Mirror Symmetry and smoothing Gorenstein toric affine 3-folds

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Mirror Symmetry and Smoothing Gorenstein toric affine 3-folds

Alessio Corti, Matej Filip and Andrea Petracci

Abstract

We state two conjectures that together allow one to describe the set of smoothing components of a Gorenstein toric affine 3-fold in terms of a combinatorially defined and easily studied set of Laurent polynomials called 0-*mutable polynomials*. We explain the origin of the conjectures in mirror symmetry and present some of the evidence.

1.1 Introduction

We explore mirror symmetry for smoothings of a 3-dimensional Gorenstein toric affine variety V. Specifically, we try to imagine what consequences mirror symmetry may have for the classification of smoothing components of the deformation space $\operatorname{Def} V$. Conjecture A makes the surprising statement that the set of smoothing components of $\operatorname{Def} V$ is in bijective correspondence with a set of easily defined and enumerated 2-variable Laurent polynomials, called 0-mutable polynomials. Our Conjecture B — in the strong form stated in Remark 1.4.2 — asserts that these smoothing components are themselves smooth, and computes their tangent spaces from the corresponding 0-mutable polynomials.

As is customary in toric geometry, V is associated to a strictly convex 3-dimensional rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, where N is a 3-dimensional lattice; the Gorenstein condition means that the integral generators of the rays of σ all lie on an integral affine hyperplane $(u = 1)$ for some $u \in M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We denote by F the convex hull of the integral generators of the rays of σ, i.e.

$$F := \sigma \cap (u = 1);$$

this is a lattice polygon (i.e. a lattice polytope of dimension 2) embedded in the affine 2-dimensional lattice $(u = 1)$. The isomorphism class of the toric variety V depends only on the affine equivalence class of the polygon F.

If V has an isolated singularity, then $\operatorname{Def} V$ is finite dimensional and we know from the work of Altmann [4] that there is a 1-to-1 correspondence between the set of irreducible components of $\operatorname{Def} V$ and integral maximal Minkowski decompositions of F. Altmann also shows that, when taken with their *reduced* structure, these components are all themselves smooth.

We are interested in the case when V has non-isolated singularities. Very little is known at this level of generality, but examples show that the picture for non-isolated singularities is very different from the one just sketched for isolated singularities. Our main reason for wanting to work with non-isolated singularities is the Fanosearch project: we wish to prove a general criterion for smoothing a toric Fano variety, and Conjecture A here is just the local case.

Conjecture A characterizes *smoothing components* of $\operatorname{Def} V$ in terms of the combinatorics of the polygon F. Specifically, we define the set $\mathfrak{B}$ of 0-mutable Laurent polynomials with Newton polygon F, and the conjecture states that there is a canonical bijective correspondence $\kappa\colon \mathfrak{B} \to \mathfrak{A}$, where $\mathfrak{A}$ is the set of smoothing components of $\operatorname{Def} V$.

At first sight the formulation of the conjecture seems strange; however, the statement makes sense in the context of mirror symmetry, where (conjecturally) the 0-mutable polynomials are the mirrors of the corresponding smoothing components.

In Section 1.4, we state a new Conjecture B,[1] which implies the existence of a map $\kappa\colon \mathfrak{B} \to \mathfrak{A}$ — see Remark 1.4.1. In that section, we also explain how to (conjecturally) construct a deformation directly from a 0-mutable polynomial in the spirit of the intrinsic mirror symmetry of Gross–Siebert [19, 18] and work of Gross–Hacking–Keel [21].

The coefficients of the 0-mutable Laurent polynomials that appear in our conjecture ought themselves to enumerate certain holomorphic discs in the corresponding smoothing, and we would love to see a precise statement along these lines.

In our view, the conjectures together are nothing other than a statement of mirror symmetry as a one-to-one correspondence between two sets of objects, similar to the conjectures made in [2] in the context of orbifold del Pezzo surfaces, and the correspondence between Fano 3-folds and Minkowski polynomials discovered in [10].

In Section 1.3 we give some equivalent characterisations of 0-mutable polynomials and begin to sketch some of their general properties. These properties make it very easy to enumerate the 0-mutable polynomials with given Newton polygon. The material here is rather sketchy — full details will appear elsewhere; it serves for context, but it is not logically necessary for the statement of the conjectures.

The suggestion that there is a simple structure to the set of smoothing components is surprising in a subject that — as all serious practitioners know — is marred by Murphy's law. In fact, there is a substantial body of direct and circumstantial evidence for the conjectures, some of which we present in Section 1.5.

In the final Section 1.6 we compute in detail the deformation space of the variety V_F associated to the polygon F of Example 1.2.12, giving evidence for the conjectures. Some of the reasons for choosing this particular example are:

(1) The variety V_F is of codimension 5 and hence it lies outside the —

[1] The statement of Conjecture B comes after Sec. 1.3 but does not logically depend on it: if you wish, you can skip directly from Sec. 1.2.3 to Sec. 1.4.1.

still rudimentary but very useful — structure theory of codimension-4 Gorenstein rings [30], see also [9, 8];

(2) For this reason, V_F is a good test of the technology of [12, 13] as a tool for possibly proving the conjectures;

(3) The polygon F appears as a facet of some of the 3-dimensional reflexive polytopes and hence it is immediately relevant for the Fanosearch project.

As things stand, we are some distance away from being able to prove the conjectures. We had a tough time even with the example of Section 1.6: while a treatment based on [12, 13] seems possible, the task became so tedious that we decided instead to rely on Ilten's Macaulay2 [17] package *Versal deformations and local Hilbert schemes* [25]. That package makes it possible to test the conjectures in many other examples in codimension ≥ 5.

Notation and conventions

We work over $\mathbb{C}$, but everything holds over an algebraically closed field of characteristic zero. We refer the reader to [15] for an introduction to toric geometry. All the toric varieties we consider are normal. We use the following notation.

F	a lattice polygon
V	the Gorenstein toric affine 3-fold associated to the cone over F put at height 1
∂V	the toric boundary of V
$\overline{X}$	the projective toric surface associated to the normal fan of F
$\overline{B}$	the toric boundary of $\overline{X}$
A	the ample line bundle on $\overline{X}$ given by F
X	the cluster surface associated to F (the non-toric blowup of $\overline{X}$ constructed in Section 1.3)
B	the strict transform of $\overline{B}$ in X
$W^\flat$	the toric 3-fold constructed in Section 1.4
$\overline{W}$	the toric blowup of $W^\flat$ constructed in Section 1.4
W	the mirror cluster variety (the non-toric blowup of $\overline{W}$ constructed in Section 1.4)

Thanks to Bill from AC

I was lucky to be a L.E. Dickson Instructor at the University of Chicago in the years 1993–1996, where I worked in the group led by Bill Fulton. I had studied deformations of singularities in a seminar held in 1988–89 at the University of Utah, but in Bill's seminar I learnt about Chow groups and quantum cohomology. It was a wonderful time in my work life thanks largely to Bill. This paper features many of the ideas that I learnt in Bill's seminar and it is very nice to see that they are so relevant in the study of deformations of singularities.

Acknowledgements

We thank Klaus Altmann, Tom Coates, Mark Gross, Paul Hacking, Al Kasprzyk, Giuseppe Pitton, and Thomas Prince for many helpful conversations.

We particularly thank Al Kasprzyk for sharing with us and allowing us to present some of his unpublished ideas on maximally mutable Laurent polynomials [26].

We discussed with Mark Gross some of our early experiments with 0-mutable polynomials.

We owe special thanks to Paul Hacking who read and corrected various mistakes in earlier versions of the paper — the responsibility for the mistakes that are left is of course ours.

Giuseppe Pitton ran computer calculations that provide indirect evidence for the conjectures.

The idea that mirror Laurent polynomials are characterised by their mutability goes back to Sergey Galkin [16].

It will be clear to all those familiar with the issues that this paper owes a very significant intellectual debt to the work of Gross, Hacking and Keel [21] and the intrinsic mirror symmetry of Gross and Siebert [19, 18].

Last but not least, it is a pleasure to thank the anonymous referee who read our manuscript very carefully, and told us about Ilten's Macaulay2 package.

1.2 Conjecture A

1.2.1 Gorenstein toric affine varieties

Consider a rank-n lattice $N \simeq \mathbb{Z}^n$ (usually $n = 3$) and, as usual in toric geometry, its dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. The n-dimensional torus

$$\mathbb{T} = \operatorname{Spec} \mathbb{C}[M]$$

is referred to simply as "the" torus.[2] Consider a strictly convex full-dimensional rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ and the corresponding affine toric variety

$$V = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M].$$

This is a normal Cohen–Macaulay n-dimensional variety.

By definition V is Gorenstein if and only if the pre-dualising sheaf

$$\omega_V^0 = \mathcal{H}^{-n}(\omega_V^\bullet)$$

is a line bundle. Since our V is Cohen–Macaulay, this is the same as insisting that all the local rings of V are local Gorenstein rings. It is known and not difficult to show that V is Gorenstein if and only if there is a vector $u \in M$ such that the integral generators $\rho_1, \ldots, \rho_m$ of the rays of the cone σ all lie on the affine lattice $\mathbb{L} = (u = 1) \subset N$. Such vector u is called the *Gorenstein degree*.

If V is Gorenstein, then the toric boundary of V is the following effective reduced Cartier divisor on V:

$$\partial V = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]/(x^u).$$

If V is Gorenstein, we set

$$F := \sigma \cap (u = 1);$$

this is an $(n-1)$-dimensional lattice polytope embedded in the affine lattice $\mathbb{L} = (u = 1)$ and it is the convex hull of the integral generators of the rays of the cone σ. One can prove that the isomorphism class of V depends only on the affine equivalence class of the lattice polytope F in the affine lattice $\mathbb{L}$. Therefore we will say that V is associated to the polytope F.

This establishes a 1-to-1 correspondence between isomorphism classes of Gorenstein toric affine n-folds without torus factors and $(n-1)$-dimensional lattice polytopes up to affine equivalence.

[2] Sometimes we denote this torus by $\mathbb{T}_N$, that is, the commutative group scheme $N \otimes_{\mathbb{Z}} \mathbb{G}_{\mathrm{m}}$ such that for all rings R $\mathbb{T}_N(R) = N \otimes_{\mathbb{Z}} R^\times$.

1.2.2 Statement of Conjecture A

In this section we explain everything that is needed to make sense of the following:

Conjecture A. *Consider a lattice polygon F in a 2-dimensional affine lattice $\mathbb{L}$. Let V be the Gorenstein toric affine 3-fold associated to F.*

Then there is a canonical bijective function $\kappa\colon \mathfrak{B} \to \mathfrak{A}$ where:

- *$\mathfrak{A}$ is the set of smoothing components of the miniversal deformation space* $\operatorname{Def} V$*,*
- *$\mathfrak{B}$ is the set of 0-mutable polynomials $f \in \mathbb{C}[\mathbb{L}]$ with Newton polygon F.*

Remark 1.2.1. It is absolutely crucial to appreciate that we are not assuming that V has an isolated singularity at the toric 0-stratum. If V does not have isolated singularities, $\operatorname{Def} V$ is infinite-dimensional. A few words are in order to clarify what kind of infinite dimensional space $\operatorname{Def} V$ is.

In full generality, there is some discussion of this issue in the literature on the analytic category, see for example [23, 24].[3] In the special situation of interest in this paper, we take a naïve approach, which we briefly explain, based on the following two key facts:

(i) If V is a Gorenstein toric affine 3-fold, then T^2_V is finite dimensional. Indeed V has transverse $A_\star$-singularities in codimension two, hence it is unobstructed in codimension two, hence T^2_V is a finite length module supported on the toric 0-stratum. In fact, there is an explicit description of T^2_V as a representation of the torus, see [6, Section 5], an example of which is in Lemma 1.6.4 below. This shows that $\operatorname{Def} V$ is cut out by finitely many equations.

(ii) On the other hand, the known explicit description of T^1_V as a representation of the torus [5, Theorem 4.4], together with the explicit description of T^2_V just mentioned, easily implies that each of the equations can only use finitely many variables.[4]

Thus we can take $\operatorname{Def} V$ to be the Spec of a non-Noetherian ring, that is, the simplest kind of infinite-dimensional scheme.[5]

[3] We thank Jan Stevens for pointing out these references to us. We are not aware of a similar discussion in the algebraic literature.

[4] The key point is to show that for all fixed weights $\mathbf{m} \in M$, the equation $\mathbf{m} = \sum m_i \mathbf{v}_i$ has finitely many solutions for $m_i \in \mathbb{N} \setminus \{0\}$ and $\mathbf{v}_i \in M$ a weight that appears non-trivially in T^1_V. This type of consideration is used extensively in the detailed example discussed in Section 1.6.

[5] The situation is not so simple for the universal family $\mathscr{U} \to \operatorname{Def} V$. Indeed the

Remark 1.2.2. Conjecture A does not state what the function κ is, nor what makes it "canonical." The existence of a function $\kappa\colon \mathfrak{B} \to \mathfrak{A}$ is implied by Conjecture B stated in Section 1.4.1 below, see Remark 1.4.1.

Remark 1.2.3. Let V be a Gorenstein toric affine 3-fold and let ∂V be the toric boundary of V. Let $\operatorname{Def}(V, \partial V)$ be the deformation functor (or the base space of the miniversal deformation) of the pair $(V, \partial V)$. There is an obvious forgetful map $\operatorname{Def}(V, \partial V) \to \operatorname{Def} V$. In this case, since ∂V is an effective Cartier divisor in V and V is affine, this map is smooth of relative dimension equal to the dimension of coker $\big(H^0(\theta_V) \to H^0(N_{\partial V/V})\big)$, where θ_V is the sheaf of derivations on V and $N_{\partial V/V} = \mathcal{O}_{\partial V}(\partial V)$ is the normal bundle of ∂V inside V. In particular, this implies that $\operatorname{Def} V$ and $\operatorname{Def}(V, \partial V)$ have exactly the same irreducible components. One can also see that a smoothing component in $\operatorname{Def} V$ is the image of a component of $\operatorname{Def}(V, \partial V)$ where also ∂V is smoothed.

In other words, we can equivalently work with deformations of V or deformations of the pair $(V, \partial V)$. The right thing to do in mirror symmetry is to work with deformations of the pair $(V, \partial V)$; however, the literature on deformations of singularities is all written in terms of V. In most cases it is not difficult to make the translation but this paper is not the right place for doing that. Thus when possible we work with deformations of V. The formulation of Conjecture B in Section 1.4 requires that we work with deformations of the pair $(V, \partial V)$.

1.2.3 The definition of 0-mutable Laurent polynomials

This subsection is occupied by the definition of 0-mutable Laurent polynomials. The simple key idea — and the explanation for the name "0-mutable" — is that an irreducible Laurent polynomial f is 0-mutable if and only if there is a sequence of mutations

$$f \mapsto f_1 \mapsto \cdots \mapsto f_p = \mathbf{1}$$

starting from f and ending with the constant monomial $\mathbf{1}$. The precise definition is given below after some preliminaries on mutations. The definition is appealing and it is meaningful in all dimensions, but it is not immediately useful if you want to study 0-mutable polynomials. Indeed, for example, to prove that a given polynomial f is 0-mutable one must produce a chain of mutations as above and it may not be obvious

equations of $\mathscr{U}$ naturally involve all the infinitely many coordinate functions on T^1_V. Thus, $\mathscr{U}$ is a bona fide ind-scheme. The language to deal with this exists but it is not our concern here.

where to look. It is even less clear how to prove that f is not 0-mutable. In Section 1.3, Theorem 1.3.5, we prove two useful characterizations of 0-mutable polynomials in two variables. The first of these states that a polynomial is 0-mutable if and only if it is rigid maximally mutable. From this property it is easy to check that a given polynomial is 0-mutable, that it is not 0-mutable, and to enumerate 0-mutable polynomials with given Newton polytope. The second characterization states that a (normalized, see below) Laurent polynomial in two variables is 0-mutable if and only if the irreducible components of its vanishing locus are -2-curves on the cluster surface: see Section 1.3 for explanations and details.

Let $\mathbb{L}$ be an affine lattice and let $\mathbb{L}_0$ be its underlying lattice.[6]

In other words, $\mathbb{L}_0$ is a free abelian group of finite rank and $\mathbb{L}$ is a set together with a free and transitive $\mathbb{L}_0$-action. We denote by $\mathbb{C}[\mathbb{L}]$ the vector space over $\mathbb{C}$ whose basis is made up of the elements of $\mathbb{L}$. For every $l \in \mathbb{L}$ we denote by x^l the corresponding element in $\mathbb{C}[\mathbb{L}]$. Elements of $\mathbb{C}[\mathbb{L}]$ will be called (Laurent) polynomials. It is clear that $\mathbb{C}[\mathbb{L}]$ is a rank-1 free module over the $\mathbb{C}$-algebra $\mathbb{C}[\mathbb{L}_0]$. The choice of an origin in $\mathbb{L}$ specifies an isomorphism $\mathbb{C}[\mathbb{L}] \simeq \mathbb{C}[\mathbb{L}_0]$.

Definition 1.2.4. A Laurent polynomial $f \in \mathbb{C}[\mathbb{L}]$ is *normalized* if for all vertices $v \in \operatorname{Newt} f$ we have $a_v = 1$ where a_v is the coefficient of the monomial x^v as it appears in f.

In this paper all polynomials are assumed to be normalized unless explicitly stated otherwise.

If $f \in \mathbb{C}[\mathbb{L}]$ then we say that f *lives* on the smallest saturated affine sub-lattice $\mathbb{L}' \subseteq \mathbb{L}$ such that $f \in \mathbb{C}[\mathbb{L}']$. The property of being 0-mutable only depends on the lattice where f lives — here it is crucial that we only allow saturated sub-lattices. We say that f is an r-variable polynomial if f lives on a rank-r affine lattice.

Let us start by defining 0-mutable polynomials in 1 variable.

Definition 1.2.5. Let $\mathbb{L}$ be an affine lattice of rank 1 and let $\mathbb{L}_0$ be its underlying lattice. Let $v \in \mathbb{L}_0$ be one of the two generators of $\mathbb{L}_0$. A polynomial $f \in \mathbb{C}[\mathbb{L}]$ is called 0-*mutable* if

$$f = (1 + x^v)^k x^l$$

[6] In our setup $\mathbb{L} = (u = 1) \subset N$ does not have a canonical origin. We try to be pedantic and write $\mathbb{L}_0$ for the the underlying lattice – in our example, $\mathbb{L}_0 = \operatorname{Ker} u$. In practice this distinction is not super-important and you are free to choose an origin anywhere you want.

for some $l \in \mathbb{L}$ and $k \in \mathbb{N}$.

(It is clear that the definition does not depend on the choice of v.)

If f is a Laurent polynomial in 1 variable and its Newton polytope is a segment of lattice length k, then f is 0-mutable if and only if the coefficients of f are the $k+1$ binomial coefficients of weight k.

The definition of 0-mutable polynomials in more than 1 variable will be given recursively on the number of variables. Thus from now on we fix $r \geq 2$ and we assume to know already what it means for a polynomial of $< r$ variables to be 0-mutable. Before we can state what it means for a polynomial f of r variables to be 0-mutable, we need to explain how to mutate f.

If $\mathbb{L}$ is an affine lattice, we denote by $\mathrm{Aff}(\mathbb{L}, \mathbb{Z})$ the lattice of affine-linear functions $\varphi\colon \mathbb{L} \to \mathbb{Z}$. If $\mathbb{L}_0$ denotes the underlying lattice of $\mathbb{L}$, φ has a well-defined *linear part* which we denote by $\varphi_0\colon \mathbb{L}_0 \to \mathbb{Z}$.

Definition 1.2.6. Let $r \geq 2$ and fix a rank-r affine lattice $\mathbb{L}$.

A *mutation datum* is a pair (φ, h) of a non-constant affine-linear function $\varphi\colon \mathbb{L} \to \mathbb{Z}$ and a 0-mutable polynomial $h \in \mathbb{C}[\operatorname{Ker} \varphi_0]$.

Given a mutation datum (φ, h) and $f \in \mathbb{C}[\mathbb{L}]$, write (uniquely)

$$f = \sum_{k \in \mathbb{Z}} f_k \qquad \text{where} \qquad f_k \in \mathbb{C}[(\varphi = k) \cap \mathbb{L}]$$

We say that f is (φ, h)-*mutable* if for all $k < 0$ h^{-k} divides f_k (equivalently, if for all $k \in \mathbb{Z}$ $h^k f_k \in \mathbb{C}[\mathbb{L}]$). If f is (φ, h)-mutable, then the *mutation* of f, with respect to the mutation datum (φ, h), is the polynomial:

$$\mathrm{mut}_{(\varphi,h)} f = \sum_{k \in \mathbb{Z}} h^k f_k.$$

Remark 1.2.7. The notion of mutation goes back (at least) to Fomin–Zelevinsky [14]. We first learned of mutations from the work of Galkin–Usnich [16] and Akhtar–Coates–Galkin–Kasprzyk [1]. This paper owes a significant intellectual debt to the interpretation of mutations developed in work by Gross, Hacking and Keel, for instance [22, 20, 21].[7]

The following is a recursive definition. The base step is given by Definition 1.2.5.

[7] Many mathematicians work on mutations from different perspectives and we apologize for not even trying to quote all the relevant references here.

Definition 1.2.8. Let $\mathbb{L}$ be an affine lattice of rank $r \geq 2$. We define the set of 0-mutable polynomials on $\mathbb{L}$ in the following recursive way.

(i) If $\mathbb{L}'$ is a saturated affine sub-lattice of $\mathbb{L}$ and $f \in \mathbb{C}[\mathbb{L}']$ is 0-mutable, then f is 0-mutable in $\mathbb{C}[\mathbb{L}]$.
(ii) If $f = f_1 f_2$ is reducible, then f is 0-mutable if both factors f_1, f_2 are 0-mutable.[8]
(iii) If f is irreducible, then f is 0-mutable if a mutation of f is 0-mutable.

Equivalently, the set of 0-mutable polynomials of $\leq r$ variables is the smallest subset of $\mathbb{C}[\mathbb{L}]$ that contains all 0-mutable polynomials of $< r$ variables and that is closed under the operations of taking products (in particular translation) and mutations of irreducible polynomials.

Remark 1.2.9. It follows easily from the definition that 0-mutable polynomials are normalized. Indeed:

(1) The polynomials in Definition 1.2.5 (the base case) are 0-mutable;
(2) The product of two normalized polynomials is normalized;
(3) The mutation of a normalized polynomial is normalized.

Example 1.2.10. Let $\mathbb{L}$ be an affine lattice and let v be a primitive vector in the underlying lattice $\mathbb{L}_0$. Definition 1.2.5 and Definition 1.2.8(i) imply that $(1 + x^v)^k x^l$ is 0-mutable for all $l \in \mathbb{L}$ and $k \in \mathbb{N}$.

Example 1.2.11. Consider the triangle

$$F = \operatorname{conv}\left((0,0), (3,0), (3,2)\right) \tag{1.1}$$

in the lattice $\mathbb{L} = \mathbb{Z}^2$. Let us identify $\mathbb{C}[\mathbb{L}]$ with $\mathbb{C}[x^{\pm}, y^{\pm}]$. One can prove that there are exactly two 0-mutable polynomials with Newton polytope F, namely:

$$\begin{gathered}(1+x)^3 + 2(1+x)x^2y + x^3y,\\ (1+y)^2x^3 + 3(1+y)x^2 + 3x + 1.\end{gathered}$$

In Figure 1.1 we have written the coefficients of these two polynomials next to the lattice point of F associated to the corresponding monomial.

It is shown in [9] that $\operatorname{Def} V$ has two components, and that they are both smoothing components, confirming our conjectures. The calculation there goes back to unpublished work by Jan Stevens, but see also [5].

[8] In order to take the "product" $f_1 f_2$ one has to choose an origin in $\mathbb{L}$. This choice makes no difference to the definition.

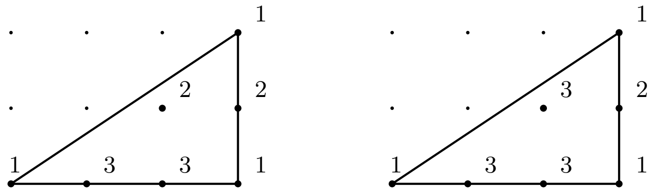

Figure 1.1 The two 0-mutable polynomials whose Newton polytope is the triangle F defined in (1.1)

Example 1.2.12. Consider the quadrilateral

$$F = \operatorname{conv}\left((-1,-1),(2,-1),(1,1),(-1,2)\right) \tag{1.2}$$

in the lattice $\mathbb{L} = \mathbb{Z}^2$. Let us identify $\mathbb{C}[\mathbb{L}]$ with $\mathbb{C}[x^{\pm}, y^{\pm}]$. Consider the polynomial

$$g = \frac{(1+x)^3 + (1+y)^3 - 1 + x^2y^2}{xy},$$

which is obtained by giving binomial coefficients to the lattice points of the boundary of F and by giving zero coefficient to the interior lattice points of F. By Lemma 1.3.1(2) every 0-mutable polynomial with Newton polytope F must coincide with g on the boundary lattice points of F. One can prove that there are exactly three 0-mutable polynomials with Newton polytope F, namely:

$$\begin{aligned}
\alpha = g + 5 + 2x + 2y &= \frac{(1+x+2y+y^2)(1+2x+x^2+y)}{xy},\\
\beta = g + 6 + 3x + 4y &= \frac{(1+x)^3 + 3y(1+x)^2 + y^2(1+x)(3+x) + y^3}{xy},\\
\gamma = g + 6 + 4x + 3y &= \frac{(1+y)^3 + 3x(1+y)^2 + x^2(1+y)(3+y) + x^3}{xy}.
\end{aligned}$$

In Figure 1.2 we have written down the coefficients of these three polynomials. The polynomial α is reducible and it is easy to show that its factors are 0-mutable.

Let us consider the following affine-linear functions $\mathbb{L} = \mathbb{Z}^2 \to \mathbb{Z}$:

$$\begin{aligned}
-m^1_{2,1} &\colon (a,b) \mapsto b - 1,\\
-m^1_{3,1} &\colon (a,b) \mapsto b - 2,\\
-m^1_{3,2} &\colon (a,b) \mapsto 2b - 1.
\end{aligned}$$

The level sets of these three affine-linear functions are depicted in Figure 1.3. We now consider the mutation data $(-m^1_{2,1}, 1+x)$, $(-m^1_{3,2}, 1+$

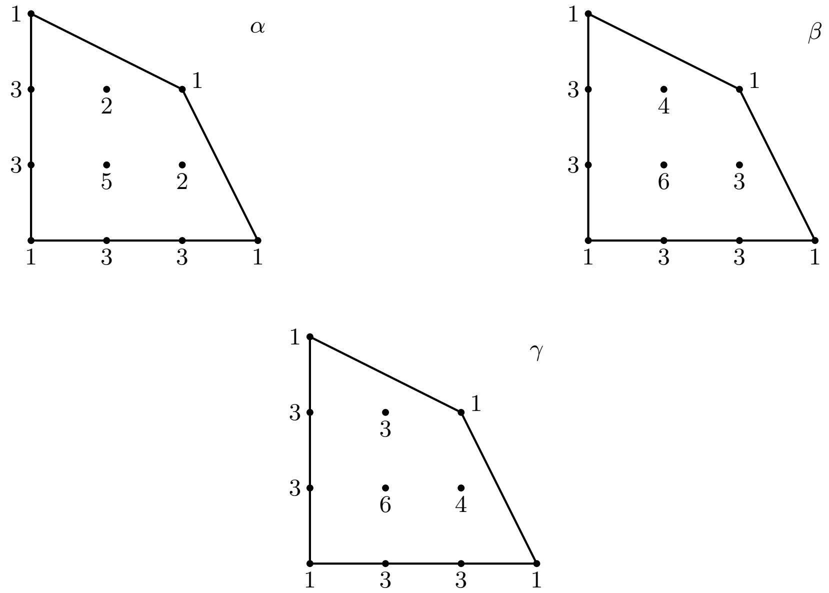

Figure 1.2 The three 0-mutable polynomials α, β and γ whose Newton polytope is the quadrilateral F defined in (1.2)

x) and $(-m^1_{3,1}, 1+x)$. The polynomial α is mutable with respect to $(-m^1_{2,1}, 1+x)$ and to $(-m^1_{3,2}, 1+x)$ and

$$\begin{aligned}
\mathrm{mut}_{(-m^1_{2,1},1+x)}\,\alpha &= \frac{1+x}{xy} + \frac{3+2x}{x} + \frac{y(3+2x+x^2)}{x} + \frac{y^2(1+x)}{x},\\
\mathrm{mut}_{(-m^1_{3,2},1+x)}\,\alpha &= \frac{1}{xy} + \frac{3+2x}{x} + \frac{y(3+5x+3x^2+x^3)}{x} + \frac{y^2(1+x)^3}{x},
\end{aligned}$$

but α is not mutable with respect to $(-m^1_{3,1}, 1+x)$. The polynomial β is mutable with respect to all three mutation data $(-m^1_{2,1}, 1+x)$, $(-m^1_{3,2}, 1+x)$ and $(-m^1_{3,1}, 1+x)$, and the mutations are:

$$\begin{aligned}
\mathrm{mut}_{(-m^1_{2,1},1+x)}\,\beta &= \frac{1+x}{xy} + \frac{3(1+x)}{x} + \frac{y(3+x)(1+x)}{x} + \frac{y^2(1+x)}{x},\\
\mathrm{mut}_{(-m^1_{3,1},1+x)}\,\beta &= \frac{1}{xy} + \frac{3}{x} + \frac{y(3+x)}{x} + \frac{y^2}{x} = \frac{(1+y)^3+xy^2}{xy},\\
\mathrm{mut}_{(-m^1_{3,2},1+x)}\,\beta &= \frac{1}{xy} + \frac{3(1+x)}{x} + \frac{y(3+x)(1+x)^2}{x} + \frac{y^2(1+x)^3}{x}.
\end{aligned}$$

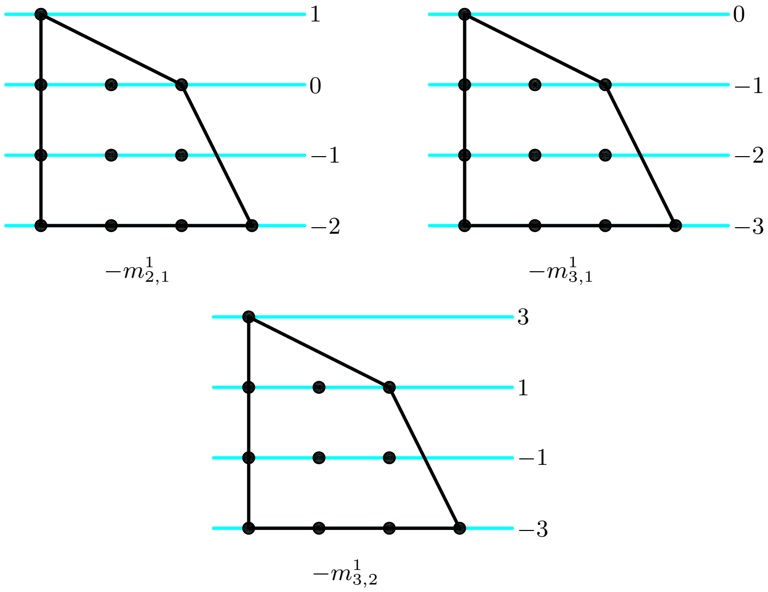

Figure 1.3 The level sets of the three affine-linear functions $\mathbb{L} = \mathbb{Z}^2 \to \mathbb{Z}$ considered in Example 1.2.12

The polynomial γ is not mutable with respect to any of the mutation data $(-m^1_{2,1}, 1+x)$, $(-m^1_{3,2}, 1+x)$, $(-m^1_{3,1}, 1+x)$.

1.3 Properties of 0-mutable polynomials

1.3.1 Some easy properties

If $\mathbb{L}$ is an affine lattice, $f \in \mathbb{C}[\mathbb{L}]$ and $F = \operatorname{Newt} f$, then we write

$$f = \sum_{l \in F \cap \mathbb{L}} a_l x^l \qquad \text{with} \qquad a_l \in \mathbb{C}.$$

For every subset $A \subseteq \mathbb{L}_{\mathbb{R}}$, we write

$$f|_A = \sum_{l \in A \cap \mathbb{L}} a_l x^l.$$

Lemma 1.3.1. *(1) (Non-negativity and integrality) If f is 0-mutable, then every coefficient of f is a non-negative integer.*

(2) (Boundary terms) If $f \in \mathbb{C}[\mathbb{L}]$ is 0-mutable and $F \leq \operatorname{Newt} f$ is a face, then $f|_F$ is 0-mutable.

Sketch of proof (1) is obvious due to the recursive definition of 0-mutable polynomials. Also (2) is easy because it is enough to observe that mutations and products behave well with respect to restriction to faces of the Newton polytope. □

Remark 1.3.2. A 0-mutable polynomial may have a zero coefficient at a lattice point of its Newton polytope, e.g. $(1 + x)(1 + xy^2)$.

1.3.2 Rigid maximally mutable polynomials

From now on we focus on the two-variable case $r = 2$. In what follows, we give two equivalent characterizations of 0-mutable polynomials, one geometric in terms of the associated cluster variety and one combinatorial in terms of rigid maximally mutable polynomials.[9]

Let $\mathbb{L}$ be an affine lattice of rank 2. For a Laurent polynomial $f \in \mathbb{C}[\mathbb{L}]$, we set

$$\mathscr{S}(f) = \Big\{ \text{mutation data } s = (\varphi, h) \;\Big|\; f \text{ is } s\text{-mutable} \Big\}.$$

Conversely, if $\mathscr{S}$ is a set of mutation data, we denote by

$$L(\mathscr{S}) = \Big\{ f \in \mathbb{C}[\mathbb{L}] \;\Big|\; \forall s \in \mathscr{S},\, f \text{ is } s\text{-mutable} \Big\}$$

the vector space of Laurent polynomials f that are s-mutable for all the mutation data $s \in \mathscr{S}$. For every polynomial $f \in \mathbb{C}[\mathbb{L}]$, it is clear that $f \in L(\mathscr{S}(f))$.

Definition 1.3.3 (Kasprzyk)**.** Let $\mathbb{L}$ be an affine lattice of rank 2, and $f \in \mathbb{C}[\mathbb{L}]$.

(i) If $f = f_1 f_2$ is the product of normalized polynomials f_1, f_2, then f is rigid maximally mutable if both factors f_1, f_2 are rigid maximally mutable;

(ii) If f is normalized and irreducible, then f is rigid maximally mutable if

$$L\left(\mathscr{S}(f)\right) = \{\lambda f \mid \lambda \in \mathbb{C}\}.$$

[9] The concept of rigid maximally mutable is due to Al Kasprzyk [26]. We thank him for allowing us to include his definition here.

1.3.3 Cluster varieties

Definition 1.3.4. A *Calabi–Yau (CY) pair* is a pair (Y, ω) of an n-dimensional quasiprojective normal variety Y and a degree n rational differential $\omega \in \Omega^n_{k(X)}$, such that

$$D = -\operatorname{div}_Y \omega \geq 0$$

is an effective reduced Cartier divisor on Y.[10]

A *torus chart* on (Y, ω) is an open embedding

$$j\colon (\mathbb{C}^\times)^n \hookrightarrow Y \setminus D \qquad \text{such that} \qquad j^\star(\omega) = \frac{1}{(2\pi \mathrm{i})^n} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}.$$

A *cluster variety* is an n-dimensional CY pair (Y, ω) that has a torus chart.

In our situation, the pair (Y, D) will always be log smooth.

1.3.4 The cluster surface

We construct a cluster surface from a lattice polygon.

Let $\mathbb{L}$ be an affine lattice of rank 2. There is a canonical bijection between the set of lattice polygons $F \subset \mathbb{L}_\mathbb{R}$ up to translation and the set of pairs $(\overline{X}, A)$ of a projective toric surface $\overline{X}$ and an ample line bundle A on $\overline{X}$: the torus in question is $\operatorname{Spec} \mathbb{C}[\mathbb{L}_0]$; the fan of the surface $\overline{X}$ is the normal fan of F, and it all works out such that there is a natural 1-to-1 correspondence between $F \cap \mathbb{L}$ and a basis of $H^0(\overline{X}, A)$.

Fix a lattice polygon F in $\mathbb{L}$ and consider the corresponding polarised toric surface $(\overline{X}, A)$. Denote by $\overline{B}$ the toric boundary of $\overline{X}$. For each edge $E \leq F$, let $\ell(E)$ be the lattice length of E and let $\overline{B}_E$ be the prime component of $\overline{B}$ corresponding to E; we have that $\overline{B}_E$ is isomorphic to $\mathbb{P}^1$ and the line bundle $A|_{\overline{B}_E}$ has degree $\ell(E)$. Denote by $x_E \in \overline{B}_E$ the point $[1 : -1] \in \mathbb{P}^1$.

For all edges $E \leq F$, blow up $\ell(E)$ times above x_E in the proper transform of $\overline{B}_E$ and denote by

$$p\colon (X, B) \longrightarrow (\overline{X}, \overline{B})$$

the resulting surface, where $B \subset X$ is the proper transform of the toric boundary $\overline{B} = \sum_{E \leq F} \overline{B}_E$. We call the pair (X, B) the *cluster surface* associated to the lattice polygon F.

[10] Like most people, we mostly work with the pair (Y, D) and omit explicit reference to ω.

Theorem 1.3.5. *Let $\mathbb{L}$ be a rank-2 lattice. The following are equivalent for a normalized Laurent polynomial $f \in \mathbb{C}[\mathbb{L}]$:*

(1) f is 0-mutable;
(2) f is rigid maximally mutable;
(3) Let $p\colon (X,B) \to (\overline{X}, \overline{B})$ be the cluster surface associated to the polygon $F = \operatorname{Newt} f$. Denote by $Z \subset \overline{X}$ the divisor of zeros of f and by $Z' \subset X$ the proper transform of Z. Every irreducible component $\Gamma \subset Z'$ is a smooth rational curve with self-intersection $\Gamma^2 = -2$. (Necessarily then $B \cdot \Gamma = 0$ hence Γ is disjoint from the boundary B.)

Remark 1.3.6. The support of Z' is not necessarily a normal crossing divisor. The irreducible components need not meet transversally, and ≥ 3 of them may meet at a point.

Sketch of proof In proving all equivalences we may and will assume that the polynomial f is irreducible hence Z is reduced and irreducible.

The proof uses the following ingredients, which we state without further discussion or proof:

(i) To give a torus chart $j\colon \mathbb{C}^{\times 2} \hookrightarrow X \setminus B$ in X is the same as to give a *toric model* of (X,B), that is a projective morphism $q\colon (X,B) \to (X',B')$ where (X',B') is a toric pair and q maps $j(\mathbb{C}^{\times 2})$ isomorphically to the torus $X' \setminus B'$;
(ii) The work of Blanc [7] implies that any two torus charts in X are connected by a sequence of mutations between torus charts in X;
(iii) A set $\mathscr{S}$ of mutation data specifies a line bundle $\mathcal{L}(\mathscr{S})$ on X such that $H^0\big(X, \mathcal{L}(\mathscr{S})\big) = L(\mathscr{S})$ and, conversely, every line bundle on X is isomorphic to a line bundle of the form $\mathcal{L}(\mathscr{S})$.

Let us show first that (1) implies (3). To say that an irreducible polynomial f is 0-mutable is to say that there exists a sequence of mutations that mutates f to the constant polynomial $\mathbf{1}$. This sequence of mutations constructs a new torus chart $j_1\colon \mathbb{T} = \mathbb{C}^{\times 2} \hookrightarrow X \setminus B$ such that the proper transform Z' – which is, by assumption, irreducible – is disjoint from $j_1(\mathbb{T})$. This new toric chart gives a new toric model $p_1\colon (X,B) \to (X_1,B_1)$ that maps $j_1(\mathbb{T})$ isomorphically to the torus $X_1 \setminus B_1$ and hence contracts Z' to a boundary point. Z' is not a -1-curve, because those are all p-exceptional, hence Z' is a -2-curve.

To show that (3) implies (1), by Lemma 1.3.7 below, there is a new toric model $p_1\colon (X,B) \to (X_1,B_1)$ that contracts Z' to a point in the boundary. The new toric model then gives a new torus chart $j_1 \colon \mathbb{T} \hookrightarrow X$ such that Z' is disjoint from $j_1(\mathbb{T})$. By Blanc the induced birational map of tori

$$j_1^{-1} j \colon \mathbb{T} \dashrightarrow \mathbb{T}$$

is a composition of mutations that mutates f to the constant polynomial.

Let us now show that (3) implies (2). By some tautology, Z' is the zero divisor of the section f of the line bundle $\mathcal{L}(\mathscr{S})$ on X specified by the set of mutation data $\mathscr{S} = \mathscr{S}(f)$, and $H^0(X, Z') = L(\mathscr{S})$. Since Z' is a -2-curve, $L(\mathscr{S})$ is 1-dimensional, which is to say that f is rigid maximally mutable.

Finally we show that (2) implies (3). Denote by $\mathcal{L} = \mathcal{L}(\mathscr{S})$ the line bundle on X specified by the set of mutations $\mathscr{S} = \mathscr{S}(f)$, so that Z' is the zero-locus of a section of $\mathcal{L}$. Note that:

$$h^2(X, \mathcal{L}) = h^0(X, K_X - Z') = h^0(Y, -\overline{B} - Z') = 0$$

Riemann–Roch and the fact that f is rigid give:

$$1 = h^0(X, Z') = h^0(X, \mathcal{L}) \geq \chi(X, \mathcal{L}) = 1 + \frac{1}{2}({Z'}^2 + Z'\overline{B})$$

and hence, because $Z'\overline{B} \geq 0$, we conclude that ${Z'}^2 \leq 0$ and:

$$2p_a(Z') - 2 = \deg \omega_{Z'} = {Z'}^2 - Z'\overline{B} \leq 0$$

so either:

(i) $\deg \omega_{Z'} < 0$ and then Z' is a smooth rational curve, and then as above Z' is not a -1 curve therefore it is a -2-curve, or

(ii) $\omega_{Z'} = \mathcal{O}_{Z'}$ and ${Z'}^2 = Z'\overline{B} = 0$. It follows that Z' is actually disjoint from $\overline{B}$ and $\mathcal{O}_{Z'}(Z') = \mathcal{O}_{Z'}$. The homomorphism

$$H^0(X, Z') \to H^0\left(Z', \mathcal{O}_{Z'}(Z')\right) = \mathbb{C}$$

is surjective, hence actually $h^0(X, Z') = 2$, a contradiction.

This means that we must be in case (i) where Z' is a -2-curve. □

Lemma 1.3.7. *Let (Y, D) be a cluster surface, and $Z' \subset Y$ an interior -2-curve. Then there is a toric model $q \colon (Y, D) \to (X', B')$ that contracts Z'.*

Sketch of proof First contract Z' to an interior node and then run a MMP. There is a small number of cases to discuss depending on how the MMP terminates. □

1.4 Conjecture B

In this section we state a new conjecture — Conjecture B — which implies, see Remark 1.4.1, the existence of the map $\kappa\colon \mathfrak{B} \to \mathfrak{A}$ of Conjecture A. In the strong form stated in Remark 1.4.2, together with Conjecture A, Conjecture B asserts that the smoothing components of $\mathrm{Def}(V, \partial V)$ are themselves smooth, and computes their tangent space explicitly as a representation of the torus. We conclude by explaining how the two Conjectures A and B originate in mirror symmetry. This last discussion is central to how we arrived at the formulation of the conjectures, but it is not logically necessary for making sense of their statement. We work with the version of mirror symmetry put forward in [21] and [18, 19].[11]

For the remainder of this section fix a lattice polygon F and denote, as usual, by V the corresponding Gorenstein toric affine 3-fold with toric boundary ∂V.

In this section we always work with the space $\mathrm{Def}(V, \partial V)$. Also fix a 0-mutable polynomial $f \in \mathbb{C}[\mathbb{L}]$ with $\mathrm{Newt}\, f = F$. Conjecture B associates to f a $\mathbb{T}$-equivariant family $(\mathscr{U}_f, \mathscr{D}_f) \to \mathscr{M}_f$ of deformations of the pair $(V, \partial V)$.

In the last part of this section we construct from f a 3-dimensional cluster variety (W, D), conjecturally the *mirror* of $\mathscr{M}_f$, and hint at an explicit conjectural construction of the family $\mathscr{U}_f \to \mathscr{M}_f$ from the degree-0 quantum log cohomology of (W, D).

Denote by $\sigma \subset N_\mathbb{R}$ the cone over F at height 1. As usual, $u \in M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$ denotes the Gorenstein degree, so $F = \sigma \cap \mathbb{L}$ where $\mathbb{L} = (u = 1)$.

1.4.1 Statement of Conjecture B

As in Sec. 1.3.2, denote by $\mathscr{S}(f)$ the set of mutation data of f. Recall that an element of $\mathscr{S}(f)$ is a pair (φ, h) consisting of an affine function $\varphi \in \mathrm{Aff}(\mathbb{L}, \mathbb{Z})$ and a Laurent polynomial $h \in \mathbb{C}[\mathrm{Ker}\, \varphi_0]$. Using the restriction isomorphism $M \simeq \mathrm{Aff}(\mathbb{L}, \mathbb{Z})$, when it suits us we view a mutation datum (φ, h) as a pair of an element $\varphi \in M$ and a polynomial $h \in \mathbb{C}[\mathbb{L}_0 \cap \mathrm{Ker}\, \varphi] \subseteq \mathbb{C}[N]$.

The most useful mutation data are those where φ is strictly negative

[11] We thank Paul Hacking for several helpful discussions on mirror symmetry and for correcting earlier drafts of this section. We are of course responsible for the mistakes that are left.

somewhere on F, and then the minimum of φ on F is achieved on an edge $E \leq F$.[12] In the present discussion we want to focus on these mutation data:

$$\mathscr{S}_-(f) = \left\{ (\varphi, h) \in \mathscr{S}(f) \;\middle|\; \begin{array}{c} \text{for some edge } E \leq F, \\ \varphi|_E \text{ is constant and } < 0 \end{array} \right\}.$$

We want to define a seed $\widetilde{\mathscr{S}}(f)$ on N, that is, a set of pairs (φ, h) of a character $\varphi \in M$ and a Laurent polynomial $h \in \mathbb{C}[\operatorname{Ker}\varphi]$. The seed we want is

$$\widetilde{\mathscr{S}}(f) = \mathscr{S}_-(f) \cup \left\{ (-ku, h) \;\middle|\; h \text{ is a prime factor of } f \text{ of multiplicity } k \right\}.$$

Conjecture B. *In this statement, if U is a representation of the torus $\mathbb{T} = \operatorname{Spec}\mathbb{C}[M]$ and $m \in M$ is a character of $\mathbb{T}$, we denote by $U(m)$ the direct summand of U on which $\mathbb{T}$ acts with pure weight m.*

Let $F \subset \mathbb{L}$ be a lattice polygon, V the corresponding Gorenstein toric affine 3-fold, and $f \in \mathbb{C}[\mathbb{L}]$ a 0-mutable polynomial with $\operatorname{Newt} f = F$. For every integer $k \geq 1$, denote by n_k the number of prime factors of f of multiplicity $\geq k$.

Then there is a $\mathbb{T}$-invariant submanifold $\mathscr{M}_f \subset \operatorname{Def}(V, \partial V)$ such that

$$\dim T_0 \mathscr{M}_f(m) = \begin{cases} 1 & \textit{if } m \notin \langle -u \rangle_+ \textit{ and there exists } (\varphi, h) \in \widetilde{\mathscr{S}}(f) \\ & \textit{such that } m = \varphi, \\ n_k & \textit{if } m = -ku \textit{ for some integer } k \geq 1, \\ 0 & \textit{otherwise}, \end{cases}$$

and the general fibre of the family over $\mathscr{M}_f$ is a pair consisting of a smooth variety and a smooth divisor.

Remark 1.4.1. If Conjecture B holds then by openness of versality a general point of $\mathscr{M}_f$ lies in a unique component of $\operatorname{Def}(V, \partial V)$ and this gives the map $\kappa\colon \mathfrak{B} \to \mathfrak{A}$ in the statement of Conjecture A (see also Remark 1.2.3).

Remark 1.4.2 (Strong form of Conjecture B)**.** A strong form of Conjecture B states that the families $\mathscr{M}_f$ are precisely the smoothing components of $\operatorname{Def}(V, \partial V)$.

Example 1.4.3 (Example 1.2.12 continued)**.** Consider the quadrilateral F in $\mathbb{L} = \mathbb{Z}^2$ defined in (1.2) and the three 0-mutable polynomials α, β

[12] Recall that in Definition 1.2.6 we explicitly assume that φ is not constant on F.

and γ with Newton polytope F. Set $N = \mathbb{L} \oplus \mathbb{Z} = \mathbb{Z}^3$ and consider the following linear functionals in $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) = \mathbb{Z}^3$:

$$\begin{aligned} -m^1_{2,1} &= (0,1,-1) & \qquad -m^2_{2,1} &= (1,0,-1)\\ -m^1_{3,1} &= (0,1,-2) & \qquad -m^2_{3,1} &= (1,0,-2)\\ -m^1_{3,2} &= (0,2,-1) & \qquad -m^2_{3,2} &= (2,0,-1) \end{aligned}$$

and $-u = (0,0,-1)$. The names $-m^1_{2,1}$, $-m^1_{3,1}$ and $-m^1_{3,2}$ are compatible with the affine-linear functions considered in Example 1.2.12 via the restriction isomorphism $M \simeq \mathrm{Aff}(\mathbb{L}, \mathbb{Z})$. Set $x = x^{(1,0,0)} \in \mathbb{C}[N]$ and $y = x^{(0,1,0)} \in \mathbb{C}[N]$. Then we have:

$$\begin{aligned} \widetilde{\mathscr{S}}(\alpha) &\supseteq \left\{(-u, 1+x+2y+y^2), (-u, 1+y+2x+x^2)\right\},\\ \widetilde{\mathscr{S}}(\alpha) &\supseteq \left\{(-m^1_{2,1}, 1+x), (-m^1_{3,2}, 1+x), (-m^2_{2,1}, 1+y), (-m^2_{3,2}, 1+y)\right\},\\ \widetilde{\mathscr{S}}(\beta) &\supseteq \left\{(-u, \beta), (-m^1_{2,1}, 1+x), (-m^1_{3,1}, 1+x), (-m^1_{3,2}, 1+x)\right\},\\ \widetilde{\mathscr{S}}(\gamma) &\supseteq \left\{(-u, \gamma), (-m^2_{2,1}, 1+y), (-m^2_{3,1}, 1+y), (-m^2_{3,2}, 1+y)\right\}. \end{aligned}$$

Let V be the Gorenstein toric affine 3-fold associated to F. Conjecture B states that there are three submanifolds $\mathscr{M}_\alpha$, $\mathscr{M}_\beta$ and $\mathscr{M}_\gamma$ of $\mathrm{Def}(V, \partial V)$ such that the dimensions of $T_0\mathscr{M}_\alpha(m)$, $T_0\mathscr{M}_\beta(m)$ and $T_0\mathscr{M}_\gamma(m)$ for $m \in \{-u, -m^1_{2,1}, -m^1_{3,1}, -m^1_{3,2}, -m^2_{2,1}, -m^2_{3,1}, -m^2_{3,2}\}$ are written down in the table below.

	$-u$	$-m^1_{2,1}$	$-m^1_{3,1}$	$-m^1_{3,2}$	$-m^2_{2,1}$	$-m^2_{3,1}$	$-m^2_{3,2}$
$\dim T_0\mathscr{M}_\alpha(m)$	2	1	0	1	1	0	1
$\dim T_0\mathscr{M}_\beta(m)$	1	1	1	1	0	0	0
$\dim T_0\mathscr{M}_\gamma(m)$	1	0	0	0	1	1	1

1.4.2 Mirror symmetry interpretation

Denote by $\mathscr{U}_f \to \mathscr{M}_f$ the deformation family of V induced by the composition $\mathscr{M}_f \hookrightarrow \mathrm{Def}(V, \partial V) \twoheadrightarrow \mathrm{Def}\, V$. We sketch a construction of $\mathscr{U}_f$ in the spirit of intrinsic mirror symmetry [18, 19].

Let $\sigma^\vee \subset M_{\mathbb{R}}$ be the dual cone, and let $s_1, \dots, s_r$ be the primitive generators of the rays of $\sigma^\vee$. Denote by $W^\flat$ the toric variety — for the dual torus $\mathbb{T}_M = \mathrm{Spec}\, \mathbb{C}[N]$ — constructed from the fan consisting of the cones $\{0\}$, $\langle -u \rangle_+$, the $\langle s_j \rangle_+$, and the two-dimensional cones

$$\langle -u, s_j \rangle_+$$

(for $j = 1, \dots, r$), and let $D^\flat \subset W^\flat$ be the toric boundary.

Now, the set of edges $E \leq F$ is in 1-to-1 correspondence with the set $\{s_1, \dots, s_r\}$, where E corresponds to s_j if $s_j|_E = 0$. If $(\varphi, h) \in \widetilde{\mathscr{S}}(f)$ and $\varphi \notin \langle -u \rangle_+$, then there is a unique j such that φ is in the cone $\langle -u, s_j \rangle_+$ spanned by $-u$ and s_j.

Let

$$(\overline{W}, \overline{D}) \longrightarrow (W^\flat, D^\flat)$$

be the toric variety obtained by adding the rays $\langle \varphi \rangle_+ \subset M_\mathbb{R}$ whenever $(\varphi, h) \in \widetilde{\mathscr{S}}(f)$, and (infinitely many) two-dimensional cones subdividing the cones $\langle -u, s_j \rangle_+$. Note that $\overline{W}$ is not quasi-compact and not proper.

Finally, we construct a projective morphism

$$(W, D) \longrightarrow (\overline{W}, \overline{D})$$

by a sequence of blowups.

In what follows for all $(\varphi, h) \in \widetilde{\mathscr{S}}(f)$ we denote by $D_{\langle \varphi \rangle_+} \subset \overline{W}$ the corresponding boundary component, and set

$$Z_h = (h = 0) \subset D_{\langle \varphi \rangle_+}.$$

The following simple remarks will be helpful in describing the construction.

(a) For all positive integers k, $(k\varphi, h) \in \mathscr{S}_-(f)$ if and only if $(\varphi, h^k) \in \mathscr{S}_-(f)$.

(b) By construction, if $(\varphi, h) \in \mathscr{S}_-(f)$, then one has $h = (1+x^e)^k$ (up to translation) for some positive integer k, where $e \in M$ is a primitive lattice vector along the edge $E \leq F$ where φ achieves its minimum.

Let $R \subset M_\mathbb{R}$ be a ray of the fan of $\overline{W}$ other than $\langle -u \rangle_+$, and let $\varphi \in M$ be the primitive generator of R. It follows from the remarks just made that there is a largest positive integer k_R such that $(k_R\varphi, 1+x^e) \in \mathscr{S}_-(f)$.

Our mirror W is obtained from $\overline{W}$ by:

(1) First, as R runs through all the rays of the fan of $\overline{W}$ other than $\langle -u \rangle_+$ in some order, blow up k_R times above $(1 + x^e = 0)$ in the proper transform of D_R. It can be seen, and it is a nontrivial fact, that after doing all these blowups the Z_h in the proper transforms of the $D_{\langle -u \rangle_+}$ are smooth;

(2) Subsequently, if $f = \prod h^{k(h)}$ where the h are irreducible, blow up in any order $k(h)$ times above Z_h in the proper transform of $D_{\langle -u \rangle_+}$.

The resulting CY pair (W, D) is log smooth, because we have blown up a sequence of smooth centres. This (W, D) is the *mirror* of $\mathscr{M}_f$.

One can see that the Mori cone $\mathrm{NE}(W/\overline{W})$ is simplicial, and hence there is an identification:

$$\mathscr{M}_f = \operatorname{Spec} \mathbb{C}[\mathrm{NE}(W/\overline{W})]$$

Mirror symmetry suggests — modulo issues with infinite-dimensionality — that the ring $QH^0_{\log}(W, D; \mathbb{C}[\mathrm{NE}(W/\overline{W})])$ has a natural filtration and that one recovers the universal family of pairs $(\mathscr{U}_f, \mathscr{D}_f) \to \mathscr{M}_f$ from this ring out of the Rees construction.

Remark 1.4.4. In the context of Conjecture B, it would be very nice to work out an interpretation of the coefficients of the 0-mutable polynomial f as counting certain holomorphic disks on the general fibre of the family $\mathscr{U}_f \to \mathscr{M}_f$.

1.5 Evidence

We have already remarked that the variety V of Example 1.2.11 confirms Conjecture A. Here we collect some further evidence. Section 1.6 is a study of $\operatorname{Def} V$ where V is the variety of Example 1.2.12 and Example 1.4.3.

1.5.1 Isolated singularities

Here we fix a lattice polygon F with unit edges, i.e. edges with lattice length 1. Let V be the Gorenstein toric affine 3-fold associated to F; we have that V has an isolated singularity.

Altmann [4] proved that there is a 1-to-1 correspondence between the irreducible components of $\operatorname{Def} V$ and the maximal Minkowski decompositions of F. This restricts to a 1-to-1 correspondence between the smoothing components of $\operatorname{Def} V$ and the Minkowski decompositions of F with summands that are either unit segments or standard triangles. Here a standard triangle is a lattice triangle that is $\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$-equivalent to $\operatorname{conv}((0,0),(1,0),(0,1))$.

On a polygon F with unit edges, the 0-mutable polynomials are exactly those that are associated to the Minkowski decompositions of F into unit segments and standard triangles. This confirms Conjecture A.

1.5.2 Local complete intersections

Nakajima [28] has characterised the affine toric varieties that are local complete intersection (lci for short). These are Gorenstein toric affine varieties associated to certain lattice polytopes called Nakajima polytopes. We refer the reader to [11, Lemma 2.7] for an inductive characterisation of Nakajima polytopes. From this characterisation it is very easy to see that every Nakajima polygon is affine equivalent to

$$F_{a,b,c} = \operatorname{conv}\left((0,0),(a,0),(0,b),(a,b+ac)\right) \tag{1.3}$$

in the lattice $\mathbb{Z}^2$, for some non-negative integers a, b, c such that $a \geq 1$ and $b + c \geq 1$. It is easy to show that the Gorenstein toric affine 3-fold associated to the polygon $F_{a,b,c}$ is

$$V_{a,b,c} = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4, x_5]/(x_1x_2 - x_4^c x_5^b,\ x_3x_4 - x_5^a).$$

There is a unique 0-mutable polynomial on $F_{a,b,c}$: this is associated to the unique Minkowski decomposition of $F_{a,b,c}$ into a copies of the triangle $F_{1,0,c} = \operatorname{conv}((0,0),(1,0),(0,c))$ and b copies of the segment $\operatorname{conv}((0,0),(0,1))$. On the other hand, as $V_{a,b,c}$ is lci, we have that $V_{a,b,c}$ is unobstructed and smoothable, therefore there is a unique smoothing component in the miniversal deformation space of $V_{a,b,c}$. This confirms Conjecture A.

1.6 A worked example

We explicitly compute the smoothing components of the miniversal deformation space of the Gorenstein toric affine 3-fold V associated to the polygon F of Example 1.2.12 (continued in Example 1.4.3). We saw that there exist exactly three 0-mutable polynomials with Newton polytope F: α, β and γ. We explicitly compute the miniversal deformation space of V and see that it has three irreducible components, all of which are smoothing components. This confirms our conjectures.

1.6.1 The equations of V

We consider the quadrilateral

$$F = \operatorname{conv}\left(\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right) \tag{1.4}$$

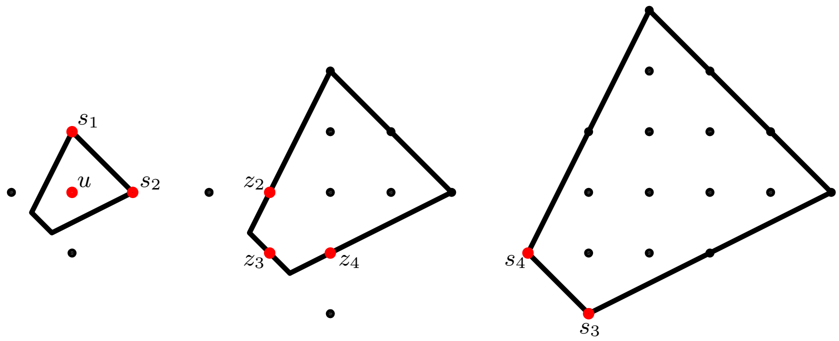

Figure 1.4 The intersections of the cone $\sigma^\vee$ with the planes $\mathbb{R}^2 \times \{1\}$, $\mathbb{R}^2 \times \{2\}$ and $\mathbb{R}^2 \times \{3\}$ in $M_{\mathbb{R}} = \mathbb{R}^3$

in the lattice $\mathbb{L} = \mathbb{Z}^2$. We consider the cone σ obtained by placing F at height 1, i.e. σ is the cone generated by

$$a_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad a_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad a_4 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

in the lattice $N = \mathbb{L} \oplus \mathbb{Z} = \mathbb{Z}^3$, and the corresponding Gorenstein toric affine 3-fold $V = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$, where $\sigma^\vee$ is the dual cone of σ in the dual lattice $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^3$.

The Gorenstein degree is

$$u = (0, 0, 1) \in M.$$

The primitive generators of the rays of the dual cone $\sigma^\vee \subseteq M_{\mathbb{R}}$ are the vectors

$$s_1 = (0, 1, 1), \quad s_2 = (1, 0, 1), \quad s_3 = (-1, 2, 3), \quad s_4 = (-2, -1, 3)$$

which are orthogonal to the 4 edges of F. The Hilbert basis of the monoid $\sigma^\vee \cap M$ is the set of the vectors

$$u,\ s_1,\ z_2 = (-1, 0, 2),\ s_4,\ z_3 = (-1, -1, 2),\ s_3,\ z_4 = (0, -1, 2),\ s_2.$$

Notice that these are the Gorenstein degree u and certain lattice vectors on the boundary of $\sigma^\vee$. The elements of the Hilbert basis of $\sigma^\vee \cap M$ are depicted in Figure 1.4.

The elements of the Hilbert basis of $\sigma^\vee \cap M$ give a closed embedding of V inside $\mathbb{A}^8$ such that the ideal is generated by binomial equations.

By using rolling factors formats (see [32] and [33, §12]), one can[13] see that these equations are:

$$\operatorname{rank}\begin{pmatrix} x_{s_1} & x_{z_2} & x_u & x_{s_2} & x_{z_4} \\ x_{z_2} & x_{s_4} & x_{z_3} & x_{z_4} & x_{s_3} \end{pmatrix} \leq 1,$$

$$x_{s_4}x_{s_3} - x_{z_3}^3 = 0 \qquad x_{z_2}x_{s_3} - x_{z_3}^2 x_u = 0,$$

$$x_{z_2}x_{z_4} - x_{z_3}x_u^2 = 0 \qquad x_{s_1}x_{z_4} - x_u^3 = 0.$$

The singular locus of V has two irreducible components of dimension 1: V has generically transverse A_2-singularities along each of these.

1.6.2 The tangent space

We consider the tangent space to the deformation functor of V, i.e. $T^1_V = \operatorname{Ext}^1_{\mathcal{O}_V}(\Omega^1_V, \mathcal{O}_V)$. This is a $\mathbb{C}$-vector space with an M-grading. For every $m \in M$ we denote by $T^1_V(-m)$ the graded component of T^1_V of degree $-m$.

Lemma 1.6.1. *We define* $J := \{(p,q) \in \mathbb{Z}^2 \mid 2 \leq p \leq 3,\ q \geq 1\}$. *For all* $p, q \in \mathbb{Z}$ *we set* $m^1_{p,q} := pu - qs_1$ *and* $m^2_{p,q} := pu - qs_2$.

Then

$$\dim T^1_V(-m) = \begin{cases} 1 & \text{if } m = u, \\ 1 & \text{if } m = m^1_{p,q} \text{ with } (p,q) \in J, \\ 1 & \text{if } m = m^2_{p,q} \text{ with } (p,q) \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Proof This is a direct consequence of [5, Theorem 4.4]. □

Some of the degrees of T^1_V are depicted in Figure 1.5.

The base of the miniversal deformation of V is the formal completion (or germ) at the origin of a closed subscheme of the countable-dimensional affine space T^1_V. We denote by t_m the coordinate on the 1-dimensional $\mathbb{C}$-vector space $T^1_V(-m)$, when $m = u$ or $m \in \{m^1_{p,q}, m^2_{p,q}\}$ with $(p,q) \in J$. Since we want to understand the structure of $\operatorname{Def} V$, we want to analyse the equations of $\operatorname{Def} V \hookrightarrow T^1_V$ in the variables t_u, $t_{m^1_{p,q}}$ and $t_{m^2_{p,q}}$ for $(p,q) \in J$.

The first observation is that each homogeneous first order deformation of V is unobstructed as we see in the following two remarks.

[13] We are obliged to the referee for suggesting these equations to us.

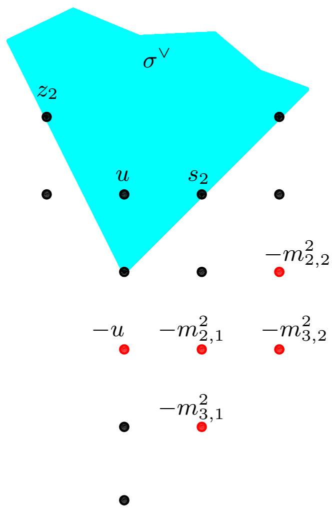

Figure 1.5 Some degrees of T^1_V in the plane $\mathbb{R} \times \{0\} \times \mathbb{R} \subseteq M_\mathbb{R} = \mathbb{R}^3$.

Remark 1.6.2. The 1-dimensional $\mathbb{C}$-vector space $T^1_V(-u)$ gives a first order deformation of V, i.e. an infinitesimal deformation of V over $\mathbb{C}[t_u]/(t_u^2)$. This deformation can be extended to an algebraic deformation of V over $\mathbb{C}[t_u]$ as follows (see [3]).

Consider the unique non-trivial Minkowski decomposition of F (see Figure 1.6):

$$F = \operatorname{conv}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right) + \operatorname{conv}\left(\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right). \quad (1.5)$$

Let $\widetilde{F}$ be the Cayley polytope associated to this Minkowski sum; $\widetilde{F}$ is a 3-dimensional lattice polytope. Let $\tilde{\sigma}$ be the cone over $\widetilde{F}$ at height 1, i.e. $\tilde{\sigma}$ is the 4-dimensional cone generated by

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}.$$

Let $\widetilde{V}$ be the Gorenstein toric affine 4-fold $\widetilde{V}$ associated to $\tilde{\sigma}$. Consider the difference of the two regular functions on $\widetilde{V}$ associated to the characters $(0,0,1,0)$ and $(0,0,0,1)$; if we consider this regular function on

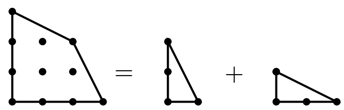

Figure 1.6 The Minkowski decomposition (1.5) of the quadrilateral F defined in (1.4)

$\widetilde{V}$ as a morphism $\widetilde{V} \to \mathbb{A}^1$, we obtain the following cartesian diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \widetilde{V} \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

which gives the wanted 1-parameter deformation of V.

Remark 1.6.3. For every $m \in \{m^1_{p,q}, m^2_{p,q}\}$ with $(p,q) \in J$, the first order deformation of V corresponding to $T^1_V(-m) \simeq \mathbb{C}$ can be extended to an algebraic deformation of V over $\mathbb{C}[t_m]$ thanks to [5, Theorem 3.4] (see also [27, 29]).

1.6.3 The obstruction space

We now consider the obstruction space of the deformation functor of V: $T^2_V = \mathrm{Ext}^2_{\mathcal{O}_V}(\Omega^1_V, \mathcal{O}_V)$. This is an M-graded $\mathbb{C}$-vector space. For all $m \in M$ we denote by $T^2_V(-m)$ the direct summand of degree $-m$.

Lemma 1.6.4. *If* $m \in \{4u-s_1,\ 4u-s_2,\ 5u-s_1-s_2,\ 6u-s_1-s_2,\ 9u-2s_1-2s_2\}$*, then* $\dim T^2_V(-m) = 1$*. Otherwise* $\dim T^2_V(-m) = 0$.

Proof This is a direct computation using formulae in [6, Section 5]. □

Remark 1.6.5. It immediately follows from the computation in [6, Section 5] that $T^2_V(-m) = 0$ if there exists a_i such that $\langle m, a_i \rangle \leq 0$.

1.6.4 Verifying the conjectures

Since $\dim T^2_V = 5$, the ideal of the closed embedding $\mathrm{Def}\, V \hookrightarrow T^1_V$ has at most 5 generators. We have the following:

Proposition 1.6.6. *The equations of the closed embedding* $\mathrm{Def}\, V \hookrightarrow T^1_V$

are

$$\begin{aligned} t_{m^1_{3,1}} t_u &= 0, \\ t_{m^2_{3,1}} t_u &= 0, \\ t_{m^1_{3,1}} t_{m^2_{2,1}} + t_{m^1_{2,1}} t_{m^2_{3,1}} &= 0, \\ t_{m^1_{3,1}} t_{m^2_{3,1}} &= 0, \\ t^2_{m^2_{3,1}} t_{m^1_{3,2}} - t^2_{m^1_{3,1}} t_{m^2_{3,2}} &= 0. \end{aligned}$$

Moreover, $\operatorname{Def} V$ *is non-reduced and has exactly 3 irreducible components; their equations inside* T^1_V *are:*

(1) $t_{m^1_{3,1}} = t_{m^2_{3,1}} = 0$,
(2) $t_u = t_{m^1_{3,1}} = t_{m^1_{2,1}} = t_{m^1_{3,2}} = 0$,
(3) $t_u = t_{m^2_{3,1}} = t_{m^2_{2,1}} = t_{m^2_{3,2}} = 0$.

Every irreducible component of $\operatorname{Def} V$ *is smooth and is a smoothing component.*

Proof The proof of the equations of $\operatorname{Def} V \hookrightarrow T^1_V$ is postponed to the next section and relies on some computer calculations performed with Macaulay2. We now assume to know these equations.

The fact that $\operatorname{Def} V$ is non-reduced and has 3 irreducible components C_1, C_2, C_3 with the equations given above can be checked by taking the primary decomposition of the ideal of $\operatorname{Def} V \hookrightarrow T^1_V$. For each $i = 1, 2, 3$, from the equations of C_i it is obvious that C_i is smooth. We need to prove that C_i is a smoothing component, i.e. the general fibre over C_i is smooth.

The component C_1 contains the 1-parameter deformation constructed in Remark 1.6.2. The singular locus of the general fibre of this deformation has 2 connected components with everywhere transverse A_2-singularities; therefore the general fibre of this deformation is smoothable.

In order to prove that the general fibre over C_2 (resp. C_3) is smooth, we prove that the general fibre over the 2-parameter deformation of V with parameters $t_{m^2_{3,1}}$ and $t_{m^1_{2,2}}$ (resp. $t_{m^1_{3,1}}$ and $t_{m^2_{2,2}}$) is smooth. This can be done by applying the jacobian criterion to the output of the computer calculations that we will describe below. □

We now illustrate Conjecture A and Conjecture B in our example. Let C_1, C_2, C_3 be the 3 irreducible components of $\operatorname{Def} V$, whose equations are given in Proposition 1.6.6. By Remark 1.2.3 $\operatorname{Def}(V, \partial V)$ has 3 irreducible components, $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}_3$, each of which lie over exactly one of C_1, C_2, C_3. For each $i \in \{1,2,3\}$ the smooth morphism $\mathscr{M}_i \to C_i$ induces a surjective linear map $T_0\mathscr{M}_i \to T_0 C_i$ of linear representations of the torus $\operatorname{Spec} \mathbb{C}[M]$.

Let α, β and γ be the three 0-mutable polynomials with Newton polytope F (see Example 1.2.12). By comparing the degrees of T_0C_1, T_0C_2, T_0C_3 with the seeds $\widetilde{\mathscr{S}}(\alpha)$, $\widetilde{\mathscr{S}}(\beta)$, $\widetilde{\mathscr{S}}(\gamma)$ in Example 1.4.3, we have that α (resp. β, resp. γ) corresponds to $\mathscr{M}_1$ (resp. $\mathscr{M}_2$, resp. $\mathscr{M}_3$).

1.6.5 Computer computations

Here we present a proof of Proposition 1.6.6 which uses the software Macaulay2 [17], in particular the package `VersalDeformations` [25, 31].

By observing the degrees of T^1_V (Lemma 1.6.1) and the degrees of T^2_V (Lemma 1.6.4) it is immediate to see that each of the 5 equations of $\operatorname{Def} V \hookrightarrow T^1_V$ can only involve the following 9 variables:

$$t_u \quad t_{m^1_{3,1}} \quad t_{m^1_{2,1}} \quad t_{m^1_{3,2}} \quad t_{m^1_{2,2}} \quad t_{m^2_{3,1}} \quad t_{m^2_{2,1}} \quad t_{m^2_{3,2}} \quad t_{m^2_{2,2}}.$$

We call the corresponding 9 degrees of T^1_V the ‘interesting’ degrees of T^1_V. This implies that there is a smooth morphism $\operatorname{Def} V \to G$, where G is a finite dimensional germ with embedding dimension 9. We now want to use the computer to determine G.

We consider the vector

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \in N = \mathbb{Z}^3.$$

This gives a homomorphism $M \to \mathbb{Z}$ and a $\mathbb{Z}$-grading on on the algebra $\mathbb{C}[\sigma^\vee \cap M] = H^0(V, \mathcal{O}_V)$, on T^1_V, and on T^2_V. We have chosen this particular $\mathbb{Z}$-grading because the corresponding linear projection is injective on the set $\{u, m^1_{3,1}, m^1_{2,1}, m^1_{3,2}, m^1_{2,2}, m^2_{3,1}, m^2_{2,1}, m^2_{3,2}, m^2_{2,2}\}$, which will allow us to identify our 9 variables above with the corresponding output of Macaulay2 below. In the following tables we write down the degrees in $\mathbb{Z}$ of the Hilbert basis of $\sigma^\vee \cap M$, of the interesting degrees of T^1_V, and of the degrees of T^2_V.

s_1	z_2	s_4	z_3	s_3	z_4	s_2	u
9	7	5	3	4	6	8	5

$-u$	$-m^1_{3,1}$	$-m^1_{2,1}$	$-m^1_{3,2}$	$-m^1_{2,2}$	$-m^2_{3,1}$	$-m^2_{2,1}$	$-m^2_{3,2}$	$-m^2_{2,2}$
-5	-6	-1	3	8	-7	-2	1	6

$4u - s_1$	$4u - s_2$	$5u - s_1 - s_2$	$6u - s_1 - s_2$	$9u - 2s_1 - 2s_2$
-11	-12	-8	-13	-11

One can see that all non-interesting summands of T^1_V have degree ≥ 9. Therefore we are interested in the summands of T^1_V with degree between -7 and 8. Now we run the following Macaulay2 code, which was suggested to us by the referee.

```
S = QQ[s1,z2,s4,z3,s3,z4,s2,u,Degrees=>{9,7,5,3,4,6,8,5}];
M = matrix {{s1,z2,u,s2,z4},{z2,s4,z3,z4,s3}};
I = minors(2,M) +
    ideal(s4*s3-z3^3,z2*s3-z3^2*u,z2*z4-z3*u^2,s1*z4-u^3);
needsPackage "VersalDeformations"
T1 = cotangentCohomology1(-7,8,I)
T2 = cotangentCohomology2(I)
(F,R,G,C) = versalDeformation(gens(I),T1,T2);
G
```

The output `T1` describes how the equations of $V \hookrightarrow \mathbb{A}^8$ are perturbed, at the first order, by the coordinates $t_1, \ldots, t_9$ of the interesting part of T^1_V. From these perturbations one can compute the degrees of these coordinates and discover the following conversion table between our notation and the output of Macaulay2.

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
$t_{m^2_{3,1}}$	$t_{m^2_{3,2}}$	$t_{m^2_{2,2}}$	$t_{m^2_{2,1}}$	$t_{m^1_{3,1}}$	$t_{m^1_{3,2}}$	$t_{m^1_{2,2}}$	$t_{m^1_{2,1}}$	t_u

The output `G` describes the miniversal deformation space of V with degrees between -7 and 8, i.e. the germ G we wanted to study. This implies that G is the germ at the origin of the closed subscheme of $\mathbb{A}^9$ defined by the following equations:

$$\begin{aligned}
t_5 t_9 &= 0, \\
t_1 t_9 &= 0, \\
t_4 t_5 + t_1 t_8 &= 0, \\
t_1 t_5 &= 0, \\
t_2 t_5^2 - t_1^2 t_6 &= 0.
\end{aligned}$$

These equations are those in Proposition 1.6.6. The output `F` gives the equations of the deformation of V over the germ G.

Remark 1.6.7. The equations of the germ G are only well defined up to a homogeneous change of coordinates whose jacobian is the identity.

In particular, the quadratic terms of these equations are well defined and can be computed by analysing the cup product $T^1_V \otimes T^1_V \to T^2_V$: this can be done via toric methods [12, 13].

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