begin{cases} 1 - m_2/2, & 2\delta - 1 \le m_2/2, \\ 2 - 2\delta, & 2\delta - 1 > m_2/2, \end{cases}$$

we complete the proof of Theorem 1.1.

3 Proof of Theorem 1.2

Let us show how Theorem 1.1 implies Theorem 1.2. Coming back to the splitting (1.5) of $p(x,\xi) \in C^s S^m_{1,0}$, now we have to negotiate between $\mu^\sharp = \mu^\sharp(\delta, m_1, m_2)$ of Theorem 1.1 for $p^\sharp$ and $s\delta$. We obtain the optimal bound $\mu = \mu^*(s)$ for

$$\mu^\sharp = s\delta.$$

We use the more precise estimates for the regularized part $p^\sharp$

$$\partial_x^\beta p^\sharp \in S^m_{1,\delta}, \ |\beta| \le s; \quad \partial_x^\beta p^\sharp \in S^{m+\delta(|\beta|-s)}_{1,\delta}, \ |\beta| > s, \tag{3.1}$$

given by Proposition 1.3.D in [9]. This means, with our notation,

$$m_1 = m_1(s) = \begin{cases} 0, & s \ge 1, \\ \delta(1-s), & 0 < s < 1, \end{cases} \tag{3.2}$$

and

$$m_2 = m_2(s) = \begin{cases} 0, & s \ge 2, \\ \delta(2-s), & 0 < s < 2. \end{cases} \tag{3.3}$$

We have $m_1 \le m_2/2$ in any case. In particular m_1 does not influence $\mu^\sharp$ and (1.10) for $p^\sharp$ reduces to

$$\mu^\sharp = \begin{cases} 1 - m_2/2, & 2\delta - 1 \le m_2/2, \\ 2(1-\delta), & 2\delta - 1 > m_2/2. \end{cases} \tag{3.4}$$

Here the best choice, if it is possible to fix δ such that $2\delta - 1 \le m_2/2$, is always $\mu^\sharp = 1 - m_2/2$.

For $s \geq 2$ we have $m_2 = 0$ in (3.3). Choosing $\delta = 1/s$, ($2\delta - 1 \leq m_2/2$ reads exactly $s \geq 2$), we have

$$\mu^\sharp = 1 - m_2/2 = 1 = s\delta \tag{3.5}$$

and the best possible bound $\mu = \mu^*(s) = 1$ is achieved in (1.12).

For $0 < s < 2$ we have $m_2 = \delta(2 - s)$ in (3.3). Choosing $\delta = 2/(s + 2)$ we have $2\delta - 1 = m_2/2$ and

$$\mu^\sharp = 1 - m_2/2 = 2s/(s + 2) = s\delta \tag{3.6}$$

that leads to $\mu = \mu^*(s) = 2s/(s + 2)$ in (1.12).

This completes the proof of Theorem 1.2.

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