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INVERSE STATIC ANALYSIS OF A PLANAR SYSTEM WITH FLEXURAL PIVOTS

Marco Carricato
Department of Mechanical Engineering - DIEM
University of Bologna
Viale Risorgimento 2, 40136 Bologna, Italy
Tel. +39 051 2093459, Fax +39 051 2093446
E-mail: marco.carricato@mail.ing.unibo.it

Joseph Duffy
Center for Intelligent Machines and Robotics
University of Florida
Gainesville, FL 32611, USA
Tel. +1 352 3920814, Fax +1 352 3921071
E-mail: cimar@cimar.me.ufl.edu

Vincenzo Parenti-Castelli
Department of Mechanical Engineering - DIEM
University of Bologna
Viale Risorgimento 2, 40136 Bologna, Italy
Tel. +39 051 2093459, Fax +39 051 2093446
E-mail: vincenzo.parenticastelli@mail.ing.unibo.it

ABSTRACT
This article presents the inverse static analysis of a two degrees of freedom planar mechanism with flexural pivots. Such analysis aims to detect the entire set of equilibrium configurations of the system once the external load is assigned. The presence of flexural pivots represents a novelty, although it remarkably complicates the problem since it causes the two state variables to appear in the solving equations as arguments of both trigonometric and linear functions. The proposed procedure eliminates one variable and leads to two equations in one unknown only. The union of the root sets of such equations constitutes the global set of solutions of the problem. Particular attention is paid to the analysis of the reliability of the final equations: critical situations, in which the solving equations may hide solutions or yield false ones, are studied. Finally, a numerical example is provided and, in the Appendix, a special design that offers computational advantages is proposed.

INTRODUCTION
In the design of conventional rigid-body mechanisms compliance is usually considered undesirable, since it introduces positional errors and vibrations. On the contrary, compliant mechanisms are intentionally flexible and achieve some or all of their motion and force transmission capabilities by virtue of the elastic deformation of some of their components.

Compliant mechanisms have been traditionally employed in mechanics as mounting and suspension devices, but interesting and promising new applications have been recently highlighted. In the field of robotics, the interposition of a passive compliant system between the end link and the end-effector of a robot enhances performance in many manipulation tasks (e.g. insertions) and represents a valuable strategy for the simultaneous control of force and displacement of the end-effector [1-3]. Flexible-link mechanisms may also serve as jointless monolithic substitutes to their rigid-body counterparts, providing simplification of assembly, diminution of lubrication need, reduction of weight, wear and backlash [4]. Such benefits render them suitable for micro-electro-mechanical systems, instrumentation, adaptive devices and special purpose appliances [5].

The kinematics of compliant mechanisms is load-dependent and affected by non-linearities introduced by large deflections of the elastic elements: analysis and synthesis are resultantly hampered [6,5]. Several approaches have been proposed. The pseudo-rigid-body model treats flexible links as an assemblage of rigid parts connected by spring-equipped joints, so that standard rigid-body kinematic analysis methods may be applied [7]. In particular, link bending pliability is simulated by means of *flexural pivots*, defined as compliant parts (very short relative to the adjoining rigid segments) allowing a rotational degree of freedom with a certain torsional stiffness [8].

Relevant results have been obtained in the *closed-form inverse static analysis* (i.s.a.) of compliant mechanisms, i.e., in the search for the entire set of equilibrium configurations that the system possesses once the acting load is given. The i.s.a. of a planar device consisting of two springs acting in parallel and connected to a common pivot has been performed by Pigoski and Duffy [9] and Hines et al. [10], whilst a tetrahedral three-spring system has been solved by Dietmaier [11] and Zhang et al. [12]; a special planar three-spring platform has been studied by Sun et al. [13]. In all cases, only *axially compliant segments* [8] have been taken into account.

This article presents for the first time the i.s.a. of a two degrees of freedom planar mechanism with *flexural pivots*. These are schematized as pin joints restrained by torsional springs (R and P stand for prismatic and revolute pairs respectively throughout the paper). As shown in Fig. 1, the device is made up of two RPR serial chains, grounded at one end and connected at the other one to a common pivot **Q**, on which a load **F** is applied. The torsional springs are installed in the grounded joints. The state variables are assumed to be the rotation angles θ_1 and θ_2, which completely determine the system configuration.

The loop-closure relations and the link equilibrium conditions thoroughly define the statics of the mechanism and yield a coupled system of two non-linear equations in two unknowns (θ_1 and θ_2). The authors' purpose is not to conceive the most efficient algorithm that numerically solves the problem, but rather to find a closed-form solution, that is a final equation containing no more than one variable possibly expressed as an explicit function of the external load and the construction parameters. Indeed, the presented procedure manages to eliminate one variable and yield a set of two independent equations in one unknown only (θ_2). The union of the root sets of such equations constitutes the global set of solutions of the problem (which are a countable infinity). These equations are too complicated to be expressed in an explicit form, mainly depending on the appearance of the unknown as an argument of both trigonometric and linear terms. In fact, while the geometry of the mechanism causes the state variables to appear as arguments of trigonometric functions, the statics of the springs produces terms that are linear in these same variables (the torque exerted by a torsional spring is proportional to its rotation): equations in which the same variables appear as arguments of both trigonometric and linear functions have in general no explicit-form solution. Thus, the obtained implicit equations are believed to express the closest result to a closed-form solution of the problem.

Particular attention has been paid to a problem that often arises when an elimination procedure is applied: equation manipulation can bring about undesirable consequences, since it may lead to final equations that do not recognize some configurations as solutions or regard some others as solutions while they are not. These configurations, hereafter named *critical*, can be reasonably located through the analysis of the elimination procedure and the mechanism's geometrical features. The behaviour of the solving equations in such critical configurations can be studied by confronting the results they offer in these special cases with the true solutions, found by performing the static analysis by means of locally-written equations of equilibrium and geometric congruence. In the discussed case, the final solving equations prove to be adequately robust, in the sense that they yield the correct solutions in the most critical circumstances, except for the *initial configuration* (when no load is applied), in which an ad-hoc set of equations is needed to thoroughly solve the problem.

Finally, the Appendix shows an interesting attribute that the device possesses if construction parameters are chosen according to a *symmetric design*. In this case, the solutions of the problem when the applied load is $[-F_x, F_y]$ directly descend from the solutions derivable when the load is $[F_x, F_y]$. This result is useful, since it allows one to limit the analysis to the cases for which F_x is positive.

1. DESCRIPTION OF THE MECHANISM

Figure 1 provides a schematic of the mechanism. As previously anticipated, it consists of two limbs, each one of which is an RPR serial chain. The limbs are grounded at one end and connected at the other one to a common pivot **Q**, on which a load **F** is applied. **Q** is also meant to be the pivot's position vector in the reference frame shown in the figure: this vector is uniquely determined by the angles θ_1 and θ_2 that the limbs form with the x-axis. The system may remain in equilibrium by virtue of the torsional springs installed in the grounded revolute pairs, which are flexural pivots. The spring constants are k_1 and k_2. The torques τ_1 and τ_2 are meant to be those exerted by the limbs upon the springs. $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces that the pivot transmits to the limbs; because of the P-pairs, $\mathbf{F}_1$ and $\mathbf{F}_2$ must be necessarily perpendicular to the link axes. By definition, l_1 and l_2 are the limb lengths, determined, except for their sign, by the distance between the centers of the two R-pairs of each serial chain. Since all joints are assumed ideal, the sliders are able to move indefinitely parallel to the P-pairs. In particular, when they move along the opposite direction with respect to that on which the links physically lie, the limb lengths are regarded as negative. The distance d between the centers of the flexural pivots is assigned. Link masses and friction are neglected. For the sake of convenience, the following arrays are explicitly defined:

$$\mathbf{Q}=\begin{bmatrix}x & y\end{bmatrix}^{\mathrm{T}}, \qquad \mathbf{Q}_0=\begin{bmatrix}x_0 & y_0\end{bmatrix}^{\mathrm{T}}$$
$$\boldsymbol{\theta}=\begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^{\mathrm{T}}, \qquad \boldsymbol{\theta}_0=\begin{bmatrix}\theta_{10} & \theta_{20}\end{bmatrix}^{\mathrm{T}} \tag{1}$$
$$\mathbf{F}=\begin{bmatrix}F_x & F_y\end{bmatrix}^{\mathrm{T}}, \quad \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}=\begin{bmatrix}k_1(\theta_1-\theta_{10})\\ k_2(\theta_2-\theta_{20})\end{bmatrix}$$

$\mathbf{Q}_0$ and $\boldsymbol{\theta}_0$ are the values that $\mathbf{Q}$ and $\boldsymbol{\theta}$ assume when the springs are not loaded: in this case, the mechanism is said to lie in its *unloaded configuration*.

The trigonometric functions of θ_1 and θ_2 are written using the following compact notation: $\sin = s$, $\cos = c$, $\tan = t$, $\theta_1 = \text{subscript } 1$, $\theta_2 = \text{subscript } 2$. For example, s_1 stands for $\sin(\theta_1)$ and $s_{1+1\text{-}2}$ stands for $\sin(\theta_1+\theta_1-\theta_2)=\sin(2\theta_1-\theta_2)$.

2. LOOP-CLOSURE AND EQUILIBRIUM EQUATIONS

The vector loop-closure equation of the mechanism, projected on the x and y axes, provides the following two scalar equations:

$$l_1c_1 = l_2c_2 + d \tag{2}$$

$$l_1s_1 = l_2s_2 \tag{3}$$

and after some manipulation:

$$l_1 = d\frac{s_2}{s_{2-1}} \tag{4}$$

$$l_2 = d\frac{s_1}{s_{2-1}} \tag{5}$$

On the other hand, the equilibrium of the moments about the grounded pivot of each limb yields:

$$F_1 = \frac{k_1(\theta_1-\theta_{10})}{l_1} \tag{6}$$

$$F_2 = \frac{k_2(\theta_2-\theta_{20})}{l_2} \tag{7}$$

Substituting Eqs. (4) and (5) in Eqs. (6) and (7) respectively gives:

$$F_1 = \beta_1\frac{s_{2-1}}{s_2} \tag{8}$$

$$F_2 = \beta_2\frac{s_{2-1}}{s_1} \tag{9}$$

where:

$$\beta_1 = \frac{k_1}{d}(\theta_1-\theta_{10}), \qquad \beta_2 = \frac{k_2}{d}(\theta_2-\theta_{20}) \tag{10}$$

The equilibrium equations of the pivot $\mathbf{Q}$ are:

$$F_x = -F_1s_1 - F_2s_2 \tag{11}$$

$$F_y = F_1c_1 + F_2c_2 \tag{12}$$

Substituting Eqs. (8) and (9) in Eqs. (11) and (12) provides:

$$F_x = -\beta_1 s_{2-1} \frac{s_1}{s_2} - \beta_2 s_{2-1} \frac{s_2}{s_1} \tag{13}$$

$$F_y = \beta_1 s_{2-1} \frac{c_1}{s_2} + \beta_2 s_{2-1} \frac{c_2}{s_1} \tag{14}$$

Equations (13) and (14) describe the statics of the mechanism and represent a system of two coupled equations in two unknowns (θ_1 and θ_2).

3. THE SOLUTION OF THE INVERSE STATIC PROBLEM

Multiplying Eq. (14) by $t_1=s_1/c_1$ (provided $s_1\neq 0$ and $c_1\neq 0$) and adding it to Eq. (13) yields the relation:

$$F_x + F_y t_1 = \beta_2 s_{2-1} \left(-\frac{s_2}{s_1} + \frac{c_2}{c_1} \right) \tag{15}$$

which can be rearranged as follows:

$$\left(F_y + \beta_2 c_2^2\right) t_1^2 + \left(F_x - 2\beta_2 s_2 c_2\right) t_1 + \beta_2 s_2^2 = 0 \tag{16}$$

Multiplying Eq. (14) by $t_2=s_2/c_2$ (provided $s_2\neq 0$ and $c_2\neq 0$), adding it to Eq. (13) and rearranging in a similar way gives instead:

$$\left(F_y - \beta_1 c_1^2\right) t_2^2 + \left(F_x + 2\beta_1 s_1 c_1\right) t_2 - \beta_1 s_1^2 = 0 \tag{17}$$

Equation (16) is a second-degree equation in t_1 whose coefficients are a function of θ_2 only. On the contrary, Eq. (17) is a second-degree equation in t_2 whose coefficients are a function of θ_1 only. Solving the two equations for t_1 and t_2 yields respectively:

$$t_1 = \frac{-F_x + 2\beta_2 s_2 c_2 \pm \sqrt{\Delta_2}}{2\left(F_y + \beta_2 c_2^2\right)} \tag{18}$$

$$t_2 = \frac{-F_x - 2\beta_1 s_1 c_1 \pm \sqrt{\Delta_1}}{2\left(F_y - \beta_1 c_1^2\right)} \tag{19}$$

where:

$$\Delta_1 = F_x^2 + 4\beta_1 s_1 \left(F_x c_1 + F_y s_1\right) \tag{20}$$

$$\Delta_2 = F_x^2 - 4\beta_2 s_2 \left(F_x c_2 + F_y s_2\right) \tag{21}$$

Equation (18) gives two expressions of θ_1 as a function of θ_2:

$$\theta_{1,a} = \arctan\left(t_{1,a}\right) + n\pi \tag{22}$$

$$\theta_{1,b} = \arctan\left(t_{1,b}\right) + n\pi \tag{23}$$

where:

$$
\begin{aligned}
t_{1,a} &= \frac{-F_x + 2\beta_2 s_2 c_2 + \sqrt{\Delta_2}}{2\left(F_y + \beta_2 c_2^2\right)} \\
t_{1,b} &= \frac{-F_x + 2\beta_2 s_2 c_2 - \sqrt{\Delta_2}}{2\left(F_y + \beta_2 c_2^2\right)} \\
n &\in \varnothing = \{\ldots, -2, -1, 0, 1, 2, \ldots\}
\end{aligned}
\tag{24}
$$

Substituting Eqs. (22) and (23) in Eq. (19) yields four equations in one unknown (θ_2). The global set of solutions of the problem consists of the union of the root sets of two of these equations only, since the other ones do not give meaningful roots, as will now be proved.

Equations (18) and (19) can be rearranged as follows:

$$2\left(F_y + \beta_2 c_2^2\right) t_1 + F_x - 2\beta_2 s_2 c_2 = \pm\sqrt{\Delta_2} \tag{25}$$

$$2\left(F_y - \beta_1 c_1^2\right) t_2 + F_x + 2\beta_1 s_1 c_1 = \pm\sqrt{\Delta_1} \tag{26}$$

Let **i** and **j** be the unit vectors of the axes x and y and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the unit vectors of the axes of the limbs. In the equilibrium configurations the following relations are valid (see Section 2):

$$
\begin{aligned}
& F_x = F_{1x} + F_{2x}, \quad F_y = F_{1y} + F_{2y} \\
& \tau_1 = F_1 l_1, \quad \tau_2 = F_2 l_2 \\
& \mathbf{F}_1 \cdot \mathbf{u}_1 = 0, \quad \mathbf{F}_2 \cdot \mathbf{u}_2 = 0 \\
& \frac{s_{2-1}}{d} = \frac{s_2}{l_1} = \frac{s_1}{l_2}
\end{aligned}
\tag{27}
$$

where:

$$
\begin{aligned}
F_{1x} &= \mathbf{F}_1 \cdot \mathbf{i} = -F_1 s_1, \quad F_{1y} = \mathbf{F}_1 \cdot \mathbf{j} = F_1 c_1 \\
F_{2x} &= \mathbf{F}_2 \cdot \mathbf{i} = -F_2 s_2, \quad F_{2y} = \mathbf{F}_2 \cdot \mathbf{j} = F_2 c_2
\end{aligned}
\tag{28}
$$

After denoting the left-hand sides of Eqs. (25) and (26) by R_1 and R_2 respectively and by using Eq. (27), it is possible to write:

$$
\begin{aligned}
c_1 R_1 &= 2\left(F_y + \beta_2 c_2^2\right) s_1 + F_x c_1 - 2\beta_2 s_2 c_2 c_1 = \\
&= -2\frac{\tau_2}{d} c_2 s_{2-1} + 2F_y s_1 + F_x c_1 = \\
&= -2F_{2y}\frac{l_2 s_{2-1}}{d} + 2F_y s_1 + F_x c_1 = \\
&= 2\left(F_y - F_{2y}\right) s_1 + F_{1x} c_1 + F_{2x} c_1 = \\
&= 2\mathbf{F}_1 \cdot \mathbf{u}_1 + \left(F_{2x} - F_{1x}\right) c_1 = \\
&= \left(F_{2x} - F_{1x}\right) c_1
\end{aligned}
\tag{29}
$$

and:

$$\begin{aligned}
c_2R_2 &= 2\left(F_y-\beta_1c_1^2\right)s_2+F_xc_2+2\beta_1s_1c_1c_2= \\
&=-2\frac{\tau_1}{d}c_1s_{2-1}+2F_ys_2+F_xc_2= \\
&=-2F_{1y}\frac{l_1s_{2-1}}{d}+2F_ys_2+F_xc_2= \\
&=2\left(F_y-F_{1y}\right)s_2+F_{1x}c_2+F_{2x}c_2= \\
&=2\mathbf{F}_2\cdot\mathbf{u}_2-\left(F_{2x}-F_{1x}\right)c_2= \\
&=-\left(F_{2x}-F_{1x}\right)c_2
\end{aligned} \tag{30}$$

Comparing Eqs. (29) and (30) yields:

$$R_1=-R_2=F_{2x}-F_{1x} \tag{31}$$

When $F_{2x}=F_{1x}$, the discriminants of Eqs. (18) and (19) are zero, so that the four mentioned equations coincide. When $F_{2x}\neq F_{1x}$, Eq. (31) states that the left-hand sides of Eqs. (25) and (26) must have opposite signs; since the equations that derive from expressions (18) and (19), if the same sign is chosen for both, cannot satisfy this condition, they cannot provide solutions. Therefore the only two meaningful solving equations are:

$$t_2=\frac{-F_x-2\beta_{1,a}s_{1,a}c_{1,a}-\sqrt{\Delta_{1,a}}}{2\left(F_y-\beta_{1,a}c_{1,a}^2\right)} \tag{32}$$

$$t_2=\frac{-F_x-2\beta_{1,b}s_{1,b}c_{1,b}+\sqrt{\Delta_{1,b}}}{2\left(F_y-\beta_{1,b}c_{1,b}^2\right)} \tag{33}$$

By means of the identity $c_1^2=1/\left(1+t_1^2\right)$, these relations can be written using t_1 as the only trigonometric function:

$$t_2+\frac{1}{2}\frac{F_xT_{1,a}^2+2\beta_{1,a}t_{1,a}+T_{1,a}\sqrt{\Delta'_{1,a}}}{F_yT_{1,a}^2-\beta_{1,a}}=0 \tag{34}$$

$$t_2+\frac{1}{2}\frac{F_xT_{1,b}^2+2\beta_{1,b}t_{1,b}-T_{1,b}\sqrt{\Delta'_{1,b}}}{F_yT_{1,b}^2-\beta_{1,b}}=0 \tag{35}$$

where:

$$\beta_{1,a}=\frac{k_1}{d}\left(\theta_{1,a}-\theta_{10}\right),\qquad \beta_{1,b}=\frac{k_1}{d}\left(\theta_{1,b}-\theta_{10}\right) \tag{36}$$

$$T_{1,a}=\left(1+t_{1,a}^2\right)^{1/2},\qquad T_{1,b}=\left(1+t_{1,b}^2\right)^{1/2} \tag{37}$$

$$\Delta'_{1,a}=T_{1,a}^2\Delta_{1,a}=F_x^2T_{1,a}^2+4\beta_{1,a}t_{1,a}\left(F_x+F_yt_{1,a}\right) \tag{38}$$

$$\Delta'_{1,b}=T_{1,b}^2\Delta_{1,b}=F_x^2T_{1,b}^2+4\beta_{1,b}t_{1,b}\left(F_x+F_yt_{1,b}\right) \tag{39}$$

Equations (34) and (35) yield real roots only in the intervals in which θ_2 satisfies the inequality:

$$\Delta_2=F_x^2-4\beta_2s_2\left(F_xc_2+F_ys_2\right)\geq 0 \tag{40}$$

For Eq. (34) and Eq. (35) it must also be respectively:

$$\Delta'_{1,a}=F_x^2T_{1,a}^2+4\beta_{1,a}t_{1,a}\left(F_x+F_yt_{1,a}\right)\geq 0 \tag{41}$$

$$\Delta'_{1,b} = F_x^2 T_{1,b}^2 + 4\beta_{1,b} t_{1,b} \left(F_x + F_y t_{1,b} \right) \geq 0 \tag{42}$$

4. CRITICAL CONFIGURATIONS

In this section, attention will be focused on a problem that often arises when an elimination procedure is applied. Manipulating equations is a very delicate operation, since incautious handling may bring about undesirable consequences. Let the elimination procedure previously shown be considered.

Since the trigonometric tangents of θ_1 and θ_2 are involved, it has been assumed from the outset $c_1 \neq 0$ and $c_2 \neq 0$, that is $\theta_1 \neq \pi/2 + h\pi$ and $\theta_2 \neq \pi/2 + k\pi$ ($h, k \in \mathbb{Z}$). Nevertheless, for some values of the external force, these can actually be equilibrium configurations of the mechanism. Furthermore, the expressions (18) and (19) are valid only if the quadratic equations (16) and (17) preserve their degree, that is:

$$F_y + \beta_2 c_2^2 \neq 0, \qquad F_y - \beta_1 c_1^2 \neq 0 \tag{43}$$

Therefore, when the external load is zero, Eqs. (34) and (35) should not be able to furnish the unloaded configuration as a solution of the problem, since in this case $F_y=0$ and $\beta_1 = \beta_2 = 0$ and condition (43) is not satisfied. Briefly, the elimination process may produce final equations that are not able to recognize some configurations as solutions of the problem.

The contrary may even happen: there can be configurations regarded as solutions while they are not. For instance, Eqs. (16) and (17) have been obtained from Eq. (14) by multiplying it respectively by t_1 and t_2. This operation is mathematically valid only if t_1 and t_2 are different from zero, because on the contrary the equations would be turned into trivial identities. In fact, Eqs. (16) and (17) are always satisfied if $\theta_1 = h\pi$ and $\theta_2 = k\pi$ ($h, k \in \mathbb{Z}$), regardless of the value assumed by the external load. If these "roots" persist in the final equations, they would certainly be *extraneous*, since, in this condition, both limbs would lie on the x-axis and no value of F_x other than zero could be counterbalanced.

In conclusion, the analysis cannot be really considered complete until these situations are studied in detail.

The adopted approach involves the following steps:

a) the reasonable identification of the critical configurations through the analysis of both the elimination procedure and the geometric peculiarities of the mechanism;
b) the study of the critical configurations by means of locally-written loop-closure and equilibrium relations: the static analysis is locally solved by a specific set of equations;
c) the study of the "behaviour" of Eqs. (34) and (35) in the critical configurations, in order to verify the correspondence between the solutions they offer and those obtained in the previous step.

For the presented mechanism, the critical configurations deserving to be taken into consideration are:

1. the initial configurations ($\mathbf{F} = \mathbf{0}$);
2. at least one limb lies on the x-axis (θ_1 or θ_2 is a multiple of π);
3. at least one limb lies in its unloaded configuration ($\theta_1 = \theta_{10}$ or $\theta_2 = \theta_{20}$);
4. at least one limb is parallel to the y-axis (θ_1 or θ_2 is an odd multiple of $\pi/2$).

The following exemplifying configurations will be analyzed here in detail:

1. the initial configurations;
2. the 2nd limb (which is the one whose length is l_2) lies on the x-axis;
3. the 1st limb (which is the one whose length is l_1) is parallel to the y-axis.

A complete analysis can be found in [14].

4.1. Initial Configurations

Posing

$$F_x = F_y = 0 \tag{44}$$

in Eqs. (34) and (35) yields:

$$t_2 = t_1 \tag{45}$$

corresponding to *any* configuration in which the two limbs are parallel. Of course, this is not the correct solution.

When $\mathbf{F}=\mathbf{0}$, the mechanism may remain in equilibrium only if:

$$\mathbf{F}_1=\mathbf{F}_2=\mathbf{0} \tag{46}$$

or

$$\mathbf{F}_1=-\mathbf{F}_2 \tag{47}$$

In the occurrence (46), if the limb lengths are required to remain finite, it must be $\tau_1=\tau_2=0$, which means $\theta_1=\theta_{10}$ and $\theta_2=\theta_{20}$. The mechanism thus lies in its unloaded configuration. This is one solution.

In the circumstance (47), the limbs must be necessarily parallel and this can happen (demanding that l_1 and l_2 be finite) only when they lie on the *x*-axis, that is when $\theta_1=h\pi$, $\theta_2=k\pi$ (h, $k \in \mathbb{Z}$). In this case, the equilibrium and loop-closure equations become:

$$F_1 l_1 = k_1\left(h\pi\text{-}\theta_{10}\right) \tag{48}$$

$$F_2 l_2 = k_2\left(k\pi\text{-}\theta_{20}\right) \tag{49}$$

$$p_h F_1 + p_k F_2 = 0 \tag{50}$$

$$p_h l_1 = d + p_k l_2 \tag{51}$$

where $p_h=\cos\left(h\pi\right)$ and $p_k=\cos\left(k\pi\right)$. Equations (48) through (51), properly rearranged, solve the problem:

$$l_1 = p_h d\left[1+\frac{k_2}{k_1}\frac{\left(k\pi-\theta_{20}\right)}{\left(h\pi-\theta_{10}\right)}\right]^{-1} \tag{52}$$

$$l_2 = -p_k d\left[1+\frac{k_1}{k_2}\frac{\left(h\pi-\theta_{10}\right)}{\left(k\pi-\theta_{20}\right)}\right]^{-1} \tag{53}$$

$$F_1 = \frac{p_h}{d}\left[k_1\left(h\pi-\theta_{10}\right)+k_2\left(k\pi-\theta_{20}\right)\right] \tag{54}$$

$$F_2 = -\frac{p_k}{d}\left[k_1\left(h\pi-\theta_{10}\right)+k_2\left(k\pi-\theta_{20}\right)\right] \tag{55}$$

Hence, there is a countable infinity of initial configurations lying on the *x*-axis, one for each pair of values (h, k). It is thus reasonable to expect an infinite number of solutions also when a non-zero load is applied.

4.2. The 2nd Limb Lies On The *x*-axis

In this condition the 1st limb can assume only two positions:

A. if $l_1\neq 0$, it must lie on the *x*-axis;
B. if $l_1=0$ (which implies $\tau_1=0$ and hence $\theta_1=\theta_{10}$), it must assume its unloaded configuration.

Case A. Since both limbs lie on the *x*-axis, it is:

$$\theta_1 = h\pi,\quad \theta_2 = k\pi \qquad \left(h,k\in\mathbb{Z}\right) \tag{56}$$

Let:

$$\begin{aligned}\beta_1=\beta_{1h}&=\frac{k_1}{d}\left(h\pi-\theta_{10}\right), &\quad \beta_2=\beta_{2k}&=\frac{k_2}{d}\left(k\pi-\theta_{20}\right)\\ p_h&=\cos\left(h\pi\right), & p_k&=\cos\left(k\pi\right)\end{aligned} \tag{57}$$

Equations (2), (6), (7), (11) and (12) yield respectively:

$$p_h l_1 = p_k l_2 + d \tag{58}$$

$$F_1 l_1 = d\beta_{1h} \tag{59}$$

$$F_2 l_2 = d\beta_{2k} \tag{60}$$

$$F_x = 0 \tag{61}$$

$$F_y = p_h F_1 + p_k F_2 \tag{62}$$

After some manipulation Eqs. (58) through (62) can be arranged in the form:

$$F_2^2 + p_k\left(\beta_{1h} + \beta_{2k} - F_y\right)F_2 - \beta_{2k}F_y = 0 \tag{63}$$

$$F_1 = p_h\,\beta_{1h}\,\frac{F_2}{F_2 + p_k\,\beta_{2k}} \tag{64}$$

$$l_1 = d\,\frac{\beta_{1h}}{F_1} \tag{65}$$

$$l_2 = d\,\frac{\beta_{2k}}{F_2} \tag{66}$$

If condition (61) is satisfied, Eqs. (63) through (66) yield, in sequence, the values of all the unknowns of the problem, in particular l_1 and l_2.

Case B. In this case it must be:

$$\theta_1 = \theta_{10}, \quad \theta_2 = k\pi \qquad (k \in \emptyset) \tag{67}$$

Let:

$$\beta_2 = \beta_{2k} = \frac{k_2}{d}\left(k\pi - \theta_{20}\right), \qquad p_k = \cos\left(k\pi\right) \tag{68}$$

Equations (3), (2), (6), (7), (11) and (12) yield respectively:

$$l_1 = 0 \tag{69}$$

$$l_2 = -p_k d \tag{70}$$

$$F_1 l_1 = 0 \tag{71}$$

$$F_2 l_2 = d\beta_{2k} \tag{72}$$

$$F_x = -F_1 s_{10} \tag{73}$$

$$F_y = F_1 c_{10} + p_k F_2 \tag{74}$$

Substituting Eq. (69) in Eq. (71) gives an identity. Substituting Eq. (70) in Eq. (72) and rearranging Eq. (73) yields respectively:

$$F_2 = -p_k\beta_{2k} \tag{75}$$

$$F_1 = -\frac{F_x}{s_{10}} \tag{76}$$

The mechanism may be in equilibrium only if the external load satisfies the following relation (obtained by substituting Eqs. (75) and (76) in Eq. (74)):

$$F_y = -\frac{F_x}{t_{10}} - \beta_{2k} \tag{77}$$

that is:

$$t_{10} = -\frac{F_x}{F_y + \beta_{2k}} \tag{78}$$

Equation (77) (or Eq. (78)) represents a family of lines parameterized by the integer k in the plane (F_x, F_y).

Equations (34) and (35) must yield, for θ_2 equal to a multiple of π ($s_2=t_2=0$), the same results obtained in the above discussion (for the sake of convenience, it will be assumed $F_x>0$).

Equation (40) yields:

$$\Delta_2 = F_x^2 \tag{79}$$

and therefore, from Eq. (24):

$$t_{1,a} = 0 \quad \Rightarrow \quad \theta_{1,a} = h\pi \quad \left(h \in \o\right) \tag{80}$$

$$t_{1,b} = -\frac{F_x}{F_y + \beta_{2k}} \tag{81}$$

Equation (34) becomes:

$$F_x = 0 \tag{82}$$

If the x-component of the external load is zero, Eq. (82) is satisfied, which means that Eq. (34) offers the solution examined in Case A: $\theta_1=h\pi$, $\theta_2=k\pi$.

If the load satisfies the condition (78), Eq. (81) yields:

$$t_{1,b} = t_{10} \quad \Rightarrow \quad \theta_{1,b} = \theta_{10} + n\pi \quad \left(n \in \o\right) \tag{83}$$

Hence, for $n=0$, Eq. (35) becomes the identity $0=0$, which means that it provides the root discussed in Case B: $\theta_1=\theta_{10}$, $\theta_2=k\pi$.

4.3. The 1st Limb Is Parallel To The *y*-axis

This condition means:

$$\theta_1 = \frac{\pi}{2} + h\pi, \quad \left(h \in \o\right) \tag{84}$$

Let:

$$\beta_{1h} = \frac{k_1}{d}\left(\frac{\pi}{2} + h\pi - \theta_{10}\right), \quad q_h = \sin\left(\frac{\pi}{2} + h\pi\right) \tag{85}$$

Equations (4), (5), (6), (7), (11) and (12) yield respectively:

$$l_1 = -q_h t_2 d \tag{86}$$

$$l_2 = -\frac{d}{c_2} \tag{87}$$

$$F_1 l_1 = d\beta_{1h} \tag{88}$$

$$F_2 l_2 = d\beta_2 \tag{89}$$

$$F_x = -q_h F_1 - F_2 s_2 \tag{90}$$

$$F_y = F_2 c_2 \tag{91}$$

Substituting Eqs. (86) and (87) in Eqs. (88) and (89) respectively gives:

$$F_1 = -\frac{q_h \beta_{1h}}{t_2} \tag{92}$$

$$F_2 = -\beta_2 c_2 \tag{93}$$

Substituting Eqs. (92) and (93) in Eqs. (90) and (91) yields:

$$F_x = \beta_{1h} \frac{c_2}{s_2} + \beta_2 s_2 c_2 \tag{94}$$

$$F_y = -\beta_2 c_2^2 \tag{95}$$

Finally, substituting for β_2 in Eq.(94) by using Eq. (95) provides:

$$F_y t_2^2 + F_x t_2 - \beta_{1h} = 0 \tag{96}$$

and hence:

$$t_2 = \frac{-F_x \pm \sqrt{F_x^2 + 4\beta_{1h} F_y}}{2F_y} \tag{97}$$

If a value of θ_2 exists as calculated from Eq. (97) which also satisfies Eq. (95), the resulting configuration is of equilibrium. Equations (94) and (95) are the parametric equations of a family of curves in the plane (F_x , F_y).

As in Section 4.2, it must be verified that Eqs. (34) and (35) yield the correct results.

θ_1 is an odd multiple of $\pi/2$ when the condition (95) is satisfied. In fact, in this case, Eq. (24) gives:

$$t_{1,a}, t_{1,b} \xrightarrow{F_y + \beta_2 c_2^2 \to 0} \infty \tag{98}$$

Equations (34) and (35) may be written in the form:

$$t_2 + \frac{1}{2} \frac{F_x\left(1+t_1^2\right) + 2\beta_1 t_1 \pm \sqrt{\left(1+t_1^2\right)\left[F_x^2\left(1+t_1^2\right) + 4\beta_1 t_1\left(F_x + F_y t_1\right)\right]}}{F_y\left(1+t_1^2\right) - \beta_1} = 0 \tag{99}$$

and furthermore:

$$t_2 + \frac{1}{2} \frac{F_x\left(1+\frac{1}{t_1^2}\right) + \frac{2\beta_1}{t_1} \pm \sqrt{\left(1+\frac{1}{t_1^2}\right)\left[F_x^2\left(1+\frac{1}{t_1^2}\right) + 4\beta_1\left(\frac{F_x}{t_1} + F_y\right)\right]}}{F_y\left(1+\frac{1}{t_1^2}\right) - \frac{\beta_1}{t_1^2}} = 0 \tag{100}$$

It is easy to verify that, for $t_1 \to \infty$, the limit of Eq. (100) is:

$$t_2 + \frac{F_x \pm \sqrt{F_x^2 + 4\beta_{1h} F_y}}{2F_y} = 0 \tag{101}$$

which is precisely the condition (97).

In conclusion, Eqs. (34) and (35) yield, as a limit result, even those solutions corresponding to configurations in which they should not be defined.

5. NUMERICAL EXAMPLE

Equations (34) and (35) can be numerically solved when both the constructive parameters (θ_{10}, θ_{20}, k_1, k_2, d) and the external forces (F_x and F_y) are assigned. Since an infinite number of solutions is expected, it is necessary to establish for θ_1 and θ_2 the intervals $\Im_1$ and $\Im_2$ inside which the solution search must be undertaken. The solving process is explained by the flowchart shown in Fig. 2.

Assume for the parameters the values:

$$d = 1, \quad k_1 = k_2 = 1, \quad \theta_{10} = \pi/4, \quad \theta_{20} = 3\pi/4 \tag{102}$$

and for the external forces:

$$F_x = F_y = 1 \tag{103}$$

Solutions are sought for in the intervals:

$$\Im_1 = [-3\pi/2, \quad 3\pi/2], \quad \Im_2 = [-\pi, \quad 2\pi] \tag{104}$$

for for θ_1 and θ_2 respectively.

All angular measures are expressed in radians. Symbols are defined in Fig. 2.

Equation (40) yields:

$$\Re_2 = [\text{-}\pi, \text{-}0.694770551] \cup [\text{-}0.115437215, 3.354019954] \cup [5.421717332, 2\pi]$$

For each value of n congruent with $\Im_1$, the following results are obtained:

- n = -1:

$$\Re_a = [\text{-}\pi, \text{-}2.97008859] \cup [\text{-}0.10182787, 0.23666567] \cup$$
$$\cup [2.25774281, 3.35401995] \cup [5.42171733, 2\pi]$$

$$\Re_b = [\text{-}0.96159000, \text{-}0.70412344] \cup [1.94209336, 3.35401995] \cup [5.42171733, 2\pi]$$

Roots of Eq. (34):

Sol. 1:
θ_2 = -0.098250, θ_1 = -3.088551
l_2 = -0.351767, l_1 = -0.650844

Sol. 2:
θ_2 = 2.906706, θ_1 = -3.166194
l_2 = -0.117847, l_1 = -1.114948

Sol. 3:
θ_2 = 5.832329, θ_1 = -3.369956
l_2 = 1.025937, l_1 = -1.974687

Roots of Eq. (35):

Sol. 4:
θ_2 = -0.706374, θ_1 = -3.971703
l_2 = -5.979608, l_1 = 5.259097

Sol. 5:
θ_2 = 2.538590, θ_1 = -3.921321
l_2 = 3.999187, l_1 = 3.225793

Sol. 6:
θ_2 = 5.927975, θ_1 = -3.712021
l_2 = -2.528515, l_1 = 1.62851

- n = 0:

$$\Re_a = [\text{-}\pi, \text{-}0.99267626] \cup [\text{-}0.11543721, 3.35401995] \cup [5.42171733, 2\pi]$$

$$\Re_b = [\text{-}\pi, \text{-}2.06634555] \cup [\text{-}1.12357750, \text{-}0.69477055] \cup [\text{-}0.11543721, 0.68600903] \cup$$
$$\cup [1.72259823, 3.35401995] \cup [5.42171733, 2\pi]$$

Roots of Eq. (34):

Sol. 1:

θ_2 = -0.695420, θ_1 = -0.915962
l_2 = -3.625690, l_1 = -2.928845

Sol. 2:

θ_2 = 2.606478, θ_1 = -0.056403
l_2 = -0.122383, l_1 = 1.107036

Sol. 3:

θ_2 = 5.582940, θ_1 = -0.446726
l_2 = 1.722473, l_1 = 2.569282

Roots of Eq. (35):

Sol. 4:

θ_2 = -0.086686, θ_1 = 0.355748
l_2 = -0.813498, l_1 = 0.202217

Sol. 5:

θ_2 = 2.796696, θ_1 = -0.721559
l_2 = 1.795871, l_1 = -0.919199

Sol. 6:

θ_2 = 6.228409, θ_1 = -0.277121
l_2 = -1.240658, l_1 = -0.248269

- $n = 1$:

$$\Re_a = [-\pi, -0.69477055] \cup [-0.11543721, 3.35308166] \cup \cup [5.42171733, 5.45954400] \cup [5.97521529, 2\pi]$$

$$\Re_b = [-\pi, -0.69477055] \cup [-0.11543721, 3.01772757] \cup [5.42171733, 5.87868841]$$

Roots of Eq. (34):

Sol. 1:

θ_2 = -0.986385, θ_1 = 1.579267
l_2 = -1.836065, l_1 = 1.531402

Sol. 2:

θ_2 = -1.565809, θ_1 = 4.141602
l_2 = -1.545457, l_1 = -1.836579

Sol. 3:

θ_2 = 1.134536, θ_1 = 3.556239
l_2 = 0.611050, l_1 = -1.374694

Sol. 4:

θ_2 = 3.352960, θ_1 = 3.011869
l_2 = 0.386707, l_1 = -0.627168

Sol. 5:

θ_2 = 5.437815, θ_1 = 2.490308
l_2 = 3.143090, l_1 = -3.879382

Roots of Eq. (35):

Sol. 6:

θ_2 = -2.452849, θ_1 = 4.354682
l_2 = 1.870982, l_1 = 1.269491

Figure 3 shows all the solutions in the plane (x, y). The mechanism, in its unloaded configuration, is drawn in dashed line.

CONCLUSIONS

This article has addressed the inverse static analysis of a two degrees of freedom planar mechanism with flexural pivots. The presence of flexural pivots renders such analysis interestingly novel, since only systems with axially compliant segments have been hitherto studied.

The union of the root sets of two solving equations in which only one state variable appears is the global set of the problem's solutions. Such equations are believed to express the closest result to a closed-form solution.

Since the adopted elimination procedure may induce the solving equations not to recognize some configurations as solutions of the problem or, on the contrary, to regard some others as solutions while they are not, particular attention has been paid to the analysis of the critical configurations, in order to prove the reliability of the solving equations. A strategy to carry out this analysis has been presented.

Finally, a particular design that yields some computational advantages has been described in the Appendix.

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APPENDIX

The mechanism is designed in a *symmetric* way when the construction parameters satisfy the following conditions:

$$k_1 = k_2 = k \tag{A.1}$$

$$\theta_{10} = \theta_0, \quad \theta_{20} = \pi - \theta_0 \tag{A.2}$$

In this case, if $S=[F_x, F_y, \theta_1, \theta_2]$ is a solution, then $S'=[-F_x, F_y, \pi-\theta_2, \pi-\theta_1]$ is a solution also. This result is useful, since it allows one to limit the analysis to the field of values for which $F_x>0$.

The demonstration is straightforward. If, by hypothesis, S is a solution, the following relations are all true:

$$l_1 = d\frac{s_2}{s_{2-1}}, \quad l_2 = d\frac{s_1}{s_{2-1}} \tag{A.3}$$

$$F_1 l_1 = k\left(\theta_1 - \theta_0\right), \quad F_2 l_2 = k\left(\theta_2 - \pi + \theta_0\right) \tag{A.4}$$

$$F_x = -F_1 s_1 - F_2 s_2, \quad F_y = F_1 c_1 + F_2 c_2 \tag{A.5}$$

Let the following values now be assumed for the state variables:

$$\theta_1' = \pi - \theta_2, \quad \theta_2' = \pi - \theta_1 \tag{A.6}$$

The limb lengths become:

$$l_1' = d\frac{s_1}{s_{2-1}}, \quad l_2' = d\frac{s_2}{s_{2-1}} \tag{A.7}$$

By confronting Eq. (A.7) with Eq. (A.3), it is clearly:

$$l_1' = l_2, \quad l_2' = l_1 \tag{A.8}$$

Analogously, the forces transmitted by the pivot become:

$$F_1' = \frac{k\left(\pi - \theta_2 - \theta_0\right)}{l_2}, \quad F_2' = \frac{k\left(-\theta_1 + \theta_0\right)}{l_1} \tag{A.9}$$

and hence:

$$F_1' = -F_2, \quad F_2' = -F_1 \tag{A.10}$$

According to Eqs. (11) and (12), S' is an equilibrium configuration only if the following equations are verified:

$$-F_x = -F_1' s_1' - F_2' s_2', \qquad F_y = F_1' c_1' + F_2' c_2' \tag{A.11}$$

By using the positions (A.6) and (A.10), the system (A.11) can be written as:

$$-F_x = F_2 s_2 + F_1 s_1, \qquad F_y = F_2 c_2 + F_1 c_1 \tag{A.12}$$

which is surely verified, since Eq. (A.5) is true by assumption.

It would also not be difficult to prove that Eqs. (34) and (35), written for the configuration *S*', are respectively identical to Eqs. (35) and (34) written for the configuration *S*.

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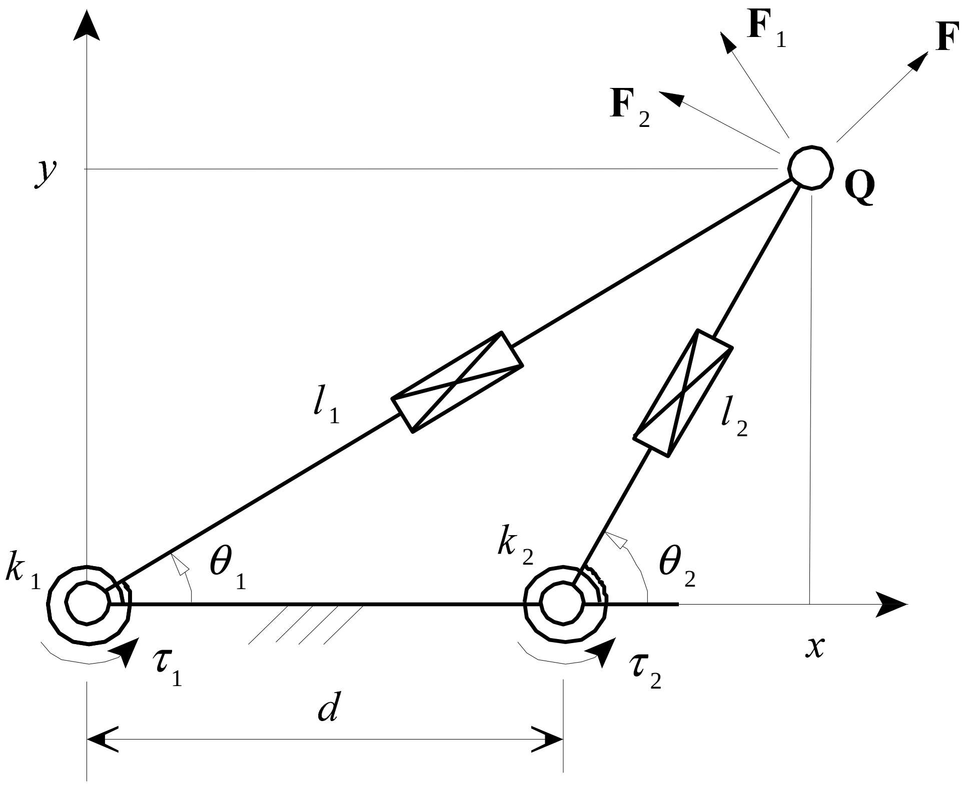

Fig. 1: Schematic of the compliant mechanism

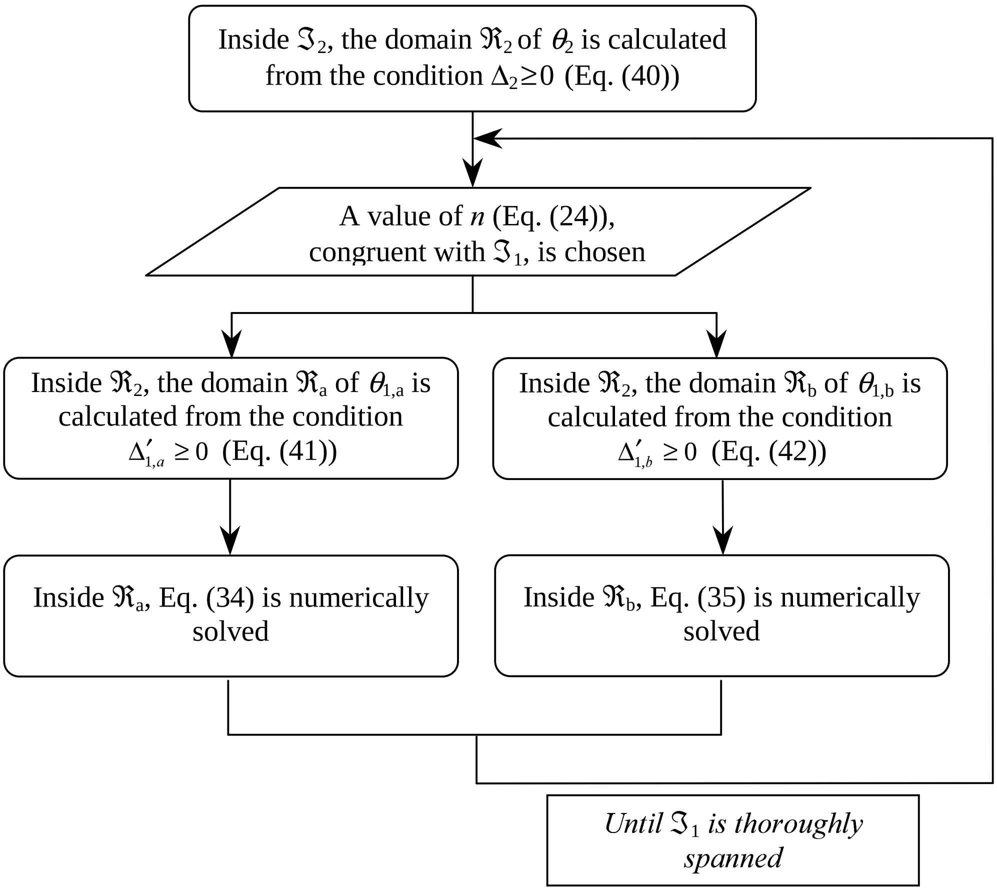

Fig. 2: Numerical example: flowchart of the solving process

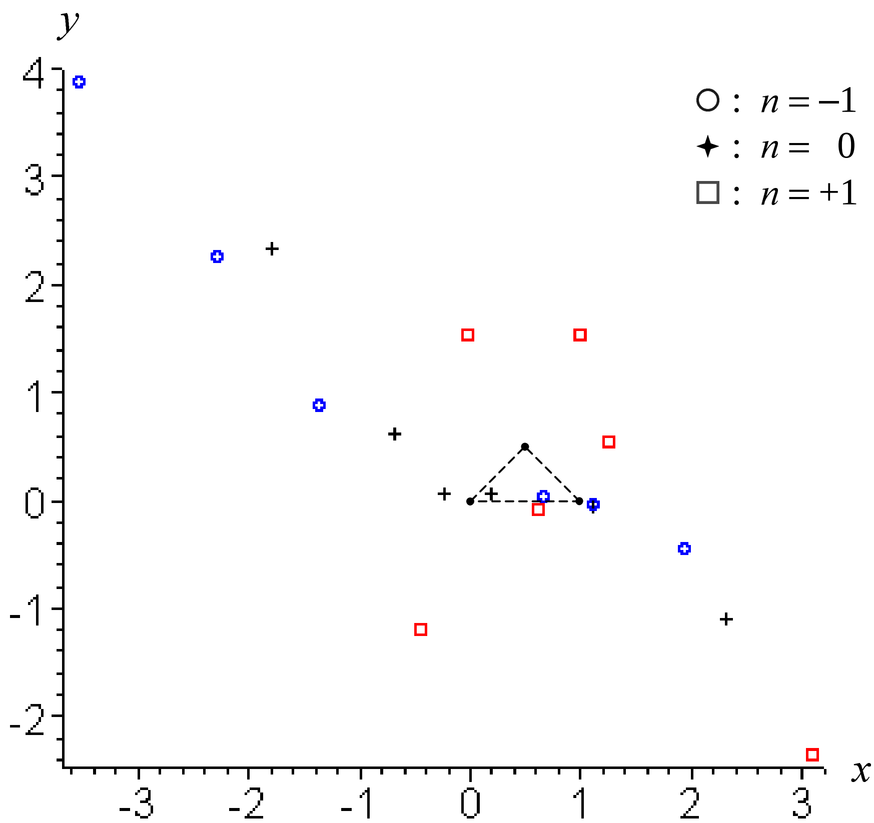

Fig. 3: Numerical example: solutions in the plane (x, y)