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Global solutions to semilinear parabolic equations driven by mixed local–nonlocal operators

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Abstract

We are concerned with the Cauchy problem for the semilinear parabolic equation driven by the mixed local–nonlocal operator $\mathcal{L} = -\Delta + (-\Delta)^s$, with a power-like source term. We show that the so-called Fujita phenomenon holds, and the critical value is exactly the same as for the fractional Laplacian.

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1 | INTRODUCTION

Let $\mathcal{L}$ be the mixed *local–nonlocal* operator $\mathcal{L} = -\Delta + (-\Delta)^s$, where $(-\Delta)^s$ stands for the fractional Laplacian of order $s \in (0,1)$. We investigate global existence and blow-up of solutions to semilinear parabolic equations driven by $\mathcal{L}$ of the following type:

$$\begin{cases} \partial_t u + \mathcal{L}u = u^p & \text{in } \mathbb{R}^N \times (0,+\infty) \\ u = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p > 1$ and u_0 is a given nonnegative initial datum.

Bibliographical notes: global existence and blow-up. Global existence and blow-up of solutions have been largely studied in the literature. Concerning the purely local case $\mathcal{L} = -\Delta$, it has been shown in [23], and in [35, 38] for the critical case, that

(a) if $1 < p \leqslant 1 + \frac{2}{N}$, any solution of (1.1) blows up in finite-time, provided that $u_0 \not\equiv 0$;

(b) if $p > 1 + \frac{2}{N}$, then there exists a global in time solution of (1.1), provided that u_0 is sufficiently small.

Such a dichotomy is known as the *Fujita phenomenon*. We refer, for example, to [1, 18, 41] and the references therein, for a complete account about blow-up and global existence of solutions in the purely local case $\mathcal{L} = -\Delta$.

This question has been addressed also on Riemannian manifolds when $\mathcal{L}$ is the Laplace–Beltrami operator; in this direction, some results can be found, for example, in [2, 33, 34, 42, 46, 47, 55, 57]. Furthermore, analogue results have also been established for local quasilinear evolution equations (see, for example, [29–32, 42–44]).

On the other hand, when $\mathcal{L} = (-\Delta)^s$ in [54] it is shown that if $p \leqslant 1 + \frac{2s}{N}$, then any solution arising from a nontrivial initial datum u_0 blows up in finite time (see also [22]). Such a result has been generalized in [40] for more general source terms. Moreover, in [37] (see also [36]), for $p > 1 + \frac{2s}{N}$, global in time solutions are considered, and the asymptotic behavior of solutions as $t \to +\infty$ has been studied.

Bibliographical notes: mixed local–nonlocal operators. Recently, the study of qualitative properties of solutions to partial differential equations, mainly of elliptic but also of parabolic type, driven by the mixed operator $\mathcal{L}$ has been attracting much attention (see [5–13, 17, 24–27]). One of the main reasons for this interest is that mixed operators of the form $\mathcal{L}$ have applications in probability; indeed, they are related to the superposition of different types of stochastic processes such as a classical random walk and a Lévy flight. Furthermore, they are exploited to model various phenomena in sciences, such as the study of optimal animal foraging strategies, see, for example, [19, 20] and references therein.

Description of our results. Along the above-described line of research, in the present paper we deal with nonnegative solutions to problem (1.1). The main result of this paper will be given in detail in the forthcoming Theorem 3.3; however, we give here a sketchy outline of this result. In particular, we show that if $p \leqslant 1 + \frac{2s}{N}$, then problem (1.1) does not admit any global solution with $u_0 \not\equiv 0$. On the other hand, if $p > 1 + \frac{2s}{N}$, then there exists a global in time solution, provided that u_0 is small enough. We point out that problem (1.1) behaves like the problem with $\mathcal{L} = (-\Delta)^s$; in other terms, for what concerns existence and nonexistence of global in time solutions the mixed local–nonlocal operator has the same character as the nonlocal operator $(-\Delta)^s$. The proof of the nonexistence of global solutions is based on a test functions argument and on suitable a priori estimates. Furthermore, the global solution is constructed by an iteration method, which exploits in a crucial way the estimates from above for the heat kernel of $\mathcal{L}$.

Plan of the paper. The paper is organized as follows. In Section 2 we fix the notation and recall some preliminary results concerning the fractional Laplacian, the operator $\mathcal{L}$ and the heat kernel of $\mathcal{L}$. In Section 3 we give the precise definition of solution to problem (1.1) and we state our main existence/nonexistence result, which is then proved in Section 4.

2 | MATHEMATICAL BACKGROUND

Notation. Throughout the paper, we will tacitly exploit all the notation listed below; we thus refer the Reader to this list for any nonstandard notation encountered.

- We denote by $\mathbb{R}^+$ (resp. $\mathbb{R}^+_0$) the interval $(0, +\infty)$ (resp., $[0, +\infty)$).

- Given any $x_0 \in \mathbb{R}^N$ and any $r > 0$, we denote by $B_r(x_0)$ the open (Euclidean) ball with center x_0 and radius r; in the particular case when $x_0 = 0$, we simply write B_r.
- Given any $0 < T \leqslant +\infty$, we denote by S_T the (infinite) strip $\mathbb{R}^N \times (0, T)$; in the particular case when $T = +\infty$, we simply write S in place of $S_{+\infty}$.
- If A is an arbitrary set in some Euclidean space $\mathbb{R}^m$ (with $m \geqslant 1$), we denote by $\mathbf{1}_A$ the usual indicator function of A, that is,

$$\mathbf{1}_A(z) = \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{if } z \notin A. \end{cases}$$

- We denote by $\mathcal{T}_0$ the set (vector space) of the functions $\varphi \in C^\infty(\overline{S})$ for which there exist numbers $r, T > 0$ (possibly depending on φ) such that

$$\varphi \equiv 0 \text{ out of } B_r \times [0, T).$$

- Given any $s \in (0, 1)$, we denote by L_s the *tail space*

$$L_s(\mathbb{R}^N) := \left\{ f : \mathbb{R}^N \to \mathbb{R} \, : \, \|f\|_{1,s} := \int_{\mathbb{R}^N} \frac{|f(x)|}{1 + |x|^{N+2s}} \, dx < +\infty \right\}.$$

- Given any open interval $I \subseteq \mathbb{R}$, any Banach space $(X, \|\cdot\|_X)$ and any $1 \leqslant \theta \leqslant \infty$, we denote by $L^\theta(I; X)$ the space of the L^θ-functions taking values in X, that is,

$$L^\theta(I; X) = \left\{ f : I \to X \, : \, \mathfrak{n}_X(f)(t) := \|f(t)\|_X \in L^\theta(I) \right\}.$$

  If $f \in L^\theta(I; X)$, we define $\|f\|_{\theta, I, X} := \|\mathfrak{n}_X(f)\|_{L^\theta(I)}$.
- If X, Y are real normed vector spaces, we denote by $B(X, Y)$ the set (vector space) of the linear, bounder operators from X into Y.
- We denote by $\mathfrak{F}$ the Fourier transform on $L^2(\mathbb{R}^N)$, normalized in such a way that it is an *isometry*; as a consequence, for every $f \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ we have

$$\mathfrak{F}(f)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\imath \langle x, \xi \rangle} f(x) \, dx.$$

As anticipated in the Introduction, in this ‘preliminary’ section we collect several definitions and known results, which will allow us to clearly state our main contribution (see Theorem 3.3 in Section 3), and to make the manuscript as self-contained as possible.

2.1 | The mixed operator $\mathcal{L} = -\Delta + (-\Delta)^s$

In order to clearly state the main result of this paper, we first need to fix some notation and to properly define what we mean by a *solution to the Cauchy problem (1.1)*; due to the *mixed nature of* $\mathcal{L}$, this will require some preliminaries.

(1) *The fractional Laplacian*. Let $s \in (0,1)$ be fixed, and let $u : \mathbb{R}^N \to \mathbb{R}$. The *fractional Laplacian* (of order s) of u at a point $x \in \mathbb{R}^N$ is defined as follows:

$$\begin{aligned}(-\Delta)^s u(x) &= C_{N,s} \cdot \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy \\ &= C_{N,s} \cdot \lim_{\varepsilon \to 0^+} \int_{\{|x-y|\geqslant \varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy,\end{aligned} \tag{2.1}$$

provided that the limit exists and is finite. Here, $C_{N,s} > 0$ is a suitable normalization constant which plays a role in the limit as $s \to 0^+$ or $s \to 1^-$, and is explicitly given by

$$C_{N,s} = \frac{2^{2s-1} 2s \Gamma((N+2s)/2)}{\pi^{N/2}\Gamma(1-s)}.$$

As it is reasonable to expect, for $(-\Delta)^s u(x)$ to be well-defined one needs to impose suitable *growth conditions* on the function u, both when $|y| \to +\infty$ and when $y \to x$. In this perspective we state the following proposition (see [39, 51] for a proof).

Proposition 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then, the following facts hold.*

(i) *If $0 < s < 1/2$ and $u \in C^{2s+\gamma}_{\mathrm{loc}}(\Omega) \cap L_s(\mathbb{R}^N)$ for some $\gamma \in (0, 1-2s)$, then*

$$\exists\ (-\Delta)^s u(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(y)-u(x)}{|x-y|^{N+2s}}\,dy \quad \textit{for all } x \in \Omega.$$

(ii) *If $1/2 < s < 1$ and $u \in C^{1,2s-1+\gamma}_{\mathrm{loc}}(\Omega) \cap L_s(\mathbb{R}^N)$ for some $\gamma \in (0, 2-2s)$, then*

$$\exists\ (-\Delta)^s u(x) = -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{N+2s}}\,dy \quad \textit{for all } x \in \Omega.$$

Moreover, in both cases (i) and (ii) *we have $(-\Delta)^s u \in C(\Omega)$.*

In the particular case when $\Omega = \mathbb{R}^N$ and $u \in \mathcal{S} \subseteq L_s(\mathbb{R}^N)$ (here and throughout, $\mathcal{S}$ denotes the usual Schwartz space of the rapidly decreasing functions), it is possible to provide an alternative expression of $(-\Delta)^s u$ (which is well defined on the whole of $\mathbb{R}^N$, see Proposition 2.1) via the Fourier Transform $\mathfrak{F}$; more precisely, we have the subsequent result.

Proposition 2.2. *Let $u \in \mathcal{S} \subseteq L_s(\mathbb{R}^N)$. Then,*

$$\exists\ (-\Delta)^s u(x) = \mathfrak{F}^{-1}\left(|\xi|^{2s}\mathfrak{F}(u)\right)(x) \quad \textit{for every } x \in \mathbb{R}^N. \tag{2.2}$$

It should be noted that, on account of (2.2), it is immediate to recognize that the Schwartz space $\mathcal{S}$ *is not preserved* by the fractional Laplacian $(-\Delta)^s$ (as $|\xi|^{2s}\mathfrak{F}u$ is not regular at $\xi = 0$), that is, one has $(-\Delta)^s(\mathcal{S}) \nsubseteq \mathcal{S}$; however, we have the following characterization of the image

$$\mathcal{S}_s = (-\Delta)^s(\mathcal{S}),$$

which will be crucial to give the definition of *solution of problem (1.1)*.

Proposition 2.3 See, for example, [53, Lemma 1]. *Setting* $\mathcal{S}_s = (-\Delta)^s(\mathcal{S})$*, we have*

$$\mathcal{S}_s = \left\{\psi \in C^\infty(\mathbb{R}^N) : (1+|x|^{N+2s})D^\alpha\psi \in L^\infty(\mathbb{R}^N) \text{ for every } \alpha \in (\mathbb{N}\cup\{0\})^N\right\}.$$

Another consequence of the 'representation formula' (2.2), which plays a fundamental role in our argument (and, in general, in the analysis of the fractional Laplace operator $(-\Delta)^s$), is the possibility of realizing this operator as a *densely defined, self-adjoint and nonnegative* operator on the Hilbert space $L^2(\mathbb{R}^N)$, whose associated heat semigroup admits a global heat kernel. Indeed, taking into account (2.2), it is natural to define

$$\begin{gathered}\mathcal{B}_s : H^s(\mathbb{R}^N) \subseteq L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \qquad \mathcal{B}_s(u) = \mathfrak{F}^{-1}\left(|\xi|^{2s}\mathfrak{F}(u)\right)\\ \text{where } H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : |\xi|^{2s}\mathfrak{F}(u) \in L^2(\mathbb{R}^N)\right\}.\end{gathered} \tag{2.3}$$

Clearly, we have $\mathcal{S} \subseteq H^s(\mathbb{R}^N)$, and thus $\mathcal{B}_s$ is densely defined; moreover, by (2.2) one has

$$\mathcal{B}_s(u) = (-\Delta)^s u \quad \text{for every } u \in \mathcal{S} \subseteq H^s(\mathbb{R}^N),$$

and this shows that $\mathcal{B}_s$ is indeed a realization of $(-\Delta)^s$ on $L^2(\mathbb{R}^N)$. We then observe that, since the map $\mathfrak{F}$ is an isometry of $L^2(\mathbb{R}^N)$, for every $u, v \in H^s(\mathbb{R}^N)$ we get

$$\begin{aligned}
\text{(i) } \langle \mathcal{B}_s(u), v\rangle_{L^2(\mathbb{R}^N)} &= \langle \mathfrak{F}(\mathcal{B}_s(u)), \mathfrak{F}(v)\rangle_{L^2(\mathbb{R}^N)} = \langle |\xi|^{2s}\mathfrak{F}(u), \mathfrak{F}(v)\rangle_{L^2(\mathbb{R}^N)}\\
&= \langle \mathfrak{F}(u), |\xi|^{2s}\mathfrak{F}(v)\rangle_{L^2(\mathbb{R}^N)} = \langle u, \mathfrak{F}^{-1}(|\xi|^{2s}\mathfrak{F}(v))\rangle_{L^2(\mathbb{R}^N)}\\
&= \langle u, \mathcal{B}_s(v)\rangle_{L^2(\mathbb{R}^N)};\\
\text{(ii) } \langle \mathcal{B}_s(u), u\rangle_{L^2(\mathbb{R}^N)} &= \langle \mathfrak{F}(\mathcal{B}_s(u)), \mathfrak{F}(u)\rangle_{L^2(\mathbb{R}^N)} = \langle |\xi|^{2s}\mathfrak{F}(u), \mathfrak{F}(u)\rangle_{L^2(\mathbb{R}^N)}\\
&= \langle |\xi|^{s}\mathfrak{F}(u), |\xi|^{s}\mathfrak{F}(u)\rangle_{L^2(\mathbb{R}^N)} \geqslant 0;
\end{aligned}$$

and thus $\mathcal{B}_s$ is self-adjoint and nonnegative. As a consequence of these facts, we are then entitled to apply [28, Theorem 4.9], ensuring that the operator $-\mathcal{B}_s$ generates a strongly continuous semigroup on the Hilbert space $L^2(\mathbb{R}^N)$, say $\{T(t)\}_{t\geqslant 0}$. By this, we mean that

(P1) for every fixed $t \geqslant 0$, we have $T(t) \in B(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$;
(P2) $T(t+\tau) = T(t)\circ T(\tau)$ for every $t, \tau \geqslant 0$;
(P3) for every fixed $t \geqslant 0$ and $f \in L^2(\mathbb{R}^N)$, we have

$$\lim_{\tau\to t} T(\tau)f = T(t)f \quad \text{in } L^2(\mathbb{R}^N);$$

(P4) for every fixed $t > 0$ and $f \in L^2(\mathbb{R}^N)$, we have $T(t)f \in H^s(\mathbb{R}^N)$ and

$$\frac{d}{dt}(T(t)f) = \lim_{h\to 0}\frac{T(t+h)f - T(t)f}{h} = -\mathcal{B}_s(T(t)f) \text{ in } L^2(\mathbb{R}^N).$$

This semigroup is called the *heat semigroup of* $-(-\Delta)^s$, and it is denoted by $(e^{-t(-\Delta)^s})_{t\geqslant 0}$.

We now observe that, starting from property (P4) and exploiting the Fourier transform (together with the very definition of $\mathcal{B}_s$), it is easy to show that the operator $e^{-t(-\Delta)^s}$ (for every $t > 0$) is actually an *integral operator on* $L^2(\mathbb{R}^N)$ *with a kernel of convolution type.*

Indeed, let $f \in L^2(\mathbb{R}^N)$ be fixed, and let

$$u : [0, +\infty) \to L^2(\mathbb{R}^N), \qquad u(t)(x) = e^{-t(-\Delta)^s} f(x).$$

Using property (P4) and applying the Fourier transform, we see that

$$\begin{aligned} *)\ \mathfrak{F}(u'(t)) &= \mathfrak{F}\Big(x \mapsto \frac{d}{dt}\Big(e^{-t(-\Delta)^s} f\Big)(x)\Big) = -\mathfrak{F}\Big(x \mapsto \mathcal{B}_s(e^{-t(-\Delta)^s} f)(x)\Big) \\ &= -|\xi|^{2s}\mathfrak{F}\Big(x \mapsto e^{-t(-\Delta)^s} f(x)\Big) = -|\xi|^{2s}\mathfrak{F}(u(t)), \\ *)\ \mathfrak{F}(u(0)) &= \mathfrak{F}\Big(x \mapsto e^{-0\cdot(-\Delta)^s} f(x)\Big) = \mathfrak{F}(f), \end{aligned}$$

which is a (formal) *first-order, linear Cauchy problem* for $t \mapsto \mathfrak{F}(u(t))(\xi)$ (for every fixed $\xi \in \mathbb{R}^N$); as a consequence, by formally solving this problem, we derive

$$\mathfrak{F}(u(t))(\xi) = \mathfrak{F}(f)(\xi)e^{-t|\xi|^{2s}} \quad \text{for all } \xi \in \mathbb{R}^N,\ t \geqslant 0.$$

Since we have expressed $\mathfrak{F}(u(t))$ as a *product of two functions*, by using the well-known properties of the Fourier transform we then conclude that

$$\begin{aligned} e^{-t(-\Delta)^s} f(x) = u(t)(x) &= \mathfrak{F}^{-1}\Big(e^{-t|\xi|^{2s}} \cdot \mathfrak{F}(f)\Big) \\ &= (\mathfrak{h}_t^{(s)} * f)(x) = \int_{\mathbb{R}^N} \mathfrak{h}_t^{(s)}(x-y) f(y)\, dy, \end{aligned} \tag{2.4}$$

where, for every $z \in \mathbb{R}^N$ and $t > 0$, we have

$$\mathfrak{h}_t^{(s)}(z) = \frac{1}{(2\pi)^{N/2}}\mathfrak{F}^{-1}\Big(e^{-t|\xi|^{2s}}\Big)(z) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{\imath\langle z,\xi\rangle - t|\xi|^{2s}}\, d\xi. \tag{2.5}$$

This function $(t, z) \mapsto \mathfrak{h}_t^{(s)}(z)$ is usually referred to as the *heat kernel of* $-(-\Delta)^s$, and it satisfies the following properties (see, for example, [3, 4, 14, 15, 54] for a complete proof):

(1) $\mathfrak{h}^{(s)} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\mathfrak{h}^{(s)} > 0$;
(2) for every $x \in \mathbb{R}^N$ and $t > 0$, we have

$$\mathfrak{h}_t^{(s)}(x) = \mathfrak{h}_t^{(s)}(-x) \quad \text{and} \quad \mathfrak{h}_t^{(s)}(x) = \frac{1}{t^{N/(2s)}}\mathfrak{h}_1^{(s)}(t^{-N/(2s)}x);$$

(3) for every fixed $x \in \mathbb{R}^N$ and $t > 0$, we have

$$\int_{\mathbb{R}^N} \mathfrak{h}_t^{(s)}(x)\, dy = 1;$$

(4) for every fixed $x \in \mathbb{R}^N$ and $t, \tau > 0$, we have

$$\int_{\mathbb{R}^N} \mathfrak{h}_t^{(s)}(x-y)\mathfrak{h}_\tau^{(s)}(y)\,dy = \mathfrak{h}_{t+\tau}^{(s)}(x);$$

(5) there exists $C \geqslant 1$ such that

$$C^{-1}\min\left\{t^{-N/(2s)}, \frac{t}{|x|^{N+2s}}\right\} \leqslant \mathfrak{h}_t^{(s)}(x) \leqslant C\min\left\{t^{-N/(2s)}, \frac{t}{|x|^{N+2s}}\right\}$$
$$\text{for every } x \in \mathbb{R}^N \text{ and every } t > 0. \tag{2.6}$$

(2) *The heat kernel of $\mathcal{L}$.* Now we have reviewed a few basic concepts on the fractional Laplace operator $(-\Delta)^s$, we spend a few words concerning the heat semigroup and the associated global heat kernel of the operator $-\mathcal{L} = \Delta - (-\Delta)^s$ (we refer, for example, to [52] for a thorough investigation on this topic); this kernel will be used to introduce the notion of *mild solution* to the Cauchy problem (1.1) (see Definition 3.1).

Our starting point is the usual realization of the operator $-\Delta$ in $L^2(\mathbb{R}^N)$: denoting by $H^2(\mathbb{R}^N)$ the classical Sobolev space $W^{2,2}(\mathbb{R}^N)$, it is very well known that the operator

$$\mathcal{A} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \qquad \mathcal{A}(u) = \mathfrak{F}^{-1}\big(|\xi|^2\mathfrak{F}(u)\big),$$

satisfies the following properties:

(a) $\mathcal{A}$ is a densely defined, positive and self-adjoint operator;
(b) $\mathcal{A}(u) = -\Delta u$ for every $u \in \mathcal{S} \subseteq H^2(\mathbb{R}^N)$

(actually, the above properties of $\mathcal{A}$ can be proved by repeating *verbatim* the computation carried out in the previous paragraph with the 'formal' choice $s = 1$, see also [21, Section 4.3]).

On the other hand, by exploiting the characterization of the Sobolev spaces $H^k(\mathbb{R}^N)$ (for $k \geqslant 1$) in terms of $\mathfrak{F}$ (see, for example, [21, Section 5.8.4]), we have

$$H^2(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N) : |\xi|^2\mathfrak{F}(u) \in L^2(\mathbb{R}^N)\big\} \subseteq H^s(\mathbb{R}^N);$$

thus, taking into account (2.3), we can define

$$\mathcal{P} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \qquad \mathcal{P}(u) = \mathcal{A}(u) + \mathcal{B}_s(u) = \mathfrak{F}^{-1}\big(|\xi|^2\mathfrak{F}(u)\big) + \mathfrak{F}^{-1}\big(|\xi|^{2s}\mathfrak{F}(u)\big).$$

Clearly, by combining the properties of $\mathcal{B}_s$ (discussed in the previous paragraph) with the properties of $\mathcal{A}$ recalled above, we immediately derive that $\mathcal{P} = \mathcal{A} + \mathcal{B}_s$ is a densely defined, positive and self-adjoint operator which realizes $\mathcal{L}$ on $L^2(\mathbb{R}^N)$: indeed, we have

$$\mathcal{P}(u) = \mathcal{L}u \quad \text{for every } u \in \mathcal{S} \subseteq H^2(\mathbb{R}^N).$$

We can then exploit once again [28, Theorem 4.9], which ensures that also the operator $-\mathcal{P}$ generates a *strongly continuous semigroup in the Hilbert space* $L^2(\mathbb{R}^N)$, which we denote by

$$(e^{-t\mathcal{L}})_{t\geqslant 0}$$

(that is, the family $(e^{-t\mathcal{L}})_{t\geqslant 0}$ satisfies the same properties (P1)–(P4) in the previous paragraph, with $-\mathcal{P}$ in place of $-\mathcal{B}_s$); this semigroup is called the *heat semigroup of* $-\mathcal{L}$.

Now, by arguing exactly as in the previous paragraph, we see that the operator $e^{-t\mathcal{L}}$ (for every fixed $t > 0$) is a integral operator with convolution-type kernel; more precisely, we have

$$e^{-t\mathcal{L}} f(x) = (\mathfrak{p}_t * f)(x) = \int_{\mathbb{R}^N} \mathfrak{p}_t(x-y) f(y)\, dy \quad (\text{for every } f \in L^2(\mathbb{R}^N)), \tag{2.7}$$

where, for every $z \in \mathbb{R}^N$ and $t > 0$, we have

$$\mathfrak{p}_t(z) = \frac{1}{(2\pi)^{N/2}} \mathfrak{F}^{-1}\Big(e^{-t(|\xi|^2+|\xi|^{2s})}\Big)(z) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{\imath\langle z,\xi\rangle - t(|\xi|^2+|\xi|^{2s})}\, d\xi \tag{2.8}$$

(note that $\mathfrak{F}(\mathcal{P}f) = (|\xi|^2 + |\xi|^{2s})\mathfrak{F}(f)$); on the other hand, by exploiting (2.5) (jointly with the explicit expression of $\mathfrak{F}^{-1}(e^{-t|\xi|^2})$ and the properties of the Fourier transform), we obtain

$$\begin{aligned}
\mathfrak{p}_t(z) &= \frac{1}{(2\pi)^{N/2}} \mathfrak{F}^{-1}\Big(e^{-t|\xi|^2} \cdot e^{-t|\xi|^{2s}}\Big)(z) \\
&= \frac{1}{(2\pi)^{N/2}} \mathfrak{F}^{-1}\Big((2\pi)^{N/2}\mathfrak{F}(\mathfrak{g}_t) \cdot (2\pi)^{N/2}\mathfrak{F}(\mathfrak{h}_t^{(s)})\Big)(z) \\
&= \mathfrak{F}^{-1}\Big((2\pi)^{N/2}\, \mathfrak{F}(\mathfrak{g}_t) \cdot \mathfrak{F}(\mathfrak{h}_t^{(s)})\Big)(z) \\
&= (\mathfrak{g}_t * \mathfrak{h}_t^{(s)})(z),
\end{aligned}$$

where $\mathfrak{g}_t(z)$ is the usual Gauss–Weierstrass heat kernel of Δ, that is,

$$\mathfrak{g}_t(z) = \frac{1}{(4\pi t)^{N/2}} e^{-|z|^2/(4t)}.$$

Summing up, we conclude that

$$\mathfrak{p}_t(z) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-|z-\zeta|^2/(4t)} \mathfrak{h}^{(s)}(\zeta)\, d\zeta \quad (z \in \mathbb{R}^N,\ t > 0). \tag{2.9}$$

This function $(t, z) \mapsto \mathfrak{p}_t(z)$ is referred to as the *heat kernel of* $-\mathcal{L}$, and it satisfies analogous properties to that of $\mathfrak{h}^{(s)}$; for a future reference, we collect these properties (which easily follow from the 'explicit' expression of $\mathfrak{p}$ in (2.8)–(2.9)) in the next theorem.

Theorem 2.4. *The heat kernel* $\mathfrak{p}$ *satisfies the following properties.*

(1) $\mathfrak{p} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ *and* $\mathfrak{p} > 0$.

(2) *For every* $x \in \mathbb{R}^N$ *and* $t > 0$, *we have*

$$\mathfrak{p}_t(x) = \mathfrak{p}_t(-x).$$

(3) *For every fixed* $x \in \mathbb{R}^N$ *and* $t > 0$, *we have*

$$\int_{\mathbb{R}^N} \mathfrak{p}_t(x-y)\, dy = 1.$$

(4) *For every fixed $x \in \mathbb{R}^N$ and $t, \tau > 0$, we have*

$$\int_{\mathbb{R}^N} \mathfrak{p}_t(x-y)\mathfrak{p}_\tau(y)\,dy = \mathfrak{p}_{t+\tau}(x).$$

Moreover, by combining (2.6) with the 'convolution-type' expression of $\mathfrak{p}_t$ in (2.9), we deduce the following upper estimate: there exists a constant $C > 0$ such that

$$0 < \mathfrak{p}_t(x) \leqslant Ct^{-\frac{n}{2s}} \quad \textit{for every } x \in \mathbb{R}^N,\ t > 0. \tag{2.10}$$

We finally point out that, starting from property (P4) of the heat semigroup $(e^{-t\mathcal{L}})_{t\geqslant 0}$, it is quite standard to prove that the *unique solution* of the 'abstract' L^2-Cauchy problem

$$\begin{cases} \partial_t u = -\mathcal{L}u + f & \text{in } \mathbb{R}^N \times (0, +\infty) \\ u(x,0) = u_0 & \text{for } x \in \mathbb{R}^N \end{cases}$$

(for any fixed $f, u_0 \in L^2(\mathbb{R}^N)$) is given by

$$u(x,t) = e^{-t\mathcal{L}}u_0(x) + \int_0^t (e^{-(t-\tau)\mathcal{L}}f)(x)\,d\tau;$$

thus, by (2.9) we can rewrite this unique solution as follows:

$$u(x,t) = \int_{\mathbb{R}^N} \mathfrak{p}_t(y)u_0(y)\,dy + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)f(y)\,dy\,d\tau. \tag{2.11}$$

Remark 2.5. It is worth mentioning that the 'convolution-type' formula (2.9) of $\mathfrak{p}$ can be easily proved by taking into account the *probabilistic interpretation of the operator $\mathcal{L}$*.

Indeed, since $\mathcal{L}$ is the sum of the two operators $-\Delta$ and $(-\Delta)^s$, it is the infinitesimal generator of a stochastic process, say $(X_t)_{t\geqslant 0}$, which is the sum of two *independent processes*, namely a Brownian motion $(W_t)_{t\geqslant 0}$ and a pure jump Lévy flight $(J_t)_{t\geqslant 0}$; thus, given any $t > 0$, we know that the law of the process X_t (which is the function $\mathfrak{p}_t$) is the convolution of the laws of W_t (the Gauss–Weierstrass heat kernel $\mathfrak{g}_t$) and of J_t (the fractional heat kernel $\mathfrak{h}_t^{(s)}$).

Remark 2.6. It is important to stress that the computations carried out in the previous paragraphs in order to obtain the 'explicit' expressions of $\mathfrak{h}_t^{(s)}$ and of $\mathfrak{p}_t$ in (2.5)–(2.9), respectively, are actually *formal computations*; however, *starting from the mentioned expressions (2.5)–(2.9)*, one can prove a posteriori that all the properties of $\mathfrak{h}^{(s)}$ and of $\mathfrak{p}$ hold.

3 | EXISTENCE AND NONEXISTENCE RESULTS

3.1 | Very weak and mild solutions to problem (1.1)

Taking into account all the facts recalled so far, we can now make precise the notion of *solution to the Cauchy problem (1.1)*. Actually, as is customary in the context of parabolic problems, we consider two different notions of solutions, that is, *very weak* and *mild*.

Definition 3.1. Let $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geqslant 0$, and let $1 \leqslant p < \infty$.

(1) (*Very weak solution*) We say that a function $u : \overline{S} \to \mathbb{R}_0^+$ is a *very weak solution* to problem (1.1) if the following properties hold:
(a)$_1$ $u \in L^p_{\mathrm{loc}}(\overline{S})$;
(b)$_1$ given any $T > 0$, we have $u \in L^\infty((0,T); L_s(\mathbb{R}^N))$;
(c)$_1$ given any $\varphi \in \mathcal{T}_0$, we have

$$\iint_S u(-\partial_t \varphi + \mathcal{L}\varphi)\, dx\, dt - \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\, dx = \iint_S u^p \varphi\, dx\, dt. \tag{3.1}$$

(2) (*Mild solution*) We say that a function $u : \overline{S} \to \mathbb{R}_0^+$ is a *mild solution* to problem (1.1) if the following properties hold:
(a)$_2$ $u \in C(\overline{S}) \cap L^\infty(S)$;
(b)$_2$ for every $(x,t) \in S$, we have the identity

$$u(x,t) = \int_{\mathbb{R}^N} \mathfrak{p}_t(x-y)u_0(y)\, dy + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)u^p(y,\tau)\, dy\, d\tau. \tag{3.2}$$

Remark 3.2. We list, for a future reference, some remarks concerning Definition 3.1.

(1) Taking into account Proposition 2.3, it is easy to check that identity (3.1) *is meaningful*, that is, for every fixed test function $\varphi \in \mathcal{T}_0$ we have
(i) $u(-\partial_t\varphi + \mathcal{L}\varphi)$, $u^p\varphi \in L^1(S)$;
(ii) $u_0\varphi(\cdot,0) \in L^1(\mathbb{R}^N)$
(provided that u satisfies properties (a)$_1$–(c)$_1$).
In fact, let $r, T > 0$ be such that $\varphi \equiv 0$ out of $B_r \times [0,T)$. First of all we observe that, since by property a)$_1$ one has $u \in L^p(B_r \times (0,T))$, we immediately get

$$\iint_S |u^p\varphi|\, dx\, dt \leqslant \|\varphi\|_{L^\infty(S)} \iint_{B_r\times(0,T)} u^p\, dx\, dt < +\infty.$$

On the other hand, recalling that $\varphi \in \mathcal{T}_0$, using Proposition 2.3 (and taking into account the explicit proof of this proposition given in [16, Theorem 9.4]) we derive that

$$|(-\Delta)^s(x \mapsto \varphi(x,t))| \leqslant \frac{c}{1+|x|^{N+2s}} \mathbf{1}_{[0,T)}(t) \quad \text{for every } (x,t) \in S,$$

for some constant $c > 0$ independent of t; as a consequence, since $-\partial_t\varphi - \Delta\varphi$ is (smooth and) supported in $B_r \times [0,T)$, and since $u \in L^\infty((0,T); L_s(\mathbb{R}^N))$, we obtain

$$\begin{aligned}
&\int_S |u(-\partial_t\varphi + \mathcal{L}\varphi|\, dx\, dt \\
&\leqslant \int_{B_r\times(0,T)} u|\partial_t\varphi + \Delta\varphi|\, dx\, dt + c\int_0^T \left(\int_{\mathbb{R}^N} \frac{u}{1+|x|^{N+2s}}\, dx\right) dt \\
&\leqslant c\left(\|u\|_{L^1(B_r\times(0,T))} + \int_0^T \|u(\cdot,t)\|_{1,s}\, dt\right) \\
&\leqslant c\left(\|u\|_{L^1(B_r\times(0,T))} + \|u\|_{\infty,(0,T),L_s(\mathbb{R}^N)}\right) < +\infty,
\end{aligned}$$

where we have used the fact that $u \in L^p(B_r \times (0,T)) \subset L^1(B_r \times (0,T))$, and $c > 0$ is a constant (possibly different from line to line) only depending on φ.

Finally, since $u_0 \in L^\infty(\mathbb{R}^N)$ and $\varphi(\cdot, 0) \in C_0^\infty(\mathbb{R}^N)$, we immediately infer that

$$u_0\varphi(\cdot,0) \in L^1(\mathbb{R}^N).$$

(2) Owing to the properties of $\mathfrak{p}$ in Theorem 2.4, it is easy to check that also identity (3.2) is *meaningful* (provided that u satisfies properties (a)$_2$–(b)$_2$). In fact, since by assumption we have $u_0 \in L^\infty(\mathbb{R}^N)$, for every $x \in \mathbb{R}^N$ we get

$$0 \leqslant \int_{\mathbb{R}^N} \mathfrak{p}_t(x-y)u_0(y)\,dy \leqslant \|u_0\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \mathfrak{p}_t(x-y)\,dy = \|u_0\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

Moreover, since by property a)$_2$ we also have $u \in L^\infty(S)$, for every $(x,t) \in S$ we get

$$\begin{aligned} 0 &\leqslant \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)u^p(y,\tau)\,dy\,d\tau \\ &\leqslant \|u\|^p_{L^\infty(S)} \int_0^t \left(\int_{\mathbb{R}^N} \mathfrak{p}_{t-\tau}(x-y)\,dy\right)d\tau = \|u\|^p_{L^\infty(S)} t < +\infty. \end{aligned} \tag{3.3}$$

We explicitly note that the definition of mild solution comes from the representation of the unique solution of the L^2-Cauchy problem for $\mathcal{L}$ discussed in the previous paragraph: indeed, our Cauchy problem (1.1) can be rewritten as

$$\begin{cases} \partial_t u = -\mathcal{L}u + f & \text{in } \mathbb{R}^N \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{for every } x \in \mathbb{R}^N; \end{cases}$$

where $f = u^p$; hence, by the 'representation formula' (2.11) we should have

$$u(x,t) = \int_{\mathbb{R}^N} \mathfrak{p}_t(y)u_0(y)\,dy + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)f(y)\,dy\,d\tau,$$

which is precisely formula (3.2) (with $f = u^p$).

(3) In the particular case when $u_0 \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, if $u \in C(\overline{S}) \cap L^\infty(S)$ is any mild solution of the Cauchy problem (1.1) it is easy to recognize that

$$u(x,0) = u_0(x) \quad \text{for every } x \in \mathbb{R}^N.$$

Indeed, since $u_0 \in L^2(\mathbb{R}^N)$, by exploiting property (P3) of the heat semigroup $(e^{-t\mathcal{L}})_{t\geqslant 0}$, together with the representation (2.7) and estimate (3.3), we get

$$\begin{aligned} &\lim_{n\to+\infty} u(x,1/n) \\ &\quad = \lim_{n\to+\infty}\left(\int_{\mathbb{R}^N} \mathfrak{p}_{1/n}(x-y)u_0(y)\,dy + \iint_{S_{1/n}} \mathfrak{p}_{1/n-\tau}(x-y)u^p(y,\tau)\,dy\,d\tau\right) \end{aligned}$$

$$= \lim_{n\to+\infty}\left(\int_{\mathbb{R}^N} \mathfrak{p}_{1/n}(x-y)u_0(y)\,dy\right)$$
$$= \lim_{n\to+\infty}(e^{-1/n\mathcal{L}}u_0)(x) = u_0(x) \qquad \text{for a.e. } x \in \mathbb{R}^N$$

(up to a sub-sequence, since $e^{-t\mathcal{L}}u_0 \to u_0$ as $t \to 0^+$ in $L^2(\mathbb{R}^N)$); thus, since $u \in C(\overline{S})$, we infer that $u(x,0) = u_0(x)$ for (a.e.) $x \in \mathbb{R}^N$. In particular, by modifying u_0 on a set of zero Lebesgue measure if needed, we conclude that

$$u_0 \in C(\mathbb{R}^N) \quad \text{and} \quad u(x,0) = u_0(x) \text{ for every } x \in \mathbb{R}^N.$$

(4) Owing to the properties of the heat kernel $\mathfrak{p}$ in Theorem 2.4, and adapting the approach in the proof of [2, Lemma 2.1], it is not difficult to recognize that *any mild solution of problem (1.1) is also a very weak solution*.

3.2 | The main result

Now we have properly introduced the two types of solutions for the Cauchy problem (1.1) we are interested in, we are finally ready to state the main result of this paper.

Theorem 3.3. *Let $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geqslant 0$, and let $1 < p < \infty$. We define*

$$\overline{p} = 1 + \frac{2s}{N}.$$

Then, the following facts hold.

(1) (Nonexistence) *If $1 < p \leqslant \overline{p}$,* there do not exist global in time very weak solutions *to the Cauchy problem (1.1) with $u_0 \not\equiv 0$.*
(2) (Global existence) *If $p > \overline{p}$, there exist $\delta_0, \tau_0 > 0$ such that the Cauchy problem (1.1) possesses at least one global in time* very weak solution, *provided that*

$$u_0(x) < \delta_0\,\mathfrak{p}_{\tau_0}(x), \quad \textit{for a.e. } x \in \mathbb{R}^N. \tag{3.4}$$

Remark 3.4. A careful inspection of the proof of Theorem 3.3-(1) will show that for $u_0 = 0$ there exists a unique very weak solution identically vanishing. We stress that uniqueness results when $u_0 \geqslant 0$ are not yet available, at least to the best of our knowledge.

Remark 3.5. As it will be clear from the proof of Theorem 3.3-(2), the solution we are able to construct in the case $p > \overline{p}$, when $u_0 \neq 0$, is actually a *mild solution* to the Cauchy problem (1.1).

4 | PROOF OF THEOREM 3.3

In this section we provide the full proof of Theorem 3.3. To ease the readability, we establish the two assertions (1) and (2) (*nonexistence* and *global existence*) separately.

Proof of Theorem 3.3-(1) (Nonexistence). Let $1 < p \leqslant \overline{p}$ be fixed, and suppose that there exists a very weak solution of the Cauchy problem (1.1) (in the sense of Definition 3.1, and for some initial condition $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geqslant 0$). We then aim at proving that

$$u \equiv 0 \text{ a.e. in } S. \tag{4.1}$$

Once we know that (4.1) holds, from (3.1) we infer that

$$\int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\,dx = \iint_S u(-\partial_t\varphi + \mathcal{L}\varphi)\,dx\,dt - \iint_S u^p\varphi\,dx\,dt = 0 \quad \forall\, \varphi \in \mathcal{T}_0,$$

for which we derive that $u_0 \equiv 0$ a.e. in $\mathbb{R}^N$. Hence, we turn to establish (4.1). To this end, it is convenient to distinguish the following two cases:

$$\text{(a) } 1 < p < \bar{p} \qquad \text{and} \qquad \text{(b) } p = \bar{p}.$$

Case (a). To begin with, we choose two functions $\zeta \in C_0^\infty(\mathbb{R}^N)$, $\psi \in C^\infty(\mathbb{R}_0^+)$ such that

(i) $\zeta \equiv 1$ on $B_{1/2}$ and $\zeta \equiv 0$ out of B_1;
(ii) $\psi \equiv 1$ on $[0, 1/2)$ and $\psi \equiv 0$ on $[1, +\infty)$;
(iii) $0 \leqslant \zeta, \psi \leqslant 1$.

Then, we arbitrarily fix $r > 1$, and we define

$$\xi_r(x) := \zeta^m\left(\frac{x}{r}\right), \qquad \phi_r(t) = \psi^m\left(\frac{t}{r^{2s}}\right)$$
$$\text{where } m := \tfrac{2p}{p-1}.$$

Since, obviously, we have $\varphi(x,t) = \xi_r(x)\phi_r(t) \in \mathcal{T}_0$, we are entitled to use this function φ as a test function in (3.1): recalling that (by assumption) $u_0 \geqslant 0$ a.e. in $\mathbb{R}^N$, this gives

$$\begin{aligned}\iint_S u^p\varphi\,dx\,dt &= \iint_S u(-\partial_t\varphi + \mathcal{L}\varphi)\,dx\,dt - \int_{\mathbb{R}^N} u_0(x)\xi_r(x)\,dx \\ &\leqslant \iint_S u(-\partial_t\varphi + \mathcal{L}\varphi)\,dx\,dt \\ &= \iint_S u(-\xi_r\partial_t\phi_r - \phi_r\Delta\xi_r + \phi_r(-\Delta)^s\xi_r)\,dx\,dt.\end{aligned} \tag{4.2}$$

We now turn to estimate the right-hand side of the above inequality.

To this aim we first observe that

$$\begin{aligned}&\text{(i) } \Delta\xi_r = mr^{-2}\left[\zeta^{m-1}\Delta\zeta + (m-1)\zeta^{m-2}|\nabla\zeta|^2\right](x/r);\\ &\text{(ii) } \partial_t\phi_r = mr^{-2s}\left[\psi^{m-1}\partial_t\psi\right](t/r^{2s}).\end{aligned} \tag{4.3}$$

Moreover, since the function $G(z) = z^m$ is *convex*, by [48, Lemma 3.2] we have

$$\begin{aligned}(-\Delta)^s\xi_r &= (-\Delta)^s(G\circ(x \mapsto \zeta(x/r))) \leqslant m\zeta^{m-1}\left(\frac{x}{r}\right)(-\Delta)^s(x \mapsto \zeta(x/r)) \\ &= \frac{m}{r^{2s}}\zeta^{m-1}(x/r)[(-\Delta)^s\zeta](x/r).\end{aligned} \tag{4.4}$$

Thus, by combining (4.3) and (4.4) (and since $r > 1$), we obtain

$$\begin{aligned} &-\xi_r\partial_t\phi_r - \phi_r\Delta\xi_r + \phi_r(-\Delta)^s\xi_r \leqslant |\xi_r\partial_t\phi_r + \phi_r\Delta\xi_r| + \phi_r(-\Delta)^s\xi_r \\ &\leqslant \mathbf{c}r^{-2s}\big(\zeta(x/r)\psi(t/r^{2s})\big)^{m-2} = \mathbf{c}r^{-2s}\varphi^{\frac{m-2}{m}} \\ &= \mathbf{c}r^{-2s}\varphi^{1/p}, \end{aligned} \tag{4.5}$$

where we have also used the fact that $(-\Delta)^s\zeta \in \mathcal{S}_s$ (as $\zeta \in C_0^\infty(\mathbb{R}^N)$, see Proposition 2.3).

With estimate (4.5) at hand, we can easily conclude the proof of (4.1): indeed, by combining the cited (4.5) with the above estimate (4.2), and by using Hölder's inequality, we get

$$\begin{aligned} \iint_S u^p\varphi\,dx\,dt &\leqslant \mathbf{c}r^{-2s}\iint_S u\varphi^{1/p}\,dx\,dt \\ &\text{(since } \varphi \text{ is supported in } B_r\times[0,r^{2s})) \\ &= \mathbf{c}r^{-2s}\int_0^{r^{2s}}\int_{B_r} u\varphi^{1/p}\,dx\,dt \\ &\leqslant \mathbf{c}r^{-2s+(2s+N)\frac{p-1}{p}}\left(\int_S u^p\varphi\,dx\,dt\right)^{1/p}; \end{aligned}$$

as a consequence, since $\varphi \equiv 1$ on $B_{r/2}\times[0,r^{2s}/2)$, we obtain

$$\int_0^{r^{2s}/2}\int_{B_{r/2}} u^p\,dx\,dt \leqslant \iint_S u^p\varphi\,dx\,dt \leqslant \mathbf{c}r^{N+2s-\frac{2sp}{p-1}}. \tag{4.6}$$

On the other hand, since we are assuming that $1 < p < \overline{p}$, we have

$$N + 2s - \frac{2sp}{p-1} < 0;$$

then, by letting $r \to +\infty$ in the above (4.6) and by using the Monotone Convergence Theorem (recall that $r > 1$ was arbitrarily fixed, and $u \geqslant 0$ a.e. in S), we derive that

$$\iint_S u^p\,dx\,dt = 0,$$

from which we conclude that $u \equiv 0$ a.e. in S, as desired.

Case (b). In this case, we use some ideas exploited in the proof of [22, Theorem 1].

First of all we observe that, if $p = \overline{p}$, we have

$$\delta := -2s + (2s+N)\frac{p-1}{p} = 0; \tag{4.7}$$

thus, by arguing as in *Case (a)*, by (4.6) and (4.7) we get

$$\int_0^{r^{2s}/2}\int_{B_{r/2}} u^p\,dx\,dt \leqslant \mathbf{c},$$

for some constant $\mathbf{c} > 0$ independent of r. In particular, by letting $r \to +\infty$ and by using the Monotone Convergence Theorem, we can infer that $u \in L^p(\mathbb{R}^N \times (0, +\infty))$.

We now define, for any $r > 1, \beta > 1$, the functions

$$\xi_{r,\beta}(x) := \zeta^m\left(\frac{x}{\beta r}\right), \qquad \phi_r(t) = \psi^m\left(\frac{t}{r^{2s}}\right),$$

where ζ, ψ and m are as in the previous case. Clearly, $\varphi(x,t) = \xi_{r,\beta}(x)\phi_r(t) \in \mathcal{T}_0$, so we can use this function φ as a test function in (3.1): since $u_0 \geqslant 0$, this gives

$$\begin{aligned}\iint_S u^p \varphi \, dx\, dt &= \iint_S u(-\partial_t \varphi + \mathcal{L}\varphi)\, dx\, dt - \int_{\mathbb{R}^N} u_0(x)\xi_{r,\beta}(x)\, dx \\ &\leqslant \iint_S u(-\partial_t \varphi + \mathcal{L}\varphi)\, dx\, dt \\ &= \iint_S u(-\xi_{r,\beta}\partial_t\phi_r - \phi_r \Delta \xi_{r,\beta} + \phi_r(-\Delta)^s \xi_{r,\beta})\, dx\, dt.\end{aligned} \tag{4.8}$$

Moreover, by arguing exactly as in *Case (a)*, we have the estimate

$$\begin{aligned}&\text{(i) } |\Delta \xi_{r,\beta}(x)| \leqslant \mathbf{c}(\beta r)^{-2}\xi_{r,\beta}^{1/p}(x) \quad \text{for every } x \in \mathbb{R}^N; \\ &\text{(ii) } (-\Delta)^s \xi_{r,\beta}(x) \leqslant \mathbf{c}(\beta r)^{-2s}\xi_{r,\beta}^{1/p}(x) \quad \text{for every } x \in \mathbb{R}^N; \\ &\text{(iii) } |\partial_t \phi_r(t)| \leqslant \mathbf{c} r^{-2s}\phi_r^{1/p}(t) \cdot \mathbf{1}_{\{r^{2s}/2<t<r^{2s}\}}(t) \quad \text{for every } t > 0;\end{aligned} \tag{4.9}$$

By combining (4.8) and (4.9), and by using Hölder's inequality, we then get

$$\begin{aligned}&\iint_S u^p \varphi\, dx\, dt \\ &\leqslant \mathbf{c} r^{-2s} \iint_S u \phi_r^{1/p} \xi_{r,\beta} \cdot \mathbf{1}_{\{r^{2s}/2<t<r^{2s}\}}(t)\, dx\, dt + \mathbf{c}(\beta r)^{-2s} \iint_S u \xi_{r,\beta}^{1/p}\phi_r(t)\, dx\, dt \\ &\leqslant \mathbf{c} r^{\delta} \beta^{\frac{N(p-1)}{p}} \left(\int_{\frac{r^{2s}}{2}}^{r^{2s}} \int_{B_{\beta r}} u^p\, dxdt\right)^{\frac{1}{p}} + \mathbf{c} r^{\delta} \beta^{-2s+\frac{N(p-1)}{p}} \left(\int_0^{r^{2s}} \int_{B_{\beta r}} u^p\, dxdt\right)^{\frac{1}{p}} \\ &= \mathbf{c} \beta^{\frac{N(p-1)}{p}} \left(\int_{\frac{r^{2s}}{2}}^{r^{2s}} \int_{B_{\beta r}} u^p\, dxdt\right)^{\frac{1}{p}} + \mathbf{c} \beta^{-2s+\frac{N(p-1)}{p}} \left(\int_0^{r^{2s}} \int_{B_{\beta r}} u^p\, dxdt\right)^{\frac{1}{p}},\end{aligned}$$

where we have used the fact that $\delta = 0$, see (4.7).

In particular, since $\varphi \equiv 1$ on $B_{\frac{r\beta}{2}} \times [0, r^{2s}/2)$, we obtain

$$\begin{aligned}\int_0^{r^{2s}/2} \int_{B_{\beta r/2}} u^p\, dx\, dt &\leqslant \iint_S u^p \varphi\, dx\, dt \\ &\leqslant \mathbf{c} \beta^{\frac{N(p-1)}{p}} \left(\int_{\frac{r^{2s}}{2}}^{r^{2s}} \int_{B_{\beta r}} u^p\, dxdt\right)^{\frac{1}{p}} + \mathbf{c} \beta^{-2s+\frac{N(p-1)}{p}} \left(\iint_S u^p dxdt\right)^{\frac{1}{p}}.\end{aligned} \tag{4.10}$$

With (4.10) at hand, we can finally complete the proof of (4.1) in this case. In fact, since we have already recognized that $u \in L^p(\mathbb{R}^N \times (0,+\infty))$, for any fixed $\beta \in (1,r)$ we have

$$\begin{aligned}
&\lim_{r\to+\infty} \int_{\frac{r^{2s}}{2}}^{r^{2s}} \int_{B_{\beta r}} u^p \, dxdt \\
&\quad = \lim_{r\to+\infty} \int_0^{r^{2s}} \int_{B_{\beta r}} u^p \, dxdt - \lim_{r\to+\infty} \int_0^{\frac{r^{2s}}{2}} \int_{B_{\beta r}} u^p \, dxdt \\
&\quad = \int_0^{+\infty} \int_{\mathbb{R}^N} u^p dxdt - \int_0^{+\infty} \int_{\mathbb{R}^N} u^p dxdt = 0\,.
\end{aligned} \tag{4.11}$$

On the other hand, since $p = \bar{p}$, we also have

$$-2s + \frac{N(p-1)}{p} = -\frac{2s(p-1)}{p} < 0\,. \tag{4.12}$$

By virtue of (4.11) and (4.12), letting $r \to +\infty$ and then $\beta \to +\infty$ in (4.10), we obtain

$$\iint_S u^p \, dx\, dt = 0,$$

from which we deduce that $u \equiv 0$ a.e. in S, as desired. ☐

Proof of Theorem 3.3-(2) (Global existence). We adapt to the present situation the line of arguments of the proof of [45, Theorem 1.1]. Let (3.4) be in force for some $\delta_0, \tau_0 > 0$ to be chosen later on, and let us introduce the following notation:

$$\tilde{u}_0(x,t) := \int_{\mathbb{R}^N} \mathfrak{p}_t(x-y) u_0(y)\, dy \tag{4.13}$$

and

$$\Phi u(x,y) := \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y) u^p(y,\tau)\, dy\, d\tau. \tag{4.14}$$

Thanks to (3.4), we have that

$$\tilde{u}_0(x,t) \leqslant \delta_0 \int_{\mathbb{R}^N} \mathfrak{p}_t(x-y)\mathfrak{p}_{\tau_0}(y)\, dy = \delta_0\, \mathfrak{p}_{t+\tau_0}(x),$$

where in the last step we used Theorem 2.4-(4).

Exploiting (4.13), we now define the recursive sequence of functions $(\tilde{u}_n)_{n\in\mathbb{N}}$ as

$$\tilde{u}_{n+1}(x,t) := \tilde{u}_0(x,t) + \Phi\tilde{u}_n(x,t). \tag{4.15}$$

By induction, we can prove that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is monotone increasing. Indeed,

$$\begin{aligned}
\tilde{u}_1(x,t) &= \tilde{u}_0(x,t) + \Phi\tilde{u}_0(x,t) \\
&= \tilde{u}_0(x,t) + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)\tilde{u}_0^p(y,\tau)\, dyd\tau \geqslant \tilde{u}_0(x,t),
\end{aligned}$$

and, assuming $\tilde{u}_n \geqslant \tilde{u}_{n-1}$, and hence $\tilde{u}_n^p \geqslant \tilde{u}_{n-1}^p$, we have

$$\tilde{u}_{n+1}(x,t) = \tilde{u}_0(x,t) + \Phi\tilde{u}_n(x,t) = \tilde{u}_0(x,t) + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)\tilde{u}_n^p(y,\tau)\,dyd\tau$$
$$\geqslant \tilde{u}_0(x,t) + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)\tilde{u}_{n-1}^p(y,\tau)\,dyd\tau = \tilde{u}_n(x,t).$$

In order to properly choose $\delta_0 > 0$, we further define the increasing (since $\delta_0 > 0$) sequence of real numbers $(\delta_n)_{n\in\mathbb{N}}$ as

$$\delta_{n+1} := \delta_0 + \delta_n^p,$$

If we choose $\delta_0 > 0$ small enough, the sequence $(\delta_n)_{n\in\mathbb{N}}$ is convergent, and therefore there exists $M \in \mathbb{R}^+$ such that

$$\delta_n \leqslant M \quad \text{for every } n \in \mathbb{N}. \tag{4.16}$$

Our next goal is to choose $\tau_0 > 0$ such that

$$\tilde{u}_n(x,t) \leqslant \delta_n\, \mathfrak{p}_{t+\tau_0}(x), \quad \text{for every } (x,t) \in S \text{ and for every } n \in \mathbb{N}. \tag{4.17}$$

Before proceeding by induction, recalling that $p > \overline{p} = 1 + \frac{2s}{N}$ and thanks to both (2.10) and Theorem 2.4(4), we note that

$$\begin{aligned}&\iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)\mathfrak{p}^p_{\tau+\tau_0}(y)\,dyd\tau\\ &\quad\leqslant C^{p-1}\,\mathfrak{p}_{t+\tau_0}(x)\int_0^{+\infty}(\tau+\tau_0)^{-N(p-1)/(2s)}d\tau < \mathfrak{p}_{t+\tau_0}(x),\end{aligned} \tag{4.18}$$

provided that $\tau_0 > 0$ is large enough, namely

$$\tau_0 > \left(C^{1-p}\left(\frac{N(p-1)}{2s} - 1\right)\right)^{2s/(2s-N(p-1))}.$$

Let us now go through the induction procedure. First,

$$\tilde{u}_1(x,t) = \tilde{u}_0(x,t) + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)u_0^p(y,\tau)\,dyd\tau$$
$$\leqslant \delta_0\,\mathfrak{p}_{t+\tau_0}(x) + \delta_0^p \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)\mathfrak{p}^p_{\tau+\tau_0}(y)\,dyd\tau$$
$$\leqslant \left(\delta_0 + \delta_0^p\right)\mathfrak{p}_{\tau+\tau_0}(x) = \delta_1\,\mathfrak{p}_{\tau+\tau_0}(x).$$

Now, assuming that (4.17) holds for a certain $n \in \mathbb{N}$, it follows that

$$\begin{aligned}\tilde{u}_{n+1}(x,t) &= \tilde{u}_0(x,t) + \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)\tilde{u}_n^p(y,\tau)\,dyd\tau \\ &\leqslant \delta_0\,\mathfrak{p}_{t+\tau_0}(x) + \delta_n^p \iint_{S_t} \mathfrak{p}_{t-\tau}(x-y)\mathfrak{p}_{t+\tau_0}^p(y)\,dyd\tau \\ &\leqslant \left(\delta_0 + \delta_n^p\right)\mathfrak{p}_{\tau+\tau_0}(x) = \delta_{n+1}\,\mathfrak{p}_{\tau+\tau_0}(x),\end{aligned}$$

where we exploited once again (4.18).

Combining (4.17) and (4.16), we find that

$$\tilde{u}_n(x,t) \leqslant M\,\mathfrak{p}_{t+\tau_0}(x), \quad \text{for every } (x,t) \in S \text{ and for every } n \in \mathbb{N}.$$

Let us now consider the function $u := \sup \tilde{u}_n$. By monotone convergence, u satisfies (3.2) and therefore u is the desired global mild solution to (1.1). In view of Remark 3.2(4), u is also a global in time very weak solution to (1.1). This closes the proof. □

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