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# Minimum length (scale) in quantum field theory, generalized uncertainty principle and the non-renormalisability of gravity

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## ABSTRACT

The notions of minimum geometrical length and minimum length scale are discussed with reference to correlation functions obtained from in-in and in-out amplitudes in quantum field theory. Whereas the in-in propagator for metric perturbations does not admit the former, the in-out Feynman propagator shows the emergence of the latter. A connection between the Feynman propagator of quantum field theories of gravity and the deformation parameter  $\delta_0$  of the generalised uncertainty principle (GUP) is then exhibited, which allows to determine an exact expression for  $\delta_0$  in terms of the residues of the causal propagator. A correspondence between the non-renormalisability of (some) theories (of gravity) and the existence of a minimum length scale is then conjectured to support the idea that non-renormalisable theories are self-complete and finite. The role played by the sign of the deformation parameter is further discussed by considering an implementation of the GUP on the lattice.

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## 1. Introduction

The idea that spacetime is endowed with a minimum (fundamental) length originated as a possible cure for the ultraviolet (UV) divergences of quantum field theory [1], and then regained notoriety with the increasing interest in quantum gravity and trans-Planckian effects (for a comprehensive review, see Ref. [2]). Many candidates for quantum gravity exhibit a minimum length, from string theory to loop quantum gravity. A minimum length, or scale, can also be shown to arise from the standard Feynman path integral for time-ordered in-out amplitudes [3], which are the ingredients for computing S-matrix elements from the Lehmann-Symanzik-Zimmermann formula. However, these amplitudes are acausal and complex since they are subjected to Feynman boundary conditions. An observable minimum length in quantum gravity should be real to arbitrary loop orders and share the statistical properties of an expectation value.

In this respect, it is therefore very important to distinguish between the use of in-out amplitudes and in-in amplitudes [4,5],

the latter being the objects which admit a proper statistical interpretation. These requirements led some of us to study the minimum length using the in-in expectation value in Ref. [6]. The in-in proper distance, which can be directly interpreted as a geometrical length, was also compared with the in-out proper “length”, which cannot be interpreted as a physical distance but sets the length scale of scattering processes: the former was shown to vanish quite generally at the coincidence limit, suggesting that a geometrical minimum length is most likely absent; the latter evaluated at the coincidence limit acquires a finite value of the order of the Planck scale under very general assumptions, indicating that a minimum length scale is very likely to exist. The implication of these results is that nothing prevents one from going, in principle, through vanishingly small distances (spacetime is not “discrete” like a lattice where points are represented by fixed nodes), but scattering experiments cannot reliably distinguish between events taking place at the Planck (length) scale or below, since any two processes differing only at trans-Planckian scales would produce the same scattering amplitudes.

A common approach to investigate the consequences of a minimum length (scale) in quantum mechanics is given by the Generalised Uncertainty Principle(s) (GUPs) [2,7–10]. GUPs, typically derived via *gedanken* experiments, are usually encoded in modified commutators for the canonical observables containing free

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parameter(s) which, in turn, determine the minimum length. Since quantum mechanics emerges in the non-relativistic limit of the one-particle sector of quantum field theory, one can be tempted to draw the origin of the modified quantum mechanical commutators to modified field commutators. However, since the emergence of a minimum length scale does not require any modification of the quantum field dynamics, it appears more natural to assume that the GUP provides an effective description of scattering processes in suitable regimes. The expression for the minimum length scale from Ref. [6] can then be used to determine the parameter(s) of a GUP for given quantum field theories of gravity.

This paper is organized as follows: in Section 2, we briefly review the main results from Ref. [6] and discuss a conjecture connecting renormalisability and (absence of) minimum length; in Section 3 we introduce the simplest example of a GUP and identify the minimum length scale emerging from generic field theories of gravity with the one determined by the GUP; in Section 4 we further support our conjecture by discussing the sign of the deformation parameter when the GUP is formulated on a lattice; in Section 5 we finally draw our conclusions.

## 2. Minimum length scale in scattering processes

In the present section, we briefly review the main results of Ref. [6], where we elaborated a model-independent argument for the absence of a minimum geometrical distance, but the possibility of a minimum length scale.

We first recall that there are different possible boundary conditions in a quantum field theory. The most prominently one used in particle physics is the Feynman boundary condition, which reflects the result of a collision process, scattering an initial state  $|0_{\text{in}}\rangle$  to a final state  $|0_{\text{out}}\rangle$ . Transition amplitudes of this type are complex and do not evolve causally. They are indeed only instrumental to the calculation of cross sections and decay rates, which are the sought after quantities at colliders. Retarded boundary conditions, on the other hand, are required when one is interested in the time evolution of a system (rather than scattering<sup>1</sup>). In this case, amplitudes are evaluated on the same state  $|0_{\text{in}}\rangle$ , yielding retarded (causal) propagators and real correlation functions.

Geometrical lengths are obviously real quantities. When the metric is promoted to an operator for the quantization of gravity, the quantum geometrical length between the points  $x^\mu$  and  $y^\mu = x^\mu + dx^\mu$  must be defined by the in-in amplitude

$$\ell_{\text{in-in}}(x, y) = \sqrt{\langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle}. \quad (2.1)$$

Feynman amplitudes cannot be interpreted geometrically or statistically, but they do provide the scale of the underlying interaction process

$$\ell_{\text{in-out}}(x, y) = \sqrt{\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle}. \quad (2.2)$$

Classically, the line element  $ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu$ , for some fiducial metric  $\bar{g}_{\mu\nu}$ , goes to zero when  $dx^\mu$  vanishes at coincident points. However, this limit in a quantum regime is subtler and turns out to depend on the boundary conditions discussed above. The reason boils down to the analytical structure of quantum amplitudes, which may develop singularities such as poles and branch cuts. In particular, propagators are typically divergent in the coincidence limit  $x^\mu \rightarrow y^\mu$ . These divergences can sometimes cancel

the vanishing classical length in the numerator, leaving out a finite and non-zero contribution. Indeed, adopting the exponential parameterization [11–13]

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\rho} \left( e^{\sqrt{\frac{32\pi\ell_p}{m_p}} h} \right)^\rho_\nu \\ &= \bar{g}_{\mu\nu} + \sqrt{\frac{32\pi\ell_p}{m_p}} h_{\mu\nu} + \frac{16\pi\ell_p}{m_p} h_{\mu\rho} h^\rho_\nu \\ &\quad + O\left((\ell_p/m_p)^{3/2}\right), \end{aligned} \quad (2.3)$$

where  $\ell_p = \sqrt{G_N \hbar}$  and  $m_p = \sqrt{\hbar/G_N}$  are the Planck length and mass, respectively, we find

$$\begin{aligned} \lim_{x \rightarrow y} \ell_{\text{in-}\tau}^2 &= \lim_{x \rightarrow y} (\langle 0_\tau | g_{\mu\nu} | 0_{\text{in}} \rangle dx^\mu dx^\nu) \\ &= \frac{16\pi\ell_p}{m_p} \lim_{x \rightarrow y} [\langle 0_\tau | h_{\mu\rho}(x) h^\rho_\nu(y) | 0_{\text{in}} \rangle dx^\mu dx^\nu] \\ &\equiv \frac{16\pi\ell_p}{m_p} \lim_{x \rightarrow y} [(G^{\text{in-}\tau})_{\mu\rho}{}^\rho{}_\nu(x, y) dx^\mu dx^\nu], \end{aligned} \quad (2.4)$$

where  $\tau \in \{\text{in}, \text{out}\}$  and

$$G_{\mu\nu\rho\sigma}^{\text{ret}} = (G^{\text{in-in}})_{\mu\nu\rho\sigma}, \quad (2.5)$$

$$G_{\mu\nu\rho\sigma}^{\text{F}} = (G^{\text{in-out}})_{\mu\nu\rho\sigma}, \quad (2.6)$$

denote the retarded and the Feynman propagators, respectively.

For simplicity, we shall take  $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ . We stress, however, that our results can be readily generalized to a curved background by adopting normal coordinates or the Schwinger proper-time representation. In momentum space, any free propagator can be written as

$$\Delta_{\mu\nu\rho\sigma}(q^2) = \sum_i \frac{\hbar P_{\mu\nu\rho\sigma}^i}{q^2 - m_i^2}, \quad (2.7)$$

where

$$P_{\mu\nu\rho\sigma}^i = \alpha_i \eta_{\mu\rho} \eta_{\nu\sigma} + \beta_i \eta_{\mu\sigma} \eta_{\nu\rho} + \gamma_i \eta_{\mu\nu} \eta_{\rho\sigma} \quad (2.8)$$

is the most general tensorial structure that can be combined into a tensor of fourth rank and is symmetric in  $\{\mu\nu\}$  and  $\{\rho\sigma\}$ . The propagator is thus parameterized by  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$ , whose values depend on the particular gravitational Lagrangian. Different integration contours result in different boundary conditions, which in practice are easily implemented via the  $i\epsilon$ -prescription. In position space, Eq. (2.7) becomes

$$\begin{aligned} G_{\mu\nu\rho\sigma}^{\text{ret}}(x, y) &= \sum_i \left[ -\frac{\theta(x^0 - y^0)}{2\pi} \delta(\ell^2) \right. \\ &\quad \left. + \theta(x^0 - y^0) \theta(\ell^2) \frac{m_i J_1(m_i \ell)}{4\pi \ell} \right] \hbar P_{\mu\nu\rho\sigma}^i, \end{aligned} \quad (2.9)$$

$$G_{\mu\nu\rho\sigma}^{\text{F}}(x, y) = \sum_i \frac{\hbar P_{\mu\nu\rho\sigma}^i}{4\pi^2 (x - y)^2} + \mathcal{O}(|x - y|), \quad (2.10)$$

where  $\ell^2 \equiv \ell^2(x, y) = \eta_{\mu\nu} dx^\mu dx^\nu$  is the background proper distance. The contraction  $P_{\mu\rho}{}^\rho{}_\nu dx^\mu dx^\nu$  will always result in a factor of  $\ell^2$  in the numerator that can potentially be cancelled by a divergence  $\ell^{-2}$  of the propagator, leaving a non-zero minimum length behind. Eq. (2.4) finally becomes

<sup>1</sup> One might nonetheless notice that all actual measurements are scattering processes (see discussion of the Heisenberg microscope at the beginning of Section 3).

$$\lim_{x \rightarrow y} \ell_{\text{in-}\tau}^2 = \begin{cases} 0 & \tau = \text{in} \\ 0 & \tau = \text{out} \text{ and } \sum_i (\alpha_i + 4\beta_i + \gamma_i) \leq 0 \\ \frac{2}{\pi} \ell_p^2 \sum_i (\alpha_i + 4\beta_i + \gamma_i) \sim \ell_p^2 & \tau = \text{out} \text{ and } \sum_i (\alpha_i + 4\beta_i + \gamma_i) > 0. \end{cases} \quad (2.11)$$

Therefore, quantizing gravity as a quantum field theory shows no sign of a minimum geometrical length, but a minimum Planckian length scale is possible whenever  $\sum_i (\alpha_i + 4\beta_i + \gamma_i) > 0$ . We stress that the above result concerns only general properties of propagators and no mention is made to specific models. The information about particular theories is contained solely in the parameters  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$ .

In general relativity, for example, the massless spin-2 field (graviton) is the only degree of freedom,

$$\hbar^{-1} \Delta_{\mu\nu\rho\sigma} = \frac{\eta_{\rho\mu} \eta_{\sigma\nu} + \eta_{\sigma\mu} \eta_{\rho\nu} - \eta_{\mu\nu} \eta_{\rho\sigma}}{q^2}. \quad (2.12)$$

In this case,  $\sum_i (\alpha_i^{\text{GR}} + 4\beta_i^{\text{GR}} + \gamma_i^{\text{GR}}) = 4$  and a minimum length scale exists which is given by

$$\ell_{\text{in-out}}^{\text{GR}}(x, x) = \sqrt{\frac{8}{\pi}} \ell_p. \quad (2.13)$$

Another interesting example is Stelle's theory [14,15], whose spectrum contains additional degrees of freedom which are needed to prove the renormalisability. In this case, the propagator reads

$$\hbar^{-1} \Delta_{\mu\nu\rho\sigma} = \frac{2 P_{\mu\nu\rho\sigma}^{(2)} - P_{\mu\nu\rho\sigma}^{(0)}}{q^2} - \frac{2 P_{\mu\nu\rho\sigma}^{(2)}}{q^2 - m_2^2} + \frac{P_{\mu\nu\rho\sigma}^{(0)}}{q^2 - m_0^2}, \quad (2.14)$$

where  $P_{\mu\nu\rho\sigma}^{(s)}$  are spin-projection operators, and one can see the additional massive degrees of freedom, namely a scalar excitation of mass  $\hbar m_0$  and a spin-2 particle of mass  $\hbar m_2$ . Surprisingly, due to the accidental cancellations of the parameters

$$\sum_i (\alpha_i^{\text{St}} + 4\beta_i^{\text{St}} + \gamma_i^{\text{St}}) = 0, \quad (2.15)$$

a minimum length scale does not exist, since Eq. (2.11) yields

$$\ell_{\text{in-out}}^{\text{St}}(x, x) = 0. \quad (2.16)$$

One should note that, unlike renormalisable theories that possess no natural scale, non-renormalisable theories always come accompanied by an intrinsic scale  $L_C$  which is then used to define the effective field theory.<sup>2</sup> For lengths  $L \sim L_C$ , the effective field theory breaks down, thus  $L_C$  "blinds" all phenomena below it. Therefore, within a non-renormalisable theory,  $L_C$  serves as a kind of a minimum length scale. In general relativity, which is non-renormalisable, we indeed find the minimum scale  $L_C \sim \ell_p$ , which turns out to be the same scale used to perform the effective field

<sup>2</sup> One should further notice the difference between the role played by the intrinsic cutoff scale in non-renormalisable theories and the scales obtained by dimensional transmutation, such as  $\Lambda_{\text{QCD}} = 220 \text{ MeV}$  in QCD. The former represents the scale where the effective field theory breaks down (and a more fundamental theory must be found). Indeed, effective theories are defined as a series in inverse powers of the cutoff, thus they are only meaningful at scales below it. The latter, on the other hand, separates different regimes of the same theory, which does not break down on either regime except, possibly, at some Landau pole. We remark that the Landau pole of QED is way beyond the Planck scale, thus it is automatically cured by the existence of a minimum length scale.

theory expansion. On the other hand, Stelle's theory is renormalisable and should not (need or) provide any intrinsic scale. The theory indeed knows nothing about the scale where it should fail and, if it were not for the ghost particle, it could be extended to arbitrary scales. Correspondingly, our calculation shows that Stelle's theory possesses no minimum scale. This suggests an interesting interplay, perhaps a correspondence, between the renormalisability of a theory (of gravity) and the non-existence of a minimum length scale.

In a different perspective, like the one assumed in the asymptotic safety scenario [16] and classicalisation [17], one might even argue that the length scale  $L_C$  does not require a new effective theory but that the (effective) theory is self-complete and simply rearranges its degrees of freedom so that no new physics appears in the UV below  $L_C$ . In particular, the minimum length scale  $\ell_p$  in general relativity can be used to treat its corresponding quantum field theory as fundamental, rather than effective, because  $\ell_p$  plays the role of a natural regulator that removes all UV divergences [1,18,19]. Differently from a hard cutoff imposed by hand, which should be removed after renormalisation,  $\ell_p$  remains finite. From this viewpoint, general relativity does not fail at the Planck scale, but rather physics beyond  $\ell_p$  becomes (operationally) meaningless.

### 3. Minimum length scale and GUP

The famous *gedanken* experiment of the Heisenberg microscope [20] shows that scattering processes are generically involved in quantum mechanical measurements. Heisenberg's original idea was to measure position and momentum of a static particle, say an electron, by using a photon as a probe. The photon scatters off the electron, and by measuring the properties of the photon after the scattering, one would like to know the exact position  $x_e$  and momentum  $p_e$  of the electron at the instant of the scattering. However, since the photon has a wavelength  $\lambda$ , from the principles of wave optics follows that the uncertainty in the position of the electron is (at least)  $\Delta x_e \simeq \lambda$ . Moreover, the photon carries a momentum  $p = h/\lambda$ , which, during the scattering, is partially transferred to the electron in an unknown magnitude and direction. This implies that, just after the scattering, the uncertainty in the electron momentum amounts to (at most)  $\Delta p_e \simeq p = h/\lambda$ . Therefore, Heisenberg concluded that

$$\Delta x_e \Delta p_e \simeq \lambda \cdot \frac{h}{\lambda} \simeq h. \quad (3.1)$$

Successively, Schrödinger and Robinson formulated the uncertainty principle for canonically conjugated variables, such as the position  $x$  and momentum  $p$  of a particle, in the form

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \quad (3.2)$$

which is the one commonly accepted today.

Heisenberg's heuristic approach paved the way to the formulation of GUPs [2,7–10], which originate from taking into account the gravitational effects in the photon-particle interaction. For example, (microscopic) black hole formation in the measuring process (or the photon-electron gravitational attraction) implies the existence of a minimum testable length below which position measurements become meaningless. Such GUPs can be mathematically encoded in modified quantum mechanical commutators, and there is the tendency to extend such modification to quantum field theory commutators. However, in the previous section we showed that a minimum length scale can be obtained without modifying the quantum field theory dynamics. This point of view implies that any GUP should emerge effectively in quantum mechanics as the

non-relativistic sector of quantum field theory of gravity without modifying the field propagators.

The simplest form of GUP is given by

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 + \frac{\delta_0}{m_p^2} \Delta p^2 \right), \quad (3.3)$$

where  $\Delta O^2 \equiv \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$  for any quantum observables  $\hat{O}$  and  $\delta_0$  is a dimensionless deforming parameter expected to emerge from candidate theories of quantum gravity. Uncertainty relations can be associated with (fundamental) commutators by means of the general inequality

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|. \quad (3.4)$$

For instance, one can derive Eq. (3.3) from the commutator<sup>3</sup>

$$[\hat{x}, \hat{p}] = i \hbar \left( 1 + \frac{\delta_0}{m_p^2} \hat{p}^2 \right), \quad (3.5)$$

for which Eq. (3.4) yields

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \frac{\delta_0}{m_p^2} (\Delta p^2 + \langle \hat{p} \rangle^2) \right]. \quad (3.6)$$

This immediately implies that the GUP (3.3) holds for any quantum state, since  $\langle \hat{p} \rangle^2 \geq 0$ . In particular, in the centre-of-mass frame of a scattering process, one can just consider the so-called mirror-symmetric states satisfying  $\langle \hat{p} \rangle = 0$ , and the inequality (3.6) coincides with the GUP (3.3).

Eq. (3.3) implies the existence of a minimum (effective) length

$$\ell = \ell_p \sqrt{\delta_0}. \quad (3.7)$$

Therefore, by comparing with Eq. (2.11) from the previous section, we obtain

$$\delta_0 = \frac{2}{\pi} \sum_i (\alpha_i + 4 \beta_i + \gamma_i), \quad (3.8)$$

namely we arrive at an exact expression for the deformation parameter of the GUP which should hold for a general class of gravity theories.

We can estimate  $\delta_0$  for various models. For general relativity, for example, one finds

$$\delta_0^{\text{GR}} = \frac{8}{\pi}, \quad (3.9)$$

whereas for Stelle's theory we have

$$\delta_0^{\text{St}} = 0. \quad (3.10)$$

We recall that, exact values of the deformation parameter  $\delta_0$  had already been obtained in the past from different approaches. For example, it was found that  $\delta_0 = 82\pi/5$  for general relativity in Ref. [21] and  $\delta_0 = 8\pi^2/9$  for models involving a maximal acceleration in Ref. [22]. All available results hence agree in order of magnitude. Experimental upper bounds on  $\delta_0$  exist [23] (see also references therein), but they are typically too weak ( $\delta_0 \lesssim 10^{36}$ ) to provide any useful information about the gravitational propagator. On the other hand, the theoretical value given by Eq. (3.8) can be viewed as a (general) lower bound.

#### 4. Renormalisability and the sign of $\delta_0$

From the discussion at the end of Section 2, one could conjecture a correspondence between the renormalisability of a theory and the absence of a minimum length scale in such a theory. In terms of  $\delta_0$ , according to Eqs. (2.11) and (3.8), such correspondence would require

$$\delta_0 \leq 0 \quad (4.1)$$

for a renormalisable theory of gravity. Interestingly, there are studies which consider this possibility for the GUP [10,24–26].

In particular, the fundamental commutator was computed on a discrete lattice, to our knowledge, for the first time in Ref. [24]. This construct can in principle be viewed as a crystal-like model of our Universe, the so called “world crystal”, when the lattice spacing  $\varepsilon$  is of the order of the Planck length. The commutator then reads

$$[\hat{x}, \hat{p}] = i \hbar \cos \left( \frac{\varepsilon}{\hbar} \hat{p} \right). \quad (4.2)$$

At low energies, or momenta  $|p| \ll \hbar/\varepsilon$ , Eq. (4.2) implies

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 - \frac{\varepsilon^2}{2\hbar^2} \Delta p^2 \right), \quad (4.3)$$

where a negative  $\delta_0 \equiv -\varepsilon^2 m_p^2 / 2\hbar^2 \sim -(\varepsilon/\ell_p)^2$  can be clearly identified. For large momenta approaching  $p \simeq \pi \hbar / 2 \varepsilon \sim m_p (\ell_p / \varepsilon)$ , Eq. (4.2) instead yields

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left( \frac{\pi}{2} - \frac{\varepsilon}{\hbar} \langle \hat{p} \rangle \right) \simeq 0. \quad (4.4)$$

This result shows that no strictly positive lower bound for the uncertainty of two conjugate observables appears when the energy reaches Planckian scale (for  $\varepsilon \sim \ell_p$ ).<sup>4</sup>

The above example allows us to provide an alternative formulation of our conjectured correspondence between renormalisability and the absence of a minimum length scale. In fact, the theory in Ref. [24] is defined on a lattice so as to be finite for any values of  $\varepsilon \geq 0$ . Hence, it is renormalisable (in the limit  $\varepsilon \rightarrow 0$ ) by construction and displays a parameter  $\delta_0 \sim -\varepsilon \leq 0$  corresponding to the absence of a GUP minimum length scale for any values of  $\varepsilon \geq 0$ . Of course, any quantum field theory should be UV finite when regularised on a lattice with a finite step  $\varepsilon > 0$ . However, this does not imply that the theory is renormalisable in the (continuum) limit  $\varepsilon \rightarrow 0$ . According to our conjecture, this should occur if the GUP parameter  $\delta_0$  computed in the regularised theory does not become positive for the (lattice) regulator  $\varepsilon \rightarrow 0$ .

Note that our conjecture makes non-renormalisable theories as good as renormalisable ones. Indeed, the presence of a minimum length scale makes the former self-complete, which does not require new physics beyond the Planck scale. From this perspective, non-renormalisable theories are, in particular, finite and predictive [1,18,19]. On the other hand, we also stress that there is nothing worrisome about a non-positive GUP parameter, thus renormalisable theories remain good candidates for describing fundamental physics.

From a physical point of view, the sign of  $\delta_0$  can give rise to a rich and varied phenomenology. A positive  $\delta_0$  is consistent with results obtained from *gedanken* experiments in high-energy

<sup>3</sup> We stress that the derivation of Eq. (3.3) only makes use of the algebraic structure of the commutator (3.5) through the general inequality (3.4) and no specific representation of the physical operators  $\hat{x}$  and  $\hat{p}$  (in whatsoever form) is needed.

<sup>4</sup> One may say that the world-crystal Universe appears to become “classical” and “deterministic” in this Planck regime.

string scatterings, which also suggest the existence of an effective minimum length. Furthermore, a GUP with a positive deforming parameter can play an important role in the Hawking radiation [27]. From Heisenberg's uncertainty relation (3.2) [corresponding to (3.3) with  $\delta_0 = 0$ ], one can show that the temperature of a spherically symmetric black hole blows up as its mass decreases during the evaporation [28]. This is in agreement with Hawking's original analysis which predicts that black holes should evaporate completely in a finite amount of time by reaching zero mass at infinite temperature. Instead, from a GUP with positive  $\delta_0$ , one finds that the evaporation process would end in a finite time with a remnant of finite mass and finite final temperature [28–30].<sup>5</sup> This result could have significant physical implications as, for example, black hole remnants are considered for potential candidates of dark matter (see, e.g. Ref. [30]). The existence of such remnants would also avoid issues like the information loss problem [37], but would raise the question of their detectability and how to avoid their excessive production in the early universe [38]. On the contrary, an interesting implication of a GUP with  $\delta_0 < 0$  is a finite final Hawking temperature and a zero mass remnant at the end of the evaporation process (see Refs. [24,39]<sup>6</sup>), which would avoid at once difficulties like the entropy/information problem, the remnant detectability issue, or their excessive production. Further evidence in favour of a negative deforming parameter  $\delta_0$  is the fact that this choice would resolve the puzzle of white dwarfs by avoiding white dwarfs of arbitrarily large mass [25]. Finally, it was shown in Ref. [26] that the equivalence between the frameworks of *corpuscular gravity* and GUPs also suggests a deforming parameter  $\delta_0 < 0$ , once the usual energy conservation is imposed.

## 5. Conclusions

In this paper, we first reviewed the idea of a minimum geometrical length in quantum gravity defined by in-in amplitudes obtained via the Schwinger-Keldysh formalism [6]. The in-in quantum proper distance can be interpreted as a truly geometrical length that happens to be real at all loop orders and satisfies a causal equation of motion. At coinciding points, the in-in proper length goes to zero at second order for any metric theory of gravity, a result that extends to all orders in perturbation theory as long as non-gravitational interactions can be neglected.

On the contrary, a minimum length scale arises from the in-out amplitudes used to derive the standard Feynman rules and propagators, and its value depends on parameters determined by the particular quantum field theory of gravity considered. We stress that this conclusion is a general, model-independent property. For general relativity, viewed as a non-renormalisable field theory, our analysis implies the existence of a minimum length scale of the order of the Planck length, as expected. In Stelle's theory of grav-

<sup>5</sup> A final positive temperature may sound puzzling. However, it has been shown within the GUP scenario that the specific heat of the remnant drops to zero when its mass reaches the minimum value (see, e.g. Refs. [29,31,32]). This result signals that any thermodynamic interaction of the black hole with the environment stops when the minimum mass is reached. It should then be possible to infer that the remnant temperature drops swiftly to zero in the very last instants of the black hole life. On this specific aspect, we can add that the evaporation models studied for example in Refs. [33,34] are more refined than those from the GUP (with  $\delta_0 > 0$ ), as they analytically predict not only a final finite mass of the remnant, but also a final temperature equal to zero. Lastly, we should notice that a vanishing final temperature agrees with the requirement of total energy conservation realised by employing a microcanonical description of the Hawking radiation [35] and black hole microstates [36].

<sup>6</sup> There, it was shown that the evaporation ends with final zero mass, when the temperature approaches a maximum positive value, at which point the specific heat also drops to zero [see Eqs. (72)–(73) in Ref. [24]]. Besides, a vanishing total energy is a feature of the classical model of point-like particles introduced in Ref. [40] (for its quantum version, see Refs. [41]).

ity, which is renormalisable, we instead showed the absence of a minimum length scale. An interesting interplay between the non-renormalisability of a (gravitational) theory and the existence of a minimum length scale thus seems to emerge, the latter being the embedded (self-)cure for the former [1].

Minimum length scales are also widely described through the so called GUPs. The minimum length scales derived from GUPs and those derived from in-out amplitudes should be the same. This suggests a deep connection between the parameters of quantum theories of gravity and the deforming parameter of GUPs emerging in the non-relativistic limit of quantum field theory. Using this identification, we find a GUP deforming parameter  $\delta_0 > 0$  and of order unity (in Planck length) for general relativity, consistently with previous evaluations [21,22] and with some models of string theory [8]. The GUP deforming parameter for Stelle's theory instead vanishes. As another example in support of the conjecture that a minimum length scale is induced by the non-renormalisability of a field theory, we discussed the GUP formulated on a lattice, where a negative deformation parameter emerges at low energy and vanishes at high energy [24]. The fact that  $\delta_0 \leq 0$  and no minimum length scale exists agrees with the expectation that any field theory (of gravity) must be UV finite, hence renormalisable, if defined on a lattice. On the other hand, when the lattice acts as a regulator for UV divergences, we expect that the theory is renormalisable if the GUP parameter is not positive in the continuum limit.

The connection between quantum field theories and the deforming GUP parameter  $\delta_0$  could be used, in principle, to rule out some theories by measuring the value of the deforming parameter from experiments. Unfortunately, current experimental upper bounds on  $\delta_0$  are still way too weak to provide any useful information on the underlying gravitational theory [23].

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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