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Smoothing effect and Strichartz estimates for some time-degenerate Schrödinger equations

Serena Federico

Abstract In this paper we present recent results about the smoothing properties of some Schrödinger operators with time degeneracies. More specifically, we will show that time-weighted smoothing and Strichartz estimates hold true for the operators under consideration. Finally, by means of the aforementioned estimates, we will derive local well-posedness results for the suitable corresponding nonlinear initial value problem.

1 Introduction

In this paper we shall investigate the smoothing properties of some time-degenerate Schrödinger operators of the form

$$\mathcal{L}_{\alpha,c} := i\partial_t + t^\alpha \Delta + c(t,x) \cdot \nabla_x \tag{1}$$

and

$$\mathcal{L}_b := i\partial_t + b'(t)\Delta, \tag{2}$$

where $\alpha > 0$, $c(t,x) = (c_1(t,x), \ldots, c_n(t,x))$ is such that, for all $j = 1, \ldots, n$, $c_j(t,x)$ is a complex valued function satisfying suitable dacy assumptions, while $b \in C^1(\mathbb{R})$ and satisfies $b(0) = b'(0) = 0$. We will go through the analysis of two kind of smoothing properties characterizing the solutions of Schrödinger equations in the Euclidean setting, that is, those described by smoothing and Strichartz estimates. More specifically, we will prove that local weighted smoothing estimates are satisfied by $\mathcal{L}_{\alpha,c}$, while local weighted Strichartz estimates are satisfied by $\mathcal{L}_b$. Once these results will be at our disposal, we will consider suitable nonlinear initial value

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problems for $\mathcal{L}_{\alpha,c}$ and $\mathcal{L}_b$, and give the corresponding local well-posedness results in each case.

Considering what previously mentioned, it should be clear that the estimates object of this work constitute a crucial tool to attack nonlinear IVPs (initial value problems) for dispersive equations.

Smoothing estimates are used to show that the solution of a certain equation gains regularity (in terms of derivatives) with respect to the regularity of the initial datum (homogeneous smoothing estimate) and/or with respect to the regularity of the inhomogeneous term of the equation (inhomogeneous smoothing estimate). Therefore these estimates are the suitable ones to be used when dealing with nonlinear problems with derivative nonlinearities.

Strichartz estimates, instead, allow to obtain a gain of integrability of the solution of a certain equation with respect to the integrability property of the initial datum (homogeneous Strichartz estimate) and/or with respect to the integrability property of the inhomogeneous term of the equation (inhomogeneous Strichartz estimate). These are the estimates to be used to solve semilinear IVPs.

Results concerning smoothing and Strichartz estimates for constant coefficient Schrödinger equations, but also for general constant coefficient dispersive equations, are by now classical (see [17, 5, 6, 20, 21, 18, 35, 3, 37, 19]). As for the the variable coefficients case where the Laplacian is replaced by a variable coefficient (elliptic) operator (the constant case with potentials is also well understood and widely studied) the situation is much different, and the results available are quite limited.

The smoothing effect of Schrödinger equations with nondegenerate space-variable coefficients was proved in [22] by Kenig et al., where the authors considered and solved the ultrahyperbolic case too. Important achievements in the study of smoothing estimates are due to Doi (see [7] and [8]), who considered the problem in the general manifold setting. As regards Strichartz estimates, Staffilani and Tataru proved in [34] the validity of such estimates for Schrödinger equations with nonsmooth coefficients (with compactly supported perturbations of the Laplacian), while in [30] Robbiano and Zuily obtained these estimates for Schrödinger equations with small perturbations of the Laplacian. Let us mention that several results have been proved for equations with potentials and in the manifold setting, and we refer the interested reader to [1, 2, 27, 9, 10, 29] and references therein.

Our analysis here, despite the aforementioned results, focuses on time-degenerate Schrödinger operators of the form (1) and (2). It is worth to mention that the class of operators (1) was first considered by Cicognani and Reissig in [4], who studied the linear problem and proved the local well-posedness of the linear IVP both in Sobolev and Gevrey spaces. The results about the local smoothing effect of the class (1), proved by the author and Staffilani in [12], will be presented below in a selfcontained way. Some results about the homogeneous smoothing effect of ***nondegenerate*** operators of the form (2) were proved by Sugimoto and Ruzhansky in [32]. As for Strichartz estimates and local well-posedness for the class $\mathcal{L}_b$ on the one and on the two-dimensional torus, and possibily generalizable to general compact Riemaniann manifolds, they were proved by the author and Staffilani in [13], where some nondegenerate space-variable coefficient Schrödinger operators on the one and on the two-dimensional torus were also studied.

Concerning the Strichartz estimates for (2) treated in this paper, they were proved by the author and Ruzhansky in [11], where some homogeneous smoothing results were also established by means of comparison principles.

Let us now conclude this introduction by giving the plan of the paper. In Section 2 we shall analyze the local smoothing effect of $\mathcal{L}_{\alpha,c}$ in two cases: when $c \equiv 0$ (in Subsection 2.1) and when c is not necessarily identically 0 (in Subsection 2.2). In each case we also give the local well-posedness result for the corresponding nonlinear IVP.

In Section 3 we focus on the class $\mathcal{L}_b$ and on the validity of local Strichartz estimates in this case. A local well-posedness result for a semilinear IVP for $\mathcal{L}_b$ will also be given.

Notations. We use the notation $A \lesssim B$ to indicate that there exists an absolute constant $c > 0$ such that $A \leq cB$. We shall denote by Λ^s the pseudo-differential operator of order s whose symbol is given by $\Lambda^s(\xi) = \langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$.

The mixed norm space $L^p_x L^q_t(\mathbb{R}^n \times [0,T])$, $1 \leq p, q \leq \infty$, is the space of functions $f(t,x)$ that are L^q in time on the interval $[0,T]$ and are L^p in space. The norm is taken in the right to left order. In a similar manner we define the spaces $L^p([0,T]; H^s(\mathbb{R}^n))$, $1 \leq p \leq \infty$, of functions that are L^p in time and in the Sobolev space $H^s(\mathbb{R}^n)$ in space. Finally we shall denote by $S^m := S^m_{1,0}$ the class of classical symbols of order $m \in \mathbb{R}$ defined by

$$S^m := \{p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n); |p|^{(j)}_{S^m} < \infty\},$$

where

$$|p|^{(j)}_{S^m} := \sup_{|\alpha+\beta|=j} \|\langle \xi \rangle^{-m+|\alpha|} \partial^\alpha_\xi \partial^\beta_x p(x,\xi)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.$$

Finally, by writing $g \not\equiv 0$ we will mean that a function $g = g(t,x)$ is not necessarily identically 0.

2 Smoothing effect and local well-posedness for the class $\mathcal{L}_{\alpha,c}$

This section is devoted to the study of the class $\mathcal{L}_{\alpha,c}$ as in (1). Below we will discuss the cases $c \equiv 0$ and $c \not\equiv 0$ separately, in Subsection 2.1 and 2.2 respectively . This distinction is done in order to show that one can use standard techniques in the first case $c \equiv 0$, and that in the more general case $c \not\equiv 0$ the usual technique does not work anymore (the case $c \equiv 0$ is always contained in the case $c \not\equiv 0$ according to our notation). For the reader convenience we shall state our main results for the class under consideration at the beginning of each subsection. As explained in the introduction, these results will be about the local smoothing and about the local well-posedness of the nonlinear IVP.

2.1 The class $\mathcal{L}_\alpha$

In the sequel we will use the notation $\mathcal{L}_\alpha := \mathcal{L}_{\alpha,0} := i\partial_t + t^\alpha \Delta_x$. The operator $W_\alpha(t,s)$ in the statements below is the operator defined as in (9) giving the solution at time t of the homogeneous IVP for $\mathcal{L}_\alpha$ with initial condition $u(s,x) = u_s(x)$ at time s. Our main results for $\mathcal{L}_\alpha$ are the following.

Theorem 1 *Let $W_\alpha(t) := W_\alpha(t,0)$, with $\alpha > 0$, then*

If $n = 1$ for all $f \in L^2(\mathbb{R})$,

$$\sup_x \|t^{\alpha/2} D_x^{1/2} W_\alpha(t) f\|^2_{L^2_t([0,T])} \lesssim \|f\|^2_{L^2(\mathbb{R})}. \tag{3}$$

If $n \geq 2$, on denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ the family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then for all $f \in L^2_x(\mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \int_0^T |t^{\alpha/2} D_x^{1/2} W_\alpha(t) f(x)|^2 dt\, dx \right)^{1/2} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \tag{4}$$

where $D_x^\gamma f(x) = (|\xi|^\gamma \widehat{f}(\xi))^\vee(x)$.

Theorem 2 *Let $n = 1$ and $g \in L^1_x L^2_t(\mathbb{R} \times [0,T])$, then*

$$\|D_x^{1/2} \int_{\mathbb{R}_+} t^{\alpha/2} W_\alpha(0,t) g(t) dt\|_{L^2_x(\mathbb{R})} \lesssim \|g\|_{L^1_x L^2_t(\mathbb{R}\times[0,T])}, \tag{5}$$

and, for all $g \in L^1_t L^2_x([0,T] \times \mathbb{R})$,

$$\|t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau\|_{L^\infty_x(\mathbb{R}) L^2_t([0,T])} \lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R})}. \tag{6}$$

If $n \geq 2$, on denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ a family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then, for all $g \in L^1_t L^2_x([0,T] \times \mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \left\| t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau \right\|^2_{L^2_t([0,T])} dx \right)^{1/2} \lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^n)}. \tag{7}$$

Theorem 3 *Let $k \geq 1$, then the IVP*

$$\begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases} \tag{8}$$

is locally well-posed in H^s for $s > n/2$ and its solution satisfies smoothing estimates.

Remark 1 Notice that Theorem 2 amounts to the validity of the homogeneous and inhomogeneous *weighted* smoothing estimates with a gain of $1/2$ derivative for $\mathcal{L}_\alpha$.

When $\alpha = 0$ one has actually an inhomogeneous smoothing effect better than the one described in (7), that is the inhomogeneous part of the solution gains 1 instead of $1/2$ derivative with respect to the inhomogeneious part of the equation (in other words, when $\alpha = 0$, one can replace $D_x^{1/2}$ by D_x^1 in (7), see [20]).

When $\alpha \neq 0$ the suitable corresponding weighted estimate still holds. This property is described in Theorem 4 part (*iii*) below for the general case $\mathcal{L}_{\alpha,c}$, with c being not necessarily identically 0, directly.

We stress that the proofs of the results of this subsection rely on the explicit knowledge of the solution of the inhomogeneous IVP for $\mathcal{L}_\alpha$. Indeed, by using classical Fourier analysis methods and Duhamel's principle (that still holds in this case, see [12]), we get that the solution of the IVP

$$\begin{cases} \mathcal{L}_\alpha u = f(t,x) \\ u(s,x) = u_s(x), \end{cases}$$

for $s < t$, is given by

$$u(t,x) = W_\alpha(t,s)u_s(x) + \int_s^t W_\alpha(t,t')f(t',x)dt',$$

where

$$W_\alpha(t,s)u_s(x) := e^{i\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}\Delta_x}u_s(x) := \int_{\mathbb{R}^n} e^{-i(\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}|\xi|^2 - x\cdot\xi)}\widehat{u_s}(\xi)d\xi \tag{9}$$

is the so called *solution operator*, that is the operator giving the solution of the homogeneous problem at time t with initial condition at time s. This is a two-parameter family of unitary operators satisfying:

1. $W_\alpha(t,t) = I$;
2. $W_\alpha(t,s) = W_\alpha(t,r)W_\alpha(r,s)$ for every $s,t,r \in [0,T]$;
3. $W_\alpha(t,s)\Delta_x u = \Delta_x W_\alpha(t,s)u$;
4. $\|W_\alpha(t,s)u_s\|_{H_x^s} = \|u_s\|_{H_x^s}$.

Let us remark that in the case $\alpha = 0$ the operator above coincides with the well known Schrödinger group.

Now we can finally give the proofs of Theorem 2 and Theorem 3.

Proof (Proof of Theorem 1) First note that (3) and (4) are true when $\alpha = 0$, that is, when $W_\alpha(t) = W_0(t) = e^{it\Delta_x}$ (see, for instance, [20]). Then it suffices to prove that

$$\|t^{\alpha/2}D_x^{1/2}W_\alpha(t)f\|^2_{L^2_t([0,T])} = \|D_x^{1/2}W_0(t)f\|^2_{L^2_t([0,T'])}.$$

To prove that the identity above is satisfied, we use the change of variables $t^{\alpha+1}/(\alpha+1) = s$, and get

$$\begin{aligned}\|t^{\alpha/2}D_x^{1/2}W_\alpha(t)f\|^2_{L^2_t([0,T])} &= \int_0^T \left|t^{\alpha/2}\int_{\mathbb{R}^n} e^{-i\left(t^{\alpha+1}|\xi|^2/(\alpha+1)-x\cdot\xi\right)}|\xi|^{1/2}\widehat{f}(\xi)d\xi\right|^2 dt\\ &= \int_0^{\frac{T^{\alpha+1}}{\alpha+1}} \left|\int_{\mathbb{R}^n} e^{-i\left(s|\xi|^2-x\cdot\xi\right)}|\xi|^{1/2}\widehat{f}(\xi)d\xi\right|^2 ds\\ &= \|D_x^{1/2}W_0(t)f\|^2_{L^2_t([0,T^{\alpha+1}/(\alpha+1)])}.\end{aligned}$$

Finally, by application of the smoothing estimates for $W_0(t) = e^{it\Delta_x}$, we conclude (3) and (4) (see [20], Theorem 2.1). □

Proof (Proof of Theorem 2) Inequality (5) follows directly from (3) by duality.

As for (6), on denoting by $L^p_x := L^p_x(\mathbb{R}^n)$, we have

$$\begin{aligned}&\|t^{\alpha/2}D_x^{1/2}\int_0^t W_\alpha(t,\tau)g(\tau)\tau\|_{L^\infty_x L^2_t([0,T])}\\ &\qquad\underset{\text{Minkowski}}{\leq} \left\|\int_0^T\left(\int_0^T\left|t^{\alpha/2}D_x^{1/2}W_\alpha(t,\tau)g(\tau)\right|^2 dt\right)^{1/2} d\tau\right\|_{L^\infty_x}\\ &\qquad\underset{\text{by (3)}}{\leq} \int_0^T \|W_\alpha(0,\tau)g(\tau)\|_{L^2_x}d\tau = \|g\|_{L^1_t([0,T])L^2_x}\end{aligned}$$

which gives (6).

To prove (7) we first observe that, by Minkowsky inequality,

$$\left\|t^{\alpha/2}D_x^{1/2}\int_0^t W_\alpha(t,\tau)g(\tau)d\tau\right\|_{L^2_t([0,T])} \leq \int_0^T \|t^{\alpha/2}D_x^{1/2}W_\alpha(t,0)W_\alpha(0,\tau)g(\tau)\|_{L^2_t([0,T])}d\tau,$$

therefore

$$\begin{aligned}&\left(\int_{Q_\beta}\left\|t^{\alpha/2}D_x^{1/2}\int_0^t W_\alpha(t,\tau)g(\tau)d\tau\right\|^2_{L^2_t([0,T])} dx\right)^{1/2}\\ &\qquad\leq \left[\int_{Q_\beta}\left(\int_0^T \|t^{\alpha/2}D_x^{1/2}W_\alpha(t)W_\alpha(0,\tau)g(\tau)\|_{L^2_t([0,T])}d\tau\right)^2 dx\right]^{1/2}\\ &\qquad\underset{\text{Minkowski}}{\leq} \int_0^T\left(\int_{Q_\beta}\|t^{\alpha/2}D_x^{1/2}W_\alpha(t,0)W_\alpha(0,\tau)g(\tau)\|^2_{L^2_t([0,T)]}dx\right)^{1/2} d\tau.\end{aligned}$$

Then we apply the $\sup_{\beta\in\mathbb{Z}^n}$ on both the RHS and the LHS of the previous inequality and get

$$\sup_{\beta\in\mathbb{Z}^n}\left(\int_{Q_\beta}\left\|t^{\alpha/2}D_x^{1/2}\int_0^t W_\alpha(t,\tau)g(\tau)d\tau\right\|^2_{L_t^2([0,T])}dx\right)^{1/2}$$

$$\leq \int_0^T \sup_{\beta\in\mathbb{Z}^n}\left(\int_{Q_\beta}\|t^{\alpha/2}D_x^{1/2}W_\alpha(t,0)W_\alpha(0,\tau)g(\tau)\|^2_{L_t^2([0,T])}dx\right)^{1/2}d\tau$$

$$\underset{\text{by (4)}}{\leq}\int_0^T \|W_\alpha(0,\tau)g(\tau)\|_{L_x^2(\mathbb{R}^n)}d\tau = \int_0^T \|g(\tau)\|_{L_x^2(\mathbb{R}^n)}d\tau,$$

which gives (7) and concludes the proof. □

We are almost ready to prove our well-posedness result, but first let us recall what we mean by saying that the IVP (8) is locally well-posed.

Definition 1 We say that the IVP (8) is locally well-posed (l.w.p) in $H^s(\mathbb{R}^n)$ if for any ball B in the space $H^s(\mathbb{R}^n)$ there exist a time T and a Banach space of functions $X \subset L^\infty([0,T], H^s(\mathbb{R}^n))$ such that, for each initial datum $u_0 \in B$, there exists a unique solution $u \in X \subset C([0,T], H^s(\mathbb{R}^n))$ of the integral equation

$$u(x,t) = W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau)|u|^{2k}u(\tau)d\tau.$$

Furthermore the map $u_0 \mapsto u$ is continuous as a map from $H^s(\mathbb{R}^n)$ into $C([0,T], H^s(\mathbb{R}^n))$.

Proof (Proof of Theorem 3) The proof is based on the standard contraction argument. We summarize below the main steps of the proof. For further details we refer the interested reader to [12].

Let us first assume that $n = 1$, and let us define the metric space X as

$$X := \{u : [0,T]\times\mathbb{R} \to \mathbb{C};\, \|t^{\alpha/2}D_x^{1/2+s}u\|_{L_x^\infty L_t^2([0,T])} < \infty,\, \|u\|_{L_t^\infty([0,T])H_x^s} < \infty\},$$

which we equip with the distance

$$d(u,v) = \|t^{\alpha/2}D_x^{1/2+s}(u-v)\|_{L_x^\infty L_t^2([0,T])} + \|u-v\|_{L_t^\infty([0,T])\dot H_x^s} + \|u-v\|_{L_t^\infty([0,T])L_x^2},$$

where $\dot H_x^s$ stands for the homogeneous Sobolev space. We then consider the map

$$\Phi : X \to X, \quad \Phi(u) = W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau)u|u|^{2k}(\tau)d\tau,$$

and prove that it is a contraction on a ball of X, that is on $B_R := \{u \in X; \|u\|_X \leq R\} \subset X$ for a suitable R.

By using the estimates in Theorem 2 and in Theorem 3 we get that

$$\|\Phi(u)\|_X \leq 3\|u_0\|_{H_x^s} + C_1 T\|u\|_X^{2k+1},$$

which, for $R = 6\|u_0\|_{H_x^s}$ and $T = \frac{1}{C_1R^{2k}}$, gives that Φ sends B_R into itself. Now, fixing $R = 6\|u_0\|_{H_x^s}$, and by using arguments similar to those used above, we can conclude that Φ is a contraction. Indeed, for all $u, v \in B_R$, we have

$$\|\Phi(u) - \Phi(v)\|_X \le C_2 T R^{2k} \|u - v\|_X,$$

therefore, by choosing T such that $T = \min\{\frac{1}{C_1 R^{2k}}, \frac{1}{C_2 R^{2k}}\}$, we obtain that Φ is a contraction, and the result follows by the fixed point theorem.

Let us now assume that $n > 1$. In this case we define X to be the space

$$X := \{u : [0,T] \times \mathbb{R}^n \to \mathbb{C}; \, \|\!| t^{\alpha/2} D_x^{s+1/2} u \|\!|_T < \infty, \, \|u\|_{L^\infty_{[0,T]} H^s_x} < \infty\},$$

where

$$\|\!| \cdot \|\!|_T = \sup_{\beta \in \mathbb{Z}^n} \| \cdot \|_{L^2_x(Q_\beta) L^2_t([0,T])},$$

and

$$d_X(u, v) = \|\!| t^{\alpha/2} D_x^{s1/2} (u - v) \|\!|_T + \|u - v\|_{L^\infty_t([0,T]) \dot{H}^s_x} + \|u - v\|_{L^\infty_t([0,T]) L^2_x}.$$

Then, considering the map Φ as before but now defined on the new space X, we can exploit the estimates in Theorem 2 and in Theorem 3 holding in the high dimensional case to get the same estimates and properties as in the case $n = 1$. The result then follows again by the fixed point theorem. For more details and explicit computations see [12]. □

Remark 2 Let us remark that the methods applied above in the case $\mathcal{L}_\alpha := \mathcal{L}_{\alpha,0}$ can also be applied to the case $\mathcal{L}_{\alpha,c} = \mathcal{L}_{\alpha,t^\alpha v}$, with v being a complex vector $v \in \mathbb{C}^n$.

2.2 The class $\mathcal{L}_{\alpha,c}$

This section focuses on the study of the more general case $\mathcal{L}_{\alpha,c}$ with c being not necessarily identically zero ($c \not\equiv 0$ in our notation). We stress that the results of this subsection hold true in the case $c \equiv 0$ as well, and that in the latter case a direct proof can be performed. However, due to the presence of the variable coefficients $c(t, x)$, whose properties will be stated soon (see Theorem 4), the strategy to be used to analyze the problem for $\mathcal{L}_{\alpha,c}$ is different than the one used before for $\mathcal{L}_\alpha$. The key tools of our analysis will be the use of the pseudodifferential calculus and the application of a lemma due to Doi in [7], that we shall call Doi's Lemma, that we recall in Lemma 2 in the Appendix.

We shall state in Theorem 4 below our result about the smoothing properties of the solution of the IVP

$$\begin{cases} \partial_t u = i t^\alpha \Delta_x u + i c(t, x) \cdot \nabla_x u + f(t, x), \\ u(0, x) = u_0(x). \end{cases} \tag{10}$$

Moreover, we will give in Theorem 5 and Theorem 6 local well-posedness results for the IVP (10) when $f = \pm|u|^{2k}u$, $k \geq 1$, and when $f = \pm t^{\beta} \sum_{j=1}^{n} (\partial_{x_j} u)u$, with $\beta \geq \alpha > 0$, respectively.

Theorem 4 *Let $u_0 \in H^s(\mathbb{R}^n), s \in \mathbb{R}$. Assume that, for all $j = 1, \dots, n$, c_j is such that $c_j \in C([0,T], C_b^{\infty}(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that*

$$|\mathrm{Im}\, \partial_x^{\gamma} c_j(t,x)|, |\mathrm{Re}\, \partial_x^{\gamma} c_j(t,x)| \lesssim t^{\alpha} \langle x \rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n. \tag{11}$$

Then, denoting by $\lambda(|x|) := \langle x \rangle^{-\sigma}$, we have the following properties:

(i) If $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ then the IVP (10) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_s \leq C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt \right);$$

(ii) If $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ then the IVP (10) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{\alpha} \left| \Lambda^{s+1/2} u \right|^2 \lambda(|x|) dx\, dt$$
$$\leq C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt \right);$$

(iii) If $\Lambda^{s-1/2} f \in L^2([0,T] \times \mathbb{R}^n; t^{-\alpha} \lambda(|x|)^{-1} dt dx)$ then the IVP (10) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{\alpha} \left| \Lambda^{s+1/2} u \right|^2 \lambda(|x|) dx\, dt$$
$$\leq C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{-\alpha} \lambda(|x|)^{-1} \left| \Lambda^{s-1/2} f \right|^2 dx\, dt \right).$$

Above we abbreviated the norm $\|f\|_{H^s(\mathbb{R}^n)} =: \|f\|_s$.

Theorem 5 *Let $\mathcal{L}_{\alpha}$ be such that condition (11) is satisfied. Then the IVP (10) with $f(t,x) = \pm|u|^{2k}u$ is locally well posed in H^s for $s > n/2$ and the solution satisfies smoothing estimates.*

Theorem 6 *Let $\mathcal{L}_{\alpha}$ be such that condition (11) is satisfied with $\sigma = 2N$ (thus $\lambda(|x|) = \langle x \rangle^{-2N}$) for some $N \geq 1$, and let $s > n + 4N + 3$ be such that $s - 1/2 \in 2\mathbb{N}$. Then, the IVP (10) with $f = \pm t^{\beta} \sum_{j=1}^{n} (\partial_{x_j} u)u$, where $\beta \geq \alpha > 0$, is locally well posed in $H_{\lambda}^s := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$ and the solution satisfies smoothing estimates.*

Remark 3 Let us stress that it is natural to require the coefficients c_j of the first order term to satisfy some decay conditions, usually called Levi conditions. Indeed such kind of conditions were proved to be necessary to have the local well-posedness of the linear IVP in the case $\alpha = 0$. To be precise, it is enough to impose some decay on $\mathsf{Re}\, \partial_x^\gamma c_j(t,x)$ only (for all $j = 1, \ldots, n$,) to conclude the local well-posedness of the linear IVP. However, the additional condition on $\mathsf{Im}\, \partial_x^\gamma c_j(t,x)$, for all $j = 1, \ldots, n$, appears in order to get estimates with "gain of derivatives", namely smoothing estimates, needed to deal with the nonlinear problem with derivative nonlinearities.

Remark 4 Notice that part (ii) and (iii) in Theorem 4 correspond to the weighted homogeneous and inhomogeneous smoothing estimate for $\mathcal{L}_{\alpha,c}$ with a gain of $1/2$ and 1 derivative, respectively. When $\alpha = 0$, these results coincide with the classical ones for $\mathcal{L}_{0,c}$ (see, for instance, [20] and [22]).

The proof of Theorem 4 is based on the results in Lemma 1 below. The proof of Lemma 1, instead, relies deeply on the use of Lemma 2, also called Doi's lemma. The crucial result due to Doi in [7] is needed to define a new norm N, equivalent to the H^s-Sobolev norm, which is used to perform the energy estimate from which the smoothing estimates are derived. We explain below the way we use Doi's lemma, that is Lemma 2, to define N.

We apply Lemma 2 on the symbol $a^w := a = a_2 + ia_1 + a_0$ with $a_2(x,\xi) = |\xi|^2$ and $a_1 = a_0 = 0$. In this case conditions (B1) and (B2) of Lemma 2 are trivially satisfied, while (A6) holds with $q(x,\xi) = x \cdot \xi \langle \xi \rangle^{-1}$. Therefore, by Lemma 2 with $\lambda'(|x|) = C' \langle x \rangle^{-\sigma}$ (see Remark 6), with C' to be chosen later, we get that there exists $p \in S^0$ and $C > 0$ such that (37) holds.

We then consider the pseudo-differential operator K with symbol $K(x,\xi) = e^{p(x,\xi)} \Lambda^s$, where $\Lambda^s := \langle \xi \rangle^s$ and $p(x,\xi)$ is the symbol given by Doi's lemma, and define the norm N on $H^s(\mathbb{R}^n)$, equivalent to the standard one (see [22] for the proof of the equivalence), as

$$N(u)^2 = \|Ku\|_0^2 + \|u\|_{s-1}^2, \tag{12}$$

where $\| \cdot \|_s$ stands for the standard norm in the Sobolev space $H^s(\mathbb{R}^n)$.

With the norm $N(\cdot)$ in (12) at our disposal we can prove Lemma 1 from which Therem 4 will follow. To prove Lemma 1 we employ the technique used in [22].

Lemma 1 *Let $s \in \mathbb{R}$, $\lambda(|x|) := \langle x \rangle^{-\sigma}$, $P_\alpha := \partial_t - it^\alpha \Delta_x - ic(t,x) \cdot \nabla_x$, and $\sigma > 1$ such that* (11) *holds. Then there exists $C_1, C_2 > 0$ such that, for all $u \in C([0,T]; H^{s+2}(\mathbb{R}^n)) \bigcap C^1([0,T]; H^s(\mathbb{R}^n))$, we have*

$$\sup_{0\le t\le T}\|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s+\int_0^T \|P_\alpha u(t,\cdot)\|_s dt\right); \tag{13}$$

$$\sup_{0\le t\le T}\|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u(\cdot,T)\|_s+\int_0^T \|P^*_\alpha u(t,\cdot)\|_s dt\right); \tag{14}$$

$$\sup_{0\le t\le T}\|u(t)\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)dx\,dt$$
$$\le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s^2+\int_0^T \|P_\alpha u(t,\cdot)\|_s^2 dt\right); \tag{15}$$

$$\sup_{0\le t\le T}\|u(t)\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)dx\,dt$$
$$\le C_1 e^{C_2\frac{T^{\alpha+1}}{\alpha+1}}\left(\|u_0\|_s^2+\int_0^T\int_{\mathbb{R}^n} t^{-\alpha}\lambda(|x|)^{-1}\left|\Lambda^{s-1/2}P_\alpha u(t,\cdot)\right|^2 dxdt\right). \tag{16}$$

□

Proof The proof is based on an enery estimate in terms of the norm $N(\cdot)$ in (12). We recall that $P_\alpha := \partial_t - it^\alpha\Delta_x - ic(t,x)\cdot\nabla_x$, $D_x = (D_{x_1},\dots,D_{x_n}) := (-i\partial_{x_1},\dots,-i\partial_{x_n})$, and that $\langle\cdot,\cdot\rangle$ stands for the $L^2(\mathbb{R}^n)$-scalar product. We then consider

$$\partial_t N(u)^2 = \partial_t\|Ku\|_0^2 + \partial_t\|u\|_{s-1}^2 = I + II,$$

and estimate I and II separately.

We start by estimating term II, for which we get

$$\begin{aligned} II &= \partial_t\|u\|_{s-1}^2 = 2\mathrm{Re}\langle\Lambda^{s-1}\partial_t u,\Lambda^{s-1}u\rangle = 2\mathrm{Re}\langle\Lambda^{s-1}P_\alpha u,\Lambda^{s-1}u\rangle \\ &= -2\mathrm{Re}\langle\Lambda^{s-1}c(t,x)\cdot D_x u,\Lambda^{s-1}u\rangle + 2\mathrm{Re}\langle\Lambda^{s-1}f,\Lambda^{s-1}u\rangle \\ &\le Ct^\alpha\|u\|_s^2 + 2\mathrm{Re}\langle\Lambda^{s-1}f,\Lambda^{s-1}u\rangle. \end{aligned}$$

Now, since

$$2\mathrm{Re}\langle\Lambda^{s-1}f,\Lambda^{s-1}u\rangle \le 2\|f\|_{s-1}\|u\|_{s-1} \le CN(f)N(u) \tag{17}$$

and

$$\begin{aligned} 2\mathrm{Re}\langle\Lambda^{s-1}f,\Lambda^{s-1}u\rangle &= 2\mathrm{Re}\langle t^{-\alpha/2}\lambda(|x|)^{-1/2}\Lambda^{s-1/2}f, t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s-3/2}u\rangle \\ &\le \|t^{-\alpha/2}\lambda(|x|)^{-1/2}\Lambda^{s-1/2}f\|_0^2 + \|t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s-3/2}u\|_0^2 \\ &\le \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f,\Lambda^{s-1/2}f\rangle + t^\alpha N(u)^2, \end{aligned} \tag{18}$$

it follows that

$$II \le Ct^\alpha N(u)^2 + C'\min\{N(f)N(u); \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f,\Lambda^{s-1/2}f\rangle\}, \tag{19}$$

with C and C' new suitable constants.

As for term I we have that

$$
\begin{aligned}
\partial_t \|Ku\|_0^2 &= 2\mathsf{Re}\langle \partial_t Ku, Ku\rangle = 2\mathsf{Re}\langle K\partial_t u, Ku\rangle \\
&= 2\mathsf{Re}\langle KP_\alpha u, Ku\rangle + 2\mathsf{Re}\langle Kf, Ku\rangle \\
&= 2\mathsf{Re}\langle it^\alpha[K,\Delta_x]u, Ku\rangle + \underbrace{2\mathsf{Re}\langle it^\alpha \Delta_x Ku, Ku\rangle}_{=0} \\
&- 2\mathsf{Re}\langle K\, b(t,x)\cdot D_x u, Ku\rangle + 2\mathsf{Re}\langle Kf, Ku\rangle \\
&= 2\mathsf{Re}\langle it^\alpha[K,\Delta_x]u, Ku\rangle - 2\mathsf{Re}\langle [K, c(t,x)\cdot D_x]u, Ku\rangle \\
&- 2\mathsf{Re}\langle c(t,x)\cdot D_x\, Ku, Ku\rangle + 2\mathsf{Re}\langle Kf, Ku\rangle, \qquad (20)
\end{aligned}
$$

therefore, in order to estimate I, it is crucial to prove suitable upper bounds for the quantities $2\mathsf{Re}\langle it^\alpha[K,\Delta_x]u, Ku\rangle$ and $2\mathsf{Re}\langle [K, c(t,x)\cdot D_x]u, Ku\rangle$ in the the fifth line of (20).

By using the pseudodifferential calculus we can compute the symbol of the commutator $[K, c(t,x)\cdot D_x]$, which is an operator of order s, and get, thanks to the properties of c (recall that $c \in C_b^\infty$ and is bounded, together with its derivatives in space, by $t^\alpha \lambda(|x|)$), that

$$-2\mathsf{Re}\langle [K, b(t,x)D_x]u, Ku\rangle \leq Ct^\alpha \|u\|_s^2.$$

For more details about how to get to this estimate see Lemma 5.0.1 in [12].

For the term $2\mathsf{Re}\langle it^\alpha[K,\Delta_x]u, Ku\rangle$, once more by using the pseudodifferential calculus, we have that $[K,\Delta_x](x,D) = [p,\Delta_x]K(x,D) + r_s(x,D)$, where r_s is an operator of order s, while $p = p(x,D)$ is the operator of order 0 appearing in the definition of the norm $N(\cdot)$.

These considerations lead to

$$
\begin{aligned}
(20) &\leq Ct^\alpha\|u\|_s^2 + 2\mathsf{Re}\langle (it^\alpha[p,\Delta_x](x,D) - c(t,x)\cdot D_x)Ku, Ku\rangle + |2\mathsf{Re}\langle it^\alpha r_s(x,D)u, Ku\rangle| \\
&\leq Ct^\alpha\|u\|_s^2 + 2\mathsf{Re}\langle (it^\alpha[p,\Delta_x](x,D) - c(t,x)D_x)Ku, Ku\rangle, \qquad (21)
\end{aligned}
$$

where C is a new suitable positive constant.

Now we denote by $Q(x,D) := it^\alpha[p,\Delta_x](x,D) - b(t,x)\cdot D_x$ the operator whose symbol satisfies

$$
\begin{aligned}
\mathsf{Re}\, Q(x,\xi) &= \mathsf{Re}\left(it^\alpha(-i)\{p, -|\xi|^2\}(x,\xi) - b(t,x)\cdot\xi\right) + r_0 \\
&\leq -t^\alpha\{p, |\xi|^2\}(x,\xi) + |\mathsf{Re}\, b(t,x)\cdot\xi| + r_0 \\
&\underset{\text{by (37)}}{\leq} -C't^\alpha\lambda(|x|)|\xi| + C_2t^\alpha + C_0t^\alpha\lambda(|x|)|\xi| + C \\
&\leq -Ct^\alpha\lambda(|x|)|\xi| + C_2t^\alpha + C_4 \\
&\leq -Ct^\alpha\lambda(|x|)(1+|\xi|^2)^{1/2} + C_3t^\alpha + C_4 \\
&= t^\alpha(-C\lambda(|x|)(1+|\xi|^2)^{1/2} + C_3) + C_4,
\end{aligned}
$$

where we chose C' (which is possible by Doi's lemma, see Remark 6) in order to have $C_0 - C' < 0$.

The property of the symbol of Q allow us to apply the sharp Gårding inequality and to conclude that

$$\begin{aligned} 2\mathrm{Re}\langle Q(x,D)Ku, Ku\rangle &\le -Ct^{\alpha}\langle \lambda(|x|)\Lambda^{1}Ku, Ku\rangle + C_3 t^{\alpha}\|Ku\|_0^2 + C_4\|Ku\|_0^2 \\ &\le -Ct^{\alpha}\langle \lambda(|x|)\Lambda^{1}Ku, Ku\rangle + C_3 t^{\alpha}\|u\|_s^2 + C_4\|u\|_s^2 \\ &\le Ct^{\alpha}\|\lambda(|x|)^{1/2}\Lambda^{1/2}Ku\|_0^2 + C_3 t^{\alpha}\|u\|_s^2 + C_4\|u\|_s^2, \end{aligned} \quad (22)$$

where $C > 0$ is a new suitable constant.

By plugging (22) in (21) we get

$$\partial_t\|Ku\|_0 \le Ct^{\alpha}N(u)^2 + C'N(u)^2 - C''t^{\alpha}\|\lambda(|x|)^{1/2}\Lambda^{1/2}Ku\|_0^2 + C'''N(f)N(u). \quad (23)$$

Finally, (19) and the equivalence of the norms $\|\cdot\|_s$ and $N(\cdot)$ (see [22] pag.390) yield

$$\begin{aligned} \partial_t N(u)^2 &= \partial_t\|Ku\|^2 + \partial_t\|u\|_{s-1}^2 \\ &\le Ct^{\alpha}N(u)^2 + C'N(u)^2 - C''t^{\alpha}\|\lambda(|x|)^{1/2}\Lambda^{1/2}Ku\|_0^2 + C'''N(f)N(u) \\ &+ C_3\min\{N(f)N(u); \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f, \Lambda^{s-1/2}f\rangle\}, \end{aligned} \quad (24)$$

□

where the constants are (eventually) new suitable constants.

Estimate (24) is now the starting point to get (13), (14) and (15).

Proof of (13). From (24) we have

$$\partial_t N(u)^2 \le C_1(t^{\alpha}+1)N(u)^2 + C_2 N(u)N(f)$$

(again with C_1 and C_2 new constants), which gives

$$2\partial_t N(u) \le C_1(t^{\alpha}+1)N(u) + C_2 N(f)$$

and

$$\partial_t\left(2e^{-\frac{1}{2}C_1(t^{\alpha+1}/(\alpha+1)+t)}N(u)\right) \le C_2 e^{-\frac{1}{2}C_1(t^{\alpha+1}/(\alpha+1)+t)}N(f).$$

Hence, by integrating in time from 0 to t and using the equivalence of the norms $N(\cdot)$ an $\|\cdot\|_s$, (13) follows.

Proof of (14).The proof of (14) follows from (13) applied to the adjoint operator and with $u(t,\cdot)$ replaced by $u(T-t,\cdot)$.

Proof of (15). Here we use the fact that there exists a pseudodifferential operator $\tilde{K}$ such that

$$I = \tilde{K}K + \Psi_{r_{-1}},$$

where $\Psi_{r_{-1}}$ is a pseudodifferential operator with symbol r_{-1} of order -1 (see [22] pag.390 for the proof of). This gives that

$$\|\lambda(|x|)^{1/2}\Lambda^{s+1/2}u\|_0 \le \|(\lambda(|x|)^{1/2}\Lambda^{1/2})(\Lambda^s\tilde{K})Ku\|_0 + O(N(u))$$

$$\le \|(\Lambda^s\tilde{K})(\lambda(|x|)^{1/2}\Lambda^{1/2})Ku\|_0 + cN(u) \le c\left(\|(\lambda(|x|)^{1/2}\Lambda^{1/2})Ku\|_0 + N(u)\right), \quad (25)$$

since $[\Lambda^s\tilde{K}, \lambda(|x|)^{1/2}\Lambda^{1/2}]K\Lambda^{1/2}$ is a pseudo-differential operator of order s. Therefore, (24) and (25) yield

$$\partial_t N(u)^2 + C_2\langle t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u, t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u\rangle \le C_1(t^\alpha+1)N(u)^2 + C_4N(f)^2.$$

Now, integrating in time from 0 to t the previous inequality, using (13) and the estimate

$$e^{\frac{1}{2}C_1(t^{\alpha+1}/(\alpha+1)+t)}\int_0^t e^{-\frac{1}{2}C_1(s^{\alpha+1}/(\alpha+1)+s)}\langle s^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u, s^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u\rangle ds$$

$$\ge \int_0^t \langle s^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u, s^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u\rangle ds,$$

(15) follows (for further details see [12]).

Proof of (16). To prove (16) we exploit the following estimate

$$\begin{aligned} 2\mathrm{Re}\langle Kf, Ku\rangle &= 2\mathrm{Re}\langle t^{\alpha/2}\lambda^{1/2}\Lambda^{1/2}Kf, t^{-\alpha/2}\lambda^{-1/2}\Lambda^{-1/2}Ku\rangle \\ &\le c_1\varepsilon\|t^{\alpha/2}\lambda^{1/2}\Lambda^{s+1/2}u\|_0^2 + c_2\frac{1}{\varepsilon}\|t^{-\alpha/2}\lambda^{-1/2}\Lambda^{s-1/2}f\|_0^2 + c_3t^\alpha\|u\|_s^2. \end{aligned} \quad (26)$$

By using (25) and (26) in (24) and the equivalence of $N(\cdot)$ and $\|\cdot\|_s$, we obtain

$$\partial_t N(u)^2 + (c_0 - c_1\varepsilon)\|t^{\alpha/2}\lambda^{1/2}\Lambda^{s+1/2}u\|_0^2 \le c_3t^\alpha N(u)^2 + c_2\frac{1}{\varepsilon}\|t^{-\alpha/2}\lambda^{-1/2}\Lambda^{s-1/2}f\|_0^2,$$

where c_j, $j = 0,1,2,3$, are new suitable constants, and where $\varepsilon > 0$ can be chosen in such a way that $c_0 - c_1\varepsilon \ge c > 0$. Finally, integrating in time from 0 to t, and arguing as in the proof of (15), the result follows. This concludes the proof. □

Proof of Theorem 4 Estimate (13) of Lemma 1 gives readly the uniqueness of the solution. In fact, let u be a solution of the homogeneous IVP for $\mathcal{L}_{\alpha,c}$ with initial datum $u_0 = 0$. Then, by (13) of Lemma 1, $u = 0$, which proves the uniqueness (even in the general inhomogeneous IVP where $f \ne 0$ and $u_0 \ne 0$).

As for the existence, it will follow by using density arguments.

Case 1: $f \in \mathcal{S}(\mathbb{R}^{n+1})$ and $u_0 \in \mathcal{S}(\mathbb{R}^n)$.
We consider the subspace $E \subset L^1([0,T]; H^{-s}(\mathbb{R}^n)$

$$E = \{P^*\varphi;\ \varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))\} = (\partial_t - it^\alpha\Delta_x + b(t,x)\cdot D_x)^*(C_0^\infty(\mathbb{R}^{n+1}))$$

and the linear functional

$$\ell^* : E \longrightarrow \mathbb{C}, \quad \ell^*(P^*\varphi) = \int_0^T \langle f, \varphi \rangle_{L^2 \times L^2} dt + \langle u_0, \varphi(\cdot, 0) \rangle_{L^2 \times L^2}.$$

Now inequality (14) of Lemma 1 (applied to φ) with s replaced by $-s$ gives, for $\eta = P^*\varphi$ and $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))$,

$$\begin{aligned}|\ell^*(\eta)| &\leq \|f\|_{(L^1[0,T];H_x^s)} \sup_{t\in[0,T]} \|\varphi\|_{H_x^{-s}} + \|u_0\|_{H_x^s} \|\varphi(0)\|_{H_x^{-s}} \\ &\leq e^{C(T^{\alpha+1}/(\alpha+1)+T)} \left(\|f\|_{L_t^1([0,T];H_x^s)} + \|u_0\|_{H_x^s} \right) \|\eta\|_{L_t^1([0,T];H_x^{-s})},\end{aligned}$$

which implies the continuity of ℓ^* on E. Then, by the Hahn-Banach theorem we can extend ℓ^* on $L^1([0,T] : H^{-s}(\mathbb{R}^n))$ and finally get the existence of $u \in L^1([0,T]; H^{-s}(\mathbb{R}^n))^* = L^\infty([0,T]; H^s(\mathbb{R}^n))$ such that

$$\ell^*(P^*\varphi) = \langle u, P^*\varphi \rangle_{L^2 \times L^2} = \int_0^T \langle f, \varphi \rangle_{L^2 \times L^2} dt + \langle u_0, \varphi(\cdot, 0) \rangle_{L^2 \times L^2},$$

and thus $Pu = f$ in the sense of distributions for $0 < t < T$.

Notice that $Pu \overset{\mathcal{D}'}{=} f$ means that $(\partial_t - it^\alpha \Delta_x + b(t,x) \cdot D_x)u \overset{\mathcal{D}'}{=} f$ (as distributions on $C_0^\infty([0,T) \times \mathbb{R}^n)$). Therefore, since $f \in \mathcal{S}(\mathbb{R}^{n+1})$, we have that $\partial_t u \in (L^\infty[0,T) : H^{s-2}(\mathbb{R}^n))$, which gives $u \in (C([0,T) : H^{s-2}(\mathbb{R}^n))$. We then use the equation once more, that is $\partial_t u = it^\alpha \Delta_x + b(t,x) \cdot D_x u + f$, and get, by the same consideration, that $u \in (C^1[0,T) : H^{s-4}(\mathbb{R}^n))$ and $u(x,0) = u_0(x)$. Finally, since $u_0 \in H^s(\mathbb{R}^n)$, repeating the previous argument with $s + 4$ in place of s we conclude that there exists a solution u of the IVP associated to (10) to which parts (i)-(iv) of Lemma 1 apply.

Case 2: $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ and $u_0 \in H^s(\mathbb{R}^n)$.
In this case we take two sequences $f_j \in \mathcal{S}(\mathbb{R}^{n+1})$, $v_j \in \mathcal{S}(\mathbb{R}^n)$ such that $f_j \to f$ in $(L^1([0,T]) : H^s(\mathbb{R}^n)$ and $v_j \to u_0$ in $H^s(\mathbb{R}^n)$.

By the arguments of case 1 we find a solution u_j of (10) with f_j and v_j in place of f and u_0 respectively. Since u_j satisfies (13) of Lemma 1, we have that u_j is a Cauchy sequence, therefore, passing to the limit, we get that $u = \lim_{j\to} u_j$ is a solution of the IVP with inhomogeneous term f and with initial datum u_0 satisfying (14) of Lemma 1, which proves part (ii) of the theorem.

Case 3: $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ and $u_0 \in H^s(\mathbb{R}^n)$.
Here we proceed as in case 2 but with $f_j \in \mathcal{S}(\mathbb{R}^{n+1})$ being such that $f_j \to f$ in $(L^2([0,T]); H^s(\mathbb{R}^n)$. Under this hypothesis we obtain point (ii) of the theorem, that is, it exists a solution $u \in (C[0,T) : H^s(\mathbb{R}^n))$ satisfying (15) of Lemma 1.

Case 4: $\Lambda^{s-1/2} f \in (L^2(\mathbb{R}^n \times [0,T]) : t^{-\alpha}\lambda(|x|)^{-1} dxdt)$ and $u_0 \in H^s(\mathbb{R}^n)$.
In this case it is possible to prove that there exists $g_j \in \mathcal{S}(\mathbb{R}^{n+1})$ such that $g_j \to \Lambda^{s-1/2} f$ in $(L^2(\mathbb{R}^n) \times [0,T] : t^{-\alpha}\lambda(|x|)^{-1} dxdt)$. Applying once again the strategy

used in case 1 with f_j replaced by $\Lambda^{-s+1/2} g_j$ in (16) of Lemma 1, and passing to the limit, we finally obtain point (iii) of Theorem 4. □

As a consequence of Theorem 4 one gets the local well-posedness results stated in Theorem 5 and in Theorem 6. We will not give a complete proof of these results here, and we refer the interested reader to [12] for detailed proofs. However, we give below a scketch of the proof listing the main ingredients of the argument.

Sketch of the proof of Theorem 5 As in the case $c \equiv 0$, the proof is based on the standard contraction argument.

According to Theorem 4 we have the local well-posedness in H^s, $s > n/2$, for the linear IVP (10) for a general function f satisfying the assumptions. We now write the solution of (10) as

$$u(t,x) = W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau) f(\tau,x)d\tau, \tag{27}$$

where $W_\alpha(t,\tau)$ is a new suitable two-parameter family of unitary operators representing the solution operator.

Because of the previous assumption, solving the IVP (10) with $f = u|u|^{2k}$ is equivalent to find the solution of the integral equation

$$u(t,x) = W_\alpha(t)u_0(x) + \int_0^t W_\alpha(t,\tau)u|u|^{2k}(\tau,x)d\tau.$$

Hence, as in the proof of Theorem 4, we look for the solution given by the fixed point of the map

$$\Phi_{u_0}(u) := W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau)u|u|^{2k}d\tau,$$

defined on

$$X_T^s := \{u : [0,T]\times\mathbb{R}^n \to \mathbb{C};\, \|u\|_{L_t^\infty H_x^s} < \infty,\, \left(\int_0^T\int_{\mathbb{R}^n} t^\alpha \lambda(|x|)|\Lambda^{s+1/2}u|^2 dx\, dt\right)^{1/2} < \infty\},$$

where, recall, $\lambda(|x|) := \langle x\rangle^\sigma$, with $\sigma > 1$ being such that (11) holds. Notice that the choice of the space X_T^s is dictated by the smoothing estimates we proved in Theorem 4. To conclude that Φ_{u_0} is a contraction on the space X_T^s, we apply the estimates in Theorem 4 together with Sobolev embeddings and a few tecnical lemmas taken from [22]. Finally, the application of the fixed point theorem then gives the result. Notice that the solution will belong to the space X_T^s, and, consequently, will satisfy smoothing estimates. □

Sketch of the proof of Theorem 6 Ther proof of this result follows by using the same arguments as before. Here the contraction argument is performed on a different space, that is, specifically, on the space

$$X_T^s := \{u : [0,T]\times\mathbb{R}^n \to \mathbb{C};\, \|u\|_{L_t^\infty H_x^s} < \infty,\, \left(\int_0^T\int_{\mathbb{R}^n} t^\alpha \lambda(|x|)|\Lambda^{s+1/2}u|^2 dx\, dt\right)^{1/2} < \infty,$$

$$\|\lambda(|x|)^{-1}u\|_{L_t^\infty H_x^{s-2N-3/2}} < \infty\},$$

where

$$\|u\|_{X_T^s}^2 = \|u\|_{L_t^\infty H_x^s}^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \lambda(|x|)|\Lambda^{s+1/2}u|^2 dx\, dt + \|\lambda(|x|)^{-1}u\|_{L_t^\infty H_x^{s-2N-3/2}}^2.$$

We repeat the assumption that the solution of (10) is given in terms of a solution operator $W_\alpha(t,s)$, so we look for the solution of the nonlinear problem as the fixed point of a map Φ_{u_0} as before, but now with $f = t^\beta \sum_{j=1}^n \partial_{\xi_j} u|u|^2$, with $\beta \geq \alpha$. We then use the smoothing estimates in Lemma 1, more precisely (16), togheter with Lemma 6.0.1 in [12] and some technical lemmas taken form [22], and conclude the result via the standard contraction argument. Once again the solution satisfies smoothing estimates. For the complete proof see [12]. □

Let us remark once again that the previous results still hold true in the case $c \equiv 0$. Moreover, more general nonlinearities can be considered in the IVP for $\mathcal{L}_{\alpha,c}$, that is, for instance, nonlinearities containing polynomials in u, in the derivatives of order one of u, and in their complex conjugates. The specific choices we made for the nonlinear terms were to keep the exposition simpler and shorter.

We finally conclude by saying that the smoothing and well-posedness results presented here are very likely still true for some generalizations of $\mathcal{L}_{\alpha,c}$, that is for equations containing first order terms in $\bar{u}$ and with time degeneracies different than t^α (for more details about these generalizations see Section 7 in [12]).

3 Strichartz estimates and local well-posedness for $\mathcal{L}_b$

This section is devoted to the study of the class $\mathcal{L}_b$ as in (2), for which, as we shall show below, local weighted Strichartz estimates hold true. Additionally, we will employ such estimates to prove the local well-posedness of a semilinear IVP associated with $\mathcal{L}_b$, where the form of the nonlinear term is dictated by the inhomogenous Strichartz estimate at our disposal. The results of this section were proved in [11] where results other than local weighted Strichartz estimates are proved. In particular, in [11] also global weighted Strichartz estimates are derived, as well as homogeneous smoothing estimates for time-degenerate operators of any order by means of comparison principles. Our choice to treat the local estimates only is due to the fact that these inequalities, because of their different form with respect to the global counterpart, are the ones to be used to get the well-posedness of the semilinear IVP. For more deatails and results about the class $\mathcal{L}_b$ we refer the interted reader to [11].

The semilinear IVP we will study in this section is

$$\begin{cases} \partial_t u + ib'(t)\Delta u = \mu|b'(t)||u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \tag{28}$$

with $p > 1$ suitable, $\mu \in \mathbb{R}$, and b satisfying the following condition (H):

(H) $b \in C^1(\mathbb{R})$, $b(0) = b'(0) = 0$, and, for any $\tilde{T} < \infty$, $\sharp\{t \in [0,\tilde{T}], b'(t) = 0\} = k < \infty$.

Since we are interested in the time-degenerate case, we assume $k \geq 1$ in condition (H), that is, $b(0) = b'(0) = 0$. However, our results are applicable in the nondegenerate case $b'(t) \neq 0$, $t \in [0,T]$, as well.

Notice that, as for $\mathcal{L}_\alpha$, the solution operator for $\mathcal{L}_b$ (giving the solution of the homogeneous IVP at time t starting at time s) can be computed explicitely, and is given, for $s < t$, by

$$e^{i(b(t)-b(s))\Delta}u_s(x) := W(t,s)u_s(x) := \int_s^t e^{ix\cdot\xi - i(b(t)-b(s))|\xi|^2}\widehat{u_s}(\xi)d\xi,$$

which coincides with the Schrödinger group $e^{i(t-s)\Delta}$ when $b(t) = t$. Moreover, Duhamel's principle still holds true in this case.

As we will make use of the so called *admissible pairs*, we recall this notion here for completeness.

Given $n \geq 1$ we shall call a pair of exponents (q,p) *admissible* if $2 \leq q,p \leq \infty$, and

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \quad \text{with} \quad (q,p,n) \neq (2,\infty,2).$$

With this definition in mind we can now state the main results of this section.

Theorem 7 (Local weighted Strichartz estimates)

Let $b \in C^1([0,T])$ be such that it satisfies condition (H). Then, on denoting by $L^q_t L^p_x := L^q([0,T];L^p(\mathbb{R}^n))$, we have that for any (q,p) admissible pair, with $2 < q,p < \infty$, the following estimates hold

$$\||b'(t)|^{1/q}e^{ib(t)\Delta}\varphi\|_{L^q_t L^p_x} \leq C\|\varphi\|_{L^2_x(\mathbb{R}^n)}, \tag{29}$$

$$\|e^{ib(t)\Delta}\varphi\|_{L^\infty_t L^2_x} \leq \|\varphi\|_{L^2_x(\mathbb{R}^n)}, \tag{30}$$

$$\||b'(t)|^{1/q}\int_0^t |b'(s)|e^{i(b(t)-b(s))\Delta}g(s)ds\|_{L^q_t L^p_x} \leq C\||b'|^{1/q'}g\|_{L^{q'}_t L^{p'}_x}, \tag{31}$$

and

$$\|\int_0^t |b'(s)|e^{i(b(t)-b(s))\Delta}g(s)ds\|_{L^\infty_t L^2_x} \leq C\||b'|^{1/q'}g\|_{L^{q'}_t L^{p'}_x}, \tag{32}$$

with $C = C(k, n, q, p)$.

Remark 5 Observe that, as opposed to the classical statement of Strichartz estimates, that is in the case when $b(t) = t$, we have estimates involving only one admissible pair (q, p), and not two arbitrary admissible pairs (q, p) and $(\tilde{q}, \tilde{p})$. However, this is enough to derive the following well-posedness result.

Theorem 8 *Let* $1 < p < \frac{4}{n} + 1$ *and* $b \in C^1([0, +\infty))$ *satisfying condition (H). Then, for all* $u_0 \in L^2(\mathbb{R}^n)$*, there exists* $T = T(\|u_0\|_2, n, \mu, p) > 0$ *such that there exists a unique solution* u *of the IVP* (28) *in the time interval* $[0, T]$ *with*

$$u \in C([0, T]; L^2(\mathbb{R}^n)) \bigcap L_t^q([0, T]; L_x^{p+1}(\mathbb{R}^n))$$

and $q = \frac{4(p+1)}{n(p-1)}$. *Moreover the map* $u_0 \mapsto u(\cdot, t)$*, locally defined from* $L^2(\mathbb{R}^n)$ *to* $C([0, T); L^2(\mathbb{R}^n))$*, is continuous.*

Proof of Theorem 7 Estimate (30) is immediate and follows by the unitaity of $e^{ib(t)\Delta}$. As for (31), we consider $0 = T_0 \leq T_1 < T_2 < \ldots < T_k \leq T_{k+1} = T$ such that $b'(T_j) = 0$ for $j = 1, \ldots k$, so that b is strictly monotone on $[T_j, T_{j+1}]$, and we have

$$\begin{aligned}
\||b'(t)|^{1/q} e^{ib(t)\Delta}\varphi\|_{L_t^q L_x^p} &= \left(\sum_{j=0}^{k} \||b'(t)|^{1/q} e^{ib(t)\Delta}\varphi\|^q_{L^q([T_j, T_{j+1}]; L_x^p)} \right)^{1/q} \\
&\leq \sum_{j=0}^{k} \||b'(t)|^{1/q} e^{ib(t)\Delta}\varphi\|_{L^q([T_j, T_{j+1}]; L_x^p)} \\
&\underset{b(t)=t'}{\leq} \sum_{j=0}^{k} \|e^{it\Delta}\varphi\|_{L^q([\tilde{T}_j, \tilde{T}_{j+1}]; L_x^p)} \leq (k+1) C(n, q, p) \|\varphi\|_{L_x^2},
\end{aligned}$$

which proves the estimate.

To prove (29) we split the time interval again, and get

$$\begin{aligned}
&\||b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds\|_{L_t^q L_x^p} \\
&\leq \sum_{j=0}^{k} \||b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds\|_{L_t^q([T_j, T_{j+1}]; L_x^p)}. \qquad (33)
\end{aligned}$$

Now, by using the changes of variables $t' = b(t)$ and $s' = b(s)$, each therm in the sum above satisfies

$$\begin{aligned}\| |b'(t)|^{1/q} \int_0^t & |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \|_{L^q_t([T_j, T_{j+1}]; L^p_x)} \\ &\leq \| \int_0^{b(t)} e^{i(t'-s')\Delta} \tilde{g}(s') ds' \|_{L^q_{t'}([\tilde{T}_j, \tilde{T}_{j+1}]; L^p_x)} \\ &= \| \int_0^{b(T)} e^{i(t'-s')\Delta} \chi(s') \tilde{g}(s') ds' \|_{L^q_{t'}([\tilde{T}_j, \tilde{T}_{j+1}]; L^p_x)},\end{aligned} \tag{34}$$

where $\tilde{g} = g \circ b^{-1}$, $\tilde{T}_j = b(T_j)$ and $\chi = 1_{[0, b(t)]}$. We then analyze the last quantity, and, by using the properties of the Schrödinger group $e^{it\Delta}$, we have

$$\begin{aligned}&\| \int_0^{b(T)} e^{i(t'-s')\Delta} \chi(s') \tilde{g}(s') ds' \|_{L^q_{t'}([\tilde{T}_j, \tilde{T}_{j+1}]; L^p_x)} \\ &\leq \| \int_0^{b(T)} \| e^{i(t'-s')\Delta} \chi(s) \tilde{g}(s') \|_{L^p_x} ds' \|_{L^q_{t'}([\tilde{T}_j, \tilde{T}_{j+1}])} \\ &\leq \| \int_0^{b(T)} \frac{1}{|t'-s'|^{n(1/2-1/p)}} \| \chi(s') \tilde{g}(s') \|_{L^{p'}_x} ds' \|_{L^q_{t'}([\tilde{T}_j, \tilde{T}_{j+1}])} \\ \underset{\text{H-L-S}}{\leq} & C(n,q,p) \| \tilde{g} \|_{L^{q'}_{t'}([\tilde{T}_j, \tilde{T}_{j+1}]; L^{p'}_x)} \underset{t=b^{-1}(t')}{\leq} C(n,q,p) \| |b'|^{1/q'} g \|_{L^{q'}_t([T_j, T_{j+1}]; L^{p'}_x)},\end{aligned}$$

where H-L-S stands for the application of the Hardy-Littlewood-Sobolev inequality. Summarizing, we have proved that

$$\begin{aligned}&\| |b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \|_{L^q_t([T_j, T_{j+1}]; L^p_x)} \\ &\lesssim \| |b'|^{1/q'} g \|_{L^{q'}_t([T_j, T_{j+1}]; L^{p'}_x)} \lesssim \| |b'|^{1/q'} g \|_{L^{q'}_t([0,T]; L^{p'}_x)},\end{aligned}$$

which, together with (33), gives

$$\| |b'(t)|^{1/q} e^{ib(t)\Delta} \varphi \|_{L^q_t L^p_x} \leq (k+1) C(n,q,p) \| |b'|^{1/q'} g \|_{L^{q'}_t L^{p'}_x},$$

and thus (31).

We are now left with the proof of (32). By using the fact that $e^{ib(t)\Delta}$ is unitary, we have

$$\begin{aligned}&\| \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \|^2_{L^2_x} = \| \int_0^t |b'(s)| e^{-ib(s))\Delta} g(s) ds \|^2_{L^2_x} \\ &= \int_{\mathbb{R}^n} \left(\int_0^t |b'(s)| e^{-ib(s)\Delta} g(s) ds \right) \overline{\left(\int_0^t |b'(s')| e^{-ib(s')\Delta} g(s') ds' \right)} dx \\ \leq & \int_0^t \| |b'(s)|^{1/q'} g(s) \|_{L^{p'}_x} \| |b'(s)|^{1/q} \int_0^t |b'(s')| e^{i(b(s)-b(s'))\Delta} g(s') ds' \|_{L^p_x} ds\end{aligned}$$

$$\underset{\text{by (31)}}{\leq} (k+1)C(n,q,p)\||b'|^{1/q'} g\|^2_{L_t^{q'} L_x^{p'}},$$

which, in particular, gives (32). This concludes the proof . □

Proof of Theorem 8 The proof is standard and based on the fixed point argument.

Here the space where the contraction argument is performed is

$$X_T := \{u \in C([0,T]; L^2(\mathbb{R}^n) \bigcap L_t^q([0,T]; L_x^{p+1}(\mathbb{R}^n)); \|u\|_{X_T} < \infty\}$$

where

$$\|u\|_{X_T} := \|u\|_{L_t^\infty L_x^2} + \||b'(t)|^{1/q} u\|_{L_t^q L_x^{p+1}},$$

with $L_t^q L_x^p := L^q([0,T]; L_x^p(\mathbb{R}^n))$, and the map Φ_{u_0} is

$$\Phi_{u_0}(u) := e^{ib(t)\Delta} u_0 + \mu \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} |u|^{p-1} u \, ds.$$

Then we take $q = \frac{4(p+1)}{n(p-1)}$ so that $(q, p+1)$ is an admissible pair, and we prove that the map above is a contraction on a suitable ball of X_T (with sufficiently small radius depending on $\|u_0\|_{L_x^2}$) by using the estimates in Theorem 7. Finally, the application of the fixed point theorem gives the reslt. For a detailed proof see [11]. □

We conclude this section by giving a few examples of operators to which Thoerem 8 for the IVP (28) applies.

Example 1 $\mathcal{L}_b = \mathcal{L}_{\frac{t^{\alpha+1}}{\alpha+1}} = \partial_t + it^\alpha \Delta, \quad \alpha \geq 0;$
Example 2 $\mathcal{L}_b = \mathcal{L}_{e^t - t - 1} = \partial_t + i(e^t - 1)\Delta;$
Example 3 $\mathcal{L}_b = \mathcal{L}_{\cos(t)} := \partial_t u - i \sin(t)\Delta.$

Notice that in the first two examples we have only one degenerate point, that is at time $t = 0$. Example 3, instead, is more interesting, since we have $k \geq 1$ degenerate points on any finite time interval $[0, T]$. Since Theorem 8 applies to all the cases listed above, this gives that, if the time of existence T in Theorem 8 is large enough, then in Example 3 we will cross more than one degenrate point.

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Appendix

We use this section to give the statement of a key result used in this paper, that is, specifically, that of the so called Doi's lemma (Lemma 2.3 in [7]). But first, let us make clear the conditions needed to apply the aforementioned lemma.

In the sequel we will use the notations used by Doi in [7], so we shall denote by (B1), (B2) and (A6) the following conditions:

Let $a^w(t,x,\xi)$ be the Weyl symbol of a pseudo-differential operator $A = A(t,x,D_x)$ (see [16]). We shall say that $a^w := a$ satisfies $(B1)$, $(B2)$ and $(A6)$ if

(B1) $a(t,x,\xi) = ia_2(x,\xi) + a_1(t,x,\xi) + a_0(t,x,\xi)$, *where* $a_2 \in S^2_{1,0}$ *is real-valued and* $a_j \in S^j_{1,0}$, for $j = 0,1$;

(B2) $|a_2(x,\xi)| \geq \delta|\xi|^2$ *with* $x \in \mathbb{R}^n$, $|\xi|^2 \geq C$, *and* $\delta, C > 0$;

(A6) *There exists a real-valued function* $q \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *such that, with* $C_{\alpha\beta}, C_1, C_2 > 0$,

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle\langle\xi\rangle^{-|\alpha|}, \quad x,\xi \in \mathbb{R}^n,$$

$$H_{a_2}q(x,\xi) = \{a_2, q\}(x,\xi) \geq C_1|\xi| - C_2, \quad x,\xi \in \mathbb{R}^n,$$

where we denoted by $S^j_{1,0} = S^j_{\rho=1,\delta=0} =: S^j$ *the standard class of pseudo-differential symbols of order* j, *and by* $\{\cdot,\cdot\}$ *the Poisson bracket.*

Lemma 2 (Doi [7], Lemma 2.3)

Assume $(B1)$, $(B2)$ *and* $(A6)$. *Let* $\lambda(s)$ *be a positive non increasing function in* $C([0,\infty))$. *Then*

1. *If* $\lambda \in L^1([0,\infty))$ *there exists a real-valued symbol* $p \in S^0$ *and* $C > 0$ *such that*

$$H_{a_2}p \geq \lambda(|x|)|\xi| - C, \quad x,\xi \in \mathbb{R}^n; \tag{35}$$

2. *If* $\int_0^t \lambda(\tau)d\tau \leq C\log(t+1) + C'$, $t \geq 0$, $C, C' > 0$, *then there exists a real-valued symbol* $p \in S^0_1(\log\langle\xi\rangle)$ *such that*

$$H_{a_2}p \geq \lambda(|x|)|\xi| - C_1\log\langle\xi\rangle - C_2, \quad x,\xi \in \mathbb{R}^n. \tag{36}$$

Remark 6 We remark that, by taking $\lambda'(|x|) = C'\lambda(|x|)$ in Doi's lemma, where C' is any positive constant and λ is as in Lemma 2, then we get that there exists a real-valued symbol $p \in S^0$ and a constant $C > 0$ such that

$$H_{a_2}p \geq C'\lambda(|x|)|\xi| - C, \quad x,\xi \in \mathbb{R}^n. \tag{37}$$

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