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The impact of intrinsic scaling on the rate of extinction for anisotropic non-Newtonian fast diffusion

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ABSTRACT

We study the decay towards the extinction that pertains to local weak solutions to fully anisotropic equations whose prototype is

$$\partial_t u = \sum_{i=1}^{N} \partial_i(|\partial_i u|^{p_i-2}\partial_i u), \qquad 1 < p_i < 2.$$

Their rates of extinction are evaluated by means of several integral Harnack-type inequalities which constitute the core of our analysis and that are obtained for anisotropic operators having full quasilinear structure. Different decays are obtained when considering different space geometries. The approach is motivated by the research of new methods for strongly nonlinear operators, hence dispensing with comparison principles, while exploiting an intrinsic geometry that affects all the variables of the solution.

1. Introduction

For an open bounded set $\Omega \subset \mathbb{R}^N$ and a positive time T, we consider anisotropic differential equations whose prototype is the following

$$\partial_t u - \Delta_{\mathbf{p}} u := \partial_t u - \sum_i \partial_i(|\partial_i u|^{p_i-2}\partial_i u) = 0, \qquad \text{weakly in} \qquad \Omega_T = \Omega \times [0,T]. \tag{1.1}$$

Differential operators as $(\partial_t - \Delta_{\mathbf{p}})$ above appear already in the seminal work [25], in the guise of the prototype example of operators obtained as the sum of monotone ones. They enjoy many interesting properties (see for instance the book [4]) whose interpretation has led to a rich mathematical theory (see for instance [6,9,29,30]). Nonetheless, even after more than half a century, the basic regularity properties of local weak solutions to Eqs. (1.1) remain an open problem (see for instance [1,8,11]). Besides the theoretical intrinsic interest and challenge, this kind of equations appear in various physical contexts (see Chap. IV of [2]), unveiling the mathematical description of diffusion processes for which the propagation has a different non-Newtonian behavior along each coordinate axis; as well as modeling electro-rheological fluids (see for instance the seminal paper [27] or the book [28]), in particular when the stress tensor is a function of an electromagnetic field that varies on each coordinate direction.

This work is developed for the so-called *fast diffusion* regime, $1 < p_i < 2$ for all $i \in \{1, \dots, N\}$, which seems to unfold very strong properties of solutions. The precise attribute we are interested in is the property of *extinction in finite time* of local weak solutions

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to (1.1), meaning that there exists a finite time $T^* < T$, called time of extinction, such that the solution u vanishes out from T^*:

$$\exists\, T^* \in [0,T]: \qquad u(\cdot,t) \equiv 0\,, \qquad \forall t \geqslant T^*.$$

This property is enjoyed by the solutions to the parabolic p-Laplacean equation

$$\partial_t u - \Delta_p u := \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad \text{weakly in} \qquad \Omega_T = \Omega \times [0,T], \tag{1.2}$$

and it affects preponderantly the nature and behavior of solutions (see [15] or, more in general, [5,14]).

For instance, in [20] the authors show that a point-wise Harnack inequality cannot be found for the solutions to (1.2) in the sub-critical range $1 < p < 2N/(N+1)$; while in the super-critical range $2N/(N+1) < p < 2$ the phenomenon of expansion of positivity is closely related to the singular character of the operator, that privileges the elliptic behavior to the diffusive one, as soon as the modulus of ellipticity $|\nabla u|^{p-2}\nabla u$ blows up.

To the very interesting properties of singular equations, the operator (1.1) adds the fascinating ones of anisotropy. In [22], the asymptotic behavior is studied through the analysis of self-similarity, showing that new mathematical methods need to be developed in order to overcome the strong non-uniqueness phenomena and to construct suitable barriers. In [3], the authors show that these anisotropic equations are, in a certain sense, richer than their p-Laplacean counterpart; indeed, for solutions to equations as (1.1) within the more relaxed condition $1 < p < 2$ (here p is an average of p_is, see Section 3) the dichotomy finite speed of propagation/extinction in finite time is no longer valid and it is replaced by conditions on the growth exponents p_is taking into account the competition between diffusions.

Solutions to singular p-Laplacean equations as (1.2), have a decay toward extinction (see [20]) that follows the law

$$\|u(\cdot,t)\|_{\infty,B_\rho} \le \gamma \left(\frac{T^*-t}{\rho^p}\right)^{\frac{1}{2-p}}, \qquad \forall \rho, t > 0\ :\ B_\rho \times ((t+T^*)/2,\, T^*] \subset \Omega_T,$$

being B_ρ the ball of radius ρ and γ a positive constant depending only on the data $\{N,p\}$. In the present work we show that the decay profile of extinction of solutions to equations of the kind of (1.1) is the same as the one to the p-Laplacean if one considers a particular space-geometry,

$$\|u(\cdot,t)\|_{\infty,\mathcal{K}_\rho(T^*-t)} \le \gamma \left(\frac{T^*-t}{\rho^p}\right)^{\frac{1}{2-p}}, \qquad \forall \rho, t > 0\ :\ \mathcal{K}_\rho(T^*-t) \times ((t+T^*)/2,\, T^*] \subset \Omega_T,$$

being γ a positive constant depending only on the data, and, for any fixed $\tau > 0$

$$\mathcal{K}_\rho(\tau) = \prod_i \left\{ |x_i| < \rho^{\frac{p}{p_i}} \left(\frac{\tau}{\rho^p}\right)^{\frac{p-p_i}{p_i(2-p)}} \right\}, \qquad \text{being} \qquad p = N \Big/ \left(\sum_i 1/p_i\right). \tag{1.3}$$

This particular space geometry, which we refer to as *intrinsic geometry* (see Section 2), has interesting features: although the cylinder $\mathcal{K}_\rho(T^*-t)$ degenerates in these directions x_i for which $p_i > p$ when t approaches T^*, it preserves its volume regardless of the time level undertaken; and more, when $p_i \equiv p$ for all $i = 1,\dots,N$, the set $\mathcal{K}_\rho(\tau)$ is the classical cube.

We also show that the decay rate of a solution u to equations of the type (1.1) can be estimated within a geometry that is non-degenerative, but at the price of a more complex rate

$$\|u(\cdot,t)\|_{\infty,\mathbb{K}_\rho} \le \gamma \sum_i \left(\frac{T^*-t}{\rho^p}\right)^{\frac{\lambda_i}{(2-p_i)\lambda}}, \qquad \text{being} \quad \lambda_i = N(p_i-2)+p,$$

$\lambda = N(p-2)+p$ (as usual) and γ a positive constant depending on the data. Here the geometry will be referred to as the *standard geometry*, being based on cubes as

$$\mathbb{K}_\rho = \prod_i \left\{ |x_i| < \rho^{\frac{p}{p_i}} \right\}, \qquad \rho > 0. \tag{1.4}$$

Unlike the *intrinsic geometry* considered before, this one does not take into account the time variable. Again, when $p_i \equiv p$ for all $i = 1,\dots,N$, the set $\mathbb{K}_\rho$ is the classic cube of hedge 2ρ. It is clear that the extinction rate in this case will depend on the smallness of T^*-t and the maximum of the exponents in the sum.

It is the precise aim of our study to carry out an analysis of these two rates of extinction within these two different underlying geometries. The method of derivation of these decay rates has its own mathematical interest: confirming the well-known principle that the run itself can be more instructive than the final destination, we obtain the above behavior of solutions from various Harnack-type estimates. These inequalities are found in three different topologic settings: $L^1_{loc}(\Omega)$, $L^1_{loc}(\Omega)$-$L^\infty_{loc}(\Omega)$ and $L^r_{loc}(\Omega)$-$L^r_{loc}(\Omega)$ backward in time, and all of them are new for solutions to operators as (1.1) (we refer to Section 2 for the precise statements).

Here below we give an example of what we mean by Harnack-type estimates in the $L^1_{loc}(\Omega)$-topology, or, in short, L^1-L^1 Harnack-type inequality.

L^1-L^1 Harnack-type inequality

Let u be a non-negative local weak solution to (1.1) in $\mathbb{R}^N \times \mathbb{R}_0^+$ and let ρ, t be positive fixed numbers. Then, the following two estimates hold true in their respective space configurations.

1 Let $\mathcal{K}_\rho(t)$ be defined as in (1.3). Then there exists a constant $\gamma(N, p_i) > 1$ such that

$$\sup_{0\le\tau\le t} \int_{\mathcal{K}_\rho(t)} u(x,\tau)\,dx \le \gamma \inf_{0\le\tau\le t} \int_{2\mathcal{K}_\rho(t)} u(x,\tau)\,dx + \gamma \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}.$$

2 Let $\mathbb{K}_\rho$ be defined as in (1.4). Then there exists a constant $\gamma(N, p_i) > 1$ such that

$$\sup_{0\le\tau\le t} \int_{\mathbb{K}_\rho} u(x,\tau)\,dx \le \gamma \inf_{0\le\tau\le t} \int_{2\mathbb{K}_\rho} u(x,\tau)\,dx + \sum_i \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{2-p_i}}.$$

Novelty and significance

Origins. To the best of our knowledge, the idea of a Harnack-type estimate in the topology of $L^1_{loc}(\Omega)$ had its first appearance in [19] for the prototype p-Laplacean equation, and it was used in [20] with the aim of giving a bound from below to its solutions in a small cylinder, so to prove a point-wise Harnack inequality. There these integral Harnack-type estimates are first used to evaluate the time of extinction of solutions; see also [7,17] for a later treatment of the sub-critical case.

The method of [19] has been reported in ([15], Chap. VII) for solutions to the prototype singular equation ($1 < p < 2$). A proof for p-Laplacean type equations with full quasilinear structure can be found first in the paper [16] and then in the monograph [18], again with the aim of obtaining a bound from below toward the determination of a point-wise Harnack-type inequality.

All these estimates are unknown for anisotropic equations such as (1.1). In contrast with the few results available in literature (see for instance [11,22]) that use crucially the invariance and comparison properties of the prototype equation, we derive here the aforementioned Harnack-type inequalities for the full-quasilinear structure operator (see definition (3.1)–(2.2)) adopting a technique that dispenses with comparison principles and treats equations that have bounded and measurable coefficients. For this whole spectrum of equations we derive the decay rate of extinction.

As anticipated, in the *cours d'oevre* for the evaluation of the extinction rate, we derive backward $L^r_{loc}(\Omega)$-$L^\infty_{loc}(\Omega)$ estimates that have their own mathematical interest (see Theorems 2.4, 2.5). For their derivation, we assume that the solutions are locally bounded: this is a crucial point for the regularity theory of anisotropic p-Laplacean equations, as a condition on the spareness of the exponents p_is is necessary already for the elliptic case (see for instance [24,26]). From the (anisotropic) parabolic point of view, the theory of local boundedness is reasonably complete, see for instance [13,21,33]. Finally, these $L^r_{loc}(\Omega)$-$L^\infty_{loc}(\Omega)$ estimates are reminiscent of the isotropic case (see for instance [20]) and are obtained through the successive application of standard $L^r_{loc}(\Omega)$-$L^\infty_{loc}(\Omega)$ estimates (Theorems 5.4, 5.1) with backwards $L^r_{loc}(\Omega)$ ones (see Theorems 5.2, 5.5). We refer to [18] and the references therein for the isotropic counterpart.

The lack of (known) regularity of solutions encumbers the research for applications on models directly intertwined with (1.1) (see [2] Chap. IV). Nonetheless, these operators reveal a very interesting picture of the underlying nonlinear analysis and competitive behavior between different diffusions.

The role of intrinsic geometry. A satisfying study of anisotropic operators as (1.1) cannot be brought on regardless of the self-similar geometry embodied in the operator itself. This is already understood in the case of the evolutionary p-Laplacean equation, where has been shown that a Harnack inequality holds true only in a particular geometry, called *intrinsic geometry*. We refer to [15,31] for insights on this topic. Roughly speaking, in the regularity theory of diffusive p-Laplacean equations, time is linked to space by a relation that takes into account the solution itself, as $t = \rho^p u_o^{2-p}$, supposing $u_o > 0$ is the value of the solution at a point. In the case of anisotropic operators behaving like (1.1), the full power of self-similar geometry is needed, and the scaling factor depending on u_o enters also the in space variables. As a concrete example, in the degenerate case and for solutions u of (1.1) in $S_\infty = \mathbb{R}^N \times \mathbb{R}_+$, a point-wise Harnack inequality takes the following form (we refer to [11]):

$$\frac{1}{\gamma} \sup_{\mathcal{K}_\rho(M)} u(\cdot, -M^{2-p}(C_2\,\rho)^p) \leqslant u_0 \leqslant \gamma \inf_{\mathcal{K}_\rho(M)} u(\cdot, M^{2-p}(C_2\,\rho)^p)$$

with $M = (u_o/C_1)$, being γ, C_1, C_2 positive constants depending only on $\{N, p_i\}$. In the available literature, L^1-L^1 Harnack-type estimates are derived for the diffusive p-Laplacean operators (see [18]) without the use of a particular intrinsic geometry. Here we overcome the difficulty of the non-homogeneity of the operator by setting an intrinsic geometry that depends also on time, as $\mathcal{K}_\rho(t)$ in (1.3), which considers self-similar space-cubes as

$$\mathcal{K}_\rho(M) = \prod_i \left\{ |x_i| < \rho^{\frac{p}{p_i}} M^{\frac{p_i-p}{p_i}} \right\}, \quad \text{with} \quad M = \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}.$$

In this case, the particular self-similar factor M depends on the radius and on the *a priori* chosen time level t, and has the interesting feature of reestablishing the homogeneity in the estimates. With a little abuse of notation, along the text we still call this geometry *intrinsic geometry*, because the quantity M here above is always related to some norm of u in applications (see for instance the use of (4.1) and (5.3)).

A last word in honor of the standard geometry $\mathbb{K}_\rho$ is due. Local integral L^1-L^∞ Harnack-type inequalities hold true also in this case (see Theorems 2.8–2.2), which is when one considers $M=1$; but the anisotropy is inevitably carried over into a sum of the quantities t/ρ^p on the right-hand side of the estimates, with different powers depending on p_is. A novel method is also used in this case, which we believe to be useful also for other nonlinear operators.

Applications and Future Perspectives. The range of application of the Harnack-type inequalities we are about to describe is very wide. As for the main purpose of the present work, they can be used to estimate the decay of the solution at the extinction time; and, assuming an integrable initial datum $\|u_0\|_{L^1(\mathbb{R}^N)}$ they imply a certain conservation of the mass of the solution in time.

In addition, not only these Harnack-type estimates are very important for the convergence of approximating solutions when dealing with the problem of the existence (see for instance [19]), but also they proved to be useful to control the measure of level sets and to give a short proof of solutions' Hölder continuity (see for instance [12] for the isotropic case).

Method. The Harnack-type estimates that are obtained throughout the paper, for each one of the mentioned geometries, have as common starting point some general energy estimates, that are collected in the Appendix. Although these energy estimates are non-trivial, they are similar to the isotropic ones (see Appendix); hence we decided to postpone their presentation so as to leave space to what is really new in the anisotropic context.

Our first step is to derive L^1-L^1 Harnack-type estimates by means of testing the equation with negative powers of the solution and a combined nonlinear iteration. In a second step, we study the L^r-L^∞ inequalities by suitably adapting the classic De Giorgi–Moser scheme; here we use the L^r-norm of the solution chained with the energy estimates provided by the equation in a certain geometry. Finally, we nest these inequalities with a backward L^r estimate to derive L^r-L^∞ inequalities in terms of the initial datum u_0; combining these with the first obtained L^1-L^1 estimates we derive the L^1-L^∞ Harnack-type estimates given by Theorems 2.7, 2.8.

Structure of the paper. In Section 2, we define the anisotropic operators with full quasilinear structure and state the main Theorems. Then, in Section 3, we give the definition of local weak solution and the proper functional spaces for it; along with the main notation used throughout the paper. In Section 4, we present the proofs of the first two Theorems, both concerning L^1-L^1 Harnack-type estimates, but specializing the geometry in each case. In a similar fashion, in Section 5, we provide the proofs of the backward L^r-L^∞ estimates, again distinguishing the two geometries. Finally, short Section 6 concludes with the main Theorems, while the last Section, Appendix, presents the main energy estimates used along our analysis and some standard iteration Lemmata.

2. Main results and applications

We consider singular parabolic nonlinear partial differential equations of the form

$$\partial_t u - \operatorname{div} A(x,t,u,Du) = B(x,t,u,Du), \qquad \text{weakly in } \Omega_T = \Omega\times[0,T], \tag{2.1}$$

where the functions $A=(A_1,\dots,A_N) : \Omega_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ and $B : \Omega_T\times\mathbb{R}^{N+1}\to\mathbb{R}$ are Caratheodory functions that satisfy the structure conditions, for $1<p_i<2$, for all $i=1,\dots,N$,

$$\begin{cases} A_i(x,t,s,\xi)\,\xi_i \geqslant C_o|\xi_i|^{p_i} - C^{p_i}, \\ |A_i(x,t,s,\xi)| \le C_1|\xi_i|^{p_i-1} + C^{p_i-1}, \\ |B(x,t,s,\xi)| \le \displaystyle\sum_i C\left(|\xi_i|^{p_i-1} + C^{p_i-1}\right), \end{cases} \tag{2.2}$$

for almost every $(x,t)\in\Omega_T$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where C_o, C_1 are positive constants and C is a non-negative constant that distinguishes between the cases when the equation to be homogeneous (when $C=0$) from when it is not.

We will say that a positive generic constant γ depends only on the data if it depends on the parameters $\{N, p_i, C_o, C_1\}$; for the summation notation we refer to Section 3.

Our main results concern the integral inequalities which, for the sake of simplicity, we state in a forward cylinder centered at the origin.

First, we state the Harnack-type inequalities for the $L^1_{loc}(\Omega)$ norm of the solution evolving in time, sorting out the case of anisotropic intrinsic geometry from the anisotropic standard one.

Theorem 2.1 (*Intrinsic L^1-L^1 Harnack-type Inequality*). *Let u be a non-negative, local weak solution to Eqs. (2.1)–(2.2) in Ω_T, $1<p_i<2$ for all $i=1,\dots,N$. Let $t,\rho>0$ be such that the inclusion*

$$\mathcal{K}_{2\rho}(t)\times[0,t]\subset\Omega_T,$$

holds true. Then, there exists a positive constant γ depending only on the data such that, either there exists an index $i\in\{1,\dots,N\}$ for which

$$C^{p_i}\rho^p > \min\{1, \nu^{p-p_i}, \nu^p\}, \qquad \textit{where} \qquad \nu = \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}, \tag{2.3}$$

or, denoting $\lambda = N(p-2)+p$, we have

$$\sup_{0\le\tau\le t}\int_{\mathcal{K}_\rho(t)} u(x,\tau)\,dx \le \gamma \inf_{0\le\tau\le t}\int_{\mathcal{K}_{2\rho}(t)} u(x,\tau)\,dx + \gamma\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}. \tag{2.4}$$

Theorem 2.2 (*Standard L^1-L^1 Harnack-type Inequality*). *Let u be a non-negative, local weak solution to Eqs. (2.1)–(2.2) in Ω_T, $1<p_i<2$ for all $i=1,\dots,N$. Let $t,\rho>0$ be such that the inclusion*

$$\mathbb{K}_{2\rho}\times[0,t]\subset\Omega_T$$

holds true. Then, there exists a positive constant γ depending only on the data such that, either there exists an index $i\in\{1,\dots,N\}$ for which

$$C^{p_i}\rho^p > \min\{1, v_\Sigma^{p_i}\}, \qquad \textit{where} \qquad v_\Sigma = \sum_k\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_k}}, \tag{2.5}$$

or, denoting $\lambda_i = N(p_i-2)+p$, we have

$$\sup_{0\le\tau\le t}\int_{\mathbb{K}_\rho} u(x,\tau)\,dx \le \gamma \inf_{0\le\tau\le t}\int_{\mathbb{K}_{2\rho}} u(x,\tau)\,dx + \gamma\sum_i\left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{2-p_i}}. \tag{2.6}$$

Remark 2.3. We remark that in Theorems 2.1 and 2.2 the constants λ,λ_i can be of either sign.

Then, considering extra local regularity assumptions on u such as local boundedness and $u\in L^r_{loc}(\Omega_T)$, for some $r>1$, we have the following L^r-L^∞ estimates, valid for exponents $p>2N/(N+r)$.

Theorem 2.4 (*Intrinsic Backwards L^r-L^∞ Estimate*). *Let u be a non-negative, locally bounded, local weak solution to (2.1)–(2.2) in Ω_T, and suppose that for some $r>1$ it satisfies both $u\in L^r_{loc}(\Omega_T)$ and*

$$\lambda_r = N(p-2)+rp>0. \tag{2.7}$$

Then, there exists a positive constant γ depending only on the data, such that for all cylinders

$$\mathcal{K}_{2\rho}(t)\times[0,t]\subset\Omega_T,$$

either there exists an index $i\in\{1,\dots,N\}$ such that (2.3) holds true, or

$$\sup_{\mathcal{K}_{\rho/2}(t)\times[t/2,t]} u \le \gamma t^{-\frac{N}{\lambda_r}}\left(\int_{\mathcal{K}_{2\rho}(t)} u^r(x,0)\,dx\right)^{\frac{p}{\lambda_r}} + \gamma\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}. \tag{2.8}$$

Theorem 2.5 (*Standard Backwards L^r-L^∞ Estimate*). *Let u be a non-negative, locally bounded, local weak solution to (2.1)–(2.2) in Ω_T and suppose additionally that, for some $r>1$, $u\in L^r_{loc}(\Omega_T)$ and*

$$\lambda_r = N(p-2)+rp>0. \tag{2.9}$$

Then, there exists a positive constant γ depending only on the data, such that for all cylinders

$$\mathbb{K}_{2\rho}\times[0,t]\subset\Omega_T,$$

either there exists an index $i\in\{1,\dots,N\}$ for which (2.5) holds true, or

$$\sup_{\mathbb{K}_{\rho/2}\times[t/2,t]} u \le \gamma t^{-\frac{N}{\lambda_r}}\left(\int_{\mathbb{K}_{2\rho}} u^r(x,0)\,dx\right)^{\frac{p}{\lambda_r}} + \gamma\sum_i\left(\frac{t}{\rho^p}\right)^{\frac{\lambda_{i,r}}{(2-p_i)\lambda_r}} + \gamma\sum_i\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}}, \tag{2.10}$$

for exponents $\lambda_{i,r} = N(p_i-2)+pr$.

Remark 2.6. In the prototype degenerate case ($p_i>2$ for all $i=1,\dots,N$) estimates (2.8)–(2.10) hold true without the second term (and third) on the right-hand side of the inequality (see for instance [10,21]). Similarly, to what discussed in [19], the distinction between the two approaches relies in the consideration of solutions that are either local or global in time. With the integral Harnack estimates derived in this paper, it is possible to embark on the path of global existence of solutions to (1.1). To this aim we observe that the first term on the right hand side of (2.8) is formally the same as in the degenerate case, while the second term on the right-hand side controls the growth of the solution for large times.

Finally, we state the main results or our analysis: Harnack-type estimates considered in the topologies $L^\infty_{loc}(\Omega)$ to $L^1_{loc}(\Omega)$, again distinguishing when the anisotropic geometry considered is intrinsic or standard.

Theorem 2.7 (*Intrinsic L^1-L^∞ Harnack-type Inequality*). *Let u be a non-negative, locally bounded, local weak solution to (2.1)–(2.2) and suppose p is in the supercritical range, i.e.*

$$\lambda = N(p-2)+p>0.$$

Then, there exists a positive constant γ depending only on the data such that, for all cylinders

$$\mathcal{K}_{2\rho}(t)\times[0,t]\subset\Omega_T,$$

either there exists $i\in\{1,\dots,N\}$ for which (2.3) holds true, or

$$\sup_{\mathcal{K}_{\rho/2}(t)\times[t/2,t]} u \le \gamma\, t^{\frac{-N}{\lambda}}\left(\inf_{0\le\tau\le t}\int_{\mathcal{K}_{2\rho(t)}} u(x,\tau)\,dx\right)^{\frac{p}{\lambda}}+\gamma\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}. \tag{2.11}$$

Theorem 2.8 (*Standard L^1-L^∞ Harnack-type Inequality*). *Let u be a non-negative, locally bounded, local weak solution to (2.1)–(2.2) and suppose p is in the supercritical range, i.e.*

$$\lambda = N(p-2)+p>0.$$

Then, there exists a positive constant γ depending only on the data such that, for all cylinders

$$\mathbb{K}_{2\rho}\times[0,t]\subset\Omega_T,$$

either there exists $i\in\{1,\dots,N\}$ for which (2.5) holds true, or

$$\sup_{\mathbb{K}_{\rho/2}\times[t/2,t]} u \le \gamma\, t^{\frac{-N}{\lambda}}\left(\inf_{0\le\tau\le t}\int_{\mathbb{K}_{2\rho}} u(x,\tau)\,dx\right)^{\frac{p}{\lambda}}+\gamma\sum_i\left(\frac{t}{\rho^p}\right)^{\frac{\lambda_i}{(2-p_i)\lambda}}+\gamma\sum_i\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}}, \tag{2.12}$$

for $\lambda_i = N(p_i-2)+p$.

Rates of Extinction. The fact that certain solutions to (2.1)–(2.2) with $C=0$ are subject to extinction in finite time has been studied in [3] and also in [4] (we refer to [5,14,15], for the isotropic case, all $p_i\equiv p$). In [3], the authors suppose u to be a solution to

$$\begin{cases}\partial_t u-\sum_i \partial_i(a_i(x,t,u)|\partial_i u|^{p_i-2}\partial_i u)=0, & (x,t)\in\Omega\times(0,T),\\ u=0 & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & x\in\Omega,\end{cases} \tag{2.13}$$

with $u_0\in L^2(\Omega)$ and where $a_i:\Omega\times(0,T)\times\mathbb{R}\to\mathbb{R}$ are Caratheodory functions satisfying $a_0\le a_i(x,t,s)\le A_0$, for $a_0,A_0>0$ structural constants. Within this framework, the authors show that if $1<p<2$, being $p=N/(\sum_i p_i{}^{-1})$ the harmonic average of the exponents p_i, then the energy solutions to (2.13) vanishes in a finite time, i.e

$$u(x,t)\equiv 0 \quad \text{for all} \quad t\geqslant T^*=\left(C_e\|u_0\|^2_{2,\Omega}\right)^{\frac{2}{2-p}}, \qquad C_e=C_e(a_0,A_0,p_i,N)>0.$$

By using a weaker definition of solution (see Definition 3.1), here we assume u is a non-negative, local weak solution to (2.1)–(2.2) in Ω_T, with $C=0$, $1<p_i<2$ for all $i=1,\dots,N$, and that there exists an extinction time $T^*<T$ for u. Then, similarly to [20], we use the L^1-L^1 Harnack-type inequalities (2.4)–(2.6) to evaluate the decay of the $L^1_{loc}(\Omega)$ norm of u toward its extinction and the L^1-L^∞ Harnack-type inequalities (2.11)–(2.12) to estimate the rate of extinction of the solution in a whole half cylinder approaching T^*. These two properties require different assumptions on the exponents p_i. We divide the cases distinguishing the underlying geometry.

Intrinsic Geometry. Let $\tau,\rho>0$ be fixed such that $\mathcal{K}_{4\rho}(T^*-\tau)\subseteq\Omega$.

- The mass decays within the law

$$\|u(\cdot,\tau)\|_{1,(\mathcal{K}_\rho(T^*-\tau))}=\int_{\mathcal{K}_{2\rho}(T^*-\tau)}u(x,\tau)\,dx\le\gamma\left(\frac{T^*-\tau}{\rho^\lambda}\right)^{\frac{1}{2-p}},$$

  for a positive constant γ depending only on the data. Hence the mass $\|u(\cdot,\tau)\|_{L^1(\mathcal{K}_\rho(T^*-\tau))}$ of the solution locally decays (to zero) as $(T^*-\tau)^{1/(2-p)}$ in a space configuration depending on time but with unchanged measure $|\mathcal{K}_\rho(T^*-\tau)|=(2\rho)^N$.
- If $\lambda=N(p-2)+p>0$, then the solution has the following vanishing rate:

$$\sup_{\mathcal{K}_\rho(T^*-\tau)\times[(T^*+\tau)/2,T^*]}u\le\gamma\left(\frac{T^*-\tau}{\rho^p}\right)^{\frac{1}{2-p}},\quad \forall\tau\in(0,T^*),$$

  for a positive constant γ depending only on the data. Choosing $T^*/2<t<T^*$, it is possible to specialize this decay to an ultra-contractive bound

$$\|u(\cdot,t)\|_{\infty,\mathcal{K}_\rho(T^*-t)}\le\gamma\left(\frac{T^*-t}{\rho^p}\right)^{\frac{1}{2-p}}.$$

This estimate shows that the rate of local decay of the L^∞-norm of the solution, in a space configuration depending on each time t, is again of the type $(T^*-t)^{1/(2-p)}$ but now for a different power of the radius ρ.

We observe that when $t \to T^*$ the time intrinsic cube $\mathcal{K}_\rho(T^*-t)$ shrinks along the directions x_k for which $p_k > p$, while in the other directions it stretches to infinity; this particular phenomenon occurs keeping the measure $|\mathcal{K}_\rho(T^*-t)|$ unchanged. Therefore, the inclusion $\mathcal{K}_{4\rho}(T^*-t) \subseteq \Omega$ degenerates according to the choice of time.

Standard Anisotropic Geometry. For a positive number ρ, let us consider the anisotropic standard cube $\mathbb{K}_\rho$ as in (1.4), for $\rho > 0$ such that $\mathbb{K}_\rho \subset \Omega$. We can estimate the local decay of its L^1 and L^∞ norms as above, but this time in a space geometry that is time independent, paying the price of having more involved estimates.

- Description of the mass decay

$$\|u(\cdot,\tau)\|_{L^1(\mathbb{K}_\rho)} = \int_{\mathbb{K}_\rho} u(x,\tau)\,dx \le \gamma \sum_i \left(\frac{T^*-\tau}{\rho^{\lambda_i}}\right)^{\frac{1}{2-p_i}}, \quad \forall \quad 0<\tau\le T^*.$$

  When considering times τ approaching T^*, the mass of the solution $\|u(\cdot,\tau)\|_{L^1(\mathbb{K}_\rho)}$ decays to zero at the rate $(T^*-\tau)^{1/(2-p_N)}$, while when considering larger times $(T^*-\tau)>1$ the rate is $(T^*-\tau)^{1/(2-p_1)}$.
- For any time $0<\tau<T^*$, and assuming that $\lambda>0$, we have a description of the local decay of the essential supremum of the solution as

$$\sup_{\mathbb{K}_\rho\times[(T^*+\tau)/2,T^*]} u \le \gamma \sum_i \left(\frac{T^*-\tau}{\rho^p}\right)^{\frac{\lambda_i}{(2-p_i)\lambda}} + \gamma \sum_i \left(\frac{T^*-\tau}{\rho^p}\right)^{\frac{1}{2-p_i}},$$

  for γ positive constant depending only on the data $\{C_o, C_1, C_2, p_i, N\}$ and being $\lambda_i = N(p_i-2)+p$. Here we observe that a decay rate towards extinction, i.e. for times $(T^*-\tau)<1$, is given from this estimate only with the extra assumption $\lambda_i = N(p_i-2)+p>0$ for all $i=1,\dots,N$, and the solution vanishes in the half-cylinder as fast as $(T^*-\tau)^{\lambda_1/[(2-p_N)\lambda]}$. This behavior is confirmed by those solutions that are constant along $N-1$ space coordinates and behave like a p_1 or p_N-Laplacian by means of the only free variable.

3. Functional setting and notation

Functional setting

We define the anisotropic spaces of locally integrable functions as

$$W^{1,\mathbf{p}}_{loc}(\Omega) = \{u \in W^{1,1}_{loc}(\Omega) \mid \partial_i u \in L^{p_i}_{loc}(\Omega)\},$$

$$L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_{loc}(\Omega)) = \{u \in L^1_{loc}(0,T;W^{1,1}_{loc}(\Omega)) \mid \partial_i u \in L^{p_i}_{loc}(0,T;L^{p_i}_{loc}(\Omega))\},$$

and the respective spaces of functions with zero boundary data

$$W^{1,\mathbf{p}}_{o}(\Omega) = \{u \in W^{1,1}_{o}(\Omega) \mid \partial_i u \in L^{p_i}_{loc}(\Omega)\},$$

$$L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_{o}(\Omega)) = \{u \in L^1_{loc}(0,T;W^{1,1}_{o}(\Omega)) \mid \partial_i u \in L^{p_i}_{loc}(0,T;L^{p_i}_{loc}(\Omega))\}.$$

It is known (see [6,32]) that when $p>N$ the embedding $W^{1,\mathbf{p}}(\Omega) \hookrightarrow C^{0,\alpha}_{loc}(\Omega)$ for Ω regular enough. Therefore in this work we will consider $p<N$.

Definition 3.1. A function

$$u \in C(0,T;L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_{loc}(\Omega))$$

is called a local weak sub(super)-solution to (2.1) in Ω_T if, for all times $0\le t_1\le t_2\le T$ and for all compact sets $K \subset\subset \Omega$, it satisfies the inequality

$$\begin{aligned}\int_K u\varphi\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \{-u\partial_\tau\varphi + \sum_i A_i(x,t,u,Du)\partial_i\varphi\}dx\,d\tau \\ \le (\geqslant) \int_{t_1}^{t_2}\int_K B(x,t,u,Du)\varphi\,dx\,d\tau,\end{aligned} \tag{3.1}$$

for all non-negative test functions $\varphi \in W^{1,2}_{loc}(0,T;L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_{o}(\Omega))$.

This last membership of the test functions, together with the structure conditions (2.2), ensures that all the integrals in (3.1) are finite. Moreover, as φ vanishes along the lateral boundary of Ω_T, its integrability increases thanks to the following known embedding theorem.

Lemma 3.2 (*Anisotropic Gagliardo–Sobolev–Nirenberg, [21]*).
Let $\Omega \subseteq \mathbb{R}^N$ be a rectangular domain, $p < N$, and $\sigma \in [1, p^*]$. For any number $\theta \in [0, p/p^*]$ define

$$q = q(\theta, \mathbf{p}) = \theta\, p^* + \sigma\,(1-\theta),$$

Then there exists a positive constant $c = c(N, \mathbf{p}, \theta, \sigma) > 0$ such that

$$\iint_{\Omega_T} |\varphi|^q\, dx\, dt \le c\, T^{1-\theta\, \frac{p^*}{p}} \left(\sup_{t\in(0,T]} \int_{\Omega} |\varphi|^{\sigma}(x,t)\, dx \right)^{1-\theta} \prod_i \left(\iint_{\Omega_T} |\partial_i \varphi|^{p_i}\, dx\, dt \right)^{\frac{\theta\, p^*}{N\, p_i}}, \tag{3.2}$$

for any $\varphi \in L^1(0,T; W_o^{1,1}(\Omega))$, being the inequality trivial when the right-hand side is unbounded.

Notation In what follows we introduce the notation we will be using along the text.

- We shorten the notation on sums and products when they are intended for all indexes $i, j, k \in \{1, \dots, N\}$,

$$\sum_i := \sum_{i=1}^N \qquad \text{and} \qquad \prod_i := \prod_{i=1}^N \quad .$$

  Only when the sum runs over a different range of exponents will be further specified.
- Exponents are ordered,

$$1 < p_1 \le p_2 \le \dots \le p_N < 2,$$

  and p stands for the harmonic average

$$p := \bar{p} = N/\Big(\sum_i 1/p_i\Big) .$$

- We denote by $\partial_i u$ the weak directional space derivatives and by $\partial_t u$ the weak time-derivative (see (A.1) for more details). Finally, $\nabla u = (\partial_1 u, \dots, \partial_N u)$.
- Our geometrical setting will distinguish between two types of N-dimensional cubes:
  - Anisotropic intrinsic cube

$$\mathcal{K}_{a\rho}(t) := \prod_i \left\{ |x_i| < a\; \rho^{\frac{p}{p_i}\frac{(2-p_i)}{(2-p)}}\, t^{\frac{(p_i-p)}{(2-p)p_i}} \right\}, \quad a > 0\,, \qquad |\mathcal{K}_\rho(t)| = (2\rho)^N$$

  - Anisotropic standard cube

$$\mathbb{K}_{a\rho} := \prod_i \left\{ |x_i| < (a\rho)^{\frac{p}{p_i}} \right\}, \quad a > 0\,, \qquad |\mathbb{K}_\rho| = (2\rho)^N.$$

- We will use two exponents for the decay rates:

$$\lambda_r = N(p-2) + rp \qquad \& \qquad \lambda_{i,r} = N(p_i - 2) + rp,$$

  when $r = 1$, the subscript r is dropped writing $\lambda = N(p-2)+p$ and $\lambda_i = N(p_i - 2) + p$.
- Given a measurable function $u : E \subset \mathbb{R}^{N+1} \to \mathbb{R}$, we denote by $\sup_E u$ ($\inf_E u$) the essential supremum (essential infimum of u) in E with respect to the Lebesgue measure.
- We denote by γ a generic positive constant that depends only on the structural data $\{p_i, N, C_o, C_1\}$ to (2.1)–(2.2), and it may vary in the estimate from line to line.
- Young's Inequality Convention. In our estimates we will repeatedly use Young's inequality in the following form: for $q > 1$ and $a, b, \epsilon > 0$ fixed, we use the well-known inequality

$$ab \le \epsilon a^q + \gamma(\epsilon) b^{q'} \,, \tag{3.3}$$

$$\text{with} \qquad q' = (1 - 1/q)^{-1}, \quad \text{and} \quad \gamma(\epsilon) = \left(\frac{q-1}{q^{1/(q-1)} q} \right) \left(\frac{1}{\epsilon} \right)^{\frac{1}{q-1}}.$$

  The constant ϵ will not be specified as long as it depends only on the data $\{p_i, N, C_o, C_1\}$.

4. Proof of L^1-L^1 Harnack estimates

In this Section we prove Theorems 2.1–2.2, dividing the argument whether the anisotropic space geometry considered is the standard or the intrinsic one.

Intrinsic anisotropic geometry: Proof of Theorem 2.1

We consider a fixed time-length $0 < t < T$, and let $\rho > 0$ be small enough to allow the inclusion

$$\mathcal{Q}_\rho(t) := \mathcal{K}_\rho(t) \times [0,t] = \prod_i \left\{ |x_i| < \rho^{\frac{p}{p_i}} v^{\frac{p_i - p}{p_i}} \right\} \times [0,t] \subseteq \Omega_T,$$

for the fixed quantity

$$v = \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}. \tag{4.1}$$

Lemma 4.1. *Let u be a non-negative local weak super-solution to (2.1) in Ω_T and $\sigma \in (0,1)$ a number. Then, there exists a positive constant γ depending only on the data such that, either (2.3) holds true for some $i = 1,\dots,N$, or we have*

$$\sum_i \frac{1}{\rho^{\frac{p}{p_i}}} \left(\frac{t}{\rho^p}\right)^{\frac{p-p_i}{p_i(2-p)}} \int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} |\partial_i u|^{p_i - 1}\, dx d\tau \le \frac{\gamma}{(1-\sigma)^p} \sum_i \left(\frac{t}{\rho^\lambda}\right)^{\frac{2-p_i}{p_i(2-p)}} \left\{ S + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}} \right\}^{\frac{2(p_i-1)}{p_i}}, \tag{4.2}$$

being

$$S = \sup_{0\le\tau\le t} \int_{\mathcal{K}_\rho(t)} u(x,\tau)\, dx \quad \textit{and} \quad \lambda = N(p-2)+p.$$

Proof. For each $i = 1,\dots,N$ we apply Hölder's inequality to the quantity to be estimated,

$$\begin{aligned}
&\int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} |\partial_i u|^{p_i-1}\, dx d\tau \\
&= \int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} \left(|\partial_i u|^{p_i-1} \tau^{\frac{1}{p_i}(\frac{p_i-1}{p_i})} (u+v)^{\frac{-2}{p_i}(\frac{p_i-1}{p_i})} \right) \left(\tau^{\frac{-1}{p_i}(\frac{p_i-1}{p_i})} (u+v)^{\frac{+2}{p_i}(\frac{p_i-1}{p_i})} \right) dx d\tau \\
&\le \left(\int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} |\partial_i u|^{p_i} \tau^{\frac{1}{p_i}} (u+v)^{\frac{-2}{p_i}}\, dx d\tau \right)^{\frac{p_i-1}{p_i}} \left(\int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} \tau^{\frac{-1}{p_i}(p_i-1)} (u+v)^{\frac{2}{p_i}(p_i-1)}\, dx d\tau \right)^{\frac{1}{p_i}} \\
&=: I_{1,i}^{\frac{p_i-1}{p_i}} I_{2,i}^{\frac{1}{p_i}}.
\end{aligned}$$

Next, we estimate $I_{2,i}$ by taking the supremum in time and then using Hölder's inequality

$$\begin{aligned}
I_{2,i} &= \int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} \tau^{\frac{1}{p_i}-1} (u+v)^{\frac{2}{p_i}(p_i-1)}\, dx d\tau \\
&\le \int_0^t \tau^{\frac{1}{p_i}-1} d\tau \left(\sup_{0\le\tau\le t} \int_{\mathcal{K}_\rho(t)} (u(\tau)+v)^{\frac{2}{p_i}(p_i-1)}\, dx \right) \\
&\le \gamma\, t^{\frac{1}{p_i}} |\mathcal{K}_\rho(t)|^{\frac{2-p_i}{p_i}} \left(\sup_{0\le\tau\le t} \int_{\mathcal{K}_\rho(t)} (u(\tau)+v)\, dx \right)^{\frac{2(p_i-1)}{p_i}} \\
&= \gamma\, t^{\frac{1}{p_i}} \rho^{N\frac{(2-p_i)}{p_i}} \left(\sup_{0\le\tau\le t} \int_{\mathcal{K}_\rho(t)} u(\tau)\, dx + v\rho^N \right)^{\frac{2(p_i-1)}{p_i}} \\
&=: \gamma \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{p_i}} \rho^{\frac{p}{p_i}} \left\{ S + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}} \right\}^{\frac{2(p_i-1)}{p_i}}.
\end{aligned}$$

In the last steps we have used the property $|\mathcal{K}_\rho(t)| = (2\rho)^N$ and the definition of v, λ_i, λ (see the statement of Theorem 2.1). Now we estimate $I_{1,i}$ using the inequalities (A.9) within the considered geometry: we test indeed repeatedly, for $i = 1,\dots,N$, Eq. (2.1) with the function

$$\varphi_i(x,\tau) = -\tau^{\frac{1}{p_i}} (u(x,\tau)+v)^{1-\frac{2}{p_i}} \zeta(x), \qquad \zeta(x) = \prod_i \zeta_i(x_i)^{p_i}, \quad \hat{\zeta}^j := \prod_{i\neq j} \zeta_i(x_i)^{p_i}$$

being ζ a smooth cut-off function between the sets $\mathcal{K}_{\sigma\rho}(t)$ and $\mathcal{K}_\rho(t)$, hence enjoying the properties

$$0 \le \zeta \le 1, \qquad \|\partial_i \zeta_i\|_\infty \le \gamma \left([(1-\sigma)\rho]^{\frac{p}{p_i}} (t/\rho^p)^{\frac{(p_i-p)}{(2-p)p_i}} \right)^{-1} = \gamma \Big/ \left([(1-\sigma)\rho]^{\frac{p}{p_i}} v^{\frac{(p_i-p)}{p_i}} \right). \tag{4.3}$$

The number $\nu \in \mathbb{R}^+$ is fixed, and by implementing (4.3) into (A.9) we obtain

$$\begin{aligned}\int_0^t \int_{\mathcal{K}_\rho(t)} \sum_j |\partial_j u|^{p_j} \tau^{\frac{1}{p_i}} (u+\nu)^{-\frac{2}{p_i}} \zeta \, dx d\tau &\le \gamma t^{\frac{1}{p_i}} \int_{\mathcal{K}_\rho(t)} (u+\nu)^{\frac{2(p_i-1)}{p_i}} \zeta \, dx \\ &+ \gamma \sum_j \frac{\nu^{p-p_j}}{[(1-\sigma)\rho]^p} \left[1 + \left(\frac{C^{p_j}\rho^p}{\nu^{p-p_j}}\right)\right] \int_0^t \int_{\mathcal{K}_\rho(t)} (u+\nu)^{p_j - \frac{2}{p_i}} \tau^{\frac{1}{p_i}} \, dx d\tau \\ &+ \gamma \sum_j C^{p_j} \int_0^t \int_{\mathcal{K}_\rho(t)} (u+\nu)^{-\frac{2}{p_i}} \tau^{\frac{1}{p_i}} \, dx d\tau =: I_1 + I_2 + I_3.\end{aligned} \tag{4.4}$$

Now we manipulate the terms of (4.4), with the aim of obtaining an homogeneous estimate similar to $I_{2,i}$.

The first term on the right is bounded from above by a similar estimate as the one for $I_{2,i}$.

The second term is the one most related with our anisotropic problem; it is here that we specialize our estimates toward homogeneity. We dominate it from above by using $p_i < 2$, with the usual trick

$$(u+\nu)^{p_j - \frac{2}{p_i}} = (u+\nu)^{\frac{2(p_i-1)}{p_i}} (u+\nu)^{p_j-2} \le (u+\nu)^{\frac{2(p_i-1)}{p_i}} \nu^{p_j-2},$$

in order to give an homogeneous estimate with respect to jth index, namely

$$\begin{aligned} I_2 &= \gamma \sum_j \frac{\nu^{p-p_j}}{[(1-\sigma)\rho]^p} \left[1 + \left(\frac{C^{p_j}\rho^p}{\nu^{p-p_j}}\right)\right] \int_0^t \int_{\mathcal{K}_\rho(t)} (u+\nu)^{p_j-\frac{2}{p_i}} \tau^{\frac{1}{p_i}} \, dx d\tau \\ &\le \gamma \sum_j \left[1 + \left(\frac{C^{p_j}\rho^p}{\nu^{p-p_j}}\right)\right] \frac{\nu^{p_j-2}}{[(1-\sigma)\rho]^p \nu^{p_j-p}} t^{1+\frac{1}{p_i}} \left(\sup_{0\le\tau\le t} \int_{K_\rho(t)} (u+\nu)^{\frac{2(p_i-1)}{p_i}} \, dx\right) \\ &\le N\gamma \left[1 + \sum_j \left(\frac{C^{p_j}\rho^p}{\nu^{p-p_j}}\right)\right] \left(\frac{t\nu^{p-2}}{\rho^p}\right) \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{p_i}} \rho^{\frac{p}{p_i}} \left\{S + \nu\rho^N\right\}^{\frac{2(p_i-1)}{p_i}},\end{aligned}$$

where $\lambda_i = N(p_i-2)+p$, for $i = 1, \ldots, N$.

Referring again to (4.4), each jth term of I_3 on the right can be estimated by

$$\begin{aligned} C^{p_j} \int_0^t \int_{\mathcal{K}_\rho(t)} & (u+\nu)^{\frac{-2}{p_i}} \tau^{\frac{1}{p_i}} \, dx d\tau \\ &\le C^{p_j} t^{1+\frac{1}{p_i}} \nu^{-2} \left(\sup_{0\le\tau\le t} \int_{\mathcal{K}_\rho(t)} (u+\nu)^{\frac{2(p_i-1)}{p_i}} \, dx\right) \\ &\le \left(\frac{C^{p_j}\rho^p}{\nu^p}\right) \left(\frac{t\nu^{p-2}}{\rho^p}\right) \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{p_i}} \rho^{\frac{p}{p_i}} \left\{S + \nu\rho^N\right\}^{\frac{2(p_i-1)}{p_i}},\end{aligned}$$

where the first inequality uses $(u+\nu)^{-2} \le \nu^{-2}$ and the last inequality is brought similarly to the one for $I_{2,i}$. Finally, collecting everything together we arrive, for each $i = 1, \ldots, N$, to the estimate

$$I_{1,i} \le \frac{\gamma\rho^{\frac{p}{p_i}}}{(1-\sigma)^p} \left\{1 + \left[1 + \sum_j \left(\frac{C^{p_j}\rho^p}{\nu^p}\right) + \left(\sum_j \frac{C^{p_j}\rho^p}{\nu^{p-p_j}}\right)\right] \left(\frac{t\nu^{p-2}}{\rho^p}\right)\right\} \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{p_i}} \left\{S + \nu\rho^N\right\}^{\frac{2(p_i-1)}{p_i}}.$$

If condition (2.3) is violated for all $i = 1, \ldots, N$, then the term in squared brackets on the right-hand side is smaller than 3, recalling (4.1). Thence we go back to the initial estimate and evaluate

$$\begin{aligned} \int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} & |\partial_i u|^{p_i-1} \, dx d\tau \le I_{1,i}^{\frac{p_i-1}{p_i}} I_{2,i}^{\frac{1}{p_i}} \\ &\le \gamma \left(\frac{\rho^{\frac{p}{p_i}}}{(1-\sigma)^p} \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{p_i}} \left\{S + \nu\rho^N\right\}^{\frac{2(p_i-1)}{p_i}}\right)^{\frac{p_i-1}{p_i}} \left(\left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{p_i}} \rho^{\frac{p}{p_i}} \left\{S + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}\right\}^{\frac{2(p_i-1)}{p_i}}\right)^{\frac{1}{p_i}} \\ &\le \gamma \frac{\rho^{\frac{p}{p_i}}}{(1-\sigma)^p} \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{p_i}} \left\{S + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}\right\}^{\frac{2(p_i-1)}{p_i}}\end{aligned}$$

and thereby

$$\sum_i \frac{1}{\rho^{\frac{p}{p_i}}} \left(\frac{t}{\rho^p}\right)^{\frac{p-p_i}{p_i(2-p)}} \int_0^t \int_{\mathcal{K}_{\sigma\rho}(t)} |\partial_i u|^{p_i-1} \, dx d\tau \le \frac{\gamma}{(1-\sigma)^p} \sum_i \left(\frac{t}{\rho^\lambda}\right)^{\frac{2-p_i}{p_i(2-p)}} \left\{S + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}\right\}^{\frac{2(p_i-1)}{p_i}}. \qquad \square$$

Proof of Theorem 2.1 concluded

Now we perform an iteration on $\sigma \in (0,1)$: we define the increasing radii

$$\rho_{n,i} := \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^p}\right)^{\frac{(p_i-p)}{(2-p)p_i}} \left(\sum_{k=0}^n 2^{-k}\right), \qquad \rho_{n+1,i} - \rho_{n,i} = 2^{-(n+1)} \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^p}\right)^{\frac{(p_i-p)}{(2-p)p_i}},$$

and consider the family of concentric intrinsic anisotropic cubes

$$\mathcal{K}_n = \prod_i \left\{ |x_i| < \rho_{n,i} \right\}, \qquad \tilde{\mathcal{K}}_n = \prod_i \left\{ |x_i| < \frac{\rho_{n+1,i} + \rho_{n,i}}{2} \right\}, \qquad \text{with}$$

$$\mathcal{K}_\rho(t) = \mathcal{K}_0 \subset \mathcal{K}_n \subset \tilde{\mathcal{K}}_n \subset \mathcal{K}_{n+1} \subset \mathcal{K}_\infty = \mathcal{K}_{2\rho}(t) = \prod_i \left\{ |x_i| < 2\rho^{\frac{p}{p_i}} (t/\rho^p)^{\frac{(p_i-p)}{(2-p)p_i}} \right\}.$$

For every $n \in \mathbb{N} \cup \{0\}$, consider time-independent cut-off functions ζ_n as in (A.2) between $\mathcal{K}_n$ and $\tilde{\mathcal{K}}_n$, hence satisfying

$$\|\partial_i \zeta_n\|_\infty \leq \frac{\gamma}{|\rho_{n+1,i} - \rho_{n,i}|} \leq \gamma 2^{n+1} \nu^{\frac{p-p_i}{p_i}} / \rho^{\frac{p}{p_i}}.$$

We test Eq. (2.1) with ζ_n and we integrate over $\tilde{\mathcal{K}}_n \times [\tau_1, \tau_2]$, for arbitrary time levels $0 \leq \tau_1 < \tau_2 \leq t$, to get

$$\begin{aligned}\int_{\tilde{\mathcal{K}}_n} u(x,\tau_1)\,dx \leq{}& \int_{\tilde{\mathcal{K}}_n} u(x,\tau_2)\,dx \\ &+ \gamma 2^{n+1} \sum_i \left(\frac{\nu^{(p-p_i)}}{\rho^p} \right)^{\frac{1}{p_i}} \left(C_1 + \left(\frac{C^{p_i}\rho^p}{\nu^{p-p_i}} \right)^{\frac{1}{p_i}} \right) \int_{\tau_1}^{\tau_2} \int_{\tilde{\mathcal{K}}_n} |\partial_i u|^{p_i-1}\,dx\,d\tau \\ &+ \gamma \sum_i 2^{n+1} \left(C^{p_i-1} \left(\frac{\nu^{p-p_i}}{\rho^p} \right)^{\frac{1}{p_i}} + C^{p_i} \right) \int_{\tau_1}^{\tau_2} \int_{\tilde{\mathcal{K}}_n} dx d\tau.\end{aligned} \tag{4.5}$$

Assume condition (2.3) is contradicted for all $i \in \{1, \dots, N\}$; then the second term in parenthesis on the right of (4.5) is bounded above by $C_1 + 1$, while the third term is estimated by

$$\begin{aligned}\gamma 2^{n+1} \sum_i \left(C^{p_i-1} \left(\frac{\nu^{p-p_i}}{\rho^p} \right)^{\frac{1}{p_i}} + C^{p_i} \right) t\rho^N \\ &= \gamma 2^{n+1} \sum_i \left[\left(\frac{C^{p_i}\rho^p}{\nu^p} \right)^{\frac{(p_i-1)}{p_i}} + \left(\frac{C^{\frac{p_i}{p}}\rho}{\nu} \right)^{p-1} (C^{\frac{p_i}{p}}\rho) \right] \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} \\ &\leq \gamma 2^{n+1} \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}}.\end{aligned}$$

Putting all the pieces together we obtain the estimate

$$\begin{aligned}\int_{\tilde{\mathcal{K}}_n} u(x,\tau_1)\,dx \leq{}& \int_{\tilde{\mathcal{K}}_n} u(x,\tau_2)\,dx \\ &+ \gamma 2^n \sum_i \left(\frac{\nu^{(p-p_i)}}{\rho^p} \right)^{\frac{1}{p_i}} \int_{\tau_1}^{\tau_2} \int_{\tilde{\mathcal{K}}_n} |\partial_i u|^{p_i-1}\,dx\,d\tau + \gamma 2^n \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}}.\end{aligned} \tag{4.6}$$

By continuity of u as a map $[0,T] \to L^2_{loc}(\Omega)$, we take τ_2 as the time level in $[0,t]$ such that

$$\mathcal{I} = \inf_{0 \leq \tau \leq t} \int_{2\mathcal{K}_\rho(t)} u(x,\tau)\,dx = \int_{2\mathcal{K}_\rho(t)} u(x,\tau_2)\,dx,$$

and τ_1 as the time level satisfying

$$S_n := \sup_{0 \leq \tau \leq t} \int_{\mathcal{K}_n} u(x,\tau)\,dx d\tau = \int_{\mathcal{K}_n} u(x,\tau_1)\,dx d\tau.$$

It is precisely for this choice of ordering between τ_1 and τ_2 that we need u to be a *solution*, and not only a super-solution. Now we evaluate the second term in (4.6) with the inequality (4.2) applied to the pair of cylinders $\tilde{\mathcal{K}}_n \times [0,t] \subset \mathcal{K}_{n+1} \times [0,t]$ and develop the definition of ν to write

$$\begin{aligned}S_n &\leq \mathcal{I} + \gamma b^n \sum_i \nu^{\frac{p-p_i}{p_i}} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} \left\{ S_{n+1} + \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} \right\}^{\frac{2(p_i-1)}{p_i}} + \gamma 2^n \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} \\ &\leq \mathcal{I} + \gamma b^n \sum_i \left(\frac{t}{\rho^\lambda} \right)^{\frac{(2-p_i)}{(2-p)p_i}} \left\{ S_{n+1} + \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} \right\}^{\frac{2(p_i-1)}{p_i}} + \gamma 2^n \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}}, \qquad b = 2^{p+1} > 1.\end{aligned}$$

By using Young's inequality on each ith term with exponents $\frac{2(p_i-1)}{p_i} + \frac{2-p_i}{p_i} = 1$ we get

$$S_n \leq \sum_i \frac{\epsilon}{N} \left[S_{n+1} + \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} \right] + \sum_i c(\epsilon,\gamma) b^n \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} + \mathcal{I} \leq \epsilon S_{n+1} + \gamma b^n \left\{ \mathcal{I} + \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} \right\}, \tag{4.7}$$

and the conclusion follows from the classical iteration of Lemma A.8.

Standard anisotropic geometry: Proof of Theorem 2.2

Let $0 < t < T$ and $\rho > 0$ such that the following inclusion is satisfied,

$$\mathbb{Q} := \mathbb{K}_\rho \times [0,t] \subset \Omega_T.$$

To consider intermediate cylinders, for a fixed $\sigma \in (0,1]$ we define

$$\mathbb{Q}_\sigma = \mathbb{K}_{\sigma\rho} \times [0,t] = \prod_i \left\{ |x_i| < (\sigma\rho)^{\frac{p}{p_i}} \right\} \times [0,t], \qquad \text{and} \qquad \mathbb{Q} = \mathbb{Q}_1.$$

Moreover, for such fixed t, ρ, we define the quantity

$$v_\Sigma = \sum_k \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p_k}}. \tag{4.8}$$

Lemma 4.2. *Let u be a non-negative local weak super-solution to (2.1) in Ω_T and $\sigma \in (0,1)$ a number. Then, there exists a positive constant γ, depending on the data, such that, either there exists an $i \in \{1,\dots,N\}$ for which (2.5) is valid, or for all $i \in \{1,\dots,N\}$ we have*

$$\sum_i \rho^{-\frac{p}{p_i}} \iint_{\mathbb{Q}_\sigma} |\partial_i u|^{p_i} \, dx d\tau \le \frac{\gamma}{(1-\sigma)^p} \sum_i \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} \left(S + v_\Sigma \rho^N \right)^{\frac{2(p_i-1)}{p_i}}, \tag{4.9}$$

with $\lambda_i = N(p_i - 2) + p$ and being

$$S = \sup_{0 \le \tau \le t} \int_{\mathbb{K}_\rho} u(x,\tau)\, dx.$$

Proof. For $\sigma \in (0,1]$ we consider the cylinders

$$\mathbb{Q}_\sigma = \mathbb{K}_{\sigma\rho} \times [0,t] = \prod_i \left\{ |x_i| < (\sigma\rho)^{\frac{p}{p_i}} \right\} \times [0,t], \qquad \text{and} \qquad \mathbb{Q} = \mathbb{Q}_1.$$

We use the estimates (A.9) by testing the equation with

$$\varphi_i = \tau^{\frac{1}{p_i}} (u+v)^{1-\frac{2}{p_i}} \zeta,$$

where ζ is a cut-off function of the type (A.2), defined between $\mathbb{K}_{\sigma\rho}$ and $\mathbb{K}_\rho$, therefore verifying

$$\|\partial_i \zeta_i\|_{\infty,\mathbb{K}_\rho} \le \gamma / [(1-\sigma)\rho]^{\frac{p}{p_i}}.$$

This gives, for all $i \in \{1,\dots,N\}$, the inequalities

$$\begin{aligned}
&\iint_{\mathbb{Q}_\sigma} |\partial_i u|^{p_i} \tau^{\frac{1}{p_i}} (u+v_\Sigma)^{-\frac{2}{p_i}} \, dx d\tau \\
&\quad \le \iint_{\mathbb{Q}_\sigma} \left(\sum_j |\partial_j u|^{p_j} \right) \tau^{\frac{1}{p_i}} (u+v_\Sigma)^{-\frac{2}{p_i}} \, dx d\tau \le \gamma t^{\frac{1}{p_i}} \int_{\mathbb{K}\times\{t\}} (u+v_\Sigma)^{\frac{2(p_i-1)}{p_i}} \, dx \\
&\qquad + \frac{\gamma}{(1-\sigma)^p \rho^p} \sum_j \left[1 + C^{p_j} \rho^p \right] \iint_{\mathbb{Q}} (u+v_\Sigma)^{p_j - \frac{2}{p_i}} \tau^{\frac{1}{p_i}} \, dx d\tau \\
&\qquad + \gamma \left(\sum_j C^{p_j} \right) \iint_{\mathbb{Q}} (u+v_\Sigma)^{-\frac{2}{p_i}} \tau^{\frac{1}{p_i}} \, dx d\tau.
\end{aligned} \tag{4.10}$$

We estimate the various terms. The first integral on the right-hand side of (4.10) is manipulated as in 4.1 to get

$$\begin{aligned}
t^{\frac{1}{p_i}} \int_{\mathbb{K}} (u+v_\Sigma)^{\frac{2(p_i-1)}{p_i}} \, dx &\le t^{\frac{1}{p_i}} |\mathbb{K}|^{\frac{2-p_i}{p_i}} \left(\sup_{0\le\tau\le t} \int_{\mathbb{K}} u(x,\tau)\, dx + v_\Sigma |\mathbb{K}| \right)^{\frac{2(p_i-1)}{p_i}} \\
&\le \gamma t^{\frac{1}{p_i}} \rho^{N(\frac{2-p_i}{p_i})} \left(S + v_\Sigma \rho^N \right)^{\frac{2(p_i-1)}{p_i}} \\
&= \gamma \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} (S + v_\Sigma \rho^N)^{\frac{2(p_i-1)}{p_i}}.
\end{aligned}$$

The second term can be estimated by using that $(u+v_\Sigma)^{p_j-2} < v_\Sigma^{p_j-2}$ to get for all $i=1,\dots,N$ the inequalities

$$\sum_j \left(\frac{[1+C^{p_j}\rho^p]}{v_\Sigma^{2-p_j}} \right) \iint_{\mathbb{Q}} (u+v_\Sigma)^{\frac{2(p_i-1)}{p_i}} \tau^{\frac{1}{p_i}}\, dx d\tau$$
$$\leq \sum_j \left(\frac{[1+C^{p_j}\rho^p]}{v_\Sigma^{2-p_j}} \right) t^{1+\frac{1}{p_i}} \left(\sup_{0\leq\tau\leq t} \int_{\mathbb{K}} u(x,\tau)\,dx + v_\Sigma \rho^N \right)^{\frac{2(p_i-1)}{p_i}} \rho^{N(\frac{2-p_i}{p_i})}$$
$$\leq \sum_j \left(\frac{[1+C^{p_j}\rho^p]}{v_\Sigma^{2-p_j}} \right) \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} t \left(S + v_\Sigma \rho^N \right)^{\frac{2(p_i-1)}{p_i}}.$$

Finally the third term on the right-hand side of (4.10) is estimated, for any $i,j\in\{1,\dots,N\}$, as

$$C^{p_j} \iint_{\mathbb{Q}} \tau^{\frac{1}{p_i}} (u+v_\Sigma)^{-\frac{2}{p_i}}\, dx d\tau$$
$$\leq C^{p_j} \rho^{\frac{p}{p_i}} \left(\frac{\rho^{\frac{p}{p_j}}}{v_\Sigma} \right)^{p_j} \left(\frac{t}{\rho^p} \right) v^{p_j-2} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} (S+v_\Sigma\rho^N)^{\frac{2(p_i-1)}{p_i}}$$
$$\leq \left(\frac{C\rho^{\frac{p}{p_j}}}{v_\Sigma} \right)^{p_j} \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^p} \right) v_\Sigma^{p_j-2} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} (S+v_\Sigma\rho^N)^{\frac{2(p_i-1)}{p_i}}.$$

Collecting everything together we obtain

$$\begin{aligned} &\iint_{\mathbb{Q}_\sigma} |\partial_i u|^{p_i} \tau^{\frac{1}{p_i}} (u+v_\Sigma)^{-\frac{2}{p_i}}\, dx d\tau \\ &\leq \gamma \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} (S+v_\Sigma\rho^N)^{\frac{2(p_i-1)}{p_i}} \times \\ &\times \left\{ 1 + \sum_j \frac{t}{v_\Sigma^{2-p_j}\rho^p}[1+C^{p_j}\rho^p] + \sum_j \left(\frac{C\rho^{\frac{p}{p_j}}}{v_\Sigma} \right)^{p_j} \left(\frac{t}{\rho^p} \right) v_\Sigma^{p_j-2} \right\} \end{aligned} \tag{4.11}$$

The second factor on the right of (4.11) is smaller than 4 if (2.5) is violated for all indexes $j\in\{1,\dots,N\}$, and once we observe

$$v_\Sigma = \sum_k \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p_k}} \geqslant \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p_j}}, \qquad \forall j=1,\dots,N.$$

This allows us to evaluate

$$\rho^{-\frac{p}{p_i}} \iint_{\mathcal{Q}_{\sigma\rho}} |\partial_i u|^{p_i-1}\, dx d\tau$$
$$= \rho^{-\frac{p}{p_i}} \iint_{\mathcal{Q}_{\sigma\rho}} \left(|\partial_i u|^{p_i-1} \tau^{\frac{1}{p_i}(\frac{p_i-1}{p_i})} (u+v)^{\frac{-2}{p_i}(\frac{p_i-1}{p_i})} \right) \left(\tau^{\frac{-1}{p_i}(\frac{p_i-1}{p_i})} (u+v)^{\frac{+2}{p_i}(\frac{p_i-1}{p_i})} \right) dx d\tau$$
$$\leq \rho^{-\frac{p}{p_i}} \left(\iint_{\mathcal{Q}_{\sigma\rho}} |\partial_i u|^{p_i} \tau^{\frac{1}{p_i}} (u+v)^{\frac{-2}{p_i}}\, dx d\tau \right)^{\frac{p_i-1}{p_i}} \left(\iint_{\mathcal{Q}_{\sigma\rho}} \tau^{\frac{-1}{p_i}(p_i-1)} (u+v)^{\frac{2}{p_i}(p_i-1)}\, dx d\tau \right)^{\frac{1}{p_i}}$$
$$\leq \rho^{-\frac{p}{p_i}} \left(\frac{\gamma}{(1-\sigma)^p} \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} (S+v_\Sigma\rho^N)^{\frac{2(p_i-1)}{p_i}} \right)^{\frac{p_i-1}{p_i}} \left(\gamma \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} \rho^{\frac{p}{p_i}} \left\{ S + \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{2-p}} \right\}^{\frac{2(p_i-1)}{p_i}} \right)^{\frac{1}{p_i}}$$
$$\leq \frac{\gamma}{(1-\sigma)^p} \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} (S+v_\Sigma\rho^N)^{\frac{2(p_i-1)}{p_i}}. \quad \square$$

Proof of Theorem 2.2 concluded.

Proof. We fix $\rho>0$, define the sequence of increasing radii

$$\rho_n := \rho\sum_{k=0}^n 2^{-k}, \qquad \rho=\rho_o\leq\rho_n\leq\tilde\rho_n := \frac{\rho_n+\rho_{n+1}}{2} \leq \rho_{n+1} < \rho_\infty = 2\rho$$

and construct the family of concentric standard anisotropic cubes

$$\mathbb{K}_n = \prod_i \left\{ |x_i| < \rho_n^{\frac{p}{p_i}} \right\}, \qquad \tilde{\mathbb{K}}_n = \prod_i \left\{ |x_i| < \tilde\rho_n^{\frac{p}{p_i}} \right\},$$

verifying $\mathbb{K}_n \subset \tilde{\mathbb{K}}_n \subset \mathbb{K}_{n+1}$, and for any $\tau_1,\tau_2\in[0,t]$, we consider the family of cylinders

$$\mathbb{Q}_n = \mathbb{K}_n \times [\tau_1,\tau_2] \subset \tilde{\mathbb{Q}}_n = \tilde{\mathbb{K}}_n \times [\tau_1,\tau_2] \subset \mathbb{Q}_{n+1}.$$

For each $n \in \mathbb{N} \cup \{0\}$ chosen, consider $\zeta_n(x)$ a cut-off function of the form (A.2) between $\mathbb{K}_n$ and $\tilde{\mathbb{K}}_n$ that is time-independent and verifies

$$0 \le \zeta_n \le 1, \qquad (\zeta_n)_{|\partial \tilde{\mathbb{K}}_n} = 0, \qquad \|\partial_i \zeta_n\|_{\infty, \tilde{\mathbb{K}}_n} \le \gamma \left(\frac{2^n}{\rho}\right)^{\frac{p}{p_i}}.$$

Testing (2.1)–(2.2) with such a ζ_n we obtain

$$\begin{aligned} \int_{\mathbb{K}_n} u(x,\tau_1)\,dx \le \int_{\tilde{\mathbb{K}}_n} u(x,\tau_2)\,dx + \sum_i \Big(\|\partial_i \zeta_n\|_\infty C_1 + C \Big) \iint_{\tilde{\mathbb{Q}}_n} |\partial_i u|^{p_i-1}\,dx\,d\tau \\ + \sum_i (C^{p_i-1}\|\partial_i \zeta_n\|_\infty + C^{p_i}) \iint_{\tilde{\mathbb{Q}}_n} dx\,d\tau\,. \end{aligned} \tag{4.12}$$

for arbitrary time levels $\tau_1, \tau_2 \in [0,t]$. Again, by the continuity of u as a map $[0,T] \to L^2_{loc}(\Omega)$, we take τ_2 as the time level in $[0,t]$ such that

$$\mathcal{I} = \inf_{0\le\tau\le t} \int_{\mathbb{K}_{2\rho}} u(x,\tau)\,dx = \int_{\mathbb{K}_{2\rho}} u(x,\tau_2)\,dx,$$

and set

$$S_n := \sup_{0\le\tau\le t} \int_{\mathbb{K}_n} u(x,\tau)\,dx\,.$$

Since τ_1 is arbitrary, (4.12) yields

$$S_n \le \mathcal{I} + \gamma 2^{\frac{p}{p_1}n} \sum_i \rho^{-\frac{p}{p_i}} \iint_{\tilde{\mathbb{Q}}_n} |\partial_i u|^{p_i-1}\,dx\,d\tau + \gamma 2^{\frac{p}{p_1}n} \sum_i \Big(C^{p_i-1}\rho^{-\frac{p}{p_i}} + C^{p_i} \Big) \iint_{\tilde{\mathbb{Q}}_n} dx\,d\tau.$$

The last term on the right-hand is dominated as follows:

$$\begin{aligned} \Big(C^{p_i-1}\rho^{-\frac{p}{p_i}} + C^{p_i} \Big) \iint_{\tilde{\mathbb{Q}}_n} dx\,d\tau &\le \gamma \left[\left(\frac{C\rho^{\frac{p}{p_i}}}{\nu_\Sigma} \right)^{p_i-1} + \left(\frac{C^{p_i}\rho^p}{\nu_\Sigma^{p_i-1}} \right) \right] \left(\sum_j \left(\frac{t}{\rho^{\lambda_j}} \right)^{\frac{1}{2-p_j}} \right) \\ &\le \gamma \sum_j \left(\frac{t}{\rho^{\lambda_j}} \right)^{\frac{1}{2-p_j}}, \end{aligned}$$

recalling $t < \nu_\Sigma^{2-p_i}\rho^p$, for all $i = 1,\dots,N$, and assuming that condition (2.5) is violated for all indexes. Therefore, by applying first Lemma 4.2 to the pair of cylinders $\mathbb{Q}_n$ and $\tilde{\mathbb{Q}}_n$, for which $1-\sigma \ge 2^{-(n+4)}$, and then Young's inequality one gets

$$\begin{aligned} S_n &\le \mathcal{I} + \gamma 2^{n\frac{p}{p_1}} \sum_i \rho^{-\frac{p}{p_i}} \iint_{\tilde{\mathbb{Q}}_n} |\partial_i u|^{p_i-1}\,dx\,d\tau + \gamma 2^{n\frac{p}{p_1}} \sum_i \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{2-p_i}} \\ &\le \mathcal{I} + \gamma b^n \sum_i \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{p_i}} \Big\{ S_{n+1} + \nu_\Sigma \rho^N \Big\}^{\frac{2(p_i-1)}{p_i}} + \gamma 2^{n\frac{p}{p_1}} \sum_i \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{2-p_i}} \\ &\le \epsilon S_{n+1} + \gamma(\epsilon) b^n \left\{ \mathcal{I} + \sum_i \left(\frac{t}{\rho^{\lambda_i}} \right)^{\frac{1}{2-p_i}} \right\}, \qquad b > 1. \end{aligned}$$

A standard iteration finishes the proof as in the case of (4.7) □

5. Proof of the backward L^r-L^∞ estimates

The proof of Theorems 2.4–2.5 rely on two estimates: L^r-L^∞ estimates combined with a L^r estimates backward in time; the presentation is done separately for the intrinsic and the standard geometries.

Intrinsic anisotropic geometry: Proof of Theorem 2.4

Theorem 5.1 (L^r_{loc}-L^∞_{loc} *Estimates*). *Suppose u is a non-negative, locally bounded, local weak sub(super)-solution to (2.1)–(2.2) in Ω_T. Let $r \geqslant 1$ and $\lambda_r = N(p-2) + rp > 0$. Then, there exists a positive constant γ, depending only on the data, such that*

$$\forall t > 0, \quad \forall \rho > 0 \ : \ \mathcal{K}_{4\rho}(t) \times (0,t) \subset \Omega_T,$$

either (2.3) holds for some $i \in \{1,\dots,N\}$ or

$$\sup_{\mathcal{K}_{\rho/2}(t)\times[t/2,t]} u \le \gamma \left(\frac{t}{\rho^p} \right)^{\frac{-N}{\lambda_r}} \left(\sup_{0\le\tau\le t} \fint_{\mathcal{K}_\rho(t)} u^r(x,\tau)\,dx \right)^{\frac{p}{\lambda_r}} + \gamma \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p}}. \tag{5.1}$$

Proof. Assume condition (2.3) does not hold for every $i \in \{1,\dots,N\}$. Let $\sigma \in (0,1)$ be fixed and consider the decreasing sequences of radii, for each $i \in \{1,\dots,N\}$,

$$\rho_i := \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^p}\right)^{\frac{(p_i-p)}{(2-p)p_i}}, \qquad \rho_{n,i} := \rho_i \left(\sigma + \frac{1-\sigma}{2^n}\right)^{\frac{p}{p_i}},$$

and of time levels

$$\sigma t = t_\infty < t_n := t\left(\sigma + \frac{1-\sigma}{2^n}\right) \le t_0 = t$$

from which one constructs the sequence of nested and shrinking cylinders

$$Q_n = \mathcal{K}_n \times (t-t_n, t), \qquad \text{for} \qquad \mathcal{K}_n = \prod_i \left\{ |x_i| < \rho_{n,i} \right\}.$$

For each $n \in \mathbb{N}$, let $\zeta_n(x,t) = \prod_i \zeta_i^{p_i}(x_i)\eta(t)$ be a cut-off function as in (A.2) therefore verifying

$$\zeta_i(x_i) = \begin{cases} 1, & |x_i| < \rho_{(n+1),i} \\ 0, & |x_i| \ge \rho_{n,i} \end{cases}, \qquad \|\partial_i \zeta_i\|_\infty \le \left(\frac{2^{n+1}}{(1-\sigma)\rho}\right)^{\frac{p}{p_i}} \left(\frac{t}{\rho^p}\right)^{\frac{(p-p_i)}{(2-p)p_i}},$$

for all $i = 1,\dots,N$, and

$$\eta(\tau) = \begin{cases} 0 & , 0 \le \tau \le t-t_n \\ 1 & , t-t_{n+1} \le \tau \le t \end{cases}, \qquad |\partial_\tau \eta| \le \frac{2^{n+1}}{(1-\sigma)t}.$$

In the weak formulation (3.1), for each $n \in \mathbb{N}$, consider the test function $\varphi_n = (u-k_{n+1})_+\xi_n$, over the cylinders Q_n, for the truncation levels

$$0 \le k_n = k\left(1 - \frac{1}{2^n}\right) < k, \qquad n \in \mathbb{N} \cup \{0\},$$

where k is a positive real number to be determined. By the classical energy estimate (A.4) we obtain the following bound on the energy

$$\begin{aligned}
\mathcal{E}_n &:= \sup_{t-t_n \le \tau \le t} \int_{\mathcal{K}_n \times \{\tau\}} (u-k_{n+1})_+^2 \xi_n \, dx + \sum_i \iint_{Q_n} |\partial_i[(u-k_{n+1})_+\xi_n]|^{p_i} \, dx\, d\tau \\
&\le \gamma \|\partial_\tau \eta\|_\infty \iint_{Q_n} (u-k_{n+1})_+^2 \, dx d\tau + \\
&+ \gamma \left\{ \sum_i \left(\|\partial_i \zeta_i\|_\infty^{p_i} + C^{p_i} \right) \iint_{Q_n} (u-k_{n+1})_+^{p_i} \, dx d\tau + C^{p_i} \iint_{Q_n} \chi_{[u>k_{n+1}]} \, dx d\tau \right\} \\
&\le \frac{\gamma 2^n}{(1-\sigma)t} \iint_{Q_n} (u-k_{n+1})_+^2 \, dx d\tau + \\
&+ \gamma \sum_i \left(\frac{2^{np_i}}{(1-\sigma)^p \rho^p} \left(\frac{t}{\rho^p}\right)^{\frac{p_i-p}{2-p}} + C^{p_i} \right) \iint_{Q_n} (u-k_{n+1})_+^{p_i} \, dx d\tau \\
&+ \gamma \sum_i \frac{1}{t}(tC^{p_i}) \iint_{Q_n} \chi_{[u>k_{n+1}]} \, dx d\tau \\
&\le \frac{\gamma 2^{2n}}{(1-\sigma)^p t} \left\{ \iint_{Q_n} (u-k_{n+1})_+^2 \, dx d\tau + \sum_i \left(\frac{t}{\rho^p}\right)^{\frac{2-p_i}{2-p}} \iint_{Q_n} (u-k_{n+1})_+^{p_i} \, dx d\tau \right. \\
&\left. + \left(\frac{t}{\rho^p}\right)^{\frac{2}{2-p}} \iint_{Q_n} \chi_{[u>k_{n+1}]} \, dx d\tau \right\},
\end{aligned}$$

where first we implemented the construction of the cut-off function ζ and then we have used that for each $i \in \{1,\dots,N\}$ the condition (2.3) is violated.

The case $\max\left\{1, \frac{2N}{N+2}\right\} < p < 2$

We estimate the energy $\mathcal{E}_n$ from above in terms of the L^2-norm of the truncations $(u-k_n)_+$. Observe that for all $s = 0, 1, \dots, N$, having defined $p_0 = 2$, it holds

$$\begin{aligned}
\iint_{Q_n} (u-k_n)_+^2 \, dx d\tau &\ge \iint_{Q_n \cap [u>k_{n+1}]} (u-k_n)_+^{2-p_s} (u-k_n)_+^{p_i} \, dx d\tau \\
&\ge \left(\frac{k}{2^{n+1}}\right)^{2-p_s} \iint_{Q_n \cap [u>k_{n+1}]} (u-k_n)_+^{p_s} \, dx d\tau \\
&\ge \left(\frac{k}{2^{n+1}}\right)^{2-p_s} \iint_{Q_n} (u-k_{n+1})_+^{p_s} \, dx d\tau.
\end{aligned}$$

Hence we have

$$\mathcal{E}_n \le \frac{\gamma 2^{2n}}{(1-\sigma)^p t}\left\{1+\sum_i \left(\frac{t}{\rho^p}\right)^{\frac{2-p_i}{(2-p)}} \frac{2^{n(2-p_i)}}{k^{2-p_i}} + \left(\frac{t}{\rho^p}\right)^{\frac{2}{2-p}} \frac{2^{2n}}{k^2}\right\} \iint_{Q_n} (u-k_n)_+^2 \, dx d\tau, \tag{5.2}$$

and taking into account as a further condition

$$k \ge \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} , \tag{5.3}$$

the right hand side of (5.2) now reads

$$\mathcal{E}_n \le \frac{\gamma 2^{4n}}{(1-\sigma)^p t} \iint_{Q_n} (u-k_n)_+^2 \, dx d\tau. \tag{5.4}$$

Now we want to put in a chain the estimate of $\mathcal{E}_n$ obtained in terms of $\|(u-k_n)_+\|^2_{L^2(Q_n)}$ with the anisotropic Sobolev embedding 3.2.

Here we take advantage of exponent p being in the super-critical range, $p > \max\{1, 2N/(N+2)\}$: indeed, in such a range, the number $q = p(N+2)/N$ is greater than 2 and we can use Hölder inequality on $\|(u-k_{n+1})_+\|^2_{L^2(Q_{n+1})}$ to allow the aforementioned chaining procedure. In the embedding 3.2 we make the choices

$$q = \frac{p(N+2)}{N}, \qquad \text{and} \qquad \theta = \frac{p}{p^*}, \qquad \sigma = 2,$$

to get

$$\begin{aligned}
&\iint_{Q_{n+1}} (u-k_{n+1})_+^2 \xi_n^2 \, dx d\tau \\
&\le \left(\iint_{Q_n} ((u-k_{n+1})_+\xi_n)^{p(\frac{N+2}{N})} \, dx d\tau\right)^{\frac{2N}{p(N+2)}} |Q_n \cap [u > k_{n+1}]|^{1-\frac{2N}{p(N+2)}} \\
&\le \gamma \left[\left(\sup_{t-t_n \le \tau \le t} \int_{\mathcal{K}_n \times \{\tau\}} (u-k_{n+1})_+^2 \xi^2 \, dx\right)^{p/N} \left(\prod_i \iint_{Q_n} |\partial_i((u-k_{n+1})_+\xi)|^{p_i} \, dx d\tau\right)^{\frac{p}{Np_i}}\right]^{\frac{2N}{p(N+2)}} \\
&\qquad \times |Q_n \cap [u > k_{n+1}]|^{1-\frac{2N}{p(N+2)}} \\
&\le \gamma \left[\mathcal{E}_n^{\frac{p}{N}} \prod_i \mathcal{E}_n^{\frac{p}{Np_i}}\right]^{\frac{2N}{p(N+2)}} |Q_n \cap [u > k_{n+1}]|^{1-\frac{2N}{p(N+2)}} \\
&\le \gamma \mathcal{E}_n^{(\frac{p+N}{N+2})(\frac{2}{p})} \left(\frac{2^{2n}}{k^2} \iint_{Q_n} (u-k_n)_+^2 \, dx d\tau\right)^{\frac{N(p-2)+2p}{p(N+2)}} \\
&\le \frac{\gamma b^n}{[(1-\sigma)^p t]^{(\frac{N+p}{N+2})(\frac{2}{p})} k^{(\frac{2}{p})\frac{N(p-2)+2p}{N+2}}} \left(\iint_{Q_n} (u-k_n)_+^2 \, dx d\tau\right)^{1+\frac{2}{N+2}}, \qquad \text{for} \quad b > 1.
\end{aligned}$$

By setting $X_n = |Q_n|^{-1} \|(u-k_n)_+\|^2_{2,Q_n}$, from the previous estimate we derive

$$X_{n+1} \le \frac{\gamma b^n}{[(1-\sigma)^p]^{(\frac{N+p}{N+2})(\frac{2}{p})} k^{(\frac{2}{p})\frac{\lambda_2}{N+2}}} \left(\frac{\rho^p}{t}\right)^{\frac{2N}{p(N+2)}} X_n^{1+\frac{2}{N+2}}, \tag{5.5}$$

with $\lambda_2 = N(p-2)+2p$. By choosing $k > 0$ such that

$$⨌_{Q_0} u^2 \le \gamma^{-\frac{N+2}{2}} b^{-\left(\frac{N+2}{2}\right)^2} (1-\sigma)^{(N+p)} \left(\frac{t}{\rho^p}\right)^{\frac{N}{p}} k^{\frac{\lambda_2}{p}},$$

the Fast Converge Lemma A.7, ensures $X_n \to 0$ as $n \to \infty$, meaning that

$$\sup_{\mathcal{K}_{\sigma\rho}(t)\times[\sigma t, t]} u \le k \le \frac{\gamma}{(1-\sigma)^{\frac{p(N+p)}{\lambda_2}}} \left(\frac{t}{\rho^p}\right)^{-\frac{N}{\lambda_2}} \left(⨌_{\mathcal{K}_\rho(t)\times[0,t]} u^2 \, dx d\tau\right)^{\frac{p}{\lambda_2}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}},$$

and then

$$\begin{aligned}
\sup_{\mathcal{K}_{\sigma\rho}(t)\times[\sigma t, t]} u &\le \frac{\gamma}{(1-\sigma)^{\frac{p(N+p)}{\lambda_2}}} \left(\frac{t}{\rho^p}\right)^{-\frac{N}{\lambda_2}} \left(⨌_{\mathcal{K}_\rho(t)\times[0,t]} u^2 \, dx d\tau\right)^{\frac{p}{\lambda_2}} + \gamma \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} \\
&\le \frac{\gamma}{(1-\sigma)^{\frac{p(N+p)}{\lambda_2}}} \left(\frac{t}{\rho^p}\right)^{-\frac{N}{\lambda_2}} \left(\sup_{\mathcal{K}_\rho(t)\times[0,t]} u\right)^{\frac{p(2-r)}{\lambda_2}} \left(⨌_{\mathcal{K}_\rho(t)\times[0,t]} u^r \, dx d\tau\right)^{\frac{p}{\lambda_2}} + \gamma \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} ,
\end{aligned}$$

for every $1 \le r \le 2 < q$ for which (and for sure) $\lambda_r = N(p-2) + rp > 0$.

Here we observe that *a priori* information on the boundedness of u was not necessary in order to get the first sup-estimate in this case.

Finally, we perform a cross-iteration on $\sigma \in (0,1)$ as follows. Still referring to radii ρ_i as in the construction above, we now consider the increasing sequences, for $n \in \mathbb{N} \cup \{0\}$,

$$\tilde{\rho}_{0,i} = \sigma \rho_i \ , \qquad \tilde{\rho}_{n,i} = \rho_i \left(\sigma + (1-\sigma) \sum_{j=1}^{n} 2^{-j} \right),$$

$$\tilde{t}_0 = \sigma t \ , \qquad \tilde{t}_n = t \left(\sigma + (1-\sigma) \sum_{j=1}^{n} 2^{-j} \right),$$

$$\tilde{\mathcal{K}}_n = \prod_i \left\{ |x_i| < \tilde{\rho}_{n,i} \right\}, \qquad \tilde{Q}_n = \tilde{\mathcal{K}}_n \times (t - \tilde{t}_n, t),$$

and define

$$S_n = \sup_{\tilde{Q}_n} u \ .$$

The previous estimate applied to the pair of cylinders $\tilde{Q}_n$ and $\tilde{Q}_{n+1}$ gives us

$$\begin{aligned} S_n &\le \frac{\gamma}{(1-\sigma)^{\frac{p(N+p)}{\lambda_2}}} S_{n+1}^{\frac{p(2-r)}{\lambda_2}} \left(\frac{t}{\rho^p} \right)^{-\frac{N}{\lambda_2}} \left(\fint\!\!\fint_{\tilde{Q}_{n+1}} u^r \, dx d\tau \right)^{\frac{p}{\lambda_2}} + \gamma \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p}} \\ &\le \frac{1}{2} S_{n+1} + \frac{\gamma}{(1-\sigma)^{\frac{p(N+p)}{\lambda_r}}} \left(\frac{t}{\rho^p} \right)^{-\frac{N}{\lambda_r}} \left(\fint\!\!\fint_{\tilde{Q}_\infty} u^r \, dx d\tau \right)^{\frac{p}{\lambda_r}} + \gamma \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p}} \end{aligned}$$

by means of Young's inequality with $\epsilon = 1/2$ for exponents $\mu = \frac{\lambda_2}{p(2-r)} > 1$ and $\mu' = \lambda_2/\lambda_r$. Therefore, by iteration, one gets

$$S_0 \le \left(\frac{1}{2} \right)^n S_n + \left(\sum_{j=0}^{n-1} 2^{-j} \right) \frac{\gamma}{(1-\sigma)^{\frac{p(N+p)}{\lambda_r}}} \left(\frac{t}{\rho^p} \right)^{-N/\lambda_r} \left(\fint\!\!\fint_{\tilde{Q}_\infty} u^r \, dx d\tau \right)^{\frac{p}{\lambda_r}} + \gamma \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p}} \ .$$

and, by taking $\sigma = 1/2$ and letting $n \to \infty$

$$\sup_{\mathcal{K}_{\frac{\rho}{2}}(t) \times [t/2, t]} u = \sup_{\tilde{Q}_o} u \le \gamma \left(\frac{t}{\rho^p} \right)^{-N/\lambda_r} \left(\fint\!\!\fint_{\mathcal{K}_\rho \times [0,t]} u^r \, dx d\tau \right)^{\frac{p}{\lambda_r}} + \gamma \left(\frac{t}{\rho^p} \right)^{\frac{1}{2-p}} \ .$$

The case $1 < p \le \max\left\{1, \frac{2N}{N+2}\right\}$

In this case, the conditions $\lambda_r > 0$ and $1 < p \le 2N/(N+2)$ imply $r > 2$ and also $q = p\frac{N+2}{N} \le 2 < r$. Here we need to consider the L^r-norm of the truncated functions

$$Y_n = \iint_{Q_n} (u - k_n)_+^r \, dx d\tau,$$

and supposing u locally bounded, recalling $q < 2 < r$, we apply the anisotropic embedding 3.2 to get

$$\begin{aligned} Y_{n+1} &\le \iint_{Q_n} (u - k_{n+1})_+^{r-q} (u - k_{n+1})_+^q \xi_n^q \, dx d\tau \\ &\le \left(\sup_{Q_0} u \right)^{r-q} \iint_{Q_n} (u - k_{n+1})_+^q \xi_n^q \, dx d\tau \\ &\le \gamma \left(\sup_{Q_0} u \right)^{r-q} \left(\sup_{t - t_n \le \tau \le t} \int_{K_n} (u - k_{n+1})_+^2 \xi_n^2 \, dx \right)^{\frac{p}{N}} \left(\prod_i \iint_{Q_n} |\partial_i (u - k_{n+1})_+ \xi_n|^{p_i} \, dx d\tau \right)^{\frac{p}{N p_i}} \\ &\le \gamma \left(\sup_{Q_0} u \right)^{r-q} \mathcal{E}_n^{1+\frac{p}{N}} . \end{aligned}$$

Now again we make a chain of inequalities, but this time using $\mathcal{E}_n$ and Y_n. By acting in a similar fashion as before and assuming (5.3), we get

$$\mathcal{E}_n \le \frac{\gamma 2^{n(r+2)}}{(1-\sigma)^p t} \frac{1}{k^{r-2}} \, Y_n,$$

and therefore the aforementioned chain reads

$$Y_{n+1} \le \gamma \left(\sup_{Q_0} u \right)^{r-q} \frac{b^n}{((1-\sigma)^p t)^{\frac{(N+p)}{N}} k^{\frac{(r-2)(N+p)}{N}}} \, Y_n^{1+\frac{p}{N}} , \qquad b = 2^{\frac{(r+2)(N+p)}{N}} > 1.$$

Again by the Fast Convergence Lemma A.7, if $k > 0$ is taken so that

$$Y_0 \le \gamma^{-\frac{N}{p}} b^{-\frac{N^2}{p^2}} \left(\sup_{Q_0} u\right)^{-\frac{(r-q)N}{p}} ((1-\sigma)^p t)^{\frac{(N+p)}{p}} k^{\frac{(r-2)(N+p)}{p}},$$

we obtain $u < k$ for almost every $(x,\tau) \in Q_\infty$. Therefore we choose

$$k = \gamma \left(\sup_{Q_0} u\right)^{\frac{(r-q)N}{(N+p)(r-2)}} \left(\iint_{Q_0} u^r \, dx d\tau\right)^{\frac{p}{(N+p)(r-2)}} ((1-\sigma)^p t)^{-\frac{1}{r-2}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} \tag{5.6}$$

for which we get

$$\sup_{Q_\infty} u \le \gamma \left(\sup_{Q_0} u\right)^{\frac{(r-q)N}{(N+p)(r-2)}} \frac{1}{((1-\sigma)^p t)^{\frac{1}{r-2}}} \left(\iint_{Q_0} u^r \, dx d\tau\right)^{\frac{p}{(N+p)(r-2)}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}.$$

Proceeding as before, one has

$$\begin{aligned} S_n &\le S_{n+1}^{\frac{(r-q)N}{(N+p)(r-2)}} \frac{\gamma}{((1-\sigma)^p t)^{\frac{1}{r-2}}} \left(\iint_{\tilde{Q}_{n+1}} u^r \, dx d\tau\right)^{\frac{p}{(N+p)(r-2)}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} \\ &\le \frac{1}{2} S_{n+1} + \frac{\gamma}{((1-\sigma)^p t)^{\frac{N+p}{\lambda_r}}} \left(\iint_{\tilde{Q}_\infty} u^r \, dx d\tau\right)^{\frac{p}{\lambda_r}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} \end{aligned}$$

by means of Young's inequality with $\epsilon = 1/2$ for exponent $\mu = \frac{(N+p)(r-2)}{N(r-q)} > 1$. Then by iteration, taking $\sigma = 1/2$ and letting $n \to \infty$

$$\begin{aligned} \sup_{\mathcal{K}_{\rho/2}(t) \times [t/2,t]} u &\le \gamma\, t^{-\frac{N+p}{\lambda_r}} \left(\int_0^t \int_{\mathcal{K}_\rho(t)} u^r \, dx d\tau\right)^{\frac{p}{\lambda_r}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} \\ &= \gamma \left(\frac{t}{\rho^p}\right)^{-\frac{N}{\lambda_r}} \left(⨍_0^t ⨍_{\mathcal{K}_\rho(t)} u^r \, dx d\tau\right)^{\frac{p}{\lambda_r}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}. \quad \square \end{aligned}$$

Theorem 5.2 (L^r_{loc} *Estimates Backward in Time*). *Let u be a non-negative, locally bounded, local weak solution to (2.1)–(2.2) and assume $u \in L^r_{loc}(\Omega_T)$, for some $r > 1$. Then there exists a positive constant γ, depending only on the data, such that either (2.3) is satisfied for some $i \in \{1, \dots, N\}$ or*

$$\sup_{0 \le \tau \le t} \int_{\mathcal{K}_\rho(t)} u^r(x,\tau)\, dx \le \gamma \int_{\mathcal{K}_{2\rho}(t)} u^r(x,0)\, dx + \gamma \left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{2-p}}, \tag{5.7}$$

being $\lambda_r = N(p-2) + pr$.

Proof. Assume (2.3) fails to happen for all $i \in \{1, \dots, N\}$. Fix $\sigma \in (0,1)$ and construct the cylinders

$$Q_1 = \mathcal{K}_\rho(t) \times [0,t], \qquad Q_2 = \mathcal{K}_{(1+\sigma)\rho}(t) \times [0,t].$$

With these stipulations, a cut off function ζ, such as in (A.2), between $\mathcal{K}_\rho(t)$ and $\mathcal{K}_{(1+\sigma)\rho}(t)$ satisfies

$$\|\partial_i \zeta_i\|_\infty \le \frac{1}{(\sigma\rho)^{\frac{p}{p_i}}} \left(\frac{t}{\rho^p}\right)^{\frac{(p-p_i)}{p_i(2-p)}} =: \frac{1}{\sigma^{\frac{p}{p_i}} \rho_i(t)},$$

and the estimates (A.5) with $K_1 = \mathcal{K}_\rho(t)$ and $K_2 = \mathcal{K}_{(1+\sigma)\rho}(t)$ are now written

$$\begin{aligned} \sup_{0 \le \tau \le t} \int_{\mathcal{K}_\rho(t)} u^r(x,\tau)\, dx &\le \gamma \int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,0)\, dx \\ &+ \sum_i \frac{\gamma}{\sigma^p \rho^p} \left(\frac{t}{\rho^p}\right)^{\frac{p-p_i}{2-p}} \Bigg\{ \int_0^t \int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^{r+p_i-2} \, dx d\tau + \\ &+ \left[\left(C \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^p}\right)^{\frac{p_i-p}{p_i(2-p)}} \right)^{p_i-1} + \left(C \rho^{\frac{p}{p_i}} \left(\frac{t}{\rho^p}\right)^{\frac{p_i-p}{p_i(2-p)}} \right)^{p_i} \left(1 + \frac{1}{M_r}\right) \right] \int_0^t \int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^{r-1} \, dx d\tau \Bigg\}, \end{aligned}$$

being

$$M_r = \left(\sup_{0 \le \tau \le t} ⨍_{\mathcal{K}_\rho(t)} u^r(x,\tau)\, dx\right)^{\frac{1}{r}}.$$

Without loss of generality one can assume that, for all $i = 1, \dots, N$,

$$C\rho^{\frac{p}{p_i}}\left(\frac{t}{\rho^p}\right)^{\frac{p_i-p}{p_i(2-p)}} \le M_r.$$

In fact, if for some index $i = 1, \dots, N$

$$C\rho^{\frac{p}{p_i}}\left(\frac{t}{\rho^p}\right)^{\frac{p_i-p}{p_i(2-p)}} > M_r,$$

implying that

$$\sup_{0\le\tau\le t}\int_{\mathcal{K}_\rho(t)} u^r(x,\tau)\,dx < 2^N\rho^N\left(C\rho^{\frac{p}{p_i}}\nu^{\frac{p_i-p}{p_i}}\right)^r < 2^N\rho^N\left(\nu^{\frac{p}{p_i}}\nu^{\frac{p_i-p}{p_i}}\right)^r = \gamma\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{2-p}}$$

and then (5.7) comes immediately. Hence

$$\sup_{0\le\tau\le t}\int_{\mathcal{K}_\rho(t)} u^r(x,\tau)\,dx \le \gamma\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,0)\,dx +$$
$$+\gamma\sum_i \frac{\gamma}{\sigma^p\rho^p}\left(\frac{t}{\rho^p}\right)^{\frac{p-p_i}{2-p}}\left\{\int_0^t\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^{r+p_i-2}\,dxd\tau + M_r^{p_i-1}\int_0^t\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^{r-1}\,dxd\tau\right\}.$$

We estimate the second integral on the right-hand side by applying Hölder's inequality,

$$\begin{aligned}\sum_i \frac{t}{\rho^p}\left(\frac{t}{\rho^p}\right)^{\frac{(p_i-p)}{(2-p)}}&\left(\sup_{0\le\tau\le t}\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^{p_i+r-2}(x,\tau)\,dx\right)\\ &\le \gamma\sum_i\left(\frac{t}{\rho^p}\right)^{\frac{2-p_i}{2-p}}\left(\sup_{0\le\tau\le t}\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,\tau)\,dx\right)^{\frac{p_i+r-2}{r}}\rho^{\frac{N(2-p_i)}{r}}\\ &= \gamma\sum_i\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{2-p_i}{r(2-p)}}\left(\sup_{0\le\tau\le t}\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,\tau)\,dx\right)^{\frac{p_i+r-2}{r}}.\end{aligned}$$

The last integral on the right-hand side is dominated as follows

$$\begin{aligned}&\gamma\sum_i\frac{\gamma}{\sigma^p\rho^p}\left(\frac{t}{\rho^p}\right)^{\frac{p-p_i}{2-p}} M_r^{p_i-1}\int_0^t\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^{r-1}\,dxd\tau\\ &\le \gamma\sum_i\frac{\gamma}{\sigma^p\rho^p}\left(\frac{t}{\rho^p}\right)^{\frac{p-p_i}{2-p}} M_r^{p_i-1}t\left(\sup_{0\le\tau\le t}\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,\tau)\,dx\right)^{\frac{r-1}{r}}(2\rho)^{\frac{N}{r}}\\ &\le \frac{\gamma}{\sigma^p}\sum_i\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{2-p_i}{r(2-p)}}\left(\sup_{0\le\tau\le t}\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,\tau)\,dx\right)^{\frac{p_i+r-2}{r}}\end{aligned}$$

using Hölder inequality and noticing that

$$M_r = \left(\sup_{0\le\tau\le t}\fint_{\mathcal{K}_\rho(t)} u^r(x,\tau)\,dx\right)^{\frac{1}{r}} < \left(\sup_{0\le\tau\le t}\int_{\mathcal{K}_{\rho(1+\sigma)}(t)} u^r(x,\tau)dx\right)^{\frac{1}{r}}(2\rho)^{-\frac{N}{r}}.$$

Putting the estimates all together we finally get

$$\begin{aligned}&\sup_{0\le\tau\le t}\int_{\mathcal{K}_\rho(t)} u^r(x,\tau)\,dx\\ &\le \gamma\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,0)\,dx + \sum_i\frac{\gamma}{\sigma^p}\left(\sup_{0\le\tau\le t}\int_{\mathcal{K}_{(1+\sigma)\rho}(t)} u^r(x,\tau)\,dx\right)^{\frac{p_i+2-r}{r}}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{2-p_i}{r(2-p)}}.\end{aligned} \tag{5.8}$$

Now we perform an iteration on σ: fix $\rho > 0$ and for $n \in \mathbb{N}\cup\{0\}$ consider the increasing sequence of radii

$$\rho_i(t) \le \rho_{n,i} := \rho_i(t)\sum_{s=0}^n 2^{-s} \quad \text{so that} \quad \rho_{n+1,i} = (1+\sigma_n)\rho_{n,i}, \quad \text{for} \quad \sigma_n = \frac{\rho_{n+1,i}-\rho_{n,i}}{\rho_{n,i}} \ge \frac{1}{2^{n+2}}.$$

By setting

$$S_n = \sup_{0\le\tau\le t}\int_{\mathcal{K}_{\rho_n}(t)} u^r(x,\tau)\,dx\ ,$$

estimate (5.8) now reads

$$S_n \le \int_{\mathcal{K}_{2\rho}(t)} u^r(x,0)\,dx + \gamma\sum_i 2^{np}\left(S_{n+1}\right)^{\frac{p_i+r-2}{r}}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{2-p_i}{r(2-p)}}.$$

We use Young's inequality in each ith term of the sum

$$\left[\gamma 2^{np}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{2-p_i}{r(2-p)}}\right]\left(S_{n+1}\right)^{\frac{p_i+r-2}{r}} \le \epsilon S_{n+1} + \gamma(\epsilon) b^n \left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{2-p}},$$

for a constant $b > 1$ depending only on the data, and with these stipulations we arrive at

$$S_n \le \epsilon\ S_{n+1} + \gamma(\epsilon) b^n \left(\int_{\mathcal{K}_{2\rho}(t)} u^r(x,0)\,dx + \left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{2-p}} \right).$$

A simple iteration shows

$$S_0 \le \epsilon^n\ S_n + \gamma(\epsilon) \sum_{k=1}^{n-1} (\epsilon\ b)^k \left(\int_{\mathcal{K}_{2\rho}(t)} u^r(x,0)\,dx + \left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{2-p}} \right),$$

and proof is completed once we choose $\epsilon = 1/2b < 1$ and let $n \to \infty$ as usual. □

Remark 5.3. Here the exponent $\lambda_r = N(p-2) + pr$ can be of either sign.

5.1. Proof of Theorem 2.4 concluded

Proof. We plug inequality (5.7) into (5.1) to obtain

$$\begin{aligned}\|u\|_{\infty,\mathcal{K}_\rho(t)\times[t/2,t]} &\le \gamma t^{-\frac{N}{\lambda_r}} \left(\int_{\mathcal{K}_{2\rho}(t)} u^r(x,0)\,dx + \left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{2-p}} \right)^{\frac{p}{\lambda_r}} + \gamma \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} \\ &\le \gamma t^{-\frac{N}{\lambda_r}} \left(\int_{\mathcal{K}_{2\rho}(t)} u^r(x,0)\,dx \right)^{\frac{p}{\lambda_r}} + \gamma \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}. \quad \square\end{aligned}$$

Standard anisotropic geometry: Proof of Theorem 2.5

Theorem 5.4. (L^r_{loc}- L^∞_{loc} **estimates**) *Let u be a non-negative, locally bounded, local weak sub(super)-solution to (2.1)–(2.2) in Ω_T. Let $r \ge 1$ be such that*

$$\lambda_r = N(p-2) + rp > 0\ . \tag{5.9}$$

Then there exists a positive constant γ, depending only on the data such that, for all $\mathbb{K}_\rho \times [0,t] \subset \Omega_T$, either for some $i \in \{1,\dots,N\}$ condition (2.5) is satisfied or

$$\sup_{\mathbb{K}_{\rho/2}\times[t/2,t]} u \le \gamma \left(\frac{t}{\rho^p}\right)^{-\frac{N}{\lambda_r}} \left(\sup_{0\le\tau\le t} \fint_{\mathbb{K}_\rho} u^r(x,\tau)\,dx \right)^{\frac{p}{\lambda_r}} + \sum_i \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}}\ . \tag{5.10}$$

Proof. Assume condition (2.5) is violated for all indexes $i \in \{1,\dots,N\}$. Let $\sigma \in (0,1)$ be fixed and consider the decreasing sequences

$$\sigma\rho = \rho_\infty < \rho_n = \rho\left(\sigma + \frac{1-\sigma}{2^n}\right) \le \rho_0 = \rho$$

and

$$\sigma t = t_\infty < t_n = t\left(\sigma + \frac{1-\sigma}{2^n}\right) \le t_0 = t$$

from which one constructs the sequence of nested and shrinking cylinders

$$\mathbb{Q}_n = \mathbb{K}_n \times (t - t_n, t)$$

where, as usual in the standard anisotropic geometry,

$$\mathbb{K}_n = \prod_i \left\{ |x_i| < \rho_n^{\frac{p}{p_i}} \right\}.$$

Define cutoff function $\zeta_n(x,t) = \zeta_n(x)\xi(\tau)$, as in (A.3), verifying

$$\zeta_{n,i}(x_i) = \begin{cases} 1 & , |x_i| < \rho_{n+1} \\ 0 & , |x_i| \ge \rho_n \end{cases} \quad , \qquad \|\partial_i \zeta_n\|_\infty \le \left(\frac{2^{n+1}}{(1-\sigma)\rho}\right)^{\frac{p}{p_i}}$$

and

$$\xi(\tau) = \begin{cases} 0 & , 0 \le \tau \le t - t_n \\ 1 & , t - t_{n+1} \le \tau \le t \end{cases} \quad , \qquad \|\partial_t \xi\|_\infty \le \frac{2^{n+1}}{(1-\sigma)t}\ .$$

In the weak formulation (3.1) we consider test functions $\varphi_n = (u-k_{n+1})_+\zeta_n$, over the cylinders $\mathbb{Q}_n$, for the truncation levels

$$0 \le k_n = k\left(1-\frac{1}{2^n}\right) < k\ , n = 0, 1, \dots$$

where k is a positive real number to be determined (along the proof). By the energy estimates (A.4) we get

$$\begin{aligned}\mathcal{E}_n = \sup_{t-t_n\le\tau\le t}\int_{\mathbb{K}_n\times\{\tau\}}(u-k_{n+1})_+^2\zeta_n\,dx + \sum_i \iint_{\mathbb{Q}_n}|\partial_i\left((u-k_{n+1})_+\zeta_n\right)|^{p_i}\,dxd\tau \\ \le \gamma\frac{2^n}{(1-\sigma)t}\iint_{\mathbb{Q}_n}(u-k_{n+1})_+^2\,dxd\tau \\ + \gamma\frac{2^{np}}{(1-\sigma)^p\rho^p}\sum_i\left(1+(C^{p_i}\rho^p)\right)\iint_{\mathbb{Q}_n}(u-k_{n+1})_+^{p_i}\,dxd\tau \\ + \gamma\sum_i C^{p_i}\iint_{\mathbb{Q}_n}\chi_{[u>k_{n+1}]}\,dxd\tau.\end{aligned} \tag{5.11}$$

As in the proof of Theorem 5.1, from now on we distinguish between the case where p is in the super and the sub-critical ranges. We will only present how to proceed when p is in the super-critical range; the sub-critical range is treated analogously to what was done for the anisotropic intrinsic geometry but now taking into account take we are working under the assumptions related to the anisotropic standard setting.

Consider $\max\{1, \frac{2N}{N+2}\} < p < 2$. By observing that $\rho^p C^{p_i} \le 1$, for all $i \in \{1, \dots, N\}$,

$$\iint_{\mathbb{Q}_n}(u-k_n)_+^2\,dxd\tau \ge \left(\frac{k}{2^{n+1}}\right)^2\iint_{\mathbb{Q}_n}\chi_{[u>k_{n+1}]}\,dxd\tau\ ,$$

$$\iint_{\mathbb{Q}_n}(u-k_n)_+^2\,dxd\tau \ge \left(\frac{k}{2^{n+1}}\right)^{2-p_i}\iint_{\mathbb{Q}_n}(u-k_{n+1})_+^{p_i}\,dxd\tau\ ,$$

and choosing $k \ge \nu_\Sigma$, from the previous estimate (5.11) one gets

$$\begin{aligned}\mathcal{E}_n &\le \gamma\frac{2^{(p+2)n}}{(1-\sigma)^p t}\left\{1+\frac{t}{\rho^p}\sum_i k^{p_i-2}+\frac{t}{\rho^p}\sum_i\frac{\rho^p C^{p_i}}{k^2}\right\}\iint_{\mathbb{Q}_n}(u-k_n)_+^2\,dxd\tau \\ &\le \gamma\frac{2^{(p+2)n}}{(1-\sigma)^p t}\iint_{\mathbb{Q}_n}(u-k_n)_+^2\,dxd\tau\ .\end{aligned}$$

Although the geometry is different, we derive a similar estimate to (5.5) by means of Hölder's inequality, so to obtain

$$\begin{aligned}&\sup_{\mathcal{K}_{\sigma\rho}(t)\times[\sigma t,t]} u \le \\ &\le \frac{\gamma}{(1-\sigma)^{\frac{p(N+p)}{\lambda_2}}}\left(\frac{t}{\rho^p}\right)^{-\frac{N}{\lambda_2}}\left(\sup_{\mathcal{K}_\rho(t)\times[0,t]}u\right)^{\frac{p(2-r)}{\lambda_2}}\left(\fint\!\!\fint_{\mathcal{K}_\rho(t)\times[0,t]}u^r\,dxd\tau\right)^{\frac{p}{\lambda_2}} + \gamma\sum_i\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}}\end{aligned}$$

An analogous iteration procedure is applied considering the radius to be ρ rather than ρ_i, completing thereby the proof for the super-critical range of p. □

Theorem 5.5. *(L^r_{loc} **estimates backward in time**) Let u be a non-negative, locally bounded, local weak solution to (2.1)–(2.2) in Ω_T. Assume that $u \in L^r_{loc}(\Omega_T)$, for some $r > 1$. Then there exists a positive constant γ, depending on the data, such that either (2.5) is verified for some $i \in \{1, \dots, N\}$, or*

$$\sup_{0\le\tau\le t}\int_{\mathbb{K}_\rho}u^r(x,\tau)\,dx \le \gamma\int_{\mathbb{K}_{2\rho}}u^r(x,0)\,dx + \gamma\sum_i\left(\frac{t^r}{\rho^{\lambda_{i,r}}}\right)^{\frac{1}{2-p_i}}, \tag{5.12}$$

where $\lambda_{i,r} = N(p_i-2)+pr$.

Proof. Assume (2.5) is not verified for all $i \in \{1, \dots, N\}$. Fix $\sigma \in (0,1)$ and construct the cylinders

$$\mathbb{Q} = \mathbb{K}_\rho \times [0,t] = \prod_i\left\{|x_i| < \rho^{\frac{p}{p_i}}\right\}\times[0,t], \qquad \mathbb{Q}_\sigma = \mathbb{K}_{(1+\sigma)\rho}\times[0,t].$$

Using (A.5) with $Q_1 = \mathbb{Q}$ and $Q_2 = \mathbb{Q}_\sigma$, and a time-independent cut-off function ζ is as in (A.2) defined in $\mathcal{K}_{(1+\sigma)\rho}$ and verifying

$$\|\partial_i\zeta\|_\infty \le \gamma/(\sigma\rho)^{\frac{p}{p_i}}, \quad \text{for all} \quad i = 1, \dots, N,$$

while considering

$$M_r = \left(\sup_{0\le\tau\le t}\fint_{\mathbb{K}_\rho}u^r\,dx\right)^{1/r} > C\rho^{\frac{p}{p_i}}\ , \qquad \forall i = 1, \dots, N \tag{5.13}$$

we obtain

$$\begin{aligned}\sup_{0\le\tau\le t}\int_{\mathbb{K}_\rho} u^r(x,\tau)\,dx \;&\le\; \gamma\int_{\mathbb{K}_{\sigma\rho}} u^r(x,0)\,dx \\ &\quad+\frac{\gamma}{(\sigma\rho)^p}\sum_i\Big(1+(C^{p_i}\rho^p)\Big)\iint_{\mathbb{Q}_\sigma} u^{r+p_i-2}\,dxd\tau \\ &\quad+\frac{\gamma}{(\sigma\rho)^p}\sum_i\Big[(C\rho^{\frac{p}{p_i}})^{p_i-1}+C^{p_i}\rho^p\Big(1+\frac{1}{M_r}\Big)\Big]\iint_{\mathbb{Q}_\sigma} u^{r-1}\,dxd\tau \\ &\le \gamma\int_{\mathbb{K}_{\sigma\rho}} u^r(x,0)\,dx \\ &\quad+\frac{\gamma}{(\sigma\rho)^p}\Big\{\sum_i\iint_{\mathbb{Q}_\sigma} u^{r+p_i-2}\,dxd\tau+\sum_i M_r^{p_i-1}\iint_{\mathbb{Q}_\sigma} u^{r-1}\,dxd\tau\Big\}\end{aligned} \tag{5.14}$$

Observe that (5.13) is a natural assumption: if it is violated then, for some $i\in\{1,\dots,N\}$, then

$$\Big(\sup_{0\le\tau\le t}\fint_{\mathbb{K}_\rho} u^r(x,\tau)\,dx\Big)^{\frac1r}\le C\rho^{\frac{p}{p_i}}\le\sum_k\Big(\frac{t}{\rho^p}\Big)^{\frac{1}{2-p_k}}=v_\Sigma$$

$$\iff\quad \sup_{0\le\tau\le t}\int_{\mathbb{K}_\rho} u^r(x,\tau)\,dx\le\sum_k\Big(\frac{t^r}{\rho^{\lambda_{k,r}}}\Big)^{\frac{1}{2-p_k}}$$

and (5.12) is found. Then, as in Theorem 5.2, we estimate the various terms as follows

$$\begin{aligned}\sum_i\frac{1}{\rho^p}\iint_{\mathbb{Q}_\sigma} u^{r+p_i-2}\,dxd\tau &\le \sum_i\Big(\frac{t}{\rho^p}\Big)\Big(\sup_{0\le\tau\le t}\int_{\mathbb{K}_{(1+\sigma)\rho}} u^{r+p_i-2}(x,\tau)\,dx\Big)\\ &\le\sum_i\Big(\frac{t}{\rho^p}\Big)\Big(\sup_{0\le\tau\le t}\int_{\mathbb{K}_{(1+\sigma)\rho}} u^{r+p_i-2}(x,\tau)\,dx\Big)^{\frac{p_i+r-2}{r}}(2\rho)^{\frac{N(2-p_i)}{r}}\\ &=\sum_i\Big(\frac{t^r}{\rho^{\lambda_{i,r}}}\Big)^{\frac1r}\Big(\sup_{0\le\tau\le t}\int_{\mathbb{K}_{(1+\sigma)\rho}} u^r(x,\tau)\,dx\Big)^{\frac{p_i+r-2}{r}},\end{aligned}$$

for $\lambda_{i,r}=N(p_i-2)+pr$, while the second term in the parenthesis of (5.14) is managed as follows

$$\begin{aligned}\sum_i\frac{M_r^{p_i-1}}{\rho^p}\iint_{\mathbb{Q}_\sigma} u^{r-1}\,dxd\tau &\le\sum_i\Big(\frac{t}{\rho^p}\Big)M_r^{p_i-1}\Big(\sup_{0\le\tau\le t}\int_{\mathbb{K}_{(1+\sigma)\rho}} u^r(x,\tau)\,dx\Big)^{\frac{r-1}{r}}(2\rho)^{\frac Nr}\\ &\le\sum_i\Big(\frac{t^r}{\rho^{\lambda_{i,r}}}\Big)^{\frac1r}\Big(\sup_{0\le\tau\le t}\int_{\mathbb{K}_{(1+\sigma)\rho}} u^r(x,\tau)\,dx\Big)^{\frac{p_i+r-2}{r}}.\end{aligned}$$

Plugging these estimates into (5.14) we obtain, and applying Young's inequality in each term of the sum, we get

$$\begin{aligned}\sup_{0\le\tau\le t}\int_{\mathbb{K}} u^r(x,\tau)\,dx &\le\gamma\int_{\mathbb{K}_{(1+\sigma)\rho}} u^r(x,0)\,dx+\sum_i\frac{\gamma}{\sigma^p}\Big(\sup_{0\le\tau\le t}\int_{\mathbb{K}_{(1+\sigma)\rho}} u^r(x,\tau)\,dx\Big)^{\frac{p_i+r-2}{r}}\Big(\frac{t^r}{\lambda_{i,r}}\Big)^{\frac1r}\\ &\le\gamma\int_{\mathbb{K}_{(1+\sigma)\rho}} u^r(x,0)\,dx+\epsilon\sup_{0\le\tau\le t}\int_{\mathbb{K}_{(1+\sigma)\rho}} u^r(x,\tau)\,dx+\gamma(\epsilon)\sum_i\Big(\frac{t^r}{\rho^{\lambda_{i,r}}}\Big)^{\frac{1}{2-p_i}}\end{aligned} \tag{5.15}$$

From this point on, we perform a standard iteration on σ: for fixed $\rho>0$ and $n\in\mathbb{N}\cup\{0\}$, we consider the increasing sequence of radii

$$\rho_n:=\rho\sum_{j=0}^n 2^{-j}\geqslant\rho\quad\text{so that}\quad\rho_{n+1}=(1+\sigma_n)\rho_n,\quad\text{for}\quad\sigma_n=\frac{\rho_{n+1}-\rho_n}{\rho_n}\ge\frac{1}{2^{n+2}},$$

by setting

$$S_n=\sup_{0\le\tau\le t}\int_{\mathbb{K}_{\rho_n}} u^r(x,\tau)\,dx,$$

estimate (5.15) now reads

$$S_n\le\gamma\left\{\epsilon\,S_{n+1}+b^n\gamma(\epsilon)\left(\int_{\mathbb{K}_{2\rho}} u^r(x,0)\,dx+\sum_i\Big(\frac{t^r}{\rho^{\lambda_{i,r}}}\Big)^{\frac{1}{2-p_i}}\right)\right\},\qquad b>1,$$

and the proof is completed once we choose $\epsilon=1/2b<1$ and let $n\to\infty$. □

5.2. Proof of Theorem 2.5 concluded

Proof. We use (5.12) to estimate the integral term at the right-hand side of (5.10)

$$\begin{aligned}\sup_{\mathbb{K}_{\rho/2}\times[t/2,t]} u &\le \gamma t^{-\frac{N}{\lambda_r}} \left(\sup_{0\le\tau\le t} \int_{\mathbb{K}_\rho} u^r(x,\tau)\,dx \right)^{\frac{p}{\lambda_r}} + \sum_i \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}} \\ &\le \gamma t^{-\frac{N}{\lambda_r}} \left(\int_{\mathbb{K}_\rho} u^r(x,0)\,dx + \sum_k \left(\frac{t^r}{\rho^{\lambda_{k,r}}}\right)^{\frac{1}{2-p_k}} \right)^{\frac{p}{\lambda_r}} + \sum_i \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}} \\ &\le \gamma t^{-\frac{N}{\lambda_r}} \left(\int_{\mathbb{K}_\rho} u^r(x,0)\,dx \right)^{\frac{p}{\lambda_r}} + \gamma \sum_i \left[\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}} \right]^{\frac{\lambda_{i,r}}{\lambda_r}} + \sum_i \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}} \quad \square\end{aligned}$$

6. Proof of the L^1-L^∞ estimates

Intrinsic geometry. Proof of Theorem 2.7

Proof. We start by considering inequality (5.1) and then estimate the integral on its right-hand side by (2.4) to get

$$\begin{aligned}\|u\|_{\infty,\mathcal{K}_{\rho/2}(t)\times[t/2,t]} &\le \gamma t^{-\frac{N}{\lambda}} \left(\inf_{0\le\tau\le t} \int_{\mathcal{K}_{2\rho}(t)} u(x,\tau)\,dx + \gamma \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}} \right)^{\frac{p}{\lambda}} + \gamma \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} \\ &\le \gamma t^{-\frac{N}{\lambda}} \left(\inf_{0\le\tau\le t} \int_{\mathcal{K}_{2\rho}(t)} u(x,\tau)\,dx \right)^{\frac{p}{\lambda}} + \gamma \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}. \quad \square\end{aligned}$$

Standard geometry. Proof of Theorem 2.8

Proof. We combine Theorem 5.4 with $r = 1$ and Theorem 2.2 to get

$$\sup_{\mathbb{K}_{\rho/2}\times[t/2,t]} u \le \gamma t^{-\frac{N}{\lambda}} \left(\inf_{0\le\tau\le t} \int_{2\mathbb{K}_\rho} u(x,\tau)\,dx \right)^{\frac{p}{\lambda}} + \gamma \sum_i \left[\left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}} \right]^{\frac{\lambda_i}{\lambda}} + \gamma \sum_i \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}}. \quad \square$$

Data availability

All data generated or analyzed during this study are included in this article.

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Appendix

Energy Estimates. To the aim of computation, it would be technically convenient to pass from the formulation (3.1) of local weak solution to its Steklov averaged version, which allows us to perform computations under the integral sign with the approximating functions

$$u_h(x,t) = \begin{cases} \displaystyle ⨍_t^{t+h} u(\cdot,\tau)\,d\tau, & 0 < t < T-h, \quad \text{for} \quad 0 < h < T, \\ 0, & t > T-h, \end{cases} \tag{A.1}$$

defined for all $0 < t < T$. This is the same definition as the one presented in [15] (see in particular Chapter II for more details), and we refrain from specifying further this procedure, leaving space to what is really new.

Separate Variables Test Functions. For a compact set $K \subset \Omega$, we will usually test the Eqs. (3.1)–(2.2) with functions $\zeta(x) \in C_o^1(K)$ such that

$$\zeta(x) = \prod_i \zeta_i(x_i)^{p_i}, \quad \hat{\zeta}^j := \prod_{i\neq j} \zeta_i(x_i)^{p_i}, \qquad 0 \le \zeta \le 1, \tag{A.2}$$

with $\zeta_i \in C_o^1(\pi_i(K))$, being π_i the euclidean projection to the ith component. Sometimes we will use the notation

$$\zeta(x,\tau) = \xi(\tau)\zeta(x), \qquad 0 \le \xi \le 1, \tag{A.3}$$

for $\zeta(x)$ as above and $\xi(\tau) \in C^1_{loc}(0,T)$ a function to be specified at each recurrence. Let $[\tau_1,\tau_2] \subset [0,T]$ be a time interval and $Q = K \times [\tau_1,\tau_2]$ a cylinder inside Ω_T. We denote by

$$\|\partial_i\zeta\|_\infty = \|(\partial_i\zeta)\xi\|_{L^\infty(Q)} \quad \text{and} \quad \|\partial_\tau\zeta\|_\infty = \|(\partial_\tau\xi)\zeta\|_{L^\infty(Q)},$$

the essential suprema of $|\partial_i\zeta|$ and $|\partial_\tau\zeta|$ in Q.

Energy Estimates 1 - Caccioppoli-type Estimates

Lemma A.1. *Let u be a local weak sub(super)-solution to (2.1)–(2.2) and let $k \in \mathbb{R}$. Let $0 \le \tau_1 < \tau_2 \le T$ and $K \subset \Omega$ be a compact set. Then, there exists a positive constant γ, depending only on the data, such that for any $\zeta \in C^1_{loc}(0,T; C^1_o(K))$ of the kind (A.3) with $\xi(\tau_1) = 0$, we have*

$$\begin{aligned} &\sup_{\tau_1\le\tau\le\tau_2} \int_{K\times\{\tau\}} (u-k)_+^2 \zeta \, dx + C_o \sum_i \iint_Q |\partial_i (u-k)_+\zeta|^{p_i}\, dx\, d\tau \\ &\quad \le \gamma \sum_i \|\partial_i\zeta_i\|_\infty^{p_i} \left[1 + \left(\frac{C}{\|\partial_i\zeta_i\|_\infty}\right)^{p_i}\right] \iint_Q (u-k)_+^{p_i}\, dxd\tau \\ &\quad + \gamma \|\partial_\tau\zeta\|_\infty \iint_Q dxd\tau + \gamma \sum_i C^{p_i} \iint_Q \chi_{[u>k]}\, dx\, d\tau, \end{aligned} \tag{A.4}$$

where $C \geqslant 0$ and $C_o > 0$ are the structure constants of (2.2).

Proof. We test Eq. (2.1) with $\varphi = (u-k)_+\zeta$, being $\zeta \in C^1(Q)$ as in (A.3), vanishing on ∂K, for all times, and verifying $\zeta(\tau_1, x) = 0$, for all $x \in K$. So we arrive, through a standard Steklov approximation, to

$$\begin{aligned} \mathcal{I}_1 + \mathcal{I}_2 :&= \sup_{\tau_1\le\tau\le\tau_2} \int_K \frac{(u-k)_+^2\zeta}{2}\, dx + \sum_i \iint_Q A_i\Big(\partial_i(u-k)_+\zeta + (u-k)_+(\partial_i\zeta)\Big)\, dxd\tau \\ &\le \iint_Q (u-k)_+^2(\partial_\tau\zeta)\, dxd\tau + \iint_Q B(u-k)_+\zeta\, dxd\tau =: \mathcal{I}_3 + \mathcal{I}_4, \end{aligned}$$

being B, A_i,for all $i = 1,\dots,N$, the Caratheodory functions of (2.1)–(2.2).

We evaluate the terms separately, using the structure conditions (2.2) and Young's inequality (3.3) on each ith term with $q = p_i$, $q' = p_i/(p_i-1)$ to get

$$\begin{aligned} \mathcal{I}_2 &\geqslant \sum_i \iint_Q \Big(C_o|\partial_i(u-k)_+|^{p_i} - C^{p_i}\chi_{[u>k]}\Big)\zeta - \Big(C_1|\partial_i u|^{p_i-1} + C^{p_i-1}\Big)(u-k)_+|\partial_i\zeta_i|p_i\hat{\zeta}^i\zeta_i^{p_i-1}\, dxd\tau \\ &\geqslant \sum_i \iint_Q \Big(C_o - \gamma\tilde{\epsilon}_i C_1\Big)|\partial_i(u-k)_+|^{p_i}\zeta - \gamma[\tilde{\gamma}(\tilde{\epsilon}_i)C_1 + 1](u-k)_+^{p_i}|\partial_i\zeta_i|^{p_i} - \gamma C^{p_i}\zeta\chi_{[u>k]}\, dxd\tau, \end{aligned}$$

where in the last inequality we have collected the terms

$$|\partial_i\zeta_i|^{p_i}\hat{\zeta}^i = |\partial_i\zeta^{\frac{1}{p_i}}|^{p_i} \le |\partial_i\zeta_i|, \qquad \text{and} \qquad \hat{\zeta}^i\zeta_i^{p_i} = \zeta\,,$$

in order to adjust the powers of ζ. Again we use Young's inequality for each $i = 1,\dots,N$ to estimate

$$\begin{aligned} |\mathcal{I}_4| &\le \sum_i \iint_Q C\Big(|\partial_i u|^{p_i-1} + C^{p_i-1}\Big)(u-k)_+\zeta\, dxd\tau \\ &\le \gamma\sum_i \iint_Q \zeta\epsilon_i|\partial_i(u-k)_+|^{p_i} + C^{p_i}(\gamma(\epsilon_i)+1)(u-k)_+^{p_i} + C^{p_i}\chi_{[u>k]}\, dxd\tau\,. \end{aligned}$$

Choosing suitably $\tilde{\epsilon}_i$ and ϵ_i small enough for all $i = 1,\dots,N$ and joining all the previous estimates together implies, for all $k \in \mathbb{R}$,

$$\begin{aligned} &\sup_{\tau_1<\tau<\tau_2} \int_K (u-k)_+^2\, dx + C_o\sum_i \iint_Q \Big(|\partial_i[(u-k)_+\zeta]|^{p_i} - \gamma(u-k)_+^{p_i}|\partial_i\zeta_i|^{p_i}\Big)\, dxd\tau \\ &\quad \le \sup_{\tau_1<\tau<\tau_2} \int_K (u-k)_+^2\, dx + \sum_i C_o \iint_Q |\partial_i(u-k)_+|^{p_i}\zeta\, dxd\tau \\ &\quad \le \gamma\|\partial_\tau\zeta\|_\infty \iint_Q (u-k)_+^2\, dxd\tau + \gamma\sum_i \|\partial_i\zeta_i\|_\infty^{p_i} \iint_Q (u-k)_+^{p_i}\, dxd\tau \\ &\qquad + \gamma\sum_i \iint_Q C^{p_i}(u-k)_+^{p_i}\, dxd\tau + \gamma\sum_i \iint_Q C^{p_i}\chi_{[u>k]}\, dxd\tau. \quad \square \end{aligned}$$

Energy Estimates 2 - Testing with positive powers.

Lemma A.2. *Let u be a non-negative, locally bounded, local weak solution to (2.1)–(2.2) satisfying $u \in L^r_{loc}(\Omega)$ for some $r > 1$. Let $K_1 \subset K_2 \subset \Omega$ be compact sets and let $\zeta \in C^1_o(K_2)$ be a cut-off function between K_1 and K_2 as in (A.2). Let $t > 0$ be any number such that the inclusion*

$$Q_j = K_j \times [0,t] \subset \Omega_T, \qquad \forall j \in \{1,2\},$$

is preserved. Then, there exists a positive constant γ, depending only on the data, such that

$$\begin{aligned} \sup_{0\le\tau\le t} \int_{K_1} u^r(x,\tau)\,dx \le \gamma \int_{K_2} u^r(x,0)\,dx + \\ + \gamma \sum_i \|\partial_i\zeta_i\|_\infty^{p_i}\left(1+\frac{C^{p_i}}{\|\partial_i\zeta_i\|_\infty^{p_i}}\right)\iint_{Q_2} u^{r+p_i-2}\,dxd\tau \\ + \gamma \sum_i \|\partial_i\zeta_i\|_\infty^{p_i}\left[\frac{C^{p_i-1}}{\|\partial_i\zeta_i\|_\infty^{p_i-1}} + \frac{C^{p_i}}{\|\partial_i\zeta_i\|_\infty^{p_i}}\left(1+\frac{1}{M_r}\right)\right]\iint_{Q_2} u^{r-1}\,dxd\tau, \end{aligned} \tag{A.5}$$

being

$$M_r = \left(\sup_{0\le\tau\le t} \fint_{K_1} u^r\,dx\right)^{1/r}. \tag{A.6}$$

Proof. In the weak formulation (3.1) choose as a test function, defined over Q_2,

$$\varphi = f(u)\zeta = u^{r-1}\left(\frac{(u-k)_+}{u}\right)^q \zeta, \qquad \text{for} \qquad \max\{1, r-1\} < q < r,$$

being ζ as in (A.2) and $k \in \mathbb{R}^+$ to be determined. We observe that $f(u) = 0$ outside the set

$$[u > k] := \{(x,\tau) \in Q_2 : \ u(x,\tau) > k\}.$$

Now we define $F(u) = \int_k^u f(s)\,ds$ an integral function of f and we observe that

$$(r-1)u^{r-2}\left(\frac{(u-k)_+}{u}\right)^q \le f'(u) \le q u^{r-2}\left(\frac{(u-k)_+}{u}\right)^{q-1}. \tag{A.7}$$

The test function φ is an admissible one, modulo a Steklov approximation, thanks to the local boundedness of u: observe that

$$\partial_i\varphi = f(u)\partial_i\zeta + f'(u)\partial_i u\,\zeta \le \left\{\|\partial_i\zeta\|_\infty u^{r-1} + q\frac{u^{r-1}}{k}|\partial_i u|\right\}\chi_{[u>k]} \in L^{p_i}_{loc}(\Omega_T).$$

Passing to the limit the in Steklov approximation, we obtain

$$\begin{aligned} 0 = \iint_{\tilde{Q}_2} \partial_\tau F(u)\,\zeta\,dxd\tau + \sum_i \iint_{\tilde{Q}_2} A_i(\partial_i u) f'(u)\zeta\,dxd\tau \\ + \sum_i \iint_{\tilde{Q}_2} \hat{\zeta}^i f(u) A_i(\partial_i\zeta)\,dxd\tau - \iint_{\tilde{Q}_2} B f(u)\zeta\,dxd\tau =: T_1 + T_2 + T_3 + T_4, \end{aligned}$$

where $\tilde{Q}_2 = K_2 \times [0,s]$, for arbitrary $s \in (0,t]$.

The bound (A.7) and the fact that ζ is independent of time allows us to estimate

$$T_1 = \int_{K_2} F(u(x,s))\zeta(x)\,dx - \int_{K_2} F(u(x,0))\zeta(x)\,dx,$$

while the structure conditions (2.2) imply

$$\begin{aligned} T_2 &= \sum_i \iint_{\tilde{Q}_2} A_i(\partial_i u) f'(u)\chi_{[u>k]}\zeta\,dxd\tau \\ &\geqslant \sum_i \iint_{\tilde{Q}_2} \Big(C_o|\partial_i u|^{p_i} f'(u) - C^{p_i} f'(u)\Big)\zeta\,dxd\tau \\ &\geqslant \sum_i \iint_{\tilde{Q}_2} \left((r-1)C_o|\partial_i u|^{p_i} u^{r-2}\left(\frac{(u-k)_+}{u}\right)^q - qC^{p_i}u^{r-2}\left(\frac{(u-k)_+}{u}\right)^{q-1}\right)\zeta\,dxd\tau \\ &\geqslant \sum_i \iint_{\tilde{Q}_2} \left((r-1)C_o|\partial_i u|^{p_i}\frac{f(u)}{u} - qC^{p_i}u^{r-2}\left(\frac{(u-k)_+}{u}\right)^{q-1}\right)\zeta\,dxd\tau, \end{aligned}$$

$$|T_3| \le \gamma \sum_i \iint_{\tilde{Q}_2} f(u)\Big(C_1|\partial_i u|^{p_i-1}|\partial_i\zeta_i| + C^{p_i-1}|\partial_i\zeta_i|\Big)p_i\zeta_i^{p_i-1}\hat{\zeta}^i\,dxd\tau,$$

$$|T_4| \le \sum_i \iint_{\tilde{Q}_2} \Big(C|\partial_i u|^{p_i-1} f(u) + C^{p_i} f(u)\Big)\zeta\,dxd\tau.$$

Combining all the estimates we obtain, for all $s \in (0,t]$

$$\begin{aligned}\int_{K_2} F(u(x,s))\zeta(x)\,dx + (r-1)C_o \sum_i \iint_{\tilde{Q}_2} \frac{f(u)}{u}|\partial_i u|^{p_i}\zeta\,dxd\tau \le \int_{K_2} F(u(x,0))\zeta(x)\,dx\\ + \gamma \sum_i \iint_{\tilde{Q}_2}\left(C_1 + \frac{C}{|\partial_i\zeta_i|}\right) f(u)|\partial_i u|^{p_i-1}|\partial_i\zeta_i|\,\hat{\zeta}^i\zeta_i^{p_i-1}\,dxd\tau\\ + \gamma\sum_i (C^{p_i-1}\|\partial_i\zeta_i\|_\infty)\left[1+\frac{C}{\|\partial_i\zeta_i\|_\infty}\right]\iint_{\tilde{Q}_2} f(u)\,dxd\tau\\ + \gamma\sum_i C^{p_i}\iint_{\tilde{Q}_2} u^{r-2}\left(\frac{(u-k)_+}{u}\right)^{q-1}\zeta\,dxd\tau =: I_1+I_2+I_3+I_4\,.\end{aligned}$$

Here we observe that, on the set $[u>k]$, the following holds true

$$\frac{f(u)}{u} = u^{r-2}\left(\frac{(u-k)_+}{u}\right)^q \le \left(\frac{u^{r-1}}{k}\right) \qquad \text{and} \qquad f(u)\le u^{r-1} \quad ,$$

so that we estimate for each $i=1,\dots,N$,

$$\begin{aligned}I_{2,1} =& C_1\sum_i\iint_{\tilde{Q}_2} f(u)|\partial_i u|^{p_i-1}|\partial_i\zeta_i|\,\hat{\zeta}^i\zeta_i^{p_i-1}\,dxd\tau\\ &\le C_1\sum_i\epsilon_i\iint_{\tilde{Q}_2}\frac{f(u)}{u}|\partial_i u|^{p_i}\zeta\,dxd\tau + C_1\sum_i\gamma(\epsilon_i)\|\partial_i\zeta_i\|_\infty^{p_i}\iint_{\tilde{Q}_2} f(u)u^{p_i-1}\zeta\,dxd\tau\\ &\le C_1\sum_i\epsilon_i\iint_{\tilde{Q}_2}\frac{f(u)}{u}|\partial_i u|^{p_i}\zeta\,dxd\tau + C_1\sum_i\gamma(\epsilon_i)\|\partial_i\zeta_i\|_\infty^{p_i}\iint_{\tilde{Q}_2} u^{r+p_i-2}\,\chi_{[u>k]}\,dxd\tau.\end{aligned}$$

The other integral term does not involve the derivatives of the cut-off function

$$\begin{aligned}I_{2,2} =& \sum_i C\iint_{\tilde{Q}_2} f(u)|\partial_i u|^{p_i-1}\,\hat{\zeta}^i\zeta_i^{p_i-1}\,dxd\tau\\ &\le \sum_i\tilde{\epsilon}_i\iint_{\tilde{Q}_2}\frac{f(u)}{u}|\partial_i u|^{p_i}\,\zeta\,dxd\tau + \sum_i C^{p_i}\gamma(\tilde{\epsilon}_i)\iint_{\tilde{Q}_2} f(u)u^{p_i-1}\zeta\,dxd\tau\\ &\le \sum_i\tilde{\epsilon}_i\iint_{\tilde{Q}_2}\frac{f(u)}{u}|\partial_i u|^{p_i}\zeta\,dxd\tau + \sum_i C^{p_i}\gamma(\tilde{\epsilon}_i)\iint_{\tilde{Q}_2} u^{r+p_i-2}\,\chi_{[u>k]}\,dxd\tau.\end{aligned}$$

Now we estimate from above I_3, I_4 as

$$I_3+I_4 \le \sum_i \|\partial_i\zeta_i\|_\infty^{p_i}\left[C^{p_i-1}\|\partial_i\zeta_i\|_\infty^{1-p_i} + \frac{C^{p_i}}{\|\partial_i\zeta_i\|_\infty^{p_i}}\left(1+\frac{1}{k}\right)\right]\iint_{\tilde{Q}_2} u^{r-1}\,\chi_{[u>k]}\,dxd\tau.$$

Hence, choosing ϵ_i and $\tilde{\epsilon}_i$ appropriately small, we obtain for all $s\in(0,t]$

$$\begin{aligned}\int_{K_1} F(u(x,s))\,dx \le& \int_{K_2} F(u(x,s))\zeta(x)\,dx + \frac{(r-1)C_o}{4}\sum_i\iint_{\tilde{Q}_2}\frac{f(u)}{u}|\partial_i u|^{p_i}\zeta\,dxd\tau\\ \le& \int_{K_2} u^r(x,0)\,dx\\ &+ \gamma\sum_i\|\partial_i\zeta_i\|_\infty^{p_i}\left(1+\frac{C^{p_i}}{\|\partial_i\zeta_i\|_\infty^{p_i}}\right)\iint_{\tilde{Q}_2} u^{r+p_i-2}\,\chi_{[u>k]}\,dxd\tau\\ &+ \gamma\sum_i\|\partial_i\zeta_i\|_\infty^{p_i}\left[\frac{C^{p_i-1}}{\|\partial_i\zeta_i\|_\infty^{p_i-1}} + \frac{C^{p_i}}{\|\partial_i\zeta_i\|_\infty^{p_i}}\left(1+\frac{1}{k}\right)\right]\iint_{\tilde{Q}_2} u^{r-1}\,\chi_{[u>k]}\,dxd\tau.\end{aligned} \tag{A.8}$$

since

$$\int_{K_2} F(u(x,0))\,dx \le \int_{K_2}\left(\int_0^{u(x,0)} s^{r-1}\,ds\right)dx \le \int_{K_2} u^r(x,0)\,dx.$$

By choosing k appropriately depending on M_r, so that (see for instance [23] Prop. 5.1)

$$\sup_{0\le\tau\le t}\fint_{K_1} u^r(x,\tau)\,dx \le 2r\left(\sup_{0\le\tau\le t}\fint_{K_1} F(u(x,\tau))\,dx + (1+\gamma)k^r|K_1|\right) \le \gamma\sup_{0\le\tau\le t}\fint_{K_1} F(u(x,\tau))\,dx,$$

estimate (A.5) follows by estimating (A.8) from below means of this last consideration. □

Remark A.3. The constant γ determined along the proof deteriorates as $r\downarrow 1$.

Energy Estimates 3 - Testing with negative powers

Lemma A.4. *Let u be a non-negative, local weak super-solution to (2.1)–(2.2). Let $K \subset \Omega$ be a compact set and $0 < t < T$ such that $Q = K \times [0,t] \subset \Omega_T$. Then, for all number $\nu > 0$ and for all indexes $i = 1,\dots,N$ we have the following inequality*

$$\begin{aligned}\iint_Q \Big(\sum_j |\partial_j u|^{p_j}\Big)\tau^{\frac{1}{p_i}}(u+\nu)^{-\frac{2}{p_i}}\zeta\,dxd\tau \le \gamma t^{\frac{1}{p_i}} \int_K (u+\nu)^{\frac{2(p_i-1)}{p_i}}\,dx \\ + \gamma \sum_j \|\partial_j\zeta_j\|_\infty^{p_j}\Big[1+\Big(\frac{C}{\|\partial_j\zeta\|_\infty}\Big)^{p_j}\Big]\iint_Q (u+\nu)^{p_j-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\,dxd\tau \\ + \gamma\Big(\sum_j C^{p_j}\Big)\iint_Q (u+\nu)^{-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\,dxd\tau,\end{aligned} \tag{A.9}$$

for all $\zeta \in C^1(0,t;C_o^1(K))$ of the form (A.3).

Proof. We test Eq. (2.1) repeatedly for $i = 1,\dots,N$ with the following test functions

$$\varphi_i(x,\tau) = -\tau^{\frac{1}{p_i}}(u(x,\tau)+\nu)^{1-\frac{2}{p_i}}\zeta(x), \tag{A.10}$$

defined in Q; where ζ is a smooth function defined in K of the form (A.2). We observe that $\varphi_i(x,0) = 0$, for all $x \in K$, and that the function φ_i, adequately averaged in time, is admissible due to the choice of ζ and

$$|\partial_i\varphi_i| \le \Big(\frac{2-p_i}{p_i}\Big)\tau^{\frac{1}{p_i}}\nu^{-2/p_i}|\partial_i u| + \tau^{\frac{1}{p_i}}\nu^{(\frac{p_i-2}{p_i})}|\partial_i\zeta| \in L^{p_i}_{loc}(\Omega_T).$$

In the weak formulation we use Steklov averages (see for instance the monograph [18]) for the interpretation of $\partial_\tau u$, to recover by approximation

$$0 \geqslant \int_K u\varphi_i\,dx\Big|_0^t - \int_0^t\int_K u\partial_\tau\varphi_i\,dxd\tau + \sum_j\iint_Q A_j\partial_j\varphi_i\,dxd\tau - \iint_Q B\varphi_i\,dxd\tau = I_1 - I_2 + I_3 - I_4.$$

As usual in the literature, the parabolic term is estimated by means of Steklov averages thereby getting

$$\begin{aligned}I_1 - I_2 &= \int_K u\varphi_i\,dx\Big|_0^t - \iint_Q u(\partial_\tau\varphi_i)\,dxd\tau \\ &= -\frac{p_i}{2(p_i-1)}t^{\frac{1}{p_i}}\int_{K\times\{t\}}(u+\nu)^{\frac{2(p_i-1)}{p_i}}\zeta\,dx + \frac{1}{2(p_i-1)}\iint_Q (u+\nu)^{\frac{2(p_i-1)}{p_i}}\tau^{\frac{1}{p_i}-1}\zeta\,dxd\tau \\ &\ge -\frac{p_i}{2(p_i-1)}t^{\frac{1}{p_i}}\int_{K\times\{t\}}(u+\nu)^{\frac{2(p_i-1)}{p_i}}\zeta\,dx\end{aligned}$$

passing to the limit thanks to the condition $u \in C_{loc}(0,T;L^2(K))$, while all the other terms in the Steklov approximation converge to the relative integrals, thanks to the structure conditions and the bound $\nu^{-\alpha} > (u+\nu)^{-\alpha}$, $\nu,\alpha > 0$.

We estimate I_3 and $-I_4$ from below by means of Young's inequality

$$\begin{aligned}I_3 &= \sum_j\iint_Q A_j\Big[\Big(\frac{2-p_i}{p_i}\Big)\tau^{\frac{1}{p_i}}(u+\nu)^{-\frac{2}{p_i}}(\partial_j u)\zeta - \tau^{\frac{1}{p_i}}(u+\nu)^{1-\frac{2}{p_i}}(\partial_j\zeta)\Big]\,dxd\tau \\ &\geqslant \sum_j\iint_Q\Big[C_o|\partial_j u|^{p_j} - C^{p_j}\Big]\Big(\frac{2-p_i}{p_i}\Big)\tau^{\frac{1}{p_i}}(u+\nu)^{-\frac{2}{p_i}}\zeta\,dxd\tau \\ &\quad - \sum_j\iint_Q\Big[C_1|\partial_j u|^{p_j-1} + C^{p_j-1}\Big]\tau^{\frac{1}{p_i}}(u+\nu)^{1-\frac{2}{p_i}}p_j|\partial_j\zeta_j|\zeta_j^{p_j-1}\hat{\zeta}^j\,dxd\tau \\ &\geqslant \sum_j\iint_Q\Big[\Big(\frac{2-p_i}{p_i}\Big)C_o - \gamma\epsilon_j C_1\Big]|\partial_j u|^{p_j}\tau^{\frac{1}{p_i}}(u+\nu)^{-\frac{2}{p_i}}\zeta\,dxd\tau \\ &\quad - \sum_j\iint_Q \gamma(\epsilon_j)C_1(u+\nu)^{p_j-\frac{2}{p_i}}|\partial_j\zeta_j|^{p_j}\tau^{\frac{1}{p_i}}\,dxd\tau \\ &\quad - \sum_j\iint_Q\Big[\Big(\frac{2-p_i}{p_i}\Big)C^{p_j} + \gamma C^{p_j}\Big](u+\nu)^{-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\,dxd\tau \\ &\quad - \sum_j\gamma\iint_Q (u+\nu)^{p_j-\frac{2}{p_i}}|\partial_j\zeta_j|^{p_j}\tau^{\frac{1}{p_i}}\,dxd\tau.\end{aligned}$$

$$
\begin{aligned}
|I_4| &\le \iint_Q \left[\sum_j C\left(|\partial_j u|^{p_j-1}+C^{p_j-1}\right)\right](u+v)^{1-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\zeta\,dxd\tau\\
&\le \sum_j \iint_Q \left[\tilde\epsilon_j|\partial_j u|^{p_j}(u+v)^{-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\zeta\,dxd\tau+\tilde\gamma(\tilde\epsilon_j)C^{p_j}(u+v)^{p_j-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\right]dxd\tau\\
&+\sum_j \iint_Q \left[C^{p_j}(u+v)^{p_j-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\zeta+C^{p_j}(u+v)^{-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\zeta\right]dxd\tau\\
&\le \sum_j \iint_Q \tilde\epsilon_j|\partial_j u|^{p_j}(u+v)^{-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\zeta\,dxd\tau\\
&+\sum_j \iint_Q C^{p_j}\left[\tilde\gamma(\tilde\epsilon_j)+1\right](u+v)^{p_j-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\,dxd\tau\\
&+\sum_j \iint_Q C^{p_j}(u+v)^{-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\,dxdt\tau.
\end{aligned}
$$

Now, reabsorbing the terms with $\epsilon_j,\tilde\epsilon_j$ on the left-hand side, we obtain

$$
\begin{aligned}
\sum_j \iint_Q |\partial_j u|^{p_j}\tau^{\frac{1}{p_i}}(u+v)^{-\frac{2}{p_i}}\zeta\,dxd\tau &\le \gamma t^{\frac{1}{p_i}}\int_{K\times\{t\}}(u+v)^{\frac{2(p_i-1)}{p_i}}\zeta\,dx\\
&+\gamma\sum_j \|\partial_j\zeta_j\|_\infty^{p_j}\left[1+\left(\frac{C}{\|\partial_j\zeta_j\|_\infty}\right)^{p_j}\right]\iint_Q (u+v)^{p_j-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\,dxd\tau\\
&+\gamma\sum_j C^{p_j}\iint_Q (u+v)^{-\frac{2}{p_i}}\tau^{\frac{1}{p_i}}\,dxd\tau\ . \quad \square
\end{aligned}
$$

Remark A.5. The constant γ deteriorates both as soon as $p_N \uparrow 2$ and as $p_1 \downarrow 1$.

Remark A.6. We observe that all the energy estimates (A.4), (A.5), (A.9) recover, when $p_i \equiv p$, known estimates known for the isotropic p-Laplacean evolution equations (see for instance the Appendix of [16]). This is due to the simple fact that for all $\xi = (\xi_1,\dots,\xi_N)\in\mathbb{R}^N$ there exists an universal constant $\gamma=\gamma(p_i,N)>0$ such that

$$
\frac{1}{\gamma}\sum_i \xi_i^p \le \|\xi\|^p \le \gamma\sum_i \xi_i^p, \qquad \text{being} \qquad \|\xi\| = \sqrt{\sum_i \xi_i^2}.
$$

Algebraic Lemmas. Here we collect two Lemmata evolving sequences of numbers, that can both be found in [15] (see [13] for the anisotropic counterpart), useful along our proofs.

Lemma A.7 (*Fast Geometric Convergence Lemma*). *Let $(Y_n)_n$ be a sequence of positive numbers verifying*

$$
Y_{n+1} \le C b^n\ Y_n^{1+\alpha}\ ,
$$

being $C>0$, $b>1$ and $\alpha>0$ given numbers. Then the following logical implication holds true

$$
Y_o \le C^{-1/\alpha}\ b^{-1/\alpha^2} \quad \Rightarrow \quad \lim_{n\uparrow\infty} Y_n = 0.
$$

Lemma A.8 (*Iteration Lemma*). *If we have a sequence of equibounded numbers $\{Y_n\}$ such that, for constants $\mathcal{I}, b>1$ and $\epsilon\in(0,1)$*

$$
Y_n \le \epsilon Y_{n+1} + \mathcal{I} b^n\ , \tag{A.11}
$$

then, by a simple iteration, there exists $\gamma>0$ such that

$$
Y_0 \le \gamma\,\mathcal{I}.
$$

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