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# ERRATUM TO "ORLIK-SOLOMON-TYPE PRESENTATIONS FOR THE COHOMOLOGY ALGEBRA OF TORIC ARRANGEMENTS"

FILIPPO CALLEGARO, MICHELE D'ADDERIO, EMANUELE DELUCCHI, LUCA MIGLIORINI, AND ROBERTO PAGARIA

Abstract. In this short note we correct the statement of the main result of [1]. That paper presented the rational cohomology ring of a toric arrangement by generators and relations. One of the series of relations given in [1] is indexed over the set circuits in the arrangement's arithmetic matroid. That series of relations should however be indexed over all sets $X$ with $|X| = \mathrm{rk}(X)+1$. Below we give the complete and correct presentation of the rational cohomology ring.

We state the correct version of [1, Theorem 6.13]:

**Theorem 1.** *Let $\mathcal{A}$ be an essential toric arrangement. The rational cohomology algebra of the complement $H^*(M(\mathcal{A}),\mathbb{Q})$ is isomorphic to the algebra $\mathcal{E}$ with*

- *Set of generators $e_{W,A;B}$, where $W$ ranges over all layers of $\mathcal{A}$, $A$ is a set generating $W$ and $B$ is disjoint from $A$ and such that $A \sqcup B$ is an independent set; the degree of the generator $e_{W,A;B}$ is $|A \sqcup B|$.*
- *The following types of relations:*
  - *For any two generators $e_{W,A;B}$, $e_{W',A';B'}$,*
  $$e_{W,A;B}e_{W',A';B'} = 0$$
  *if $A \sqcup B \sqcup A' \sqcup B'$ is a dependent set, and otherwise*
  $$e_{W,A;B}e_{W',A';B'} = (-1)^{\ell(A\cup B, A'\cup B')} \sum_{L\in\pi_0(W\cap W')} e_{L,A\cup A';B\cup B'}. \tag{1}$$
  - *For every linear dependency $\sum_{i\in E} n_i\chi_i = 0$ with $n_i \in \mathbb{Z}$, a relation*
  $$\sum_{i\in E} n_i e_{T,\emptyset;\{i\}} = 0. \tag{2}$$
  - *For every subset $X \subseteq E$ where $\mathrm{rk}(X) = |X| - 1$ write $X = C \sqcup F$ with $C$ the unique circuit in $X$. Consider the associated (unique) linear dependency $\sum_{i\in C} n_i\chi_i = 0$ with $n_i \in \mathbb{Z}$, and for every connected*

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*component $L$ of $\cap_{i\in X} H_i$ a relation*

$$\sum_{j\in C} \sum_{\substack{A,B\subset X\\ A\supseteq F\\ X=A\sqcup B\sqcup\{j\}\\ |B| \text{ even}\\ W\supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A\cup B)} e_{W,A;B} = 0 \tag{3}$$

*where, for all $i \in C$, $c_i := \operatorname{sgn} n_i$, $c_B = \prod_{i\in B} c_i$.*

The only difference between Theorem 1 and [1, Theorem 6.13] consists in eq. (3) that hold not only for every circuit $C$ but also for all $X$ with $|X| = \operatorname{rk}(X)+1$. This difference is important as shown in the following example.

**Example 2.** Consider the central toric arrangement in $T = (\mathbb{C}^*)^3$ given by the four hypertori $H_1 = \{x = 1\}$, $H_2 = \{y = 1\}$, $H_3 = \{xy = 1\}$, and $H_4 = \{xy^{-1}z^3 = 1\}$. The zero-dimensional layers are the points $p = (1, 1, 1)$, $q = (1, 1, \zeta_3)$, and $r = (1, 1, \zeta_3^2)$. Let $W$ be the layer $H_1 \cap H_2 \cap H_3$. The relations given by (3) are the following:

$$\overline{\omega}_{W,\{1,2\}} - \overline{\omega}_{W,\{1,3\}} + \overline{\omega}_{W,\{2,3\}} + \psi_1\psi_2 - \psi_1\psi_3 - \psi_2\psi_3 = 0$$
$$\overline{\omega}_{s,\{1,2,4\}} - \overline{\omega}_{s,\{1,3,4\}} + \overline{\omega}_{s,\{2,3,4\}} + \frac{1}{3}\psi_1\psi_2\overline{\omega}_4 - \frac{1}{3}\psi_1\psi_3\overline{\omega}_4 - \frac{1}{3}\psi_2\psi_3\overline{\omega}_4 = 0$$

for all $s = p, q, r$. Notice that the relation in degree two does not imply the relations in degree three.

The proof of Theorem 1 is the same as in [1] with the following corrections (see also the preprint arXiv:1806.02195v3).

Let $X \subseteq E$ with $|X| = \operatorname{rk}(X)+1$, then $X$ can be written uniquely as $C \sqcup F$ where $C$ is a circuit and $F = X \setminus C$. Theorem 6.12 of [1] holds in a wider generality: the set $C$ does not need to be a circuit but can be any $X \subseteq E$ with $|X| = \operatorname{rk}(X) + 1$. It can be proven by choosing a suitable separating cover of $X$: for $i \in C$ define $a_i = m(X) \prod_{j\in C\setminus\{i\}} m(C \setminus \{j\})$ and $a_i = m(X)$ for $i \in F$. Let $\Lambda(X)$ be the lattice generated by $\frac{\chi_i}{a_i}$ for all $i \in X$, it defines a covering $\pi_U \colon U \to T$ of tori (cf. [1, Definition 6.6]).

Lemmas 6.3, 6.4, 6.5, 6.7, 6.8, and 6.10 of [1] should be corrected by changing $C$ with $X$ and with minor changes in their proofs. The proof of [1, Theorem 6.12] needs an extra step: let $L$ be a connected component of $\bigcap_{i\in X} H_i$ and $p$ a point in $L$, we use the relation (12) of [1] in the torus $U$ to obtain:

$$\sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} \overline{\eta}^U_{A,B}(q) c_B = 0.$$

for all $q \in \pi_U^{-1}(p)$. We multiply this equation by $\overline{\eta}^U_{F,\emptyset}(q)$, and using the equality $\overline{\eta}^U_{A,B}(q)\overline{\eta}^U_{F,\emptyset}(q) = (-1)^{|F_{\leq j}|+\ell(C,F)}\overline{\eta}^U_{A\sqcup F,B}(q)$ we obtain:

$$\sum_{j\in C} \sum_{\substack{A,B\subset X\\ A\supseteq F\\ X=A\sqcup B\sqcup\{j\}\\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} \overline{\eta}^U_{A,B}(q) c_B = 0.$$

This corrects the proof of [1, Theorem 6.12].

In the proof of Theorem 6.13 it was claimed that the old relations allow one to write each $e_{W,A;B}$ in term of generators with $A$ a no-broken-circuit set. In the following we prove the claim by using the new relations.

Indeed if $A$ contains a broken-circuit $A_1 \subseteq A$, i.e. $A_1 = C \setminus \min(C)$ for a circuit $C$, consider the relation (3) for $X = C \cup A$: it expresses the element $e_{W,A;B}$ as a linear combination of some $e_{W',A';B'}$ with $|A'| < |A|$ or with $|A'| = |A|$ and $A'$ lexicographically smaller than $A$. We have inductively proved the claim.

## References

[1] Filippo Callegaro, Michele D'Adderio, Emanuele Delucchi, Luca Migliorini, and Pagaria Roberto, *Orlik-Solomon type presentations for the cohomology algebra of toric arrangements*, Trans. Amer. Math. Soc. **373** (2019), no. 3, 1909–1940.

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