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AUTOREGRESSIVE LATENT TRAJECTORY (ALT) MODELS: AN ASSESSMENT 25 YEARS LATER

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INTRODUCTION

In 1999, the Autoregressive Latent Trajectory (ALT) model was introduced at the annual meeting of the Psychometric Society (Bollen & Curran, 1999, 2004). At the time, growth curve models were attracting considerable attention as alternatives to traditional autoregressive models (*e.g.*, Rogosa, Brandt, & Zimowski, 1982). The ALT model was proposed to integrate key features of growth curve and autoregressive approaches, while also providing an empirical framework for specializing to either model alone or to their combination. Within this framework, a number of familiar models are special cases of the ALT model such as autoregressive and cross-lagged models, and linear and nonlinear latent growth curve models with or without covariates (see Bianconcini & Bollen, 2018).

Over the last 25 years, numerous applications of the ALT have emerged. It has proven invaluable for exploring longitudinal dynamics, particularly in terms of changes in multiple variables over time. This is evidenced by its application to a diverse range of phenomena, including developmental trajectories of anxiety and depression (Dennis, 2025; ter Avest et al., 2021; Connell et al., 2021; Lee & Vaillancourt, 2020; McLaughlin & King, 2015), substance abuse and its relationship with antisocial behaviors (Sullivan and Piquero, 2010; Hussong et al, 2004), the effect of parental actions on adolescent conduct issues (Fredriksen et al., 2019; Simons Morton et al., 2008), students' achievements and their determinants (Herrmann, 2025; Tang and Dai, 2021; Hughes et al., 2011), trends in disposition and happiness (Caprara et al., 2017; Garland et al., 2017), the analysis of physiological behaviors (Honeycutt and Keaton, 2015), and the connection between aggressive actions and peer victimization in preadolescence (Yao & Enright, 2022; Sullivan et al., 2016).

During this same period, the Classic ALT model was extended in several important directions. Bollen and Curran (2004, pp. 375–376) mentioned an expanded ALT model that incorporates measurement error and latent variables, demonstrating that latent difference score models (McArdle & Hamagami, 2001) and State–Trait models (Kenny & Zautra, 2001) arise as special cases. Building on this work, Bianconcini and Bollen (2018) introduced the Latent Variable–ALT (LV-ALT) model as a formal generalization. They showed that the LV-ALT framework subsumes a wide range of commonly used longitudinal models, including latent difference score and State-Trait models, autoregressive and cross-lagged models, fixed- and random-effects models with or without lagged dependent variables, growth curve models, and simplex models. Beyond cataloging special cases, they demonstrated how the LV-ALT facilitates model comparison in settings where theory and substantive knowledge do not uniquely determine a longitudinal specification, and they illustrated its application to latent variables and multiple repeated measures. Subsequent extensions showed that the ALT framework accommodates nonlinear trajectories, such as quadratic and freed-loading models (Bauldry & Bollen, 2018), as well as models with binary and ordinal observed variables (Bollen & Gutin, 2021).[1]

Hamaker (2005) showed that a latent curve model with autoregressive errors was *statistically* equivalent to a restrictive version of the ALT model, provided the autoregressive coefficients were equal across time and had absolute values less than one. More recently, Andersen (2022) proved that the Random Intercept-Cross Lagged Panel Model (RI-CLPM; Hamaker, Kuiper, & Grasman, 2015) and the Latent Curve Model with Structured Residuals (LCM-SR; Curran, Howard, Bainter, Lane, & McGinley, 2014) are also statistically equivalent to restrictive forms of ALT models, thereby permitting comparisons between such models and less restrictive ALT models.

Accompanying the widespread use of the ALT model and its generalizations has been confusion and uncertainty about it. For instance, there are contradictory claims about the number of data waves needed to identify the ALT model. Some have questioned how to interpret its random intercept and its other parameters. Several analysts assert that the ALT model "*overfits*" the data, while others say it is difficult to fit. Others have presented longitudinal models with autoregressions and cross-lagged relations of errors, rather than

[1] Though the list of special cases of the LV-ALT model is long, it does not include *all* longitudinal models. For instance, Browne & du Toit's (1991) nonlinear structured latent curve model or some versions of Dynamic Structural Equation Modeling (Asparohouv, Hamaker and Muthén, 2018) tend to have different structures than the LV-ALT model.

observed variables, as distinct from the ALT. Finally, some question whether analysts can interpret between- and within-effects in the ALT model.

These questions and contradictory assertions point to the need for clarification on the use and interpretation of ALT models. Furthermore, some claims have accompanied the presentation of additional hybrid models proposed after the ALT, so it would also be helpful to compare the ALT models with the alternatives that followed.

With these issues in mind, our paper has several purposes. First, it will review the specification and assumptions of the Classic ALT and distinguish it from the LV-ALT. The incorrect descriptions of ALT models even in top peer-reviewed quantitative psychology journals justifies reviewing the assumptions and properties of these models as essential background. Second, the paper will provide the properties and interpretations of the parameters from these ALT models. Included in this is new material on within- and between-case components in ALT models. Third, it will present claims about the properties of the ALT model and assess their accuracy. Fourth, it will compare the ALT model to alternative longitudinal models. Fifth, it will illustrate the material with empirical examples. Finally, we will present the *lavaan* code for the empirical examples in the supplementary material. Overall, we aim to expand the current understanding of the merits and limitations of the ALT model and to correct misinterpretations.

AUTOREGRESSIVE LATENT TRAJECTORY (ALT) MODEL

As we stated in the Introduction, the ALT model comes in many forms, ranging from a single repeated measure to a series of multiple latent variables over time. In practice, one or two repeated measures are the most common, with or without covariates. In this section, we focus on these simpler situations, while later we will discuss more general settings. First, we present the Classic ALT model in the Predetermined Wave 1 (PW1) and Endogenous Wave 1 (EW1) forms. We also incorporate covariates into this model, resulting in the Conditional ALT model. Second, we provide comments on extensions to two or more repeated measures, allowing for measurement errors in these models. Finally, we turn to the Latent Variable ALT (LV-ALT) model in both its Predetermined and Endogenous Wave 1 forms, as well as these models with covariates. Many researchers assume that there is only one form of the ALT model and these can mislead descriptions of the properties of the ALT model. In this section, we show the variety of forms of the ALT models. Furthermore, we present interpretations of the ALT models that are not always known by researchers.

CLASSIC ALT MODEL

The Classic ALT model with a single repeated measure is the most common application. Suppose that Y_{it} is a repeated continuous measure for the ith case at the tth time, where $i = 1, 2, 3, \ldots, N,$ with N cases in the sample, and $t = 1, 2, 3, \ldots, T,$ with T waves of data[2]. The Classic ALT model is

$$Y_{it} = \alpha_i + \lambda_t \beta_i + \rho_{t,t-1} Y_{i,t-1} + \varepsilon_{it}, \tag{1}$$

for $t = 2, 3, \ldots, T$. α_i is the random intercept, β_i is the random slope, λ_t is the *time trend* variable, equal to $(t-1)$ in linear models, whereas some λ_ts are freely estimated for some nonlinear trajectories. The term $\rho_{t,t-1}$ represents the autoregressive coefficient of the expected impact of $Y_{i,t-1}$ on $Y_{i,t}$, while ε_{it} is the equation error term. We assume that the error variable has a mean of zero and is uncorrelated with the Right-Hand Side (RHS) variables [*i.e.*, $E(\varepsilon_{it}) = 0$, $C(\varepsilon_{it}, Y_{i,t-1}) = 0$, $C(\varepsilon_{it}, \alpha_i) = 0$, and $C(\varepsilon_{it}, \beta_i) = 0$, where $E(.)$ is the expected value and $C(.)$ is the covariance of the quantities in parentheses]. Furthermore, we assume that the error variables for different cases are uncorrelated [$C(\varepsilon_{it}, \varepsilon_{jt}) = 0$ for all t and $i \neq j$], that each case at each time point has the same error variance [$C(\varepsilon_{it}, \varepsilon_{it}) = V(\varepsilon_t)$ where $V(.)$ is the variance], and that there are no autocorrelated errors [i.e., $C(\varepsilon_{it}, \varepsilon_{jt+k}) = 0$ for all i, j and $k \neq 0$][3].

The random intercept and slope components are expressed as follows

$$\alpha_i = \mu_\alpha + \varepsilon_{\alpha i}, \tag{2}$$

$$\beta_i = \mu_\beta + \varepsilon_{\beta i}, \tag{3}$$

where μ_α and μ_β are the mean intercept and slope across all cases, and the $\varepsilon_{\alpha i}$ and $\varepsilon_{\beta i}$ are disturbances with means of zero, uncorrelated with ε_{it}, but can correlate with each other.

Interpretations of the ALT components

[2] For simplicity, we assume a balanced longitudinal dataset in this study, meaning each individual is measured at the same T time points. However, this assumption can be relaxed. If some individuals have missing observations at certain waves, the ALT model can still be estimated under a Missing At Random (MAR) assumption using Full Information Maximum Likelihood (also called Casewise ML) or multiple imputation, which effectively accommodates unbalanced panels due to dropouts or intermittent missingness. Alternatively, continuous-time extensions of the ALT (also known as continuous-time ALT or CALT) can handle measurement occasions that are irregularly spaced. We discuss these continuous-time specifications later in the paper. Importantly, the autoregressive coefficients in the ALT model are indexed by the time periods to which they refer. Thus, when measurement intervals are unequal, imposing equality constraints on these coefficients is generally not appropriate in a discrete-time specification.

[3] Though later, we discuss models with autoregressive and cross-lagged errors and their equivalences to restrictive versions of the ALT model.

The RHS of equation (1) contains both the growth curve and the autoregressive parts of the ALT. The ($\alpha_i + \lambda_t \beta_i$) component contains a random intercept and random slope of the growth curve, though, as detailed below, the random intercept (α_i) *cannot* be interpreted as the initial value of the growth curve due to the presence of the lagged term $\rho_{t,t-1} Y_{i,t-1}$. To better understand this, it is helpful to contrast its interpretation with that of a standard growth curve model. Spcification of the latter is,

$$Y_{it} = \alpha_i + \lambda_t \beta_i + \varepsilon_{it}, \tag{4}$$

for $t=1,2,.., T$, with $\lambda_t = 0, 1, 2, \ldots, (T-1)$, being a linear trend variable. The α_i term is the initial value of the latent trajectory for the ith case, and the β_i is the rate of change or slope for a one-unit difference in time for the ith case.

Now consider the ALT model given in eq. (1). Because of the presence of $\rho_{t,t-1} Y_{i,t-1}$ term, we *cannot* interpret α_i as the initial value. However, rewrite the ALT model as follows

$$Y^{\#}_{it} = \alpha_i + \lambda_t \beta i + \varepsilon_{it}, \tag{5}$$

with $t=2,\ldots,T$. Equation (5) looks like the usual latent growth curve model, and α_i and β_i have a classic growth curve interpretation, except that the repeated measure is $Y^{\#}_{it}$ rather than Y_{it}. That is, α_i is the intercept of the latent trajectory of $Y^{\#}_{it,}$ and β_i is its rate of change. But what is $Y^{\#}_{it}$? For the ALT model, $Y^{\#}_{it}$ is

$$Y^{\#}_{it} = Y_{it} - \rho_{t,t-1} Y_{i,t-1}. \tag{6}$$

The $Y^{\#}_{it}$ results from subtracting $\rho_{t,t-1} Y_{i,t-1}$ from both sides of $Y_{it} = \alpha_i + \lambda t \beta i + \rho_{t,t-1} Y_{i,t-1} + \varepsilon_{it}$. The interpretation is that after removing the lagged effect, a growth curve model remains that characterizes the *transformed* repeated measures (Bianconcini & Bollen, 2018, pages 805-806). Though the RHS in eq. (5) indicates that a random intercept and random slope shape each case's trajectory even after controlling for the lagged effect of the repeated measures. This lagged effect of $\rho_{t,t-1} Y_{i,t-1}$ allows the prior value of the dependent variable to influence its current value (Y_{it}), taking into account the random intercept and random slope ($\alpha_i + \beta_i \lambda_t$). Indeed, an additional perspective results from creating Y^{+}_{it} by subtracting the growth curve part of the model, resulting in

$$Y^{+}_{it} = \rho_{t,t-1} Y_{i,t-1} + \varepsilon_{it}, \tag{7}$$

where $Y^{+}_{it} = Y_{it} - (\alpha_i + \beta_i \lambda_t)$. This transformation shows that even after removing the growth curve effects on the repeated measures, the lagged repeated measure still has an influence of

$\rho_{t,t-1}$. Specifically, a one-unit difference in $Y_{i,t-1}$ is expected to have a $\rho_{t,t-1}$ effect on the repeated measures after removing the growth curve effect ($\alpha_i + \beta_i \lambda_t$).

Including both the growth curve and lagged effects on the right-hand side does not guarantee their statistical or substantive significance. If the process is purely a growth curve, we would expect $\rho_{t,t-1}$ to fall within sampling fluctuations of zero. Alternatively, if the data do not support the latent growth component, that is, if the means and variances of the random intercept (α_i) and random slope (β_i) are negligible, then an autoregressive process should suffice, with or without time-specific intercepts to account for potential systematic mean shifts. However, if the longitudinal data reflect both a growth curve and an autoregressive process, then terms from both components should be included.

Note that in the Classic ALT model, such time-specific intercepts are omitted to prevent identification issues with the mean growth structure within the model. But in the latent variable ALT model that we discuss later, time-specific intercepts are part of the model.

Between- and within-case components in ALT models. When analyzing longitudinal data, distinguishing within- and between-case components is common (*e.g.*, Raudenbush & Bryk, 2002). Some studies question the applicability of this distinction in ALT models (*e.g.*, Curran & Bauer, 2011; Usami, 2021; Andersen, 2022). In this subsection, we introduce how to separate between- and within-case components in the ALT model. To begin, we need to define our use of between- and within-case components, and how some definitions create problems. For instance, sometimes between-case components are described as *stable* features among cases (*e.g.*, Schuurman, 2023). However, the term "*stable*" suggests unchanging values of the components, while some differences between cases are differences in trends, and trends do not imply stability. Rather, they connote nonstationarity and change.

A common approach to separating between- and within-case components uses the case-specific over time mean of a variable as the between-component with the case-specific deviation from its mean as the within-component. For Y_{it}, the decomposition would be the identity of

$$Y_{it} = \bar{Y}_i + (Y_{it} - \bar{Y}_i),$$

where $\bar{Y}_i = \frac{1}{T}\sum_{t=1}^{T} Y_{it}$ is the case-specific mean (between component), and $(Y_{it} - \bar{Y}_i)$ is the deviation from that mean at time t (within component).

One issue with this expression is that it ignores the sampling error in $\bar{Y}_i$. In this definition, the population mean μ_{Y_i} should be the between-component with $\bar{Y}_i$ as a sample estimator of it (Curran et al., 2012, pp. 241-244). To reflect this change, we rewrite the identity to,

$$Y_{it} = \mu_{Y_i} + (Y_{it} - \mu_{Y_i}),$$

where μ_{Y_i} is the between-component and $(Y_{it} - \mu_{Y_i})$ is the within-component. Indeed, this maps nicely onto an intercept-only growth curve model of

$$Y_{it} = \alpha_i + \varepsilon_{it},$$

where α_i is the random intercept of case *i* and ε_{it} is the random error for case *i* at time *t*. By equating the RHS of both equations, we see that $\alpha_i = \mu_{Y_i}$ is a *between*-component and $\varepsilon_{it} = (Y_{it} - \mu_{Y_i})$ is the *within* one.

Although this definition creates a clean separation of between- and within-case components, it *assumes* an intercept-only growth curve model. Suppose that a latent growth curve model (LGCM) characterizes Y_{it} so that,

$$Y_{it} = \alpha_i + \lambda_t \beta_i + \varepsilon_{it}\,. \tag{8}$$

Now $\alpha_i = \mu_{Y_i}$ holds under special conditions, such as balanced data with centering time on the middle time point (Curran & Bauer, 2011, p. 15). But, under more general conditions, this equivalency no longer holds nor equates to the between-component because we have omitted the $\lambda_t \beta_i$ term. However, we can still decompose the between- and within-components in the LGCM in eq. (8). Clearly, α_i and β_i differ across cases and in this sense are variables that represent *between* cases' components on Y_{it}. The error $\varepsilon_{it} = Y_{it} - (\alpha_i + \lambda_t \beta_i)$ is the deviation of case *i* at time *t* from the growth curve and hence is a *within*-case component. Thus, the LGCM decomposes into between- and within-effects on Y_{it}.

What happens when we move to the Classic ALT model? The ALT model as given in eq. (1) is,

$$Y_{it} = \alpha_i + \lambda_t \beta_i + \rho_{t,t-1} Y_{i,t-1} + \varepsilon_{it}.$$

The α_i and β_i remain the same over time, but differ over cases and are the between-case components in Y_{it}. The $Y_{i,t-1}$ is a Time-Varying Covariate (TVC) that the ALT model adds to the LGCM. When TVCs are added to a model, it is common to suggest that if the TVC enters in raw form (neither centered nor detrended), its coefficient reflects a mixture of between- and within-effects, unless the between-effects are taken into account. However, a characteristic of the ALT model is that the TVC $Y_{i,t-1}$ is an endogenous observed variable that correlates with α_i and β_i. This cross-level covariance structure ensures that the coefficient $\rho_{t,t-1}$ is an accurate estimate of the within-effect of the lagged variable on the current Y_{it}.[4] Hence, the term $(\alpha_i + \lambda_t \beta_i)$ accounts for or controls the between-component in Y_{it}. This means that the autoregressive coefficient $\rho_{t,t-1}$ captures within-effects of the lagged variable on the current Y_{it}, while controlling for the between-component in the same variable. We could more formally represent this as $\rho_{t,t-1} Y_{i,t-1} | (\alpha_i + \lambda_t \beta_i)$ providing within-case component where "|" signifies "given" and denotes that the between components in $Y_{i,t-1}$ are controlled. The remaining within-effects component is ε_{it}.

In brief, in the ALT model eq. (1) the between-case components are reflected in $(\alpha_i + \lambda_t \beta_i) | Y_{i,t-1}$, while the within-case components are captured by $\rho_{t,t-1} Y_{i,t-1} | (\alpha_i + \lambda_t \beta_i)$ and ε_{it}. The straightforward division into the between- and within-components runs counter to claims of the difficulties of this decomposition in ALT models, an incorrect claim to which we return later. Like all such decompositions, this separation into between- and within-case components assumes that the model specification is correct. If there are omitted time-invariant or time-varying variables, the random slope is excluded, or if analysts use the wrong functional form, then this division of between- and within-case components is ambiguous, a point to which we return. Later, we will discuss the between- and within-effects when additional covariates are added to the ALT model.

The initial condition problem

Equation (1) is the Classic ALT model for wave two and later. At wave 1, because of the inclusion of lagged variables in the latent curve model, an initial condition problem arises. The variable at the start of the observation period, Y_{i1}, should be influenced by the random intercept and slope, as well as the *unavailable presample* response, Y_{i0}. To address this issue, Bollen and Curran (1999; 2004) proposed two variations of the ALT model, which, along with equation (1), differ in their model specification for Y_{i1}: the *Predetermined Wave 1* (PW1) and the *Endogenous Wave 1* (EW1).

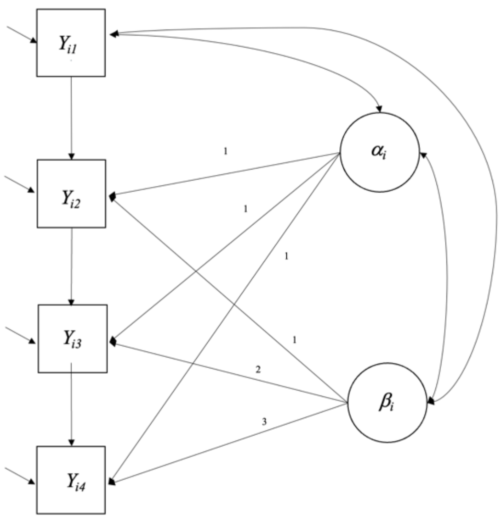
Predetermined Wave 1 (PW1). In the *Predetermined Wave 1* (*PW1*) model, the first wave equation (Y_{i1}) is

[4] In the context of conditional latent growth models, Hori and Miyazaki (2025) formally demonstrate that correlating a time-varying covariate with the latent growth components (*e.g.*, intercept and slope factors) isolates the within-person effect of that covariate. Although their analyses focus on exogenous time-varying predictors rather than autoregressive lagged dependent variables, the same principle applies: the cross-level covariance approach effectively disentangles within-person from between-person effects. Accordingly, their findings support interpreting $\rho_{t,t-1}$ in the ALT model as a consistent estimator of the within-person effect of $Y_{i,t-1}$ on Y_{it}.

$$Y_{i1} = \mu_1 + \varepsilon_{i1}, \tag{9}$$

where μ_1 is the mean of the first repeated measure $Y_{i1,}$ and ε_{i1} is the deviation from the mean of the ith case in the first wave Y_{i1}. The first wave is taken as given, meaning it is not explained within the model and correlates with the random intercept and random slope. This is why it is called the "*Predetermined Wave 1*" (PW1) version of the Classic ALT model.

Figure 1. *Path Diagram of Predetermined Wave 1 Model* (*PW1*)



[Figure 1 About Here]

Figure 1 is a path diagram of the PW1 form of the ALT model for four waves of data. The random intercept (α_i) and random slope (β_i) are latent variables (Bollen, 2002) in that we do not directly observe them, but we can trace their influence through their effects on the repeated measures. As is typical in Structural Equation Models (SEMs), they are enclosed in ellipses or circles to signify their latent status. The repeated measures are shown in boxes, which represent observed variables. The single-headed arrows symbolize the direct effects of the variable at the base of the arrow on the variable at the head of the arrow. Thus, a series of single-headed arrows originates from the random intercept and random slope and points toward the repeated measures. The error variables (ε_{it}) contain all other influences on the repeated measures that are not explicitly part of their equations. Although not drawn in Figure 1, their presence is indicated by the single-headed arrows pointing to each observed endogenous variable Y_{it}. The first wave Y_{i1} variable is treated as given and is correlated with the random intercept and random slope, as shown by the curved two-headed arrows connecting them.[5]

Identification of the PW1 form of the ALT model. Model identification is necessary for the successful estimation of the Classic ALT model. Here we assume a linear growth model with $\lambda_t = (t-1),\ t = 1, 2, \ldots, T$. Identification refers to whether it is possible to find unique values of all parameters in the model. The parameters are constants and include the means and variances of the random intercept [$\mu_\alpha, V(\alpha_i)$], random slope [$\mu_\beta, V(\beta_i)$], and Y_{i1} [$\mu_1, V(\varepsilon_{i1})$], the covariances among α_i, β_i, and Y_{i1} [$C(\alpha_i, \beta_i), C(\alpha_i, Y_{i1}), C(\beta_i, Y_{i1})$], the autoregressive coefficients ($\rho_{t,t-1}$), and the error variances [$V(\varepsilon_{it}),\ t=1, \ldots, T$]. These parameters are identified if it is possible to find unique solutions for each that are functions of one or more means, variances, or covariances of the observed repeated measures.[6]

The number of time points of data in the PW1 model helps to determine whether it is identified. Bollen and Curran (2004, pages 362-363) prove that the PW1 form of the ALT model is identified with four waves of data *if* the autoregressive coefficient is the same over time ($\rho_{t,t-1} = \rho$ and $|\rho| < 1$) and a linear growth curve applies. With five or more waves of data, the autoregressive coefficients could differ over time, and whether they are equal is testable.

Endogenous Wave 1 (EW1). The *Endogenous Wave 1* (EW1) is a second form of the Classic ALT model. Under this specification, the ALT equation for the first time period is given by

$$Y_{i1} = \Lambda_{1\alpha}\, \alpha_i + \Lambda_{1\beta}\, \beta_i + \varepsilon_{i1}, \tag{10}$$

where $\Lambda_{1\alpha}$ is a coefficient that gives the effect of α_i on Y_{i1}, and $\Lambda_{1\beta}$ is a coefficient that indicates the effect of β_i on Y_{i1}. In general, we can interpret $\Lambda_{1\alpha}$ as the total effect (reduced form) coefficient of α_i's impact on Y_{i1} and the earlier waves of Y_i, and similarly, $\Lambda_{1\beta}$ as the total effect (reduced form) coefficient of β_i's effects on Y_{i1} and previous values.

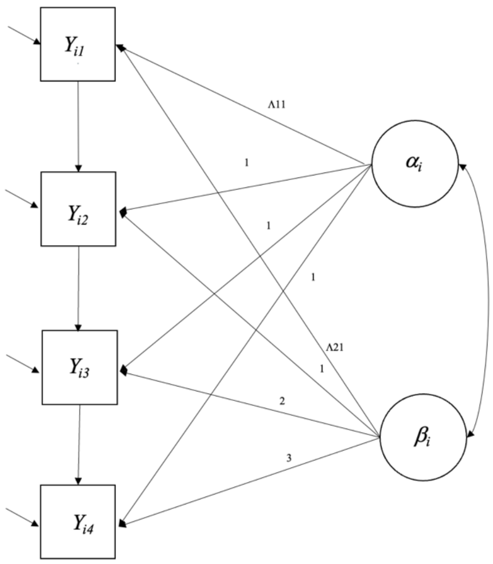
The two new coefficients for α_i and β_i in wave one differ for the *Endogenous Wave 1* (EW1) equation compared to the other waves in equation (1). Another difference is that eq. (10) excludes the $\rho_{t,t-1}Y_{i,t-1}$ term that is present in eq. (1). A practical reason for this exclusion is that the lagged value of $Y_{i,t-1}$ is not available before the first wave of data.

[5] We recently became aware of a model from the statistical literature called the Autoregressive Linear Mixed Effects model from Funatogawa et al. (2007). Their model was proposed several years after the ALT model and comes from the perspective of mixed effects models. Their model is a restrictive form of the classic ALT model with a stationary, equal autoregressive coefficient with a linear trend in the PW1 form of this section.

[6] Theoretically, it is possible to use higher-order moments than the means, variances, and covariances of the observed variables to help identify model parameters. However, in practice this is rarely done.

Equation (10) for wave 1 contrasts with equation (9) for wave 1 in PW1 in several ways. As their names suggest, Y_{i1} is dependent (endogenous) on the random intercept (α_i) and random slope (β_i) in the first wave, similar to wave 2 and later waves of Y_{it} in equation (1). This explains the name "*Endogenous Wave 1*" for this form of the ALT model, in which the first wave of the repeated measure is explained within the model. In contrast, in the PW1 form of the ALT model, Y_{i1} is treated as given and not explained by α_i and β_i. Furthermore, in the PW1 form, Y_{i1}, α_i, and β_i correlate with one another, while in the EW1 model, only α_i and β_i correlate as exogenous variables, and their association with Y_{i1} results from the influence of α_i and β_i on Y_{i1}, not by correlating these three variables.

Figure 2. *Path Diagram of Endogenous Wave 1 (EW1) Model*



[Figure 2 About Here]

Figure 2 is the path diagram of the EW1 form of the ALT model. The symbols and notation in the path diagram were defined in Figure 1. In many ways, Figure 2 is like Figure 1 for the PW1 form. The primary difference is that the first wave of Y_{i1} is treated as endogenous, like the other waves in the repeated measures. Labeling the paths of the random intercepts and random slopes to $\Lambda_{1\alpha}$ and $\Lambda_{1\beta}$ contrasts with the fixed constants for the paths to the other repeated measures.

A special and commonly applied formulation of eq. (10) arises under the assumption that the autoregressive coefficients are time-invariant, $\rho_{t,t-1}=\rho$, and in absolute value less than one ($|\rho|<1$). Bollen and Curran (2004, pages 363-365) prove that this is equivalent to assuming that equation (1) holds not only for the observational period of $t = 2, 3, \ldots, T$, but also extends to the past, such that the first wave (Y_{i1}) results

$$Y_{i1} = (1\text{-}\rho)^{-1}\, \alpha_i \,\text{-}\rho(1\text{-}\rho)^{-2}\, \beta_i + \varepsilon_{i1}, \tag{11}$$

that is, $\Lambda_{1\alpha}= (1\text{-}\rho)^{-1}$ and $\Lambda_{1\beta}=\text{-}\rho(1\text{-}\rho)^{-2}$.

Another special case of the EW1 is the so-called *History Independent* (HI1) specification of the classic ALT process for the first wave (Ou et al., 2017). It consists of setting $\Lambda_{1\alpha}$ to 1 and $\Lambda_{1\beta}$ to 0, such that

$$Y_{i1} = \alpha_i + \varepsilon_{i1}, \tag{12}$$

where the intercept, α_i, has the same conceptual meaning as the intercept term in a latent growth model. No explicit assumptions of stationarity and time invariance of the ALT process at the subsequent waves are required.

Following Ou et al. (2017), it can be demonstrated that the EW1 ALT model, along with its special cases, is nested within the predetermined ALT model. The constraints required to specify such nested relationships are detailed in the supplementary material.

Identification of the Endogenous Wave 1 model. As we stated earlier, identification concerns the possibility of finding unique values for each model parameter. We addressed that question for the PW1 form in a previous section. Here, we examine the EW1 form while maintaining the assumption of a linear growth model [$\lambda_t =(t\text{-}1),\ t= 1,\ 2,\ ...,\ T$]. The unknown parameters are the means and variances of the random intercept [$\mu_\alpha,\ V(\alpha_i)$], random slope [$\mu_\beta,\ V(\beta_i)$], and their covariance [$C(\alpha_i,\ \beta_i)$]. Additional parameters include the factor loadings of Y_{i1} [$\Lambda_{1\alpha}$ and $\Lambda_{1\beta}$], the autoregressive coefficients ($\rho_{t,t\text{-}1}$), and the error variances [$V(\varepsilon_{it}),\ t=1,\ ...,\ T$]. In the case of four waves of data, assuming a linear growth applies, the EW1 ALT model has 14 free parameters, which exactly equals the 14 observed means, variances, and covariances. The model is just identified in that all model parameters have a complex closed-form expression based on the observed moments (see supplementary material). If the autoregressive coefficients are constrained to be equal over time ($\rho_{t,t\text{-}1} = \rho$), the model becomes overidentified, allowing us to formally test whether the data support this assumption.

Empirical example

We present an example using empirical data from previous studies (Usami et al., 2019; Usami, 2021) that compared the performance of the ALT with that of alternative models. We choose this empirical example because it was used in critiques of the ALT model. By using it we show that even an alleged problematic case does well when researchers properly apply the ALT model. These previous studies examined the reciprocal relationship between adolescents' perceived exposure to smoking in movies (*X*) and their smoking intensity (*Y*), using data from the Minnesota Adolescent Community Cohort (MACC) Study, 2000-2013. The MACC study is a prospective cohort study aimed at better understanding the transition from non-smoking to smoking during adolescence and evaluating the influence of state and local tobacco prevention and control programs on youth in Minnesota (Choi et al., 2012).

Consistent with the studies mentioned earlier, we analyzed a sample of 4670 adolescents who were surveyed every six months from ages 15 to 20. When participants completed two surveys in a year, only the responses from the first survey were used to create the dataset across six waves. More detailed information about the study design and the MACC population can be found in Choi et al. (2012) and on the website of the Interuniversity Consortium for Political and Social Research (https://www.icpsr.umich.edu/icpsrweb/ICPSR/studies/36282).

At each wave, participants reported their smoking intensity, which indicates how frequently they smoked. To keep our analysis comparable with prior research (Usami et al., 2019), we treated this four-level ordinal measure as continuous. The data included two sources of missing data: missing by design and attrition. Following Usami et al. (2019), we used all available data to estimate the parameters of the ALT models using the full-information maximum likelihood (Casewise ML) estimation method. All analyses were conducted using the *lavaan* package (Rosseel, 2012) in R (R Core Team, 2016), and the source code is available in the supplementary material.

We fitted six variants of a univariate classic ALT model to the smoking intensity trajectories. These models differ in how the initial observation (wave 1) is treated - PW1, EW1, and HI1 – and whether the autoregressive effect $\rho_{t,t-1}$ is equal across time or allowed to vary. Each model includes a latent intercept α_i and a linear slope β_i, which capture differences between individuals in baseline smoking and changes over time, as well as autoregressive effects $\rho_{t,t-1}$ that reflect the within-person carryover of smoking from one wave to the next. All models converged without warnings, and the chi-square overidentification test and fit indices are reported in Table 1.

Table 1. *Fit indices (N = 4670)*

	PW1ALT with time-varying $\rho_{t,t-1}$	*PW1 ALT with $\rho_{t,t-1} = \rho$*	*EW1 ALT with time-varying $\rho_{t,t-1}$*	*EW1 ALT with $\rho_{t,t-1} = \rho$*	*HI1 ALT with time-varying $\rho_{t,t-1}$*	*HI1 ALT with $\rho_{t,t-1} = \rho$*
χ^2	4.977	16.998	11.435	28.593	33.283	56.175
df	8	12	9	15	11	15
p-value	0.760	0.150	0.247	0.018	0.000	0.000
CFI	1.000	0.999	1.000	0.998	0.997	0.994
TLI	1.001	0.999	0.999	0.998	0.995	0.994
RMSEA	0.000	0.009	0.008	0.014	0.021	0.024
BIC	-62.61	-84.39	-64.61	-98.14	-26.37	-70.56

The PW1 and EW1 ALT models with time-varying $\rho_{t,t-1}$ have exceptional fit to the data, while the HI1 variants in comparison showed a decline in fit. For the PW1 and EW1 ALT models, the χ^2 goodness-of-fit tests were nonsignificant, and the approximate fit indices were excellent (*e.g.*, PW1: $\chi^2(8) = 4.98$, $p = 0.760$; $CFI = 1.000$; $RMSEA = 0.000$; $BIC = -62.61$). In contrast, the HI1 model exhibited a significant chi-square ($\chi^2(11) = 33.28$, $p = 0.000$) and lower fit indices ($CFI/TLI = 0.997$, $RMSEA = 0.021$; $BIC = -26.37$.

Constraining the autoregressive coefficients to be equal worsens the model fit on all but the Bayesian Information Criterion (BIC) across all models. Likelihood ratio chi-square difference tests comparing each constrained ρ model to its freely estimated version were statistically significant (*e.g.,* PW1 with ρ vs. PW1 with $\rho_{t,t-1}$: $\Delta\chi^2(4) = 12.09$, $p = 0.02$; EW1 with ρ vs. EW1 with $\rho_{t,t-1}$: $\Delta\chi^2(6) = 17.16$, $p = 0.01$; HI1 with ρ vs. HI1 with $\rho_{t,t-1}$: $\Delta\chi^2(4) = 22.89$, $p = 0.00$), suggesting variation of the autoregression parameter over time. However, the BIC, which accounts for model complexity, favored the simpler, time-invariant ρ models. Nevertheless, all other fit indices and the chi-square difference tests indicate improved fit with unequal autoregressive coefficients. Therefore, we focus on the results from the time-varying $\rho_{t,t-1}$ models, recognizing that the BIC values would support the equality constraints.

Table 2. *Parameter estimates for ALT models with time-varying $\rho_{t,t-1}$*

Parameter	*PW1ALT*	*EW1 ALT*	*HI1 ALT*
ρ_{21}	0.115 (0.024)	0.127 (0.024)	0.024 (0.006)
ρ_{32}	0.102 (0.017)	0.112 (0.018)	0.053 (0.009)
ρ_{43}	0.073 (0.016)	0.080 (0.016)	0.065 (0.013)
ρ_{54}	0.048 (0.021)	0.053 (0.020)	0.080 (0.018)
ρ_{65}	0.021 (0.029)	0.023 (0.028)	0.092 (0.022)
$C(\alpha_i, Y_{i1})$	0.169 (0.012)	—	—
$C(\beta_i, Y_{i1})$	–0.012 (0.003)	—	—

$\Lambda_{1\alpha}$	—	1.160 (0.050)	1.000
$\Lambda_{1\beta}$	—	0.429 (0.558)	0.000
μ_1	1.861 (0.010)		
μ_α	1.602 (0.062)	1.577 (0.062)	1.854 (0.010)
μ_β	0.066 (0.021)	0.070 (0.020)	-0.013 (0.009)
$V(\alpha_i)$	0.163 (0.018)	0.145 (0.018)	0.209 (0.008)
$V(\beta_i)$	0.005 (0.001)	0.005 (0.001)	0.006 (0.001)
$C(\alpha_i, \beta_i)$	–0.012 (0.004)	–0.010 (0.004)	–0.021 (0.002)

Table 2 summarizes key parameter estimates (with standard errors in brackets) from all three models with unequal ρs. The variance of the intercept is estimated at approximately 0.15–0.21 and is significant in all models. In contrast, the latent slope factor (β_i) shows a very small but significant variance (0.005). This suggests that adolescents differ much more in their overall smoking levels (α_i) than in their linear rates of change over time. Substantively, some youths consistently smoked more frequently than others throughout adolescence, while the rate at which their smoking frequency increased was relatively small. The intercept–slope correlation is negative across all models (ranging from –0.593 to –0.371). This negative correlation indicates that adolescents with higher α_i levels of smoking tend to increase at a slightly slower rate than those who begin with lower levels. This may reflect a ceiling effect - those already smoking frequently at 15 have less room to increase - or heterogeneity in smoking uptake - high baseline smokers may have initiated earlier and reached a plateau.

The autoregressive coefficients highlight the within-person carryover of smoking from one wave to the next. In the PW1 and EW1 ALT models, these coefficients showed a decreasing trend over time: approximately $\rho_{21} \approx 0.13$ between waves 1 and 2, then decreasing and not being significant, to 0.02 by waves 5 and 6. Conversely, the HI1 ALT model had an autoregressive coefficients that started very low (ρ_{21}= 0.02) and increased over time to 0.09. But recall that the HI1 ALT model had a poorer fit than the other models.

Comparing the models shows how the treatment of the initial condition, Y_{i1}, influences parameter estimates and interpretation. In the PW1 ALT model, Y_{i1} is not modeled as a function of the latent factors; instead, it correlates with them. The estimated correlation between Y_{i1} and the intercept factor is positive and significant (*e.g.,* $Corr(\alpha_i, Y_{i1})$=0.674), $Corr(\beta_i, Y_{i1})$ is negative (–0.273), suggesting that those with higher initial smoking have lower slopes. The latent intercept α_i captures the individual underlying tendency to smoke across adolescence. The latent slope β_i similarly reflects the linear change net of baseline influences.

In the EW1 ALT model, the initial observation is explicitly included as part of the trajectory. Here, Y_{i1} is regressed on the latent intercept and slope with coefficients $\Lambda_{1\alpha}$ and $\Lambda_{1\beta}$.

We found $\Lambda_{1\alpha}$ to be significantly greater than 1 (≈1.16). In contrast, the non-significant $\Lambda_{1\beta}$ suggests that the linear slope factor did not contribute to smoking at the initial time point in this sample.

Finally, the HI1 ALT model sets $\Lambda_{1\alpha} = 1$ and $\Lambda_{1\beta} = 0$, making $Y_{i,}$ entirely determined by the intercept factor. The latent intercept represents the initial true score, similar to a traditional growth curve model. This assumption significantly impacts the estimates. First, the intercept variance in the HI1 ALT model is the largest among all models ($V(\alpha_i) = 0.20$). The intercept–slope correlation $Corr(\alpha_i, \beta_i)$ is also larger (–0.593) than in other models. These differences suggests that the HI1 ALT model forces any variation related to initial status into α_i, increasing its variability and its (negative) relationship with β_i. More clearly, the autoregressive estimates in the HI1 ALT model differ from those of the other models, as described earlier, and the overall fit worsens.

Conditional Classic ALT models

Conditional Classic ALT models include covariates in addition to the lagged repeated measure. There are two main types of covariates: *time-varying* and *time-invariant.* Time-varying variables can take different values, whereas time-invariant variables remain constant. We will discuss each type of covariate.

Time-varying covariates. Time-varying covariates are dynamic predictors that fluctuate over time within individuals. In our earlier empirical illustration, an example is the adolescent's perceived exposure to smoking in movies. When analyzing its influence on adolescent smoking intensity (Y_{it}), researchers typically incorporate it in eq. (1), allowing it to have a contemporaneous influence on the outcome. Denoting X_{it} as a time-varying covariate for the *i*th case, equation (1) results

$$Y_{it} = \alpha_i + \lambda_t \beta_i + \rho_{t,t-1} Y_{i,t-1} + \gamma_{yt} X_{it} + \varepsilon_{it}, \tag{13}$$

for $t = 2, 3, \ldots, T,$ where γ_{yt} is the coefficient representing the expected effect of X_{it} on Y_{it} while controlling for all other variables on the RHS. Note that $Y_{i,t-1}$ is a time-varying variable, so X_{it} is the second time-varying variable in the equation. In the previous ALT model, the coefficient $\rho_{t,t-1}$ was a within-case effect because ($\alpha_i + \lambda_t \beta_i$) controlled the between effects of $Y_{i,t-1}$. The inclusion of an additional time-varying covariate in the model changes the interpretation of the model parameters. Since X_{it} directly influences the outcome variable, the growth curve parameters (*e.g.,* intercept and rate of change) should be interpreted as *net* of the effect of this time-varying covariate. Additionally, the time-specific effects of X_{it} on Y_{it} also need to be interpreted as effects net of the influence of the growth process.

Can time-varying explanatory variables enter the random intercept and random slope equations? Generally, the answer is no. This can be best illustrated with an example. Suppose the random intercept equation includes the time-varying covariate at wave three (X_{i3}) as

$$\alpha_i = \mu_\alpha + \gamma_{\alpha 3} X_{i3} + \varepsilon_{\alpha i.} \tag{14}$$

In equation (1) α_i directly affects the repeated measures, Y_{i2}, Y_{i3}, Y_{i4}, …. Any variable that *directly* affects α_i would *indirectly* affect these same repeated measures. There is no temporal problem for the Ys that come at wave 3 or later, but this is an issue for Y_{i2} and earlier values in that this implies that a future value of the covariate at wave 3 (X_{i3}) indirectly influences a wave 2 variable (Y_{i2}) from the past.

Between- and within-effects in time-varying covariates. Several authors have emphasized the importance of disentangling two distinct sources of variability when incorporating time-varying covariates X_{it} into longitudinal models (Curran & Bauer, 2011; Curran et al., 2012; Hoffman & Stawski, 2009; Muniz-Terrera et al., 2017; Hori & Miyazaki, 2025). In the earlier subsection, we explained how in the ALT model $\rho_{t,t-1}Y_{i,t-1}|(\alpha_i + \lambda_t\beta_i)$ and ε_{it} capture the within components while $(\alpha_i + \lambda_t\beta_i)|Y_{i,t-1}$ is the between component.

How does the decomposition into between- and within-components differ if we introduce the TVC variable X_{it}, into the ALT model? Consider,

$$Y_{it} = \alpha_{Yi} + \lambda_{Yt}\beta_{Yi} + \rho_{t,t-1}Y_{i,t-1} + \gamma_{xt}X_{it} + \gamma_{\alpha X}\alpha_{Xi} + \gamma_{\beta X}\beta_{Xi} + \varepsilon_{it} \tag{15}$$

which is the previous ALT model with the addition of X_{it}, α_{Xi}, β_{Xi}, and their corresponding coefficients of γ_{xt}, $\gamma_{\alpha X}$, and $\gamma_{\beta X}$. We are assuming that the TVC X_{it} follows a linear growth curve model with α_{Xi} its random intercept and β_{Xi} its random slope. We could simplify this if only the random intercept were needed, or complicate it if more complicated functional forms described X_{it}'s trajectory. But the linear trajectory is sufficient to make our points. The α_{Xi}, β_{Xi}, α_{Yi}, and β_{Yi} are correlated exogenous variables. If analysts specify the predetermined variable ALT form, then the first wave of the repeated measures, X_{i1} and Y_{i1}, correlate with these growth parameters. The α_{Xi} and β_{Xi} are constant over time but differ over cases. They are the between components of X_{it}, and their coefficients give the between-effects of X_{it} on Y_{it} as captured by α_{Xi} and β_{Xi} while holding the other variables in the equation constant. Because we are controlling for the between components (*i.e.*, α_{Xi}, β_{Xi}) of X_{it}, the coefficient of X_{it} is the within-effects of $X_{it}|(\alpha_{Xi} + \lambda_t\beta_{Xi})$. Note also that equation (15) contains the original (uncentered) X_{it}.

Equation (15) assumes that the between-components of X_{it} have *direct effects* on Y_{it}[7]. What happens when these direct effects are zero ($\gamma_{\alpha X} = \gamma_{\beta X} = 0$)? This implies that the $\gamma_{\alpha X}\, \alpha_{Xi}$ and $\gamma_{\beta X}\, \beta_{Xi}$ are unnecessary in equation (15), but X_{it} maintains a direct effect and remains in the equation. As before, X_{it} correlates with the other RHS variables except the error. Under this scenario, the coefficient γ_{xt} remains the same with or without α_{Xi} and β_{Xi} in the equation. This implies that γ_{xt}, the coefficient of X_{it}, continues to give the within-effects even using the raw rather than the deviation score of X_{it}.

Time-invariant covariates. When analyzing individuals, time-invariant variables include characteristics like race, ethnicity, gender, or place of birth. For each case, the time-invariant variables assume the same value over the observation period covered by the longitudinal study. At the same time, the time-invariant variables can take different values across cases.

In the ALT model, the random intercept α_i and random slope β_i become the dependent variables with respect to the time-invariant covariates, leading to

$$\alpha_i = \mu_\alpha + \gamma_{\alpha Z} Z_{i1} + \varepsilon_{\alpha i} \tag{16}$$

$$\beta_i = \mu_\beta + \gamma_{\beta Z} Z_{i1} + \varepsilon_{\beta i}, \tag{17}$$

where the μs are intercept constants, and the γs are regression coefficients. As in eqs. (2) and (3), the errors $\varepsilon_{\alpha i}$ and $\varepsilon_{\beta i}$ have means of zero and can correlate with each other, but do not correlate with the time-invariant Z_{i1} variable. The covariate Z_{i1} can impact the random intercept and random slope, and these are between-effects in that Z_{i1} takes the same value for each case over time but typically differs over cases.

When using the PW1 form of the ALT model, not only the random intercepts (α_i) and random slopes (β_i), but also Y_{i1} are dependent variables concerning the time-invariant covariates. In other words, together with eqs. (16) and (17), we add

$$Y_{i1} = \mu_1 + \gamma_{Y1} Z_{i1} + \varepsilon_{i1}, \tag{18}$$

[7] In the multilevel modeling literature, the between-person components of X_{it}, namely α_{Xi} and β_{Xi}, are typically specified in the Level-2 equations (2) and (3), rather than directly in the Level-1 equation. A detailed discussion of alternative approaches for including the between-case components of time-varying covariates lies beyond the scope of this paper. For comprehensive discussions, see Hamaker & Muthén (2020) and Curran et al. (2012). Importantly, regardless of whether α_{Xi} and β_{Xi} exert their influence on Y_{it} directly or indirectly, the coefficient γ_{xt} consistently represents the *within-person effect* of X_{it} on Y_{it}.

where γ_{Y1} is a regression coefficient, and the error ε_{i1} has zero mean, is uncorrelated with the time-invariant Z_{i1} variable. However, ε_{i1}, $\varepsilon_{\alpha i}$, and $\varepsilon_{\beta i}$ correlate with each other. In other words, the errors of these three equations are associated. The label "*Predetermined Wave 1*" (PW1) is less accurate once we introduce covariates to predict $Y_{i1,}$ and hence Y_{i1} becomes a dependent variable. However, to avoid confusing matters, we continue to use the term "*predetermined*" and the acronym PW1 to refer to this form of the ALT model.

Bianconcini and Bollen (2018) highlight a difference in how time-invariant covariates enter the model within the growth curve model literature compared to fixed and random effects models in economics and other social sciences. Equations (16) and (17) are typical in latent growth curve models, where time-invariant covariates directly influence the growth parameters. In contrast, in many econometric - specifically random effects - models (*e.g.*, Bollen & Brand, 2010), time-invariant variables enter the model by directly affecting the repeated measures. For instance, in wave 2 and beyond, the equation for the repeated measure would be,

$$Y_{it} = \alpha_i + \lambda t\, \beta i + \rho_{t,t-1}\, Y_{i,t-1} + \gamma_{zt} Z_{i1} + \varepsilon_{it}, \tag{19}$$

for $t = 2, 3, \ldots, T,$ where the time-invariant variable, Z_{i1}, has a direct influence of γ_{zt} on the repeated measures Y_{it}, rather than having a direct effect on the growth parameters α_i and β_i. Often, the effect of the time-invariant variable is constrained to be equal over time ($\gamma_{zt} = \gamma_z$). Furthermore, the definition of between- and within-case components requires modifications in equation (19). Now ($\alpha_i + \lambda_t\, \beta_{i1} + \gamma_{zt} Z_{i1}$) contains between-components and ($\rho_{t,t-1} Y_{i,t-1} + \varepsilon_{it}$) contains within-case components.

The decision of whether to include the time-invariant variable in equation (19) or in eqs. (16) and (17) is not straightforward. It raises the question of whether the analyst believes all effects of the time-invariant variables are *indirect* through the growth parameters of α_i and β_i, or if these variables directly influence the repeated measures even after accounting for the growth curve parameters. Thus, analysts need to decide where the time-invariant covariates should enter the model.

Two or more repeated measures

As discussed in the previous section, time-varying covariates typically appear in equation (1) of the Classic ALT model as exogenous variables that directly influence the repeated measures.

However, sometimes, these time-varying variables form a second repeated measures series, and they both influence and are influenced by the first repeated measures.

There are many possible specifications to relate the two repeated measures. We chose a relatively simple model while recognizing that many modifications are plausible. In our simplified two series model, each repeated measure has a lag-one autoregression and a lagged effect from the other repeated measures series. Furthermore, a random intercept and random slope are part of the specification. Two equations to capture these relations are in (20) and (21),

$$Y_{it} = \mu_{yt} + \alpha_{yi} + \Lambda_{yt2}\,\beta_{yi} + \rho_{yt,y(t-1)} Y_{i,t-1} + \rho_{yt,x(t-1)} X_{i,t-1} + \varepsilon_{yit} \tag{20}$$

$$X_{it} = \mu_{xt} + \alpha_{xi} + \Lambda_{xt2}\,\beta_{xi} + \rho_{xt,x(t-1)} X_{i,t-1} + \rho_{xt,y(t-1)} Y_{i,t-1} + \varepsilon_{xit} \tag{21}$$

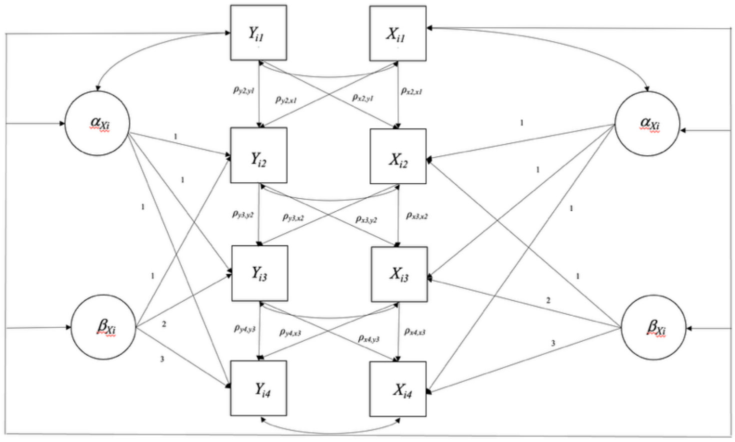
for $t = 2, 3, \ldots, T$. The two repeated measures series are Y_{it} and X_{it}. The autoregressive and cross-lagged effects have ρs as their coefficients, with their subscripts listing the "*dependent*" and "*explanatory*" relations they capture. For instance, $\rho_{yt,x(t-1)}$ represents the coefficient that indicates the expected effect of $X_{i,t-1}$ on Y_{it}, after controlling for the other variables in the equation. The μ_{yt} and μ_{xt} are regression intercept constants at time t, typically set to zero when growth components are included in the model. The Λ_{yt2} and Λ_{xt2} coefficients of eqs. (20) and (21) give the coefficients of the random slopes of β_{yi} and β_{xi}. These could be linear trends [$\Lambda_{yt2} = (t-1)$ and $\Lambda_{xt2} = (t-1)$] or another pattern of change. The random intercepts and slopes of the two series are expressed as

$$\alpha_{yi} = \mu_{\alpha y} + \varepsilon_{\alpha yi} \quad \text{and} \quad \beta_{yi} = \mu_{\beta y} + \varepsilon_{\beta yi}, \tag{22}$$

$$\alpha_{xi} = \mu_{\alpha x} + \varepsilon_{\alpha xi} \quad \text{and} \quad \beta_{xi} = \mu_{\beta x} + \varepsilon_{\beta xi}, \tag{23}$$

where the errors $\varepsilon_{\alpha yi}$, $\varepsilon_{\beta yi}$, $\varepsilon_{\alpha xi}$, and $\varepsilon_{\beta xi}$ have zero means, and are assumed to be all correlated with each other. Finally, ε_{yit} and ε_{xit} are the error variables of each series. We make the usual assumptions about the error terms as before, though we allow ε_{yit} and ε_{xit} to correlate at each wave of the data.

Figure 3. *Path Diagram of the ALT model for two series of repeated measures*



[Figure 3 About Here]

Figure 3 is the path diagram of eqs. (20) - (23) for two repeated measures using the PW1 form of the ALT model. First wave variables Y_{i1} and X_{i1} are predetermined and correlate with each other and with α_{yi}, α_{xi}, $\beta_{yi,}$ and β_{xi}. This model assumes that the contemporaneous errors for the Y and X series correlate and that there is a lagged effect of each series on the other. The autoregressive and cross-lagged effects Y_{it} and X_{it} are hypothesized to remain even after accounting for the effects of the random intercepts and random slopes. Although we have drawn the same type of longitudinal model for each series, this need not be. For example, it could be that the repeated measure for Y has no growth curve parameters ($\alpha_{yi} = 0$; $\beta_{yi} = 0$ for all i) while the series for X requires the full PW1 form of the ALT model. It would be rare to have specific theoretical or substantive knowledge of this, but the results of fitting the data might suggest simpler and possibly different longitudinal models for each repeated measure.

This basic two series model in Figure 3 could be elaborated in a variety of ways. A time-invariant observed variable (*e.g.*, sex at birth, race) could be introduced as a determinant of α_{yi}, $\alpha_{xi,}$ β_{yi}, β_{xi}, Y_{i1}, and X_{i1}. It is even possible to have the growth parameters of one series to influence the growth parameters of the other. Another modification would be to use the EW1 form of the ALT model for the repeated measures. The possibilities are extensive, mostly limited by the researcher's hypotheses and the need to create identified models.

LATENT VARIABLE ALT (LV-ALT) MODEL

A generalization of the classic ALT model is the Latent Variable ALT (LV-ALT), first introduced by Bollen & Curran (2004) and later formalized by Bianconcini & Bollen (2018).

A key feature is allowing for measurement error and including multiple indicators of the latent variable. Because of this, we begin with the measurement part of the LV-ALT model.

Let j subscript the multiple indicators ($j = 1, 2, ..., J$) that measure the same latent variable (L_{it}) for N cases in the sample at T time points. Thus, Y_{ijt} is the continuous indicator j at time t for the ith individual. At each time point t, we have a measurement model of the form

$$Y_{ijt} = \mu_{Yjt} + \Lambda_{jt}L_{it} + \varepsilon_{ijt}, \tag{24}$$

for $t = 1, ..., T; j = 1, ..., J; i = 1, ..., N$. μ_{Yjt} is the constant intercept for the jth indicator at time t, Λ_{jt} is its corresponding factor loading, and ε_{ijt} is the error in the indicator. We assume the error variables have means of zero, are uncorrelated with L_{it}, and are homoscedastic and not autocorrelated across cases.

Essentially, equation (24) is a confirmatory factor model with multiple indicators at each wave and allows the intercept (μ_{Yjt}) to differ by indicator and over time. To set a scale for the latent variable, we set one indicator's intercept to zero and its factor loading to one (Bollen, 1989, pp. 307-308), and use the same indicator to scale the latent variable at each wave (*e.g.,* $\mu_{Y1t} = 0$ and $\Lambda_{1t} = 1$ for $t = 1,...,T$). The implicit measurement model in the Classic ALT model has no measurement errors in the repeated measures, so the measures were equivalent to the latent variables with one measure per wave. The LV-ALT model allows multiple indicators, and instead of repeated measures as in the Classic ALT, we have *repeated latent variables*.

The equation for the repeated latent variables in the LV-ALT model is

$$L_{it} = \mu_{Lt} + \alpha_i + \Lambda_{2t}\beta_i + \rho_{t,t-1} L_{i,t-1} + \varepsilon_{iLt}, \tag{25}$$

for $t = 2, 3, ..., T$. μ_{Lt} is a constant intercept at time t, usually set to zero when growth components are included in the model. As before, Λ_{2t} is the *time trend* variable, equal to $(t-1)$ in linear models, while some Λ_{2t}s can be freely estimated for nonlinear trajectories. The α_i is the random intercept that differs across cases. This form of the ALT model includes a random slope, β_i, as was true for the other ALT models. The $\rho_{t,t-1}L_{i,t-1}$ term matches the Classic ALT model except the latent variable, $L_{i,t-1}$, replaces the repeated measure, $Y_{i,t-1}$.

Without repeating it now, much of our earlier discussion on interpreting ALT components within the Classic ALT model applies here. Here as there, structural misspecification - such as an incorrect functional form, an omitted random slope, or missing time-varying covariates - can bias estimates of between- and within-person effects. Similarly, our approach to handling two (or more) repeated measures series under the Classic ALT framework extends to the LV-ALT (Bianconcini & Bollen, 2018). For simplicity, we assumed error-free repeated measures in that two-series discussion, but the same substantive interetations remain valid when latent variables and measurement errors are included.

The initial condition problem

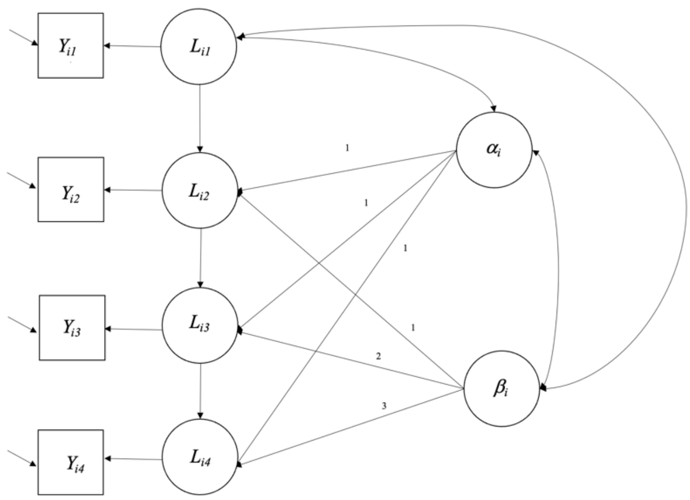
Like the Classic ALT, the LV-ALT model has three forms. We only discuss the two most common ones: the *Predetermined Wave 1 (PW1)* and the *Endogenous Wave 1 (EW1)*. Equation (25) for wave 2 and later is the same for both forms; they only differ in how they treat wave 1. The next two subsections describe these for single repeated latent variables, followed by some extensions of the model that include covariates.

Predetermined Wave 1 (PW1) in the LV-ALT model. For the PW1 form of the LV-ALT model, the wave 1 equation for L_{i1} is

$$L_{i1} = \mu_{L1} + \varepsilon_{iL1}, \tag{26}$$

where μ_{L1} is the mean of the first repeated latent variable L_{i1} and ε_{L1i} is the deviation from the mean of the ith case in the first wave. As is true for the PW1 form of the Classic ALT model, the first wave of the PW1 form of the LV-ALT is taken as given or predetermined.

Figure 4. *Path Diagram of PW1 in LV-ALT Model*



[Figure 4 About Here]

Figure 4 shows the path diagram of the PW1 form of the LV-ALT model. As before, the ellipses or circles represent latent variables such as the repeated latent variables, L_{it}, the random intercept, α_i, and the random slope, β_i. To simplify the model, there is a single indicator (Y_{it}) for the latent variable (L_{it}) at each of the four waves of data, with factor loadings set to 1 and intercepts set to 0. The coefficients on the paths from the random intercept (α_i) to the repeated latent variables ($L_{i2}, \ldots, L_{i4}$) are also fixed at one. In contrast, the paths from the random slope (β_i) to the repeated latent variables ($L_{i2}, \ldots, L_{i4}$) are fixed at ($t-1$), based on the assumption of linear change over time. Curved two-headed arrows connect the first wave of the latent variable L_{i1}, the random intercept α_i, and the random slope β_i, indicating the covariances among these three variables.

Identification of PW1 LV-ALT model. Following Bianconcini and Bollen (2018), we can apply the two-step identification rule (Bollen, 1989) to assess model identification. They demonstrate that if we assume the single indicator (Y_{it}) has its intercepts set to zero, factor loadings set to one, and equal measurement error variances [$V(\varepsilon_{it}) = V(\varepsilon_i)$], then the measurement model is identified in the first step. In the second step, they assume a linear growth curve ($\Lambda_{2t} = t-1$), and equal variances of the errors of the latent variables [$V(\varepsilon_{iLt}) = V(\varepsilon_{iL})$ for $t = 2, 3, \ldots, T$], and this, along with the first step identification, shows that the PW1 form of the LV-ALT model is identified with four waves with equal autoregressive coefficients and five or more waves of data when they are not equal.

The identification process varies depending on whether multiple indicators are included in the model or if different functional forms of the latent growth curve model are used. Nonetheless, the two-step identification rule can be helpful in those cases as well.

Endogenous Wave 1 (EW1) LV-ALT model. The equation for the first repeated latent variable (L_{i1}) in the EW1 form of the LV-ALT Model is given by

$$L_{i1} = \mu_{L1} + \Lambda_{1\alpha}\, \alpha_i + \Lambda_{1\beta}\, \beta_i + \varepsilon_{iL1}, \tag{27}$$

where $\Lambda_{1\alpha}$ and $\Lambda_{1\beta}$ are the coefficients for the random intercept and slope, respectively, for the first wave of data. Equation (27) in the EW1 form of the LV-ALT model is analogous to equation (10) in the EW1 form of the Classic ALT model. One difference is that the latent variable L_{i1} is the dependent variable in the LV-ALT rather than the observed variable Y_{i1}, as it was for the Classic ALT model.

Figure 5 shows the path diagram of the EW1 form of the LV-ALT Model. As before, we assume a single indicator for the repeated latent variable. The diagram is like the one for the PW1 form of the LV-ALT, with a few exceptions. The most noticeable difference is that the first wave of the latent variable, L_{i1}, depends on the random intercept (α_i) and random slope (β_i), rather than these three variables covarying. The rest of the path diagrams are quite similar.

As discussed for the Classic ALT model, a special case of the EW1 LV-ALT model arises by assuming the autoregressive coefficients are time-invariant, $\rho_{t,t-1}=\rho$, and in absolute value less than one ($|\rho|<1$), such that

$$L_{i1} = (1-\rho)^{-1} \alpha_i - \rho(1-\rho)^{-2} \beta_i + \varepsilon_{iL1}, \tag{28}$$

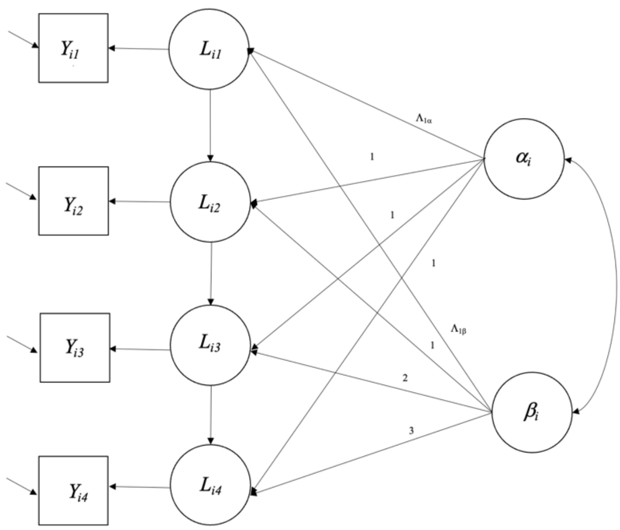
that is, $\Lambda_{1\alpha}= (1-\rho)^{-1}$ and $\Lambda_{1\beta}=-\rho(1-\rho)^{-2}$.

Also, for the LV-ALT model, the so-called *History Independent* (HI1) specification for the first wave is possible, and given by

$$L_{i1} = \alpha_i + \varepsilon_{iL1}, \tag{29}$$

where $\Lambda_{1\alpha}$ is set to 1 and $\Lambda_{1\beta}$ to 0. No explicit assumptions of stationarity and time invariance of the LV-ALT process at the subsequent waves are required.

Figure 5. *Path Diagram of EW1 in LV-ALT Model*



[Figure 5 About Here]

Identification of EW1 in LV-ALT Model. The identification conditions we apply to the measurement model for the EW1 form of the LV-ALT are the same as those for the PW1 form. Refer to the earlier section on identification of the PW1 LV-ALT. The identification of the remaining parts of the model in equations (25) and (27) is similar to that described for the EW1 form of the Classic ALT. If we assume a linear growth curve ($\Lambda_{2t} = t\text{-}1$ for $t > 1$) and that the autoregressive coefficients are consistent over time ($\rho_{t,t-1} = \rho$) and that the absolute value of the autoregression is less than one ($|\rho| < 1$), then the model is identified with three or more waves of data.

Empirical example

We now analyze the smoke intensity indicator using Latent-Variable ALT (LV-ALT) models, extending the Classic ALT analyses by treating the observed indicator as a latent variable with measurement error. Consistent with previous analyses, we specified six LV-ALT models, varying how the first latent state (at wave 1) is treated – PW1, EW1, and HI1 - and whether the autoregressive coefficients are constrained to remain constant over time or not.

As before, we estimated the LV-ALT models using full information maximum likelihood (Casewise ML) to handle missing data. All available data were used, and estimation was performed with the *lavaan* package in R. We found that nonstationary versions of the PW1, EW1, and HI1 models, and for the latter also its specification based on time-invariant autoregressive coefficients, produced inadmissible (negative) variances. Therefore, we focused on the two models that converged properly: the PW1 LV-ALT model and the EW1 LV-ALT model, both with time-invariant ρ**.**

Table 3 summarizes the corresponding chi-square and fit indices. Both the PW1 and the EW1 LV-ALT models fit the data well, with nonsignificant chi-square tests and excellent fit indices. When comparing Bayesian Information Criterion (BIC) values, the EW1 model has a slightly lower BIC (-87.306 vs. -79.293), indicating a better fit.

Table 3. *Fit statistics for LV-ALT models of smoking intensity.*

	PW1 LV-ALT with $\rho_{t,t-1} = \rho$	*EW1 LV-ALT with $\rho_{t,t-1} = \rho$*
χ^2	13.855	14.310
df	11	12
p-value	0.241	0.281
CFI	1.000	1.000
TLI	0.999	1.000
RMSEA	0.007	0.006
BIC	-79.293	-87.306

Table 4 reports parameter estimates (with standard errors) for the two models. In the PW1 LV-ALT model, the only parameters significantly different from zero are the baseline mean μ_1, which is 1.858, and the latent intercept mean μ_α, which is 1.077. Neither the average slope μ_β, the autoregressive coefficient ρ, nor the variances of the intercept $V(\alpha_i)$ or slope $V(\beta_i)$ are significantly different from zero. This pattern closely mirrors the classic ALT finding that only the intercept is substantial, while the variance of the slope and residual carry-over beyond the initial value are negligible once the baseline is modeled as exogenous.

Table 4. *Parameter estimates for LV-ALT models.*

Parameter	*PW1ALT*	*EW1 ALT*	*PW1 ALT only intercept*	*EW1 ALT only intercept*
ρ	0.434 (0.253)	0.632 (0.162)	0.745 (0.047)	0.701 (0.047)
$C(\alpha_i, L_{i1})$	0.097 (0.065)	—	0.018 (0.008)	—
$C(\beta_i, L_{i1})$	–0.008 (0.007)	—	—	—
$\Lambda_{1\alpha}$	—	2.720 (1.033)	—	3.111 (0.469)
$\Lambda_{1\beta}$	—	-17.355 (0.006)	—	—
μ_1	1.858 (0.010)	—	1.855 (0.009)	—
μ_α	1.077 (0.463)	0.714 (0.297)	0.512 (0.090)	0.596 (0.089)
μ_β	0.010 (0.007)	0.005 (0.005)	—	—
$V(\alpha_i)$	0.062 (0.067)	0.017 (0.018)	0.008 (0.003)	0.009 (0.004)
$V(\beta_i)$	0.001 (0.002)	0.000 (0.000)	—	—
$C(\alpha_i, \beta_i)$	–0.005 (0.007)	0.000 (0.000)	—	—

In contrast, in the EW1 LV-ALT model, the autoregressive coefficient is large and highly significant, ρ = 0.632, and both loadings on wave 1 are significant, with $\Lambda_{1\alpha}$ equal to 2.720 and $\Lambda_{1\beta}$ equal to –17.355, as it is the latent intercept mean μ_α, being around 0.714. The slope mean μ_β and its variance $V(\beta_i)$ are not significantly different from zero, nor is the intercept variance $V(\alpha_i)$. Therefore, the results align with the classic EW1 ALT, indicating that smoking intensity dynamics are dominated by a strong autoregressive pattern around a steady intercept, with no evidence of systematic linear growth or lasting differences in slope between individuals.

Because the mean and variance of the slope factors in both PW1 and EW1 ALT were not significantly different from zero, we re-estimated only-intercept ALT versions by fixing the slope mean and variance to zero. In *lavaan* 0.6-19, both reduced models converged normally after 50 iterations and showed an excellent fit. The PW1-only-intercept model yields $\chi^2(15) = 16.39$ ($p = 0.357$), CFI = 1.000, TLI = 1.000, RMSEA = 0.004, and BIC = -110.344. In the EW1-only-intercept model, $\chi^2(16) = 23.12$ ($p = 0.111$), CFI = 0.999, TLI = 0.999, RMSEA = 0.010, and BIC = -112.063. Focusing on these simplified models, we see that fixing the non-significant slope to zero provides more parsimonious representations of smoking trajectories without any loss of fit. In the PW1 model, constraining the slope variability shifts all persistence into the autoregressive path, raising ρ from 0.434 to 0.745, and decreasing the intercept variance from 0.062 to 0.008. This indicates that, after accounting for individual starting points, stability in smoking is almost entirely self-reinforcing rather than driven by systematic change. In the EW1 model, removing the slope factor similarly increases ρ from 0.632 to 0.701, enhances the wave 1 intercept loading ($\Lambda_{1\alpha}$ rising from 2.720 to 3.111), and maintains a reliably positive intercept mean while intercept variance remains negligible. Thus, both models support the conclusion that adolescent smoking intensity fluctuates mainly around a stable baseline, and there is no evidence of a substantial linear trend once this baseline is properly accounted for.

Conditional Latent Variable ALT (LV-ALT) models

In the subsection on conditional Classic ALT models, we presented ALT models that included covariates beyond the lagged repeated measures. We distinguished time-varying and time-invariant covariates. Furthermore, we stated that time-varying covariates nearly always enter the equation for the repeated measures rather than the equation for the random intercept and random slope. In contrast, the placement of time-invariant covariates varies according to the traditions followed by those using different longitudinal models. In growth curve models, it is common to see time-invariant covariates in the Level 2 equations for the random intercepts and random slopes. While in random effects models or autoregressive and cross-lagged models, the time-invariant variables occur in the equations for the repeated measures.

Nearly all the material we presented on time-varying and time-invariant covariates for the Classic ALT model carries over to the LV-ALT. For instance, the equation for the repeated latent variables with a single time-varying covariate is

$$L_{it} = \mu_{Lt} + \alpha_i + \Lambda_{2t}\,\beta_i + \rho_{t,t-1}\,L_{i,t-1} + \gamma_{Lt}X_{it} + \varepsilon_{iLt}, \tag{30}$$

for $t = 2, 3, \ldots, T$. γ_{Lt} is the coefficient that is the expected effect of X_{it} on L_{it} controlling for all other variables on the RHS, μ_{Lt} is the intercept constant at time t, and we have already defined the other variables and coefficients.

If we have a time-invariant variable, say, Z_{i1}, we can enter this in the latent variable equation (25) to create

$$L_{it} = = \mu_{Lt} + \alpha_i + \Lambda_{2t}\,\beta_i + \rho_{t,t-1}\,L_{i,t-1} + \gamma_{LXt}X_{it} + \gamma_{LZt}Z_{i1} + \varepsilon_{iLt}, \tag{31}$$

for $t = 2, 3, \ldots, T$. Alternatively, we can include the time-invariant Z_{i1} in the equations for the random components as follows

$$\alpha_i = \mu_\alpha + \gamma_\alpha Z_{i1} + \varepsilon_{\alpha i} \tag{32}$$

$$\beta_i = \mu_\beta + \gamma_\beta Z_{i1} + \varepsilon_{\beta i}. \tag{33}$$

As in the Classic ALT model, in the PW1 LV-ALT, Z_{i1} is also regressed on the latent variable at wave 1, that is

$$L_{i1} = \mu_1 + \gamma_{LZ1}Z_{i1} + \varepsilon_{iL1} \tag{34}$$

where γ_{LZ1} is a regression coefficient, and the error ε_{iL1} has zero mean, is uncorrelated with the time-invariant Z_{i1} variable. However, ε_{iL1}, $\varepsilon_{\alpha i}$, and $\varepsilon_{\beta i}$ correlate with each other.

As we stated in the Classic ALT model section, whether the time-invariant covariates enter the equation (31) or equations (32) and (33) must be decided by the researcher. Previously, we have included only one time-varying and one time-invariant variable. This was done to keep the equations simple. In practice, we can include multiple covariates of both types.

Relationship to popular longitudinal models

Theory and substantive research are rarely specific enough to determine which longitudinal model is best for a particular dataset. Instead, researchers often choose a longitudinal model based on factors such as familiarity, a model's popularity, or software availability. These selection methods are not scientifically based.

One valuable feature of the Classic ALT, and especially of the LV-ALT, is that by imposing constraints on their parameters we have a nested relationship that can help in choosing the appropriate longitudinal model for an application. For instance, testable restrictions on the ALT model can result in random and fixed effects models, autoregressive and cross-lagged models, linear and freed loadings growth curve models, random intercept only models, latent dual change score models, and structured residual models among other hybrid structures. We direct readers to Table 1 in Bianconcini and Bollen (2018, p. 798) for a detailed description of the constraints needed on the LV-ALT to derive these well-known longitudinal models.This means that researchers can estimate a very general ALT model and then test these constraints to determine whether the empirical evidence supports simpler structures or alternative longitudinal models that better capture the patterns in the data (*e.g.*, see Bollen & Gutin, 2021).

In this section, we use the PW1 and EW1 forms of the LV-ALT model to demonstrate that several additional popular longitudinal models are special cases of the LV-ALT.

Cross-Lagged Panel Models (CLPM). The autoregressive model is one of the simplest longitudinal models, where the current value of the repeated measure depends on its previous value and an error term. That is,

$$Y_{it} = \mu_t + \rho_{t,t-1}\, Y_{i,t-1} + \varepsilon_{it} \tag{35}$$

for $t = 2, 3, \ldots, T,$ where μ_t is a constant intercept at time $t,$ and the other terms have already been defined. At the first occasion, $Y_{i1} = \mu_1 + \varepsilon_{i1}$.

Recall that the LV-ALT, as defined in equation (25), is

$$L_{it} = \mu_{Lt} + \alpha_i + \Lambda_{2t}\, \beta_i + \rho_{t,t-1}\, L_{i,t-1} + \varepsilon_{it} \tag{36}$$

for $t = 2, 3, \ldots, T$. To derive equation (35) from equation (36), constraints are required. First, equation (35) assumes no measurement error, meaning that Y_{it} equals L_{it}. This leads to the Classic ALT model with the addition of a constant intercept at time t (μ_{Lt}).

$$Y_{it} = \mu_{Lt} + \alpha_i + \Lambda_{2t}\, \beta_i + \rho_{t,t-1}\, Y_{i,t-1} + \varepsilon_{it}, \tag{37}$$

for $t = 2, 3, \ldots, T$. Second, equation (35) assumes that there are no growth curve parameters, meaning the means and variances of α_i and β_i are zero. If this holds, then these parameters can be omitted, leading to,

$$Y_{it} = \mu_{Lt} + \rho_{t,t-1}\, Y_{i,t-1} + \varepsilon_{it}, \tag{38}$$

for $t = 2, 3, \ldots, T$, which is identical to equation (35) once we realize that $\mu_{Lt} = \mu_t$. Additionally, for the first wave of data, Y_{i1} is treated as predetermined.

Suppose we have two repeated measures and hypothesize cross-lagged effects. Two equations to represent this model are

$$Y_{it} = \mu_{yt} + \rho_{yt,y(t-1)}Y_{i,t-1} + \rho_{yt,x(t-1)}X_{i,t-1} + \varepsilon_{yit}, \tag{39}$$

$$X_{it} = \mu_{xt} + \rho_{xt,y(t-1)}Y_{i,t-1} + \rho_{xt,x(t-1)}X_{i,t-1} + \varepsilon_{xit}, \tag{40}$$

for $t = 2, 3, \ldots, T$. On the first occasion,

$$Y_{i1} = \mu_{Y1} + \varepsilon_{yi1},$$

$$X_{i1} = \mu_{x1} + \varepsilon_{xi1},$$

with the errors ε_{yit} and ε_{xit} correlating at each wave of the data, $t = 1, \ldots, T$.

The LV-ALT model in the PW1 form for the Y repeated measures is

$$L_{yit} = \mu_{Lyt} + \alpha_{yi} + \Lambda_{y2t}\,\beta_{yi} + \rho_{yt,y(t-1)}\,L_{yi,t-1} + \rho_{yt,x(t-1)}\,L_{xi,t-1} + \varepsilon_{Lit}, \tag{41}$$

for $t = 2, 3, \ldots, T$, with $L_{yi1} = \mu_{Ly1} + \varepsilon_{Li1}$. To align this more closely with equation (38), we assume each latent variable is measured with a single indicator with no measurement error ($Y_{it} = L_{yit}$, $X_{it} = L_{xit}$ for all t), resulting in the Classic PW1 ALT model for Y_{it}

$$Y_{it} = \mu_{Lyt} + \alpha_{yi} + \Lambda_{y2t}\,\beta_{yi} + \rho_{yt,y(t-1)}\,Y_{i,t-1} + \rho_{yt,x(t-1)}\,X_{i,t-1} + \varepsilon_{yit}, \tag{42}$$

for $t = 2, 3, \ldots, T$, with $Y_{i1} = \mu_{Ly1} + \varepsilon_{yi1}$.

The conversion assumes that the growth parameters are zero ($\alpha_{yi} = 0, \beta_{yi} = 0$).

$$Y_{it} = \mu_{Lyt} + \rho_{yt,y(t-1)}\,Y_{i,t-1} + \rho_{yt,x(t-1)}\,X_{i,t-1} + \varepsilon_{yit}, \tag{43}$$

for $t = 2, 3, \ldots, T$, and setting μ_{Lyt} to μ_{yt} completes it. Making analogous constraints on the X series results in both equations (39) and (40), showing that these equations are constrained versions of the LV-ALT.

Beyond demonstrating the restrictions needed to convert the LV-ALT model into the cross-lagged panel model (CLPM), these results also indicate potential modifications that could enhance the fit of the CLPM to our data. For instance, if fitting the CLPM equations (39) and (40) produces a poor fit, we could improve the model by adding only the random intercept α_{yi} or both the random intercept and slope ($\alpha_{yi} + \Lambda_{y2t}\,\beta_{yi}$) to see if this results in a better fit.

Longitudinal models with autoregressive or cross-lagged errors. Several longitudinal models specify autoregressive or cross-lagged relationships among the *errors* of repeated measures rather than directly between the observed variables. Several authors, from Hamaker (2005) to

Andersen (2022), have demonstrated that most of these models with direct relations between errors are *statistically equivalent* to *restricted* versions of ALT models.

By being statistically equivalent, we mean that a *constrained* ALT model generates the same model-implied moment matrices - and consequently the same fit statistics, indexes, and degrees of freedom - as other longitudinal models that permit complex relationships among the errors of repeated measures (Lee & Hershberger, 1990; Bianconcini & Bollen, 2025). However, statistical equivalence does not imply *substantive* equivalence. Researchers still need to justify why one model is more appropriate than another based on theory or context.

Because restricted ALT models can be statistically equivalent to models with more complex error structures, we can assess whether a simpler ALT model fits the data as well as a more flexible one. This allows us to indirectly evaluate certain autoregressive and cross-lagged error models. We explain how to perform these tests in the following sections.

Latent growth curve model with autoregressive disturbances. Latent growth curve models with autoregressive disturbances have a long history (*e.g*., Chi and Reinsel 1989; Browne and du Toit 1991; Diggle, Liang, and Zeger 1994). These models have features such as random intercepts and slopes, like an LGCM, but they also include a separate equation for the disturbance that accounts for the autoregressive relationship. Hamaker (2005) shows that the Classic linear ALT in its EW1 form – with $\rho_{t,t-1} = \rho$ and $|\rho|<1$ (Bollen & Curran, 2004), *for* $t = 2, 3, \ldots, T$,

$$Y_{it} = \alpha_i + t\beta_i + \rho Y_{i,t-1} + \varepsilon_{it}, \tag{44}$$

$$Y_{i1} = (1-\rho)^{-1}\alpha_i - \rho(1-\rho)^{-2}\beta_i + \varepsilon_{i1}, \tag{45}$$

is equivalent to an LGCM with autoregressive disturbances written as, for $t = 2, 3, \ldots, T$,

$$Y_{it} = \eta_{1i} + t\,\eta_{2i} + z_{it}, \tag{46}$$

$$z_{it} = \rho z_{i,t-1} + \delta_{it}, \tag{47}$$

where η_{1i} is a random intercept, and η_{2i} is a random slope for case i, z_{it} is a random error with a mean of zero, and an autoregressive relation to $z_{i,t-1}$, whereas the errors δ_{it}, $t=2, \ldots, T$, have zero mean, constant variance, and are uncorrelated over time. The initial value Y_{i1} is given by

$$Y_{i1} = \eta_{1i} + z_{i1}. \tag{48}$$

Hamaker (2005) shows the relationships between the variables and parameters in the Classic ALT model and those in the LGCM with autoregressive errors. She established that the random intercept from the latter model is

$$\eta_{1i} = \alpha_i(1-\rho)^{-1} - \beta_i\rho(1-\rho)^{-2},$$

the random slope equals

$$\eta_{2i} = \beta_i(1 - \rho)^{-1},$$

and the errors $z_{it} = \sum_{j=0}^{\infty} \rho^j \varepsilon_{i,t-j}$.

This result implies that the LGCM with autoregressive errors is a restrictive version of the Classic EW1 ALT model. We cannot *empirically* distinguish between a Classic ALT model with equality restrictions on the autoregressive coefficients of the lagged repeated measure and the LGCM with the same equality restrictions on the autoregressive coefficients of the lagged disturbances. The two models will have the same fit, as indicated by the usual likelihood ratio (LR) chi-square test and degrees of freedom from SEM software, when applied to the same data. The Classic ALT model does not require these restrictions, but if they are valid, then the LGCM with equal autoregressive coefficients of the errors is statistically equivalent to it.

This statistical equivalency between the LGCM with autoregressive errors and the Classic ALT - where both models have $\rho_{t,t-1} = \rho$ for all t and $|\rho| < 1$ - provides an opportunity to compare the fit of the Classic ALT model with these restrictions on ρ to one without them because they are nested and enable a LR chi-square test. Furthermore, due to the equivalence of the Classic ALT and the LGCM with autoregressive errors under the same restrictions on ρ, the LR chi-square test allows us to compare the fit of the ALT model without restrictions to the two models with restrictions. Therefore, we can empirically compare the unrestricted Classic ALT with the LGCM with restricted autoregressive errors.

Latent Curve Model with Structured Residuals (LCM-SR). The Latent Curve Model with Structured Residuals (LCM-SR, Curran et al., 2014) is the same as the LGCM with autoregressive disturbances when analyzing a single repeated measure, so the results from the previous subsection apply to the LCM-SR model with one repeated measure. However, the LCM-SR generally applies to two or more repeated measures, and the equations are

$$Y_{it} = \eta_{Y1i} + \lambda_t \eta_{Y2i} + z_{Yit}, \tag{49}$$

$$X_{it} = \eta_{X1i} + \lambda_t \eta_{X2i} + z_{Xit}, \tag{50}$$

$$z_{Yit} = \rho_{YY}\, z_{Yi,t-1} + \rho_{YX}\, z_{Xi,t-1} + \delta_{Yit}, \tag{51}$$

$$z_{Xit} = \rho_{XY}\, z_{Yi,t-1} + \rho_{XX}\, z_{Xi,t-1} + \delta_{Xit}. \tag{52}$$

For the Y and X repeated measures, the random intercepts are η_{Y1i} and η_{X1i}, the random slopes are η_{Y2i} and η_{X2i}, and the equation errors are z_{Yit} and z_{Xit}. These errors have autoregressive and cross-lagged relationships, where the subscript of ρ indicates the variable receiving and sending effects. That is, ρ_{XY} is the coefficient for the influence of Y on X.

As for the LCM with autoregressive disturbances, the LCM-SR has a restricted ALT counterpart – that is, the EW1 ALT model for two repeated measures with constant autoregressive and cross-lagged coefficients - which shares the same statistical fit and degrees of freedom (Andersen, 2022). The detailed derivation of the relationship between these two models is provided in Appendix A. It follows that a LR chi-square test can compare the unrestricted ALT with its restricted ALT counterpart and the LCM-SR as a fit diagnostic. If these restrictions and stationarity assumptions are not valid for the LCM-SR, then parameter estimates are likely to be biased.

Random Intercept – Cross Lagged Panel Model (RI–CLPM). The Random Intercept – Cross Lagged Panel Model (RI–CLPM, Hamaker, Kuiper, & Grasman, 2015) is another longitudinal model that handles errors more complexly. This applies to one or more repeated measures, but here we will focus on the most applied case of two series. A key difference between the RI–CLPM and the LCM-SR is that the RI–CLPM includes only random intercepts, not random slopes. We define the RI–CLPM for two repeated measures as

$$Y_{it} = \mu_{Yt} + \eta_{Y1i} + z_{Yit}, \tag{53}$$

$$X_{it} = \mu_{Xt} + \eta_{X1i} + z_{Xit}, \tag{54}$$

$$z_{Yit} = \rho_{YY}\, z_{Yi,t-1} + \rho_{YX}\, z_{Xi,t-1} + \delta_{Yit}, \tag{55}$$

$$z_{Xit} = \rho_{XY}\, z_{Xi,t-1} + \rho_{XX}\, z_{Xi,t-1} + \delta_{Xit}, \tag{56}$$

where, for the Y and X repeated measures, μ_{Yt} and μ_{Xt} are time-dependent intercepts, the random intercepts are η_{Y1i} and η_{X1i}, assumed to have zero means, and the equation errors are z_{Yit} and z_{Xit}. As before, the ρ coefficients represent the autoregressive and cross-lagged effects, with the first subscript indicating the error of the repeated measure receiving the effect, and the second subscript indicating the error of the variable sending the effect. Andersen (2022) demonstrates that this RI-CLPM is a restricted version of the EW1 ALT model without the random slope. Details on the relationships are reported in Appendix B. Using an analogous logic to that described in the previous section, this would enable a comparison of the unrestricted ALT model with the restricted ALT and the corresponding RI-CLPM, providing evidence on whether the unrestricted ALT better fits the data.

Two additional issues with using the RI-CLPM are how analysts can introduce bias if they use it without diagnostics and fit assessment. One issue is that the RI-CLPM *assumes* *stationarity* of the repeated measures, meaning the autoregressive coefficients for the same series are equal and have absolute values less than 1 (see Appendix B). A second issue is that the RI-CLPM typically *omits random slopes* for the repeated measures. Although it already incorporates time-varying intercepts to capture occasion-specific shifts in the overall mean, those intercepts often act as a stand-in for genuine slope heterogeneity when random slopes are absent. Time-varying intercepts absorb some of the unmodeled individual differences in change that an explicit slope term would capture, thereby reducing bias in the cross-lagged estimates. However, if substantive slope variability exists, intercepts alone can only partially correct for it, leaving residual confounding. Best practice is therefore to formally test for random slopes and, where supported by the data, include latent random slope or growth curve factors alongside time-varying intercepts to ensure both between-person and within-person dynamics are modeled without bias. This diagnostic check appears rare in practice.

Empirical application. In this section, we use the Minnesota Adolescent Community Cohort (MACC) data – where Y represents the adolescents' smoking intensity and X their perceived exposure to smoking in movies - to compare the bivariate models introduced in this section with the corresponding restricted ALT model together with more general versions of the ALT.

Table 5 illustrates how the four models provide a consistent explanation of how accounting for stable traits and systematic growth affects our estimates of autoregression, reciprocal effects (cross-lags), and individual heterogeneity. In the CLPM, which omits any trait or growth factors, both processes show high persistence: smoking intensity at one wave predicts itself with ρ_{YY} equal to 0.514, and perceived movie smoking exposure predicts itself with ρ_{XX} equal to 0.806. Small negative cross-lagged effects (ρ_{YX} = -0.013 and ρ_{XY} = -0.040) indicate slight reciprocal suppression; however, these estimates combine within-person dynamics with unmodeled between-person differences and growth.

Introducing random intercepts and expressing the autoregressive and cross-lagged on the errors rather than on the observed variables (RI-CLPM) absorbs stable between-person differences, and the apparent persistence shrinks to ρ_{XX}= 0.632 and ρ_{YY} = 0.170. Cross-lagged effects become effectively zero, indicating that the CLPM's reciprocal effects were largely artifacts of trait variance. The remaining intercept variances, equal to 0.142 and 0.652 for α_{yi} and α_{xi}, respectively, confirm substantial stable heterogeneity in both processes.

Analyzing the estimated time-varying intercepts - μ_{Y1} = 1.863, μ_{Y2} = 1.881, μ_{Y3} = 1.927, μ_{Y4} = 1.939, μ_{Y5} = 1.961, μ_{Y6} = 1.974 for Y, and μ_{X1} = 1.317, μ_{X2} = 1.432, μ_{X3} = 1.568, μ_{X4} = 1.786,

μ_{X5} = 1.851, μ_{X6} = 1.864 for X (all significantly different from zero) - reveals a clear upward trend in both processes, confirming that mean levels of Y and X increase monotonically over time.

Adding random slopes along with structured residual dynamics in the LCM-SR gives us a clearer picture of the within-person processes: residual autoregressive effects decrease to 0.322 for ρ_{XX} and to 0.092 for ρ_{YY}, while residual cross-lagged effects become essentially zero. The slope variances emerge, showing that adolescents differ significantly in their linear trends of smoking and movie exposure once trait effects are accounted for. A small negative covariance between the intercept and slope for movie exposure suggests that individuals who start at a higher level tend to increase less over time.

Finally, the unrestricted PW1 ALT reveals that within-person persistence in both processes decays over time: smoking intensity autoregressive coefficients decline from 0.367 in wave two down to 0.125 by wave six, and movie exposure autoregressive pattern from 0.115 down to 0.023. Cross-lagged effects now fluctuate around zero: the only moderately sized effect is the early influence of smoking on subsequent exposure (ρ_{x2y1}=0.060), but later intervals show small positive or negative effects that never resulted significantly different from zero. Growth heterogeneity remains evident, and interestingly, the covariance between the intercept and slope in smoking intensity flips negative, indicating that adolescents who start as more intense smokers tend to escalate slightly slower over time. These results differ from Usami et al. (2019). They also fitted a bivariate ALT model to these MAAC data to test autoregressive and cross-lagged effects between smoking intensity and exposure to smoking in movies. However, they reported that "*...the ALT model...resulted in an improper solution*." In their version, they constrained all autoregressive and cross-lagged coefficients, residual variances, and the residual covariances between smoking intensity and movie-related smoking at each wave to be time-invariant and homoscedastic - restrictions not supported by the data. By contrast, our PW1 ALT specification allows autoregressive paths, cross-lags, residual variances, and concomitant covariances to vary across waves, rather than assuming they are time-invariant. This flexible setup eliminates improper solutions throughout. Particularly, the chi-square is statistically significant while the other fit measures are excellent ($\chi^2(28)$ = 66.147, $p < .001$; CFI = 0.997; TLI = 0.994; RMSEA = 0.017; BIC = -170.153).

Table 5. *Parameter estimates (standard errors in parentheses) for the bivariate models fitted to smoke intensity (Y) and exposure to smoking in movies (X)*

<table>
<tr><th>Parameter</th><th>CLPM</th><th>LCM-SR</th><th>RI CLPM</th><th>PW1 ALT</th></tr>
<tr><td>ρ_{y2y1}</td><td rowspan="5">0.514 (0.006)</td><td rowspan="5">0.092 (0.014)</td><td rowspan="5">0.170 (0.011)</td><td>0.115 (0.000)</td></tr>
<tr><td>ρ_{y3y2}</td><td>0.092 (0.018)</td></tr>
<tr><td>ρ_{y4y3}</td><td>0.072 (0.016)</td></tr>
<tr><td>ρ_{y5y4}</td><td>0.049 (0.021)</td></tr>
<tr><td>ρ_{y6y5}</td><td>0.023 (0.030)</td></tr>
<tr><td>ρ_{x2x1}</td><td rowspan="5">0.806 (0.006)</td><td rowspan="5">0.322 (0.029)</td><td rowspan="5">0.632 (0.015)</td><td>0.367 (0.073)</td></tr>
<tr><td>ρ_{x3x2}</td><td>0.336 (0.053)</td></tr>
<tr><td>ρ_{x4x3}</td><td>0.311 (0.039)</td></tr>
<tr><td>ρ_{x5x4}</td><td>0.217 (0.040)</td></tr>
<tr><td>ρ_{x6x5}</td><td>0.125 (0.057)</td></tr>
<tr><td>ρ_{y2x1}</td><td rowspan="5">-0.013 (0.004)</td><td rowspan="5">-0.000 (0.010)</td><td rowspan="5">0.004 (0.007)</td><td>-0.002 (0.038)</td></tr>
<tr><td>ρ_{y3x2}</td><td>0.012 (0.023)</td></tr>
<tr><td>ρ_{y4x3}</td><td>-0.001 (0.015)</td></tr>
<tr><td>ρ_{y5x4}</td><td>-0.002 (0.013)</td></tr>
<tr><td>ρ_{y6x5}</td><td>-0.005 (0.017)</td></tr>
<tr><td>ρ_{x2y1}</td><td rowspan="5">-0.040 (0.013)</td><td rowspan="5">-0.017 (0.027)</td><td rowspan="5">-0.014 (0.023)</td><td>0.060 (0.048)</td></tr>
<tr><td>ρ_{x3y2}</td><td>0.016 (0.036)</td></tr>
<tr><td>ρ_{x4y3}</td><td>0.011 (0.031)</td></tr>
<tr><td>ρ_{x5y4}</td><td>-0.014 (0.039)</td></tr>
<tr><td>ρ_{x6y5}</td><td>-0.042 (0.056)</td></tr>
<tr><td>$V(\alpha_{yi})$</td><td>-</td><td>0.623 (0042)</td><td>0.142 (0.004)</td><td>0.168 (0.019)</td></tr>
<tr><td>$V(\beta_{yi})$</td><td>-</td><td>0.031 (0.004)</td><td></td><td>0.005 (0.001)</td></tr>
<tr><td>$C(\alpha_{yi}, \beta_{yi})$</td><td>-</td><td>0.013 (0.011)</td><td></td><td>–0.013 (0.004)</td></tr>
<tr><td>$V(\alpha_{xi})$</td><td>-</td><td>0.201 (0.008)</td><td>0.652 (0.032)</td><td>0.218 (0.103)</td></tr>
<tr><td>$V(\beta_{xi})$</td><td>-</td><td>0.005 (0.001)</td><td></td><td>0.026 (0.009)</td></tr>
<tr><td>$C(\alpha_{xi}, \beta_{xi})$</td><td>-</td><td>-0.016 (0.002)</td><td></td><td>0.033 (0.016)</td></tr>
</table>

Because most cross-lagged effects were non-significant across models, we also estimated constrained versions of the LCM-SR, RI-CLPM, and unrestricted PW1-ALT models in which all cross-lagged parameters were fixed to zero. The substantive conclusions based on the coefficient estimates remained unchanged.

Table 6. *Fit statistics of different bivariate models for smoking intensity (Y) and exposure to smoking in movies (X).*

	LCM-SR	*EW1 ALT equivalent to the LCM-SR*	*RI-CLPM*	*EW1 ALT equivalent to the RI-CLPM*	*Unrestricted PW1 ALT*
χ^2	174.719	174.719	395.214	395.214	71.349
df	48	48	55	55	38
p-value	0.000	0.000	0.000	0.000	0.000
CFI	0.991	0.991	0.977	0.977	0.998
TLI	0.988	0.988	0.972	0.972	0.996
RMSEA	0.024	0.024	0.036	0.037	0.014
BIC	-230.828	-230.828	-69.476	-69.476	-249.709

Table 6 shows the fit indices for the remaining models. However, unlike Usami et al. (2019), who further constrained concomitant residual covariances and repeated measure variances to be constant over time, we follow Andersen (2022) in allowing all residual covariances and variances to vary freely on each occasion, maintaining consistency with the generality of the unrestricted PW1 ALT model.

The overall fit of the CLPM was poor even allowing its cross-lagged effects (e.g., $\chi^2(56) = 1730.231$, CFI = 0.887, BIC =1256), so we do not include its overall fit statistics. Introducing random intercepts and an autoregressive pattern among the errors in the RI-CLPM markedly improves fit, demonstrating that modeling stable between-person differences removes much of the misfit. The LCM-SR further enhances model fit by adding random slopes and structured residual autoregressive paths, while still assuming time-invariant lagged effects. For both the RI-CLPM and LCM-SR, the corresponding restricted-ALT equivalent (as discussed in the Appendix) reproduces these exact fit indices, confirming their statistical equivalence. By contrast, the unrestricted PW1 ALT, which frees all autoregressive coefficients to vary across time, achieves the best overall fit statistics, underscoring the benefit of relaxing time-invariance to capture genuine within-person dynamics. Indeed, when each restrictive model is compared to the unrestricted PW1 ALT via a likelihood-ratio test, the loss of fit is highly statistically significant in every case. The RI-CLPM has a $\Delta\chi^2$ of 323.865 on $\Delta df = 17$ ($p < 0.001$), and the LCM-SR a $\Delta\chi^2$ of 103.37 on $\Delta df = 10$ ($p < 0.001$). These results confirm that each relaxation - from omitting growth (CLPM), to adding only random intercepts (RI-CLPM), to incorporating random slopes and structured residuals (LCM-SR) - makes a statistically significant contribution, and that freeing all autoregressive effects over time in the unrestricted PW1 ALT helps capture the within-person dynamics in the MACC data.

In summary, shifting from the CLPM to the RI-CLPM/EW1-ALT and then to the LCM-SR/EW1-ALT (see Table 6) gradually separates trait and growth from state dynamics, significantly reducing apparent inertia and spurious cross-lagged effects. Allowing each lag to vary in the PW1 ALT reveals the time-varying decay of persistence and the lack of strong reciprocal effects once all heterogeneity is accounted for. This would be hidden under any model that assumes constant lagged parameters.

Statistical versus substantive equivalence. In this section, we have discussed how constraints on ALT models can produce statistically equivalent versions of the Latent Curve Model with Autoregressive Errors (LCM-ARE), LCM-SR, and the RI-CLPM. This allows us to compare these restrictive ALT models to less restrictive versions and to contrast models with different structures. At the same time, we argue that *statistical* equivalence does not mean *substantive*ly equivalent. Here, we briefly describe how the ALT differs from these other models.

The LCM-SR and RI-CLPM are sometimes described as *automatically* separating the between- and within-components in longitudinal data. This description appears based on the model using either the random intercept alone (RI-CLPM) or with a random slope (LCM-SR) to represent the between-effects, while considering the error structure as the within-effects. Because the random intercepts and slopes are uncorrelated with the errors, this appears to create a clean separation of between- and within-components. However, this clear separation becomes less certain once we recognize that the uncorrelatedness of the errors and the intercepts and/or slopes is an *assumption,* not an inherent property of these models. There could be good reasons to question this assumption.

The error terms in LCM-ARE, LCM-SR, and RI-CLPM include all other influences on the repeated measures that are not part of the RHS of the model. Typically, when only the random intercepts/slopes are explicitly included as RHS variables, this implies that any time-varying variables affecting Y_{it} are part of the error. The random intercepts and random slopes are between-case components and might capture time-invariant variables affecting the repeated measures. But if there are other time-invariant influences on the repeated measures not represented by the random intercepts and slopes, then these could also be part of the error.

Could these omitted variables lead to a correlation between the errors and the random intercepts/slopes, thereby violating the assumption that errors are uncorrelated? If so, we must accept that the clear separation of between- and within-components is compromised due to potentially biased estimates of the LCM parameters. Therefore, the automatic distinctions in between- and within-components claimed by the LCM-ARE, LCM-SR, and RI-CLPM depend on the assumption that errors are uncorrelated with the random intercepts/slopes, which is questionable in many applications.

Substantively, the ALT model makes different assumptions. Its equation indicates that the current repeated measure (Y_{it}) depends on $Y_{i,t-1}$, even after controlling for the growth curve parameters. It allows $Y_{i,t-1}$ to correlate with the growth curve parameters rather than assuming no association. However, the ALT model does assume that the errors in its equation are uncorrelated with the RHS random intercepts/slopes and with $Y_{i,t-1}$. This assumption can also be violated by omitted time-varying or time-invariant variables in an ALT model. The difference is that analysts presenting the ALT model make no claim of automatic separation of between- and within-effects, whereas writings on LCM-ARE, LCM-SR, and RI-CLPM often do and view this as an advantage over the ALT. But as we have discussed, there is nothing automatic about the separation of between- and within-effects in any longitudinal models.

Summary. All three of the longitudinal models with complex errors discussed in this section share the characteristic that they equate to restrictive versions of ALT models. In fact, Andersen (2022, p. 749), who presents and illustrates these statistically equivalent model findings, states: "*The point of all this algebra is to make clear that the residual-level models are just the observation-level models [i.e., ALT models] with constraints on the relationships between the individual effects variable(s) and the initial observations*."

Furthermore, these three longitudinal models with complex errors make strong assumptions about stationarity, equal autoregressive coefficients, and ρs that meet specific conditions. A simpler model is applicable when these assumptions are valid, but biased coefficients result if they are not. Unfortunately, current applications of these models rarely test whether these assumptions are valid when testing is feasible.

Despite these issues, some argue that these models are easier to interpret. The next section of the paper examines these and other claims about the ALT and other longitudinal models.

ASSESSING LITERATURE ASSERTIONS ABOUT THE ALT MODEL

As the ALT model gained more widespread recognition and usage, so did discussions and critiques. Some of these criticisms apply to all statistical models, including the Classic ALT and the LV-ALT. These include the need to rely on unprovable assumptions, the risk of structural misspecifications such as omitted variables, the potential for using the incorrect functional form, the difficulties in establishing causal and noncausal relationships, and the approximate nature of all models. Research design, careful measurement, and causal assumptions grounded in expertise can reduce but not eliminate these issues in the ALT and all longitudinal models. These are important issues that analysts need to be aware of.

However, there are other claims about the ALT model that are either incorrect or, at best, debatable. We have read many peer-reviewed published assertions about the ALT model, and in this section, we highlight and classify seven major ones and evaluate their validity. Table 7 summarizes these assertions and includes citations to papers that document them. The citations in Table 7 are illustrative and not exhaustive. Our discussion will include additional references and more details for each point.

1. ***Identification Requirements***. There is some confusion in the literature about the minimum number of time points or waves of data needed for the ALT model to be identified. For example, Usami (2021, p. 333) states that: "*The ALT is identified if two or more variables have been measured at four or more time points when stationarity of parameters is assumed, while five or more time points are required under a non-stationarity assumption.*" The same assertion is also made by Usami et al. (2019, p. 641). However, Bollen and Curran (1999, 2004, pp. 375-376) demonstrate that the ALT model with Y_{i1} endogenous, equal autoregressive coefficients, $\rho_{t,t-1} = \rho$ for all t, and $|\rho| < 1$ is identified with three waves of data.

Even when we consider measurement error in repeated measures and only have one measure per latent variable, Bianconcini and Bollen (2018, pp. 795-796) show that, given the same conditions for the latent variables (*i.e.*, L_{i1} is endogenous, ρs are equal, and $|\rho| < 1$), it is possible to identify the LV-ALT with just three waves of data if the measurement error variances are equal and the measurement intercepts of the single indicators are zero. Therefore, the claim that at least four or five waves are always necessary contradicts earlier proofs.

Table 7. *Assertions and assessments about the autoregressive latent trajectory model*

Assertions	***Assessments***
1. ***Identification Requirements***	
- ALT requires $T \geq 4\text{-}5$ waves for identification. *(Usami, 2021, p. 33; Usami et al. 2019, p. 641).*	- $T = 3$ waves suffices if AR parameters are equal and Y_{i1} is endogenous. *(Bollen & Curran, 2004, pp. 382-384; Bollen, 2007, p. 135).*
2. ***Allowing Measurement Error***	
- ALT cannot incorporate measurement error. *(Usami et al., 2019, p. 648-649).*	- LV-ALT includes measurement error via unique factors. *(Bollen & Curran, 2004, pp. 375-376; Bianconcini & Bollen, 2018, pp. 792-794).*
3. ***Discrete Versus Continuous Time***	
- Discrete-time ALT models can misrepresent continuous processes and yield unreliable results if measurement intervals vary (*Delsing & Oud, 2008*).	- The Continuous-Time Autoregressive Latent Trajectory (CALT) model uses stochastic differential equations to represent latent processes in continuous time (*Delsing and Oud, 2008*).
4. ***ALT Parameter Interpretation***	
- ALT's intercept (α_i) and slope (β_i) cannot be interpreted like traditional LCM parameters due to autoregressive effects. *(Hamaker, 2005, pp. 409-410; Usami et al., 2019, p. 641).*	- Parameters represent growth in detrended data ($Y_{it} - \rho Y_{i,t-1}$), retaining interpretability. *(Bollen & Curran, 2004, pp. 377-378; Bianconcini & Bollen, 2018, p. 805).*
5. ***Between- & Within-Person Effects***	
- ALT conflates between-person traits and within-person fluctuations. *(Usami et al., 2019, p. 650).*	- Random intercept (α_i) and slope (β_i) model stable between-person differences. *(Bollen & Curran, 2004, pp. 372-373; Curran et al., 2014, p. 14).*
6. ***Accumulating Effects***	
- ALT's accumulating lagged effects complicate causal interpretation. *(Usami et al., 2019, p. 644).*	- Total effects are calculable; "accumulation" is inherent to AR models. *(Bollen & Curran, 2004, pp. 389--390; Bollen et al., 2007, p. 320).*
7. ***Model Selection***	
7.1 <u>ALT is too flexible</u>	
- ALT can generate diverse change trajectories, potentially confounding parameter interpretation. (Ou et al. 2017, p. 178).	- Starting from the most general model and applying robust criteria helps avoid accepting an oversimplified, incorrect model. (*Ye & Bollen, 2021).*
7.2 <u>Parsimony</u>	
- ALT is less parsimonious than LCMs. *(Hamaker, 2005, p. 414).*	- With equal ρ constraints, ALT and LCM-AR have identical parameters. *(Bollen & Curran, 2004, pp. 384--385; Bollen, 2007, p. 137).*
7.3 <u>Nonlinearity Obscuration</u>	
- ALT's AR components may mask nonlinear trends. *(Newsom, 2015, p. 209, Voelkle, 2008, p. 564).*	- Nonlinear extensions (*e.g.*, time-varying ρ) are possible. *(Bollen & Curran, 2004, p. 396; Grimm et al., 2013, p. 208).*

What about the claim from these same sources that "*two or more*" repeated measures are necessary for identification? The proofs from Bollen and Curran (2004, pp. 375-376) and Bianconcini and Bollen (2018, pp. 795-796) examine only a single repeated variable, which contradicts the claim that two or more repeated variables are required.

Since proof of identification in the three-wave model with equal autoregressive coefficients was provided in Bollen and Curran (2004), and years before these other articles, the reason for their incorrect claims remains unclear. We suspect it may be due to a failure to distinguish between the two forms of the ALT model discussed earlier in the paper. The predetermined Y_{i1} (PW1) form of the ALT requires four waves and equal autoregressive parameters for identification of a single repeated measure, whereas the endogenous (EW1) form can be identified with three.

2. ***Allowing measurement error***. A second assertion is that the ALT model does not allow measurement errors in the repeated measures. Usami et al. (2019, p. 639) refer to observed scores as equal to the sum of latent true scores and unique factors ("*measurement errors*"). They go on to classify longitudinal models that do and do not take account of unique factors (measurement errors), writing that "*the CLMP, the RI-CLMP, the ALT model, and the LCM-SR do not include unique factors*" (Usami et al., 2019, p. 643). This assertion is repeated in Usami (2021, p. 337). In contrast, over 15 years earlier, Bollen and Curran (2004, pp. 375-376) show a version of the ALT model with "*unique factors*" and latent variables and mention extending it to multiple indicators rather than single indicators of the repeated latent variables. And they state in the conclusion of the same paper (p. 377): "*…we can apply the ALT model to situations in which the repeated measure construct is operationalized via multiple indicator latent variables to evaluate dimensionality and estimate measurement error*." This preview of the LV-ALT is fully developed in Bianconcini and Bollen (2018). Thus, initial and later presentations of the ALT demonstrated how to incorporate unique factors and multiple indicators into what is now known as the LV-ALT model. Therefore, the assertion that the ALT model does not account for unique factors or measurement errors is incorrect.

3. ***Discrete versus continuous time***. The ALT model has been criticized for using discrete time analysis rather than continuous time. Discrete time analysis treats variables as they occur at points of data collection, such as the wave of data. The variables are taken as measured at these discrete time points, and these are the basis of analysis. Delsing and Oud (2008, p. 79), for example, argue that "*… discrete-time parameters provide us with only a snapshot of what happens in continuous time. These snapshots may lead to totally different conclusions in the case of different observation intervals. The stochastic differential equations approach does not* *only provide estimates of change, stabilities and reciprocities for the given time intervals like the discrete-time AR cross-lagged and ALT models do, it also enables to interpolate between observation time points and predict change, cross-lagged effects and stabilities as a function of time, putting processes in a continuous-time framework*."

This comment applies not only to the ALT model, but also to all longitudinal models based on discrete time. Delsing and Oud (2008) provide a continuous time ALT model that addresses this critique. The continuous versus discrete time issue is one that bears further investigation because there is insufficient evidence to determine its full impact for the ALT, as well as all the other longitudinal models common in practice.

4. ***ALT parameter interpretation***. Several authors assert that intercept (α_i) and slope (β_i) from the ALT model cannot be interpreted like traditional LGCM parameters due to the presence of Y_{t-1}'s autoregressive effects on Y_t. Hamaker (2005) provides an early example of this claim. Hamaker (2005) demonstrates that when the autoregressive coefficient ρ is equal across all waves and its absolute value is less than 1, the ALT model and a latent growth curve model with autoregressive errors are empirically equivalent. But she goes on to argue that the LGCM with equal *autoregressive errors* has the usual interpretations of the random intercept and random slope, whereas this is not true for the ALT model: "*Only when $\rho = 0$ do α and β serve as the intercept and slope…*" (p. 410).

Almost 15 years later, Usami et al. (2019, p. 642) make the same claim: "*Rather, the advantage of the LCM-SR [Latent Curve Model – Structured Residuals] over the ALT model is that the growth parameters have a clearer interpretation than the accumulating parameters from the ALT model. In the LCM-SR, growth parameters can be interpreted as the intercept and slope, similar to these parameters in the LCM.*" Related arguments arise in Usami (2021).

When assessing these assertions, several points come to mind. One is that none of the methodological presentations of the ALT model of which we are aware asserts that the random intercept and random slope are interpreted the same as in a growth curve model of Y_t without the presence of Y_{t-1} on the RHS of the equation. This is a "*strawman*" argument, or we could consider it a cautionary note. Second, the situation is similar in any LGCM when, in addition to the random intercept and random slope, a time-varying covariate affects the repeated measure. For instance, a researcher who specifies an equation of $Y_{it} = \alpha_i + \lambda_t \beta_i + \gamma_{yt} X_{it} + \varepsilon_{it}$ cannot interpret α_i as the intercept and β_i as the slope of the expected trajectory of Y_{it} for the *i*th case without considering the value of X_{it}. So, to generalize, the usual interpretation of α_i and β_i as describing the intercept and slope of the trajectory of the repeated measure does not hold when covariates are also in the Y_{it} equation. Third, in the first half of this paper, we argued that

α_i and β_i from the ALT model *do have* an interpretation like a LGCM if we focus on a transformation of the repeated measure. The α_i is the intercept and β_i is the slope of $Y^{\#}_{it} = (Y_{it} - \rho_{t,t-1} Y_{i,t-1})$. In other words, the usual growth curve model interpretation holds for the dependent variable once we subtract the autoregressive term. So contrary to earlier claims, the random intercept and random slope are interpretable like LGCM counterparts even when ρ is not zero for $Y^{\#}_{it}$.

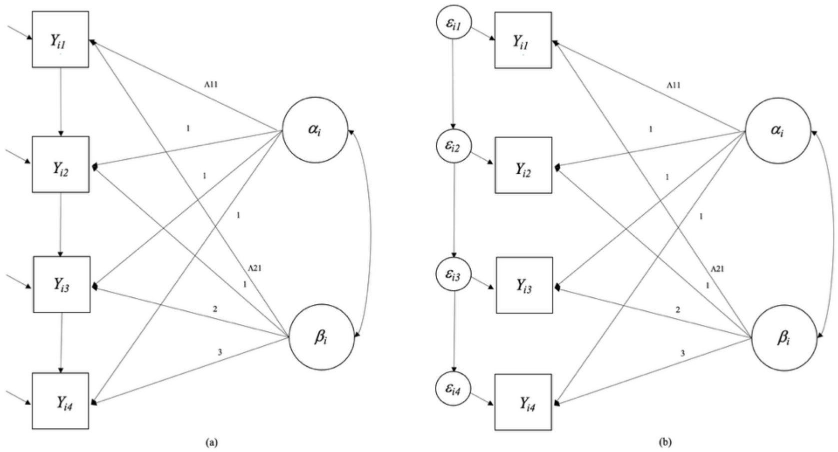
5. *Between- and within-person effects*. Another claim made in several sources is that the ALT model confuses between-person characteristics and within-person fluctuations. Curran et al. (2014, p. 884) have a clear statement of this position when discussing an ALT model that has two repeated measures with random intercepts and random slopes for each series: "*in the ALT model the repeated measures of one construct serve as mediators for the influence of the latent curves of that construct on the indicators of the other construct. Because of these mediated influences, the ALT model does not provide a pure disaggregation of the between- and within-person relations over time*."

Similarly, Usami et al. (2019, p. 650) refer to the random intercepts (αs) and random slopes (βs) of two series of repeated measures as the "*accumulating factors (A and B)*" and state that "*Accumulating factors (A and B), which are included in the ALT ... models, do not indicate stable individual differences, because their influences accumulate through the dynamics of the process over time. In these models, the lagged relations are not separated from the accumulating factors, and as a result, the accumulating factors have both direct and indirect effects on the observed scores. The critical point is that two primary analytic purposes in applying cross-lagged models (i.e., inferring reciprocal effects between variables, and describing latent trajectories of individuals) are intertwined in these models, such that the lagged relations represent both the short-term dynamics and the long-term developmental trajectories.*" See also Usami (2021) and Andersen (2022).

When these assertions are compared to our earlier discussions of between- and within-effects, there appear to be different conclusions. As previously stated in the paper, we refer to between-components as enduring differences between cases (*e.g.*, individuals), while within-components are changes around these enduring effects for individual cases. With this definition, it is clear that the random intercepts (αs) and random slopes (βs) in the ALT model represent differences between cases that persist throughout the observation period in the longitudinal study. As such, the αs and βs in the ALT model are between-case components. The lagged repeated measures are variables that can take different values over the longitudinal study, and at least some, if not nearly all, of the lagged values do change. Moreover, the random intercepts and slopes in the ALT models account for the between-components of the repeated measures, so the lagged variable is a within-component with its between component explicitly controlled by the growth parameters. Consequently, the autoregressive coefficients indicate within-effects, although in some restricted versions of the ALT we have considered, the autoregressive coefficients ($\rho_{t,t-1}$) are the same for all cases at each wave, even if the lagged variable varies over time.

Is there a way to reconcile this and the earlier discussion that distinguishes the between- and within-components in the ALT model with the view that this cannot be done as in the previous quotes? We believe so and refer to Figure 6 to help in our discussion.

Figure 6. *Path diagram of the (a) EW1 ALT model and (b) latent curve model with autoregressive errors*



[Figure 6 About Here]

Figure 6(a) is a path diagram of the Endogenous Wave 1 (EW1) version of the Classic ALT model, while Figure 6(b) is a path diagram of a Latent Growth Curve Model with Autoregressive Errors (LGCM-AE). Each diagram has one repeated measure to keep them simple, although the points raised apply to two or more repeated measures. The primary difference between the Classic ALT and the LGCM-AE models lies in the location of the autoregressive relations. In the Classic ALT, the autoregression is between the repeated measures, whereas it is between the errors in the LGCM-AE model.

The critiques argue that the separation of between- and within-components is unambiguous within the LGCM-AE, but not in the Classic ALT. Using the language of direct, indirect, and total effects from the SEM literature can clarify these two positions. Examining the LGCM-AE model the random intercept (α_i) and random slope (β_i) have direct effects on the repeated measures which correspond to the paths from them to the observed variables. The errors from this model have direct and indirect effects from errors in the earlier wave to those in later waves.

Moving to the Classic ALT model, the random intercept (α_i) and random slope (β_i) have direct effects on the repeated measures, just like in the LGCM-AE model. However, the random intercept and random slope also have *indirect* effects on the repeated measures at later periods. For instance, α_i has a direct effect on Y_{i2}, but also an indirect effect on Y_{i3} and later repeated measures. The quoted authors suggest that it is okay for the random intercepts (α_i) and random slopes (β_i) to have direct effects and to still be classified as between-components as in the LGCM-AE. However, if the random intercepts (α_i) and random slopes (β_i) also have indirect effects on the lagged repeated measures, then this rules out classifying them as between-components. We do not agree that if the growth parameters in the ALT model have both direct and indirect effects on some of the repeated measures, then they are no longer providing information on between-components, while, in contrast, if they only have direct effects on the repeated measures, then they are valid between-component variables. The question of whether a variable represents between-case influences is whether its values for the same case remain the same over the observation period, not the nature of its indirect effects.

Furthermore, researchers can separate the direct, indirect, and total effects of the random intercepts (α_i) and random slopes (β_i) on any of the repeated measures using well-established mediation analysis formulas (*e.g.*, Bollen, 1987) implemented in widely used SEM software. Or analysts using the path diagram can employ the tracing rules from path analysis to study the types of effects. So, "*disentangling*" and separating the between-component variables' influence on the repeated measures is straightforward.

As discussed earlier, there are models developed to automatically distinguish between- and within-components, such as the LCM-ARE, LCM-SR, and RI-CLPM; however, they do so by assuming that the errors are uncorrelated with random intercepts/slopes. The ALT model separates between-components as those from the growth curve and within-components as those from the lagged variable, while controlling for the between-components of the growth curve. However, it also assumes that the errors are uncorrelated with the RHS variables. Therefore, none of these models automatically separates between- and within-components if these assumptions are violated.

6. *Accumulating effects*. The "*accumulation effects*" claim about the ALT model is closely related to the between- and within-components arguments and is made by some of the same authors. Indeed, the quote from Usami et al. (2019, p. 650) illustrates their ideas of referring to the random intercepts (α_i) and random slopes (β_i) in the ALT model having accumulating effects.

7. ***Model selection*****.**

7.1. *Model selection and flexibility*. Another criticism of ALT models concerns the ambiguous model selection that arises from their hybrid nature. Because ALT combines growth and autoregressive processes, it can closely approximate a wide range of data-generating mechanisms. Ou et al. (2017, p. 178) warn that "*the ALT model is flexible enough to produce a variety of discrepant model-implied change trajectories. ... this may confound interpretations of the model's parameters*". The empirical examples from our paper illustrate how to fit and compare different longitudinal models and we have done the same with other examples on which we have worked. Furthermore, Ye and Bollen's (2021, p. 7) simulation research suggests that the ALT is useful in choosing a longitudinal model under many conditions. They conclude that "*a good empirical practice is to start with the most general model and reduce it to more restrictive forms until one reaches the most parsimonious representation with roughly or very consistent goodness-of-fit results from the most general model and applying robust criteria*". We also showed that several longitudinal models were restricted versions of the ALT model, such that likelihood ratio tests and fit indexes have been used in our empirical application to choose among alternative models. As previously discussed, we recall that this raises an additional point regarding the distinction between *statistical equivalence* and *substantive equivalence*. The choice among different model specifications should consider the underlying assumptions of each model.

7.2. *Parsimony*. Hamaker (2005, p. 414) noted that treating the first observed score in the ALT model as predetermined "*three extra parameters must be estimated*" compared to an LCM. However as we discussed earlier, under the constraints of time-invariant autoregressive coefficients $\rho_{t,t-1}= \rho$ and $|\rho| < 1$ the EW1 ALT model and the LGCM-AE have an identical number of parameters. Thus, a restricted ALT model need not be less parsimonious than an LCM with structured residuals; it only becomes more complex when one allows time-varying or higher-order autoregressive effects. Researchers should therefore test whether such constraints are tenable before adopting a more general ALT specification.

7.3. *Nonlinearity obscuration*. A common criticism in several studies is that a linear ALT model can inadvertently absorb true nonlinear growth into its autoregressive component. Voelkle (2008, p. 564) showed through a simulation study that "*the autoregressive process incorrectly accounts for part of this nonlinearity, thus rendering any substantive interpretation* *of parameter estimates virtually impossible*." Bauldry and Bollen (2018) found that fitting a linear ALT to data with underlying curvature caused the model to underestimate the intercept and overestimate the slope mean to "*compensate for an unmodeled nonlinear pattern*" (p. 285). To address this issue, these authors introduce nonlinear ALT models to "..*address a concern with linear ALT models that they may be misspecified when fit to nonlinear trajectories*" (p. 297). These nonlinear ALT specifications help researchers distinguish genuine curvature from carryover effects, reducing bias and enhancing interpretability. It also seems likely that nonlinearity could manifest in autoregressive and cross-lagged errors with the LGCM-AE, LCM-SR, and RI-CLPM when analysts fail to check for it. So, checks for nonlinearity are prudent regardless of model choice.

CONCLUSIONS

This paper provided an overview of 25 years of research on Autoregressive Latent Trajectory (ALT) models, clarified common misconceptions, and identified many widely used longitudinal models that are parameter-restricted versions of the ALT framework. We summarized the specification and assumptions of the Classic ALT and the Latent Variable ALT (LV-ALT) models, distinguished the between- and within-person components, and clarified how different treatments of the first wave (PW1, EW1, HI1) affect parameter interpretation. We also showed that many seemingly distinct models—such as latent curve models with structured residuals and random-intercept cross-lagged panel models—are statistically equivalent to restricted ALT specifications, underscoring the ALT model's unifying role.

Our empirical example demonstrated that flexible ALT specifications can avoid convergence problems that arise when unsupported homogeneity constraints are imposed. Our results showed that adolescents differ primarily in baseline smoking propensity rather than in rates of change, and that autoregressive effects weaken with age. Importantly, once the model accounted for growth and carryover processes, cross-lagged effects between smoking and exposure to smoking in movies were nonsignificant. This illustrates how ALT models can disentangle stable individual differences, developmental change, and short-term dynamics.

By clarifying the properties of ALT models, dispelling common misconceptions, and illustrating their application, we aim to support researchers in selecting longitudinal models that best reflect their theories and data. Because ALT models can specialize to purely autoregressive, purely growth curve, or hybrid processes, analysts are well served by beginning with general ALT specifications and testing restrictions suggested by theory or parsimony.

Careful parameter interpretation is essential, particularly for random intercepts and slopes when the initial observation is treated differently across model variants. Clear reporting of model assumptions, the treatment of wave 1, and whether time-varying covariates are decomposed into between- and within-person components will improve transparency and comparability across studies.

Future work can extend ALT models along several dimensions. Promising directions include further development in continuous time, the incorporation of more flexible nonlinear growth functions, and improved diagnostics for situations in which random slopes are necessary. Additional research is needed on ALT applications with categorical or zero-inflated outcomes and on strategies for modeling multiple interrelated processes. Progress in these areas will further enhance the usefulness of ALT models for a wide range of substantive questions.

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APPENDIX

A. LCM-SR and equivalent EW1 ALT model

Given eqs. (53)-(56), the LCM-SR can be rewritten in reduced form as follows

$$Y_{it} = [(1-\rho_{YY})\eta_{Y1i} - \rho_{YX}\eta_{X1i} + \rho_{YY}\eta_{Y2i} + \rho_{YX}\eta_{X2i}] + \lambda_t[(1-\rho_{YY})\eta_{Y2i} - \rho_{YX}\eta_{X2i}] + \rho_{YY}Y_{i,t-1} + \rho_{YX}X_{i,t-1} + \delta_{Yit}$$
$$X_{it} = [(1-\rho_{XX})\eta_{X1i} - \rho_{XY}\eta_{Y1i} + \rho_{XX}\eta_{X2i} + \rho_{XY}\eta_{Y2i}] + \lambda_t[(1-\rho_{XX})\eta_{X2i} - \rho_{XY}\eta_{Y2i}] + \rho_{XX}X_{i,t-1} + \rho_{XY}Y_{i,t-1} + \delta_{Xit}$$

such that the equivalent ALT model is defined as

$$Y_{it} = \alpha_{Yi} + \lambda_t\beta_{Yi} + \rho_{YY}Y_{i,t-1} + \rho_{YX}X_{i,t-1} + \varepsilon_{Yit}$$
$$X_{it} = \alpha_{Xi} + \lambda_t\beta_{Xi} + \rho_{XX}X_{i,t-1} + \rho_{XY}Y_{i,t-1} + \varepsilon_{Xit}$$

for $t=2,\ldots,T$, and where

$\alpha_{Yi} = [(1-\rho_{YY})\eta_{Y1i} - \rho_{YX}\eta_{X1i} + \rho_{YY}\eta_{Y2i} + \rho_{YX}\eta_{X2i}]$

$\beta_{Yi} = [(1-\rho_{YY})\eta_{Y2i} - \rho_{YX}\eta_{X2i}]$

$\alpha_{Xi} = [(1-\rho_{XX})\eta_{X1i} - \rho_{XY}\eta_{Y1i} + \rho_{XX}\eta_{X2i} + \rho_{XY}\eta_{Y2i}]$

$\beta_{Xi} = [(1-\rho_{XX})\eta_{X2i} - \rho_{XY}\eta_{Y2i}]$

$\varepsilon_{Yit} = \delta_{Yit}$

$\varepsilon_{Xit} = \delta_{Xit}$

Recalling that, in the LCM-SR,

$$Y_{i1} = \eta_{Y1i} + \delta_{Yi1}$$
$$X_{it} = \eta_{X1i} + \delta_{Xi1}$$

In the equivalent EW1 ALT model,

$$Y_{i1} = \Lambda_{Y\alpha Y}\alpha_{Yi} + \Lambda_{Y\beta Y}\beta_{Yi} + \Lambda_{Y\alpha X}\alpha_{Xi} + \Lambda_{Y\beta X}\beta_{Xi} + \varepsilon_{Yi1}$$
$$X_{i1} = \Lambda_{X\alpha X}\alpha_{Xi} + \Lambda_{X\beta X}\beta_{Xi} + \Lambda_{X\alpha Y}\alpha_{Yi} + \Lambda_{X\beta Y}\beta_{Yi} + \varepsilon_{Xi1}$$

where the coefficients $\Lambda_{Y\alpha Y}$, $\Lambda_{Y\beta Y}$, $\Lambda_{Y\alpha X}$, $\Lambda_{Y\beta X}$, $\Lambda_{X\alpha X}$, $\Lambda_{X\beta X}$, $\Lambda_{X\alpha Y}$, $\Lambda_{X\beta Y}$ are determined by observing that

$$\begin{pmatrix}\alpha_{Yi}\\ \alpha_{Xi}\\ \beta_{Yi}\\ \beta_{Xi}\end{pmatrix} = \begin{pmatrix} 1-\rho_{YY} & -\rho_{YX} & \rho_{YY} & \rho_{YX}\\ -\rho_{XY} & 1-\rho_{XX} & \rho_{XY} & \rho_{XX}\\ 0 \quad 0 & & 1-\rho_{YY} & -\rho_{YX}\\ 0 \quad 0 & & -\rho_{XY} & 1-\rho_{XX}\end{pmatrix}\begin{pmatrix}\eta_{Y1i}\\ \eta_{X1i}\\ \eta_{Y2i}\\ \eta_{X2i}\end{pmatrix}$$

such that

$$\begin{pmatrix}\eta_{Y1i}\\ \eta_{X1i}\end{pmatrix} = D^{-1}\begin{pmatrix}\alpha_{Yi}\\ \alpha_{Xi}\end{pmatrix} - D^{-1}BD^{-1}\begin{pmatrix}\beta_{Yi}\\ \beta_{Xi}\end{pmatrix}$$

with $D^{-1} = \frac{1}{(1-\rho_{YY})(1-\rho_{XX})-\rho_{YX}\rho_{XY}}\begin{pmatrix}(1-\rho_{XX}) & \rho_{YX}\\ \rho_{XY} & (1-\rho_{YY})\end{pmatrix}$ and $B = \begin{pmatrix}\rho_{YY} & \rho_{YX}\\ \rho_{XY} & \rho_{XX}\end{pmatrix}$.

It follows that

$\Lambda_{Y\alpha Y} = (1-\rho_{XX})/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]$

$\Lambda_{Y\beta Y} = -[(1-\rho_{XX})^2\rho_{YY} + \rho_{YX}\rho_{XY}(2-\rho_{XX})]/\,)/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]^2$

$\Lambda_{Y\alpha X} = \rho_{YX}/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]$

$\Lambda_{Y\beta X} = -\rho_{YX}[(1-\rho_{XX}\rho_{YY} + \rho_{XY}\rho_{XX}]/\,)/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]^2$

$\Lambda_{X\alpha Y} = \rho_{XY}/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]$

$\Lambda_{X\alpha X} = (1-\rho_{YY})/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]$

$\Lambda_{X\beta Y} = -\rho_{XY}[(1-\rho_{XX}\rho_{YY} + \rho_{XY}\rho_{XY}]/\,)/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]^2$

$\Lambda_{X\beta Y} = -[(1-\rho_{YY})^2\rho_{XX} + \rho_{YX}\rho_{XY}(2-\rho_{YY})]/\,)/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\rho_{XY}]^2$

This shows that the LCM-SR can be reexpressed as a corresponding EW1 ALT model in which the initial variables, Y_{i1} and X_{i1}, are each linear functions of both the random intercepts, α_{Yi} and α_{Xi}, and random slopes, β_{Yi} and β_{Xi}.

B. RI-CLPM and its equivalent EW1 ALT model

We proceed similarly for the RI-CLPM, whose reduced-form is given by

$$Y_{it} = [\mu_{Yt} - \rho_{YY}\,\mu_{Yt-1} - \rho_{YX}\,\mu_{Xt-1}] + [(1-\rho_{YY})\eta_{Y1i} - \rho_{YX}\,\eta_{X1i}] + \rho_{YY}\,Y_{i,t-1} + \rho_{YX}X_{i,t-1} + \delta_{Yit}$$
$$X_{it} = [\mu_{Xt} - \rho_{XX}\,\mu_{Xt-1} - \rho_{XY}\,\mu_{Yt-1}] + [(1-\rho_{XX})\eta_{X1i} - \rho_{XY}\,\eta_{Y1i}] + \rho_{XX}X_{i,t-1} + \rho_{XY}\,Y_{i,t-1} + \delta_{Xit}$$

The equivalent ALT model is defined as

$$Y_{it} = \tau_{Yt} + \alpha_{Yi} + \rho_{YY}\,Y_{i,t-1} + \rho_{YX}X_{i,t-1} + \varepsilon_{Yit}$$
$$X_{it} = \tau_{Xt} + \alpha_{Xi} + \rho_{XX}X_{i,t-1} + \rho_{XY}\,Y_{i,t-1} + \varepsilon_{Xit}$$

for $t=2,\ldots,T$, where $\tau_{Yt} = [\mu_{Yt} - \rho_{YY}\, \mu_{Yt-1} - \rho_{YX}\, \mu_{Xt-1}]$, $\tau_{Xt} = [\mu_{Xt} - \rho_{XX}\, \mu_{Xt-1} - \rho_{XY}\, \mu_{Yt-1}]$, $\alpha_{Yi} = [(1-\rho_{YY})\eta_{Y1i} - \rho_{YX}\, \eta_{X1i}]$ and $\alpha_{Xi} = [(1-\rho_{XX})\ \eta_{X1i} - \rho_{XY}\, \eta_{Y1i}]$.

Recalling that, in the RI-CLPM,

$$Y_{i1} = \mu_{Y1} + \eta_{Y1i} + \delta_{Yi1}$$

$$X_{i1} = \mu_{X1} + \eta_{X1i} + \delta_{Xi1}$$

the equivalent EW1 ALT model results

$$Y_{i1} = \tau_{Y1} + \Lambda_{Y\alpha Y}\, \alpha_{Yi} + \Lambda_{Y\alpha X}\, \alpha_{Xi} + \varepsilon_{Yi1}$$

$$X_{i1} = \tau_{Xt} + \Lambda_{X\alpha X}\, \alpha_{Xi} + \Lambda_{X\alpha Y}\, \alpha_{Yi} + \varepsilon_{Xi1}$$

where $\tau_{Y1} = \mu_{Y1}$, $\tau_{X1} = \mu_{X1}$, and the coefficients $\Lambda_{Y\alpha Y}$, $\Lambda_{Y\alpha X}$, $\Lambda_{X\alpha X}$, $\Lambda_{X\alpha Y}$ are determined by

$$\begin{pmatrix} \eta_{Y1i} \\ \eta_{X1i} \end{pmatrix} = \begin{pmatrix} 1-\rho_{YY} & -\rho_{YX} \\ -\rho_{XY} & 1-\rho_{XX} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{Yi} \\ \alpha_{Xi} \end{pmatrix}$$

such that

$\Lambda_{Y\alpha Y} = (1-\rho_{XX})/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\,\rho_{XY}]$

$\Lambda_{Y\alpha X} = \rho_{YX}/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\,\rho_{XY}]$

$\Lambda_{X\alpha X} = (1-\rho_{YY})/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\,\rho_{XY}]$

$\Lambda_{X\alpha Y} = \rho_{XY}/[(1-\rho_{XX})(1-\rho_{YY}) - \rho_{YX}\,\rho_{XY}]$

As for the LCM-SR, the RI-CLPM is equivalent to an EW1 ALT model in which the initial variables, Y_{i1} and X_{i1}, are each linear functions of both the random intercepts, α_{Yi} and α_{Xi}.