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On Kohn's sums of squares of complex vector fields

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Abstract. This is a survey of some recent alternative way of proving a subelliptic estimate, first proven by J. J. Kohn, for certain sums of squares of *complex* vector fields. My approach here makes it possible to extend the result also to more general families of complex vector fields, to perturbations of sums of squares operators by a first-order complex term and furthermore to a pseudodifferential setting.

Keywords: Subelliptic estimate; hypoellipticity; sums of squares of complex vector fields; Melin's inequality.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $X_1(x, D), \ldots, X_N(x, D)$ be first-order partial differential operators with smooth *real* coefficients, without zeroth-order terms ($D = -i\partial$). For each $x \in \Omega$, consider the real vector space $\mathcal{L}_X(x)$ spanned by the vector fields $iX_1, \ldots, iX_N$, and their repeated commutators $[iX_{j_1}, [iX_{j_2}, [\ldots, [iX_{j_{h-1}}, iX_{j_h}]] \ldots],]$ $1 \leq j_h \leq N$, frozen at

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the point x. Hörmander's celebrated hypoellipticity theorem for sums of squares ([2] or [3]) states the following.

Theorem 1.1. *Let $P = \sum_{j=1}^N X_j^* X_j$. Suppose that for every $x \in \Omega$ one has $\mathcal{L}_X(x) = T_x\Omega$. Then P is C^∞-hypoelliptic, that is,*

$$\operatorname{singsupp}(Pu) = \operatorname{singsupp}(u), \quad \forall u \in \mathcal{D}'(\Omega).$$

It is well-known from Fedii and Morimoto that the Lie-algebra condition $\mathcal{L}_X(x) = T_x\Omega$, for all $x \in \Omega$, is not necessary for the C^∞-hypoellipticity to hold (see the references in [8]; in what follows, I will mainly refer to the bibliography of [8] to cite some of the many important contributions, and to the bibliography of the books cited here). However, when the coefficients are analytic, Derridj proved that the Lie-algebra condition is also necessary for the C^∞-hypoellipticity. The situation regarding *analytic-hypoellipticity* is completely different (see the literature concerned with the Treves conjecture) and to this day quite open.

The main step in the proof of Theorem 1.1 is a subelliptic estimate, that is, an energy estimate of the following kind: *There exists $\varepsilon > 0$ (necessarily smaller than or equal to 1, i.e. 1/2 of the order of the operator) such that for any given compact $K \subset \Omega$ there is $C_K > 0$ such that*

$$\|u\|_\varepsilon^2 \le C_K\Big(\mathsf{Re}(Pu, u) + \|u\|_0^2\Big), \quad \forall u \in C_c^\infty(K). \tag{SE$_\varepsilon$}$$

Notice that when $P = P^*$ (i.e. P is formally self-adjoint), the real part in the inner product can be omitted. However, we prefer to keep it in the above estimate because we shall consider cases in which we add to P a not necessarily self-adjoint first-order operator Q.

It was Rothschild and Stein [9] who obtained the sharp subelliptic exponent $\varepsilon = 1/k$, where k is the number of brackets necessary to span the tangent space (iX_j has length 1, $[iX_j, iX_{j'}]$ has length 2 and so on). One has also results by J. J. Kohn (whose proof is the one followed by Treves and by Hörmander in their respective books on pseudodifferential and Fourier integral operators and on the analysis of linear partial differential operators) and by F. Treves in his study of hypoellipticity. Subsequent work by Oleinik and Radkevich extended greately the result to more general operators with real coefficients. Hörmander's theorem was microlocalized by Bolley, Camus and Nourrigat, and in the case of polynomials in the operators X_j it was Helffer and Nourrigat [1] who obtained sharp microhypoelliptic results with optimal gain. On the side of the study of subelliptic operators, one has to mention the contributions of Egorov, Hörmander, Fefferman and Phong, but the list of important contributions is still long and I have mentioned only a few of them (and I apologize about that).

To make the measurement of hypoellipticity more precise, it is now convenient to introduce the following definition.

Definition 1.2. Let P be an operator of order m. One says that P is hypoelliptic with a loss of $r \geq 0$ derivatives at $x_0 \in \Omega$ if for any given $u \in \mathcal{D}'(\Omega)$ and any given $s \in \mathbb{R}$

$$Pu \in H^s_{\text{loc}}(x_0) \Longrightarrow u \in H^{s+m-r}_{\text{loc}}(x_0).$$

For the definition of $H^s_{\text{loc}}(x_0)$ see [3] (it means that $u = u_1 + u_2$ with $u_1 \in H^s_{\text{loc}}$ and $u_2 \in C^\infty$ near x_0). Note that when $r = 0$ (i.e. we have no loss of derivatives) the operator is elliptic near x_0.

There are operators that are hypoelliptic and yet lose many derivatives, and this was already known to E. Stein, who considered the Kohn-Laplacian $\Box_b$ on the Heisenberg group $\mathbb{H}^n$ on $(0, q)$ forms with $q = 0$ or $q = n$ and showed that although in such a case $\Box_b$ *cannot* by hypoelliptic, $\Box_b + c$ is indeed hypoelliptic with a loss of 2 derivatives whatever the complex number $c \neq 0$. A theory (based on Boutet De Monvel's concept of localized operator and subsequent work by Boutet De Monvel, Helffer and Grigis, and by Helffer) to understand this phenomenon for transversally elliptic operators was developed by C. Parenti and myself in [6]. At about the same time, Kohn [4], motivated by Y. T. Siu's program [10] to use multipliers for the $\bar{\partial}$-Neumann problem to obtain an explicit construction of critical varieties that control the D'Angelo type, extended Hörmander's hypoellipticity result to *complex* operators $Z_1(x, D), \ldots, Z_N(x, D)$

(i.e. first-order partial differential operators with smooth complex coefficients and no zeroth-order terms) that span, along with their commutators of length 2, the complexified tangent space at every point. But he also showed that as soon as one needs commutators of greater length there are operators that span along with higher commutators, but whose related sum-of-squares operator cannot satisfy any subelliptic estimates and yet remains hypoelliptic (with a loss of many derivatives).

2 Kohn's theorems

Let $Z_1(x,D),\dots,Z_N(x,D)$ be first-order partial differential operators with smooth *complex* coefficients and no zeroth-order terms on Ω. Hence the iZ_j may be regarded as vector fields on Ω with complex coefficients. Let $P=\sum_{j=1}^N Z_j^* Z_j$.

Theorem 2.1 ([4], Thm. A)**.** *Suppose that*

$$\mathrm{Span}_{\mathbb{C}}\{iZ_j,\ [iZ_j,iZ_k];\ 1\le j,k\le N\}(x)=\mathbb{C}T_x\Omega,\quad \forall x\in\Omega. \qquad (K)$$

Then the subelliptic estimate $(\mathrm{SE}_{1/2})$ *holds.*

Note that no commutator of the kind $[Z_j,\bar{Z}_k]$ is considered.

However, as soon as more commutators are required the result is no longer true.

Theorem 2.2 ([4], Thm. B)**.** *For any given $k\in\mathbb{Z}_+$ there are first-order complex operators Z_1, Z_{2k} (with no zeroth-order term) defined near $0\in\mathbb{R}^3$ such that the complex vector fields iZ_1 and iZ_{2k} and their commutators of order $k+1$ (i.e. length $k+2$) span $\mathbb{C}T_0\Omega$ and when $k\ge 1$ the subelliptic estimate no longer holds. Moreover, the sum-of-squares operator P_k is hypoelliptic with a loss of $k+1$ derivatives.*

Kohn constructs the example as follows. Let

$$\bar{L}=\frac{\partial}{\partial\bar{z}_1}-iz_1\frac{\partial}{\partial x_3},\ z_1=x_1+ix_2,$$

be (a version of) the Lewy operator. Put then

$$iZ_1 = \bar{L}, \quad iZ_{2k} = \bar{z}_1^k L,$$

$$P_k = Z_1^* Z_1 + Z_{2k}^* Z_{2k} = -(L\bar{L} + \bar{L}|z_1|^{2k} L).$$

After Kohn's paper appeared, M. Christ came out with a simplified version of P_k and Parenti and myself with a general class of simplified examples [7]. The latter class can be described as follows. Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}$. Let $n, k \geq 1$ be integers. Let $\mu_1, \ldots, \mu_n > 0$ be rationally independent, and let $\gamma \in \mathbb{R}$. Consider the polynomial

$$Q(x) = \sum_{|\alpha|=k} c_\alpha x^{2\alpha},\ x \in \mathbb{R}^n,\ \sum_{|\alpha|=k} c_\alpha > 0.$$

Let $Z_j = D_{x_j} - i\mu_j x_j D_y$, $1 \leq j \leq n$, and

$$P = \sum_{j=1}^n Z_j^* Z_j + \sum_{j=1}^n Z_j(Q(x) Z_j^*) + (\gamma + \sum_{j=1}^n \mu_j) D_y.$$

Then P is hypoelliptic with a loss of exactly 1 derivative iff

$$\gamma \notin S := \{\pm(\sum_{j=1}^n h_j \mu_j + \sum_{j=1}^n \mu_j);\ h_1, \ldots, h_n \in \mathbb{Z}_+\},$$

and when $\gamma \in S$ then P is hypoelliptic with a loss of exactly $k+1$ derivatives.

An important issue in the hypoellipticity of operators of the kind sums-of-squares of homogeneous first-order differential operators is the stability of hypoellipticity by perturbations of order 1. The problem is very delicate, due to the degeneracy of the operators considered (one may look at the references [16] and [17] in [8] in the C^∞ setting, and to some recent work of P. Cordaro and his collaborators in the C^ω setting).

I will consider here only Theorem 2.1, and give an alternative approach which goes through Melin's inequality (see [5], and [3] for the strong form used here). This allows one also to deal with the above kind of perturbations and with further generalizations such as, for instance, a weakening of Kohn's condition (K), see Theorem 5.3 below.

3 Melin's inequality (in the strong form)

Suppose $P = P^*$ is an mth-order properly supported classical pseudodifferential operator on Ω. Let p_m be its principal symbol, which is a function (positively homogeneous of degree m) defined on $T^*\Omega \setminus 0$. Let $\Sigma = p_m^{-1}(0) \subset T^*\Omega \setminus 0$ be the characteristic set of P, which in general is not a manifold. On the subset $\Sigma_2 \subset \Sigma$ on which p_m vanishes at least to second order one has another invariant, which is the *subprincipal* symbol

$$p^s_{m-1}(x,\xi) = p_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=1}^n \partial_{x_j}\partial_{\xi_j} p_m(x,\xi).$$

Notice that when P is formally self-adjoint then p^s_{m-1} is real.

Moreover, at $\rho \in \Sigma_2$ one also has that $dp_m(\rho) = 0$ whence the Hessian of p_m is invariant and one may define the *fundamental matrix* $F(\rho)$ (aka *Hamilton map*) of p_m at ρ as

$$\sigma(w, F(\rho)w') = \frac{1}{2}\langle \mathrm{Hess}(p_m)(\rho)w, w'\rangle, \quad w, w' \in T_\rho T^*\Omega.$$

Here $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ is the canonical symplectic form of $T^*\Omega$. Note that $F(\rho)$ is the linearization of the Hamilton vector field

$$H_{p_m}(\rho) = \sum_{j=1}^n \Big(\frac{\partial p_m}{\partial \xi_j}(\rho)\frac{\partial}{\partial x_j} - \frac{\partial p_m}{\partial x_j}(\rho)\frac{\partial}{\partial \xi_j}\Big)$$

at ρ. Note also that F is skew-symmetric with respect to σ.

When $p_m \geq 0$ on $T^*\Omega \setminus 0$ one has that $\Sigma_2 = \Sigma$ and that $F(\rho)$, $\rho \in \Sigma$, has the following spectral structure:

- $\mathrm{Ker}\, F(\rho) \subset \mathrm{Ker}(F(\rho)^2) = \mathrm{Ker}(F(\rho)^3)$, 0 is the only generalized eigenvalue while all the others (when $F \neq 0$) are semisimple and of the form $\pm i\mu_j$ so that, *with repetitions of the latter according to multiplicities*, for some r one has

  $$\mathrm{Spec}(F(\rho)) = \{0\} \cup \{\pm i\mu_j;\ \mu_j > 0,\ 1 \leq j \leq r\}$$

  where the $\pm i\mu_j$ are all semisimple;

- One has
$$T_\rho T^*\Omega = \operatorname{Ker}(F(\rho)^2) \oplus \operatorname{Range}(F(\rho)^2);$$
- The *positive trace* of $F(\rho)$ is defined to be (it is a symplectic invariant)
$$\operatorname{Tr}^+ F(\rho) = \sum_{\substack{\mu>0 \\ i\mu\in\operatorname{Spec}(F(\rho))}} \mu.$$
It turns out that $\operatorname{Tr}^+ F$ is positively homogeneous of degree $m-1$.

Theorem 3.1 (Melin's strong inequality)**.** *Let $P = P^*$ be a properly supported mth-order classical pseudodifferential operator on Ω such that $p_m \geq 0$ and*
$$p_m(\rho) = 0 \Longrightarrow p^s_{m-1}(\rho) + \operatorname{Tr}^+ F(\rho) > 0. \tag{3.1}$$
Then for all compact $K \subset \Omega$ there are $c_K, C_K > 0$ such that
$$(Pu, u) \geq c_K\|u\|^2_{(m-1)/2} - C_K\|u\|^2_{(m-2)/2}, \quad \forall u \in C^\infty_c(K). \tag{3.2}$$

Note that no smoothness assumption is required of the characteristic set Σ.

4 The result

When $m = 2$ inequality (3.2) is exactly ($\mathrm{SE}_{1/2}$). So, the question is: *Is there a link between Kohn's Theorem 2.1 and Melin's Theorem 3.1?* The point is to understand the symplectic content of Kohn's spanning condition (K).

There is a very important issue one has to point out relative to operators that are written as sums of squares of first-order differential operators without zeroth-order terms. In the case case of real coefficients (i.e. the case of Hörmander's Theorem 1.1 with generation of length 2), the subprincipal part vanishes on the characteristic manifold $\bigcap_{j=1}^N X_j^{-1}(0)$, whereas in the case of complex coefficients (i.e. the case of Kohn's Theorem 2.1), the subprincipal term on the characteristic manifold

$$\Sigma = \bigcap_{j=1}^{N} Z_j^{-1}(0)$$

in general does not vanish identically and hence may spoil the subelliptic estimate.

I will systematically write $Z_j = Z_j(x,\xi)$ for the symbol of $Z_j(x,D)$. Recall also that for (scalar and properly supported) pseudodifferential operators $a(x,D)$ and $b(x,D)$, the symbol of the commutator $[a(x,D), b(x,D)]$ starts with $-i\{a,b\}(x,\xi)$, where the Poisson bracket $\{\cdot,\cdot\}$ is given by

$$\{a,b\} = \sum_{j=1}^{n}\Big(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j}\Big) = H_a b.$$

One has that

$$p_2(x,\xi) = \sum_{j=1}^{N} |Z_j(x,\xi)|^2, \quad p_1^s\big|_\Sigma = -\frac{i}{2}\sum_{j=1}^{N}\{\bar{Z}_j, Z_j\}\big|_\Sigma,$$

and for $\rho \in \Sigma$ and $w \in T_\rho T^*\Omega$

$$F(\rho)w = \sum_{j=1}^{N}\Big(\sigma(w, H_{2j-1}(\rho))H_{2j-1}(\rho) + \sigma(w, H_{2j}(\rho))H_{2j}(\rho)\Big),$$

where I write $H_{Z_j} = H_{2j-1} + iH_{2j}$, $1 \le j \le N$, and σ, recall, is the canonical symplectic form on $T^*\Omega$.

Hence, a main problem is how to control p_1^s on Σ since condition (K), seemingly, does not provide information on $\{\bar{Z}_j, Z_k\}$.

In the first place one has the following crucial observation.

Proposition 4.1. *Let $x \in \Omega$ be such that $\pi^{-1}(x) \cap \Sigma \neq \emptyset$, where $\pi\colon T^*\Omega \setminus 0 \longrightarrow \Omega$ denotes the canonical projection. Kohn's condition (K) at $x \in \Omega$ yields the existence of j,k with $1 \le j < k \le N$ and $0 \neq \xi \in T_x^*\Omega$ such that*

$$\{Z_j, Z_k\}(x,\xi) \neq 0.$$

Hence, the main point is to see whether Melin's condition (3.1) holds. This requires a study of $\operatorname{Spec}(F(\rho))$ when $\rho \in \Sigma$. Therefore we must study the eigenvalue equation $F(\rho)w = i\mu w$, $\mu > 0$, with $0 \neq w \in \mathbb{C}\mathrm{Range}(F(\rho)^2)$. One has the following, to me remarkable, result (see [8]).

Theorem 4.2. *Let $P = \sum_{j=1}^N Z_j^* Z_j$. The sum-of-squares form of P yields always*

$$\mathrm{Tr}^+ F(\rho) \geq \Big(p_1^s(\rho)^2 + \max_{1\leq j<k\leq N} |\{Z_j, Z_k\}(\rho)|^2\Big)^{1/2} =: \kappa(\rho), \quad \forall \rho \in \Sigma. \tag{4.1}$$

In particular, when (K) holds, one has then

$$\mathrm{Tr}^+ F(\rho) > |p_1^s(\rho)|, \quad \forall \rho \in \Sigma.$$

Note that $\kappa\colon \Sigma \longrightarrow [0, +\infty)$ is continuous and positively homogeneous of degree 1. Consider the functions on Σ

$$-p_1^s \pm \kappa\colon \Sigma \longrightarrow \mathbb{R}.$$

Then $-p_1^s \pm \kappa$ are continuous and positively homogeneous of degree 1, and one has $-p_1^s - \kappa \leq 0 \leq -p_1^s + \kappa$ on Σ.

We have the following result, which generalizes Kohn's Theorem 2.1.

Theorem 4.3. *Suppose condition (K) for P. Then the strong Melin inequality holds and one has the subelliptic estimate* $(\mathrm{SE}_{1/2})$. *Moreover, if Q is a first-order properly supported classical pseudodifferential operator on Ω, the operator $P+Q$ keeps satisfying* $(\mathrm{SE}_{1/2})$ *provided the real part q_1 of the principal symbol of Q fulfills*

$$-p_1^s(\rho) - \kappa(\rho) < q_1(\rho) < -p_1^s(\rho) + \kappa(\rho), \quad \forall \rho \in \Sigma.$$

In particular, in case Q is a partial differential operator, when

$$|q_1(\rho)| < \min\{p_1^s(\rho) + \kappa(\rho), -p_1^s(\rho) + \kappa(\rho)\} = \kappa(\rho) - |p_1^s(\rho)|, \quad \forall \rho \in \Sigma.$$

Note that no smoothness assumption on Σ is required.

5 Further generalizations

By Theorem 4.2 there is no obstruction to making the above result pseudodifferential. Let $P_1, \dots, P_N$ be properly supported, classical mth-order pseudodifferential operators with *complex symbols* on $\Omega \subset \mathbb{R}^n$. Let $P = \sum_{j=1}^{N} P_j^* P_j$, and let p_j be the principal symbol of P_j.

Definition 5.1. I say that the system $(P_1, \dots, P_N)$ satisfies condition (K_Σ) if

$$\sum_{1 \le j < k \le N} |\{p_j, p_k\}(\rho)| > 0, \quad \forall \rho = (x, \xi) \in \Sigma, \text{ with } |\xi| = 1,$$

where

$$\Sigma = \bigcap_{j=1}^{N} p_j^{-1}(0)$$

is the characteristic set of P.

Note that this time the function κ in (4.1) is positively homogeneous of degree $2m - 1$.

One has the following result.

Theorem 5.2. *If the system $(P_1, \dots, P_N)$ satisfies condition (K_Σ), then the sum-of-squares operator P, of order $2m$, fulfills the following subelliptic estimate $(\mathrm{SE}_{m-\frac{1}{2}})$: For any given compact $K \subset \Omega$ there is $C_K > 0$ such that*

$$\|u\|^2_{m-1/2} \le C_K \big(\mathsf{Re}(Pu, u) + \|u\|^2_{m-1}\big), \quad \forall u \in C_c^\infty(K).$$

Moreover, if Q is a $(2m-1)$st-order properly supported classical pseudodifferential operator on Ω, the operator $P + Q$ keeps satisfying the above subelliptic estimate provided the real part q_{2m-1} of the principal symbol of Q fulfills

$$q_{2m-1}(\rho) \in \big(-p^s_{2m-1}(\rho) - \kappa(\rho), -p^s_{2m-1}(\rho) + \kappa(\rho)\big), \quad \forall \rho \in \Sigma.$$

In particular, in case Q is a partial differential operator, when

$$|q_{2m-1}(\rho)| < \kappa(\rho) - |p^s_{2m-1}(\rho)|, \quad \forall \rho \in \Sigma.$$

Note that no smoothness assumption on Σ is required.

As a further consequence of this approach (namely of Theorem 4.2 and Theorem 5.2) we have the following result, which shows an extent to which Kohn's condition (K) can be weakened.

Theorem 5.3. *Let $Z_1, \dots, Z_N$ be first-order partial differential operator with smooth complex coefficients on $\Omega \subset \mathbb{R}^n$ (and no zeroth-order term). For each $x \in \Omega$ let*

$$V_x := \mathrm{Span}_{\mathbb{C}}\{Z_j, [Z_j, Z_k];\ 1 \le j, k \le N\}(x)$$

(where we think of the Z_j and $[Z_j, Z_k]$ as vector fields with complex coefficients). Suppose that

$$V_x + \overline{V_x} = \mathbb{C}T_x\Omega, \quad \forall x \in \Omega. \tag{5.1}$$

Then $P = \sum_{j=1}^N Z_j^* Z_j$ satisfies $(\mathrm{SE}_{1/2})$. Moreover, the perturbation result of Theorem 4.3 holds and, in particular, the theorem is true also for the operator $\sum_{j=1}^N (Z_j + \alpha_j)^*(Z_j + \alpha_j)$, with $\alpha_1, \dots, \alpha_N \in C^\infty(\Omega; \mathbb{C})$.

Remark 5.4. Note that only the points $x \in \Omega$ for which $\pi^{-1}(x) \cap \Sigma \neq \emptyset$ matter for the subelliptic estimate.

Proof. The proof consists in showing that condition (K_Σ) is fulfilled. For $x \in \Omega$ such that $\pi^{-1}(x) \cap \Sigma \neq \emptyset$ consider the system

$$\begin{cases} Z_j(x,\xi) = 0,\ 1 \le j \le N, \\ \{Z_j, Z_k\}(x,\xi) = 0,\ 1 \le j < k \le N, \end{cases} \tag{5.2}$$

for the unknown $\xi \in T_x^*\Omega$, $\xi \neq 0$. Since ξ is real, system (5.2) is fulfilled by some ξ_0 iff the complex conjugate system ($\overline{5.2}$) is fulfilled by the same ξ_0. Therefore on the one hand one must have $(x, \xi_0) \in \Sigma \subset T^*\Omega \setminus 0$ and on the other

$$\xi_0 \in V_x^\perp \cap \overline{V_x}^\perp = (V_x + \overline{V_x})^\perp = \{0\}.$$

Therefore, when $\Sigma \neq \emptyset$ then for all $x \in \Omega$ for which $\pi^{-1}(x) \cap \Sigma \neq \emptyset$ we have that if (5.1) holds then

$$(x,\xi) \in \Sigma \Longrightarrow \{Z_j, Z_k\}(x,\xi) \neq 0$$

for some j, k with $1 \leq j < k \leq N$ and for some $\xi \neq 0$, whence condition (K_Σ) is satisfied. $\square$

I end the paper by giving a concrete example of Theorem 5.3.

For $x \in \Omega \subset \mathbb{R}^3$ with $0 \in \Omega$, let

$$Z_1(x,\xi) = \xi_1 + ix_1\xi_2, \quad Z_2(x,\xi) = x_1(\xi_2 + i\xi_3).$$

Then

$$\{Z_1, Z_2\}(x,\xi) = \xi_2 + i\xi_3.$$

In this case

$$\Sigma = \{(x,\xi);\ \xi_1 = x_1\xi_2 = x_1\xi_3 = 0,\ (\xi_2,\xi_3) \neq (0,0)\}.$$

Kohn's condition (K) does not hold, but condition (5.1) does. In fact, for any given $x \in \Omega$

$$V_x = \mathrm{Span}_{\mathbb{C}}\{\begin{bmatrix} 1 \\ ix_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_1 \\ ix_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}\},$$

so that $\dim_{\mathbb{C}} V_x = 2$ for all $x \in \Omega$ and Kohn's condition (K) does not hold. However,

$$V_x + \overline{V_x} = \mathrm{Span}_{\mathbb{C}}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}\} = \mathbb{C}^3 = \mathbb{C}T_x\Omega.$$

Equivalently, one may also directly see that condition (K_Σ) holds. In fact, consider

$$\Sigma_1 = \{(x,\xi);\ \{Z_1,Z_2\}(x,\xi) = 0,\ \xi \neq 0\} = \{(x,\xi);\ \xi_2 = \xi_3 = 0,\ \xi_1 \neq 0\}.$$

Then

$$\Sigma \cap \Sigma_1 = \emptyset$$

which proves the claim.

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