

Scalar amplitudes from fiber bundle geometry

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We compute tree-level n -point scattering amplitudes in scalar field theories in terms of geometric invariants on a fiber bundle. All 0- and 2-derivative interactions are incorporated into a metric on this bundle. The on-shell amplitudes can be efficiently pieced together from covariant Feynman rules, and we present a general closed formula for obtaining the n -point amplitude in this way. The covariant Feynman rules themselves can be derived using a generalization of the normal coordinate expansion of the fiber bundle metric. We demonstrate the efficiency of this approach by computing the covariant Feynman rules up to $n = 10$ points, from which one can obtain the full amplitudes using our general formula. The formalism offers a prototype for obtaining geometric amplitudes in theories with *higher*-derivative interactions, by passing from the fiber bundle to its jet bundles.

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I. INTRODUCTION

The use of geometrical methods to characterize quantum field theories (QFTs) of scalar fields ϕ^i is rooted in the observation that the kinetic terms plus interactions with exactly two spacetime derivatives can be cast in a universal form $\mathcal{L} \supset \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^j g_{ij}(\phi)$, that defines a metric with components g_{ij} on a Riemannian manifold \mathcal{M} [1–3]. A key insight of this field space geometry formalism is that on-shell scattering amplitudes can be expressed using geometric objects which are covariant under nonderivative field redefinitions. This feature makes it attractive for the characterization of theories with nonlinear field representations [4–8], notably the Higgs effective field theory (HEFT) [9–14]. This has triggered a revival of geometric techniques, that have been extended to gauge and fermion fields [15–20], applied to the derivation of soft theorems [21–24], to EFT renormalization [25–29] and EFT matching [30].

Arguably, this field space geometry approach suffers from two main limitations: (i) interactions with more (or fewer) than two derivatives are not assigned a geometric interpretation, and (ii) covariance is not manifest under

derivative field redefinitions. Overcoming these two challenges would be a big step forward in applying geometrical methods to truly general EFTs, and would solidify the interpretation of geometric objects (such as Riemann tensors and their derivatives) as measurable quantities. Several resolutions have been recently put forth, notably the functional geometry approach which introduces a momentum-dependent metric [16,22,31–35]. Other approaches modify the fundamental field space, considering Lagrange spaces [36] or jet bundles [37,38].

In this work we extend the jet bundle formalism we proposed in Ref. [38], by applying it to the concrete computation of scattering amplitudes. In fact, to make progress in this direction we retreat from the use of metrics on the r th-jet bundle (for some $r \in \mathbb{Z}_{\geq 0}$), which was shown to capture all scalar EFT Lagrangians with up to $2(r+1)$ -derivatives [38], and focus on the simplest case of the 0-jet bundle. In this case, the scalar field is a section of a fiber bundle whose base is Minkowski spacetime Σ and whose fiber $F \cong \mathcal{M}$. By passing from \mathcal{M} to the bundle, we keep track of the spacetime dependence of the field, and we capture both the scalar potential and the full 2-derivative action via a metric on this bundle.

Our main goal in this paper is to showcase the power of this formalism, even at the 0-jet order, by presenting new expressions for general n -point scattering amplitudes as functions of geometric tensors on the fiber bundle (Sec. IV). We show how tree-level on-shell amplitudes can be efficiently computed via covariant Feynman rules, an old notion [4,39,40] that was recently revived in [31], removing significant computational hurdles associated to the nontensorial components of traditional Feynman rules. These covariant Feynman rules can be efficiently derived in

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our formalism from the Taylor expansion of the fiber bundle metric. We push our explicit results to $n = 10$ points, and present a new formula for obtaining the full n -point amplitude from its covariant building blocks [41].

In future work, we will extend this formalism beyond tree-level, and to higher r -jet order. In that setting, the entire EFT basis up to arbitrary fixed order in the derivative expansion is captured by a fiber bundle metric [38]. Thus, we expect the same techniques we develop here, for passing from bundle metric to amplitudes, should deliver geometric formulas for amplitudes in theories with higher derivatives. The present paper lays important technical groundwork toward achieving this goal.

II. FORMALISM

In this section we review general properties of scattering amplitudes in scalar theories, taking care to highlight covariance properties. This motivates our fiber bundle formalism for describing theories with 0- and 2-derivative interactions; the fact that all couplings are subsumed consistently into a single metric tensor provides an efficient formalism for building amplitudes out of all appropriate covariants that we can derive from that metric.

III. AMPLITUDES AND THEIR COVARIANCE

Consider the quantum dynamics of a set of scalar fields $\phi^i(x)$, $i = 1 \dots N$, that we assume can be specified by an action functional $S[\phi^i(x)]$. For an n -particle scattering $\phi^{a_1}(p_1) \dots \phi^{a_m}(p_m) \rightarrow \phi^{a_{m+1}}(-p_{m+1}) \dots \phi^{a_n}(-p_n)$, where a_i are fixed flavor indices valued in $1 \dots N$ and p_i are the ingoing four-momenta, the \mathcal{S} -matrix element is given by the Lehmann-Symanzik-Zimmermann (LSZ) formula,

$$\langle p_n \dots p_{m+1} | i\mathcal{T} | p_1 \dots p_m \rangle_{a_1 \dots a_n} = (2\pi)^4 \delta^{(4)} \left(\sum p_i \right) \mathcal{A}_{a_1 \dots a_n} \quad (1)$$

where $\mathcal{S} = 1 - i\mathcal{T}$ and $\mathcal{A}_{a_1 \dots a_n}$ is the connected, amputated n -point function. The standard procedure to calculate $\mathcal{A}_{a_1 \dots a_n}$ is to first obtain the effective action $\Gamma[\phi] = S[\phi] + (i/2) \log \det (-\delta^2 S / \delta\phi\delta\phi) + \dots$. From there, the one-particle-irreducible (1PI) k -point functions are the functional derivatives $\delta^k(i\Gamma) / \delta\phi^{i_1}(x_1) \dots \delta\phi^{i_k}(x_k) |_{\phi=\phi_{\text{cl}}}$, where $\phi_{\text{cl}} = \langle 0 | \phi | 0 \rangle$ is the classical field configuration defined by the vacuum condition $\delta\Gamma / \delta\phi |_{\phi=\phi_{\text{cl}}} = 0$. Of particular importance, at $k = 2$ we have that

$$\frac{\delta^2(i\Gamma)}{\delta\phi^i(x_i)\delta\phi^j(x_j)} \Big|_{\phi=\phi_{\text{cl}}} =: -(D_F)_{ij}^{-1}(x_i - x_j) \quad (2)$$

defines the inverse Feynman propagator in position space. Under a field redefinition $\phi^i(x) \mapsto \psi^i(x)$, twice applying the chain rule tells us

$$\frac{i\delta^2\Gamma[\psi]}{\delta\psi^a\delta\psi^b} \Big|_{\psi_{\text{cl}}} = U^i{}_a U^j{}_b \frac{i\delta^2\Gamma[\phi]}{\delta\phi^i\delta\phi^j} \Big|_{\phi_{\text{cl}}}, \quad U^i{}_a := \frac{\delta\phi^a}{\delta\psi^i} \Big|_{\psi_{\text{cl}}}, \quad (3)$$

iff we impose the vacuum condition $\delta\Gamma / \delta\psi |_{\psi=\psi_{\text{cl}}} = 0$ [42]. Thus the propagator transforms as a symmetric tensor under field redefinitions, up to the vacuum condition [33].

Starting from propagators and the 1PI k -point functions, the connected n -point functions $\mathcal{A}_{a_1 \dots a_n}$ are then built by Wick-contracting 1PI k -point functions (with $3 \leq k \leq n$) via D_F internal lines, forming all possible tree-level topologies that make diagrams with n external legs. Unlike inverse propagators, 1PI k -point functions with $k \geq 3$ are not tensors. However, the amputated connected functions $\mathcal{A}_{a_1 \dots a_n}$ do transform as tensors of type $(0, n)$ under field redefinitions, i.e.,

$$\mathcal{A}_{a_1 \dots a_n} = U^i{}_a \dots U^j{}_n \mathcal{A}_{i_1 \dots i_n}, \quad (4)$$

provided we impose (i) the vacuum condition $\delta\Gamma / \delta\psi |_{\psi_{\text{cl}}} = 0$, and (ii) the on-shell condition for all the external legs, loosely that (see Sec. IV A 1)

$$\frac{\delta^2\Gamma[\psi]}{\delta\psi\delta\psi} \Big|_{\psi=\psi_{\text{cl}}, p^2=m^2} = 0. \quad (5)$$

This highly nontrivial property follows from an intricate set of cancellations of the nontensorial parts between different Feynman graphs. As will be shown in [41], and in accordance with the recursive argument of [33], these cancellations occur at arbitrary finite n essentially because the Feynman graph expansion—itsself a property of the path integral—is a nested diagrammatic representation of the Faà di Bruno formula for applying the chain rule k times

$$\frac{\delta^k}{\delta\phi^k} \Gamma(\psi(\phi)) = \sum_{l=0}^k \Gamma^{(l)}(\psi) \cdot B_{k,l}(\psi^{(1)}, \dots, \psi^{(k-l+1)}), \quad (6)$$

where $f^{(n)}(x) \equiv \delta^n f(x) / \delta x^n$, and $B_{n,k}$ are the partial exponential Bell polynomials. In Appendix C we illustrate how this works in the stripped-down context of a 0-d scalar theory. Inserting (4) into the LSZ formula (1) ensures that, for fixed indices $a_1 \dots a_n$, i.e., fixed physical external states, the matrix element is *invariant* under field redefinitions—a textbook result in QFT [43,44].

The same combinatoric technology set out in [41] can be used to establish the important corollaries:

- (i) That the on-shell, tree-level amplitudes can be equivalently derived using *covariant Feynman rules*, as introduced in [4,31,39,40], that coincide with canonical Feynman rules up to nontensorial terms;
- (ii) That the on-shell n -point amplitude $\mathcal{A}_{a_1 \dots a_n}$, henceforth abbreviated to \mathcal{A}_n , can be obtained from these covariant Feynman rules via the following closed formula, that is to our knowledge new [41]

$$\mathcal{A}_n = \sum_{k=1}^{n-2} \frac{(n+k-2)!}{(n-2)!} \Delta^{1-k} \times B_{n-2,k} \left(\frac{1}{2} \mathcal{R}_3, \dots, \frac{1}{n-k} \mathcal{R}_{n-k+1} \right) \quad (7)$$

where Δ is the momentum space propagator (see Sec. IV A 1), $B_{n,k}$ are the same partial Bell polynomials appearing in the Faà di Bruno formula above, and \mathcal{R}_l denotes the l -point covariant Feynman rule, whose l indices (and how they are contracted with the propagator indices) we have suppressed for brevity. A symmetrization over these indices is implied. Special cases of this general formula have been discussed in, e.g., [45,46].

A central result of the present paper (see Sec. IV) is to reveal the structure of the covariant Feynman rules $\mathcal{R}^{(l)}$ for scalar theories with 0- and 2-derivative terms, and to efficiently put them together into amplitudes with many external legs using (7).

A. Geometric theories via fiber bundles

The statements so far regarding covariance hold for general QFTs. They become especially powerful for theories in which we can systematically build objects that are covariant under (possibly a subset of) field redefinitions. This is where geometrical methods enter the game. In short, if the couplings of a QFT can be packaged into a small number of tensors on some auxiliary target space, then we can hope to build covariant objects (under field redefinitions) out of tensors on that target space, and thence build amplitudes. A theory of Goldstone bosons admits such a treatment: at leading order in the derivative expansion, the relevant interactions feature two spacetime derivatives and can be obtained from a metric on a Riemannian manifold \mathcal{M} . The amplitudes are built from the associated Riemann tensor and covariant derivatives thereof [9,10,21–24].

In [38] we showed that, via a generalization of this formalism, one can incorporate 0-derivative interactions (and masses) into a target geometry. To wit, we describe N scalar fields $\phi^i(x)$ using a fiber bundle (E, Σ, π) ; here E is the total space, Σ the base space which we take to be flat Minkowski spacetime, and $\pi: E \rightarrow \Sigma$ is a surjective submersion called the projection map (see Fig. 1). For an open neighborhood $\mathcal{U}_x \subset \Sigma$ containing a point $x \in \Sigma$, there exists a local trivialization $\varphi: \pi^{-1}(\mathcal{U}_x) \mapsto \mathcal{U}_x \times \mathcal{M}$, where \mathcal{M} is itself a manifold called the *fiber*, which is the target space of our scalar fields. The dimension of \mathcal{M} equals the number of real scalar degrees of freedom. We introduce local fibered coordinates (x^μ, u^i) on $\pi^{-1}(\mathcal{U}_x)$, with x^μ being coordinates on the base Σ and u^i coordinatizing \mathcal{M} .

A field configuration is a section of this bundle, that is a smooth map $\phi: \Sigma \rightarrow E$ such that $\pi \circ \phi = \text{id}_\Sigma$. We let $\Gamma(\pi)$ denote the set of all sections of π . It is the

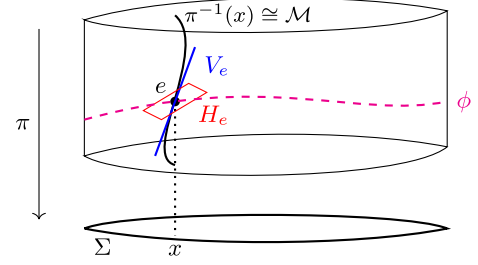


FIG. 1. Illustration of the fiber bundle geometry (E, π, Σ) , with a section ϕ labeled that passes through a point $e \in \pi^{-1}(x)$ in the bundle, at which the vertical space V_e and a choice of horizontal space H_e are also sketched.

(infinite-dimensional) space of all smooth field configurations. The arguments of Sec. III imply the amplitudes $\mathcal{A}_{a_1 \dots a_n}$ ought to be tensors on $\Gamma(\pi)$, up to imposing the vacuum and on-shell conditions.

To build a Lagrangian, we equip the fiber bundle (E, Σ, π) with a pseudo-Riemannian metric g . In our fibered coordinate system, g takes the block diagonal form

$$g = g_{\mu\nu}(u) dx^\mu \otimes dx^\nu + g_{ij}(u) du^i \otimes du^j. \quad (8)$$

Poincaré invariance has been used to (i) set the mixed components $\sim dx^\mu \otimes du^i$ vanish, and (ii) enforce $\partial_\mu g_{\rho\sigma} = \partial_\mu g_{ij} = 0$. The Lagrangian density, evaluated for a particular field configuration, i.e., section ϕ , is then obtained by pulling back g from E to spacetime Σ along ϕ before contracting with the inverse spacetime metric

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \langle \eta^{-1}, \phi^*(g) \rangle \\ &= \frac{\Lambda^4}{2} \eta^{\mu\nu} g_{\mu\nu}(\phi(x)) + \frac{1}{2} g_{ij}(\phi(x)) \partial_\rho \phi^i(x) \partial^\rho \phi^j(x), \end{aligned} \quad (9)$$

where $\phi^i := u^i \circ \phi$. Then the action functional is $S[\phi] = \int_\Sigma \mathcal{L} \mu_\Sigma$, where $\mu_\Sigma = d^4x$ is the volume form on (Σ, η) . Choosing $g_{\mu\nu}(u_i) = -\frac{1}{2} \eta_{\mu\nu} V(u_i)$ ensures the first term equals $-\Lambda^4 V(\phi_i)$ for some smooth potential function V . Equation (9) is then the most general scalar EFT Lagrangian with ≤ 2 derivatives. Indeed, we showed in [38] that by extending the bundle E to its jet bundles $J^r(E)$, which are a sequence of fiber bundles measuring higher derivatives of sections ϕ , the entire EFT Lagrangian with up to $2(r+1)$ derivatives can be obtained by pulling back a metric, as in (9). Many of the techniques we develop in this paper for passing from a bundle metric to amplitudes should find a home in this higher-derivative setting.

We hereon assume a normalization $V(0) = 1$ for the cosmological constant, but our results generalise to other values. [Note, however, that $V(0) \neq 0$ is required for g to be nonsingular at $u^i = 0$.] The vacuum condition $\delta\Gamma/\delta\phi|_{\phi_{\text{cl}}} = 0$ implies $\partial_i g_{\mu\nu} = 0 \Rightarrow \partial_i V|_{\phi_{\text{cl}}} = 0$, that ϕ_{cl} should minimize the potential.

B. Amplitude decomposition via vertical tensors

Now that we have a geometric theory, with all couplings packaged into a metric g on E , we can systematically build covariant objects on $\Gamma(\pi)$ out of tensors on E . As an illustrative example (not yet related to an amplitude), from g we can define a metric G on $\Gamma(\pi)$ by pointwise evaluation followed by integration over Σ : declare $G_\phi(\delta_1\phi, \delta_2\phi) := \int_\Sigma g_{\phi(x)}(\delta_1\phi(x), \delta_2\phi(x))\mu_\Sigma$ for any pair of tangent vectors $\delta_1\phi, \delta_2\phi \in T_\phi\Gamma(\pi)$ [47].

Via the same procedure, covariant amplitudes \mathcal{A}_n can be built from tensors on the fiber bundle E . Also Taylor expanding in the Mandelstams $s_{ij} = p_i \cdot p_j$ to $\mathcal{O}(s)$ and imposing $\Gamma^{(1)} = \Gamma^{(2)} = 0$, we can write

$$\begin{aligned} \mathcal{A}_n(\delta_1\phi, \dots, \delta_n\phi; s) &= \sum_\alpha \int_\Sigma T_{\phi(x)}^{0,\alpha}(\delta_1\phi(x), \dots, \delta_n\phi(x))\mu_\Sigma \\ &+ \sum_{ij} s_{ij} \sum_\alpha \int_\Sigma T_{\phi(x)}^{ij,\alpha}(\delta_1\phi(x), \dots, \delta_n\phi(x))\mu_\Sigma \quad (10) \end{aligned}$$

where each α sums over some tensor structures T on E with the appropriate mass dimension. Specifically, Lorentz invariance tells us that it is only *vertical tensors* on E that can appear in \mathcal{A}_n , which we define in the Appendix D. Colloquially speaking, vertical tensors only have non-vanishing components in the fiber directions (see Fig. 1); any μ indices must be contracted.

Poincaré invariance moreover implies $\partial_\mu g_{\rho\sigma} = \partial_\mu g_{ij} = 0$, so we can further restrict to vertical tensors obtained by pulling back tensors on the fiber \mathcal{M} . The integrals over Σ in (10) are then redundant, simply averaging over constant functions—indeed the classical configuration we evaluate on is also constant, meaning we only pick out the value of each vertical tensor at the vacuum point.

Despite restricting to vertical tensors, including the base degrees of freedom in our bundle E allows more geometric invariants to be constructed than if we only had a map to field space and a metric thereon. This is because the Lagrangian (9) encodes the potential V through its $g_{\mu\nu}$ components: we can build vertical, x^μ -independent tensors from g that nevertheless depend on V . For example, consider the vertical tensor with components

$$\tilde{R}_{ij} = R^\mu{}_{\mu j} = \frac{1}{2\Lambda^4} \text{Tr}[\eta] \partial_i \partial_j V, \quad (11)$$

i.e., the mass squared matrix, which will appear in the propagator (see Sec. IV). The precise meaning of the index contraction is explained in the Appendix D. We see from (11) how, in this formalism, derivatives of V measure curvature components in the fiber bundle, and so capture aspects of the geometry of E . In this sense, the potential acquires a geometric meaning.

The sum over such vertical tensors in (10) captures all contributions to \mathcal{A}_n from theories with ≤ 2 derivatives. The explicit tensors that appear, along with their coefficients, will be presented in Sec. IV. This decomposition dictates, by covariance, the dependence of all \mathcal{A}_n on $V(\phi)$ and its derivatives.

IV. TREE-LEVEL ON-SHELL AMPLITUDES

We have so far been schematic in describing how amplitudes can be built from vertical tensors on E . Now we make this concrete, explaining how covariant Feynman rules offer an efficient tool for deriving \mathcal{A}_n to large n .

A. Covariant Feynman rules

As anticipated in Sec. III, tree-level on-shell scattering amplitudes \mathcal{A}_n can be efficiently computed using the general formula (7). The input we need to evaluate this finite sum is the set of *covariant Feynman rules*, that we denote $\mathcal{R}_{a_1\dots a_k}$ (or \mathcal{R}_k for brevity), for each integer $k \in [3, n]$. This notion was revived in [31] using functional geometry; in that language, \mathcal{R}_k can be defined as the k th covariant functional derivative of Γ . The \mathcal{R}_k coincide with the usual Feynman rules $F_{a_1\dots a_k}$ up to nontensor terms $N_{a_1\dots a_k}$

$$F_{a_1\dots a_k}(p_i) = \mathcal{R}_{a_1\dots a_k}(p_i) + N_{a_1\dots a_k}(p_i), \quad (12)$$

where p_i represent the external momenta.

Computing the covariant Feynman rules $\mathcal{R}_{a_1\dots a_n}$ becomes especially tractable for a geometric theory such as ours. In the fiber bundle formalism, the $\mathcal{R}_{a_1\dots a_n}$ are vertical tensors on (E, Σ, π) , while $N_{a_1\dots a_n}$ contain various terms proportional to Christoffel symbols and their derivatives, which can be further decomposed as

$$N_{a_1\dots a_n} = N_{a_1\dots a_n}^{(1)} + N_{a_1\dots a_n}^{(2)} + \bar{N}_{a_1\dots a_n}, \quad (13)$$

where $N^{(1)}$ vanishes at the vacuum, $N^{(2)}$ vanishes upon imposing on-shell conditions on all p_i , while \bar{N} remains nonzero. When computing tree-level on-shell scattering amplitudes \mathcal{A}_n via F insertions, the \bar{N} terms and the $N^{(2)}$ terms associated to legs contracted in internal propagators cancel among each other, so the final result depends only on the \mathcal{R}_n structures as in (7). We show how these cancellations play out in practice for the case of $2 \rightarrow 2$ scattering in Sec. IV B below.

The power of our general formula (7) for amplitudes, which holds independently of the fiber bundle formalism, hinges on how easily the \mathcal{R}_n structures can be identified. The fiber bundle picture offers a systematic procedure to determine each \mathcal{R}_n , including all 0- and 2-derivative contributions, via a single Taylor expansion of the metric on the bundle, the details of which are relegated to Appendix A.

1. Covariant propagators

The Feynman propagator $iD_F^{ij}(x-y)$ in Eq. (2) is the Green's function of the Klein-Gordon equation

$$(g_{ij}\square_x + \Lambda^4 R^\mu{}_{i\mu j}/2)iD_F^{jk}(x-y) = \delta_i^k \delta^4(x-y). \quad (14)$$

As a Feynman rule in momentum space, the tree-level propagator and its inverse are

$$\Delta^{a_1 a_2}(p^2) = i(p^2 g_{a_1 a_2} + \Lambda^4 R^\mu{}_{a_1 \mu a_2}/2)^{-1}, \quad (15)$$

$$\Delta_{a_1 a_2}^{-1}(p^2) = -i(p^2 g_{a_1 a_2} + \Lambda^4 R^\mu{}_{a_1 \mu a_2}/2), \quad (16)$$

in a notation such that $(A_{ij})^{-1} = (A^{-1})^{ij}$ for a $(0, 2)$ tensor A . In general Δ^{ij} is neither diagonal nor canonically normalized, but it is always possible to bring it to a canonical form via a coordinate transformation as in Eq. (3). An on-shell condition can only be defined for mass eigenstates, i.e., it is $U_k^i U^j{}_k \Delta_{ij}^{-1} = 0$ for the asymptotic state ϕ_k . In Eqs. (14)–(16) and in all the results of this section, the geometric objects $(g_{a_1 a_2}, R^\mu{}_{a_1 \mu a_2} \dots)$ are understood to be evaluated at the vacuum.

2. Covariant vertices

Using the fiber bundle formalism, we can straightforwardly compute the covariant Feynman rules \mathcal{R}_n to high order, by applying known techniques in differential geometry for Taylor expanding a metric tensor and decomposing the result into tensor and nontensor pieces [48]. We summarize various simplifying tricks that follow from Poincaré invariance in Appendix A. We note that the same results could be obtained with the traditional field space formalism by treating the potential V as a separate function and taking covariant derivatives thereof [31]. In passing to the bundle, all contributions come from a single metric expansion. The results up to $n \leq 5$ are

$$\mathcal{R}_{a_1 a_2 a_3} = i \frac{\Lambda^4}{2} \nabla_{a_1} R^\mu{}_{a_2 \mu a_3}, \quad (17)$$

$$\mathcal{R}_{a_1 a_2 a_3 a_4} = \frac{i}{4!} \sum_{S_4} \left[\frac{\Lambda^4}{2} \nabla_{a_1} \nabla_{a_2} R^\mu{}_{a_3 \mu a_4} - 2\Lambda^4 R^\mu{}_{a_1 \nu a_2} R^\nu{}_{a_3 \mu a_4} - 2s_{12} R_{a_1 a_3 a_4 a_2} \right] \quad (18)$$

$$\mathcal{R}_{a_1 a_2 a_3 a_4 a_5} = \frac{i}{5!} \sum_{S_5} \left[\frac{\Lambda^4}{2} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} R^\mu{}_{a_4 \mu a_5} - 7\Lambda^4 R^\mu{}_{a_1 \nu a_2} \nabla_{a_3} R^\nu{}_{a_4 \mu a_5} - 5s_{12} \nabla_{a_5} R_{a_1 a_4 a_3 a_2} \right] \quad (19)$$

where $s_{ij} = (p_i + p_j)^2$, with p_i the incoming 4-momentum corresponding to a_i . The sums run over all permutations of

the external states S_n , acting simultaneously on flavor indices $a_1 \dots a_n$ and momenta $p_1 \dots p_n$. Note $\mathcal{R}_{a_1 a_2 a_3}^{(3)}$ is invariant under such permutations due to the second Bianchi identity. The expressions for vertices with $n \leq 10$ are provided in Appendix A.

We pause to make several comments. First, we emphasize that both the s -dependent and s -independent parts of the amplitude appear on an equal footing, in that both are covariant derivatives of the Riemann tensor on the fiber bundle. The potential contributions, like those from the kinetic term, can be identified with invariants of the bundle geometry. Second, our covariant Feynman rules in general feature several distinct terms corresponding to $\nabla^n V$, all of which are separately invariant—in that sense, the bundle provides us with a finer set of invariant quantities also known as observables.

B. Covariant tree-level amplitudes

The covariant Feynman rules can be composed into tree-level amplitudes using the general formula (7) that sums over all Feynman diagrams. As a word of caution, note that the compact form of (7) includes a sum over terms with the same \mathcal{R}_n insertions but different index contractions, that correspond to different Feynman diagrams coming with different multiplicities. The details of the combinatoric coefficients that enter, which we record in Fig. 2 for $n = 4, 5, 6$, will be presented in [41]. To illustrate, the first few examples are

$$\mathcal{A}_{a_1 a_2 a_3} = \mathcal{R}_{a_1 a_2 a_3} \quad (20)$$

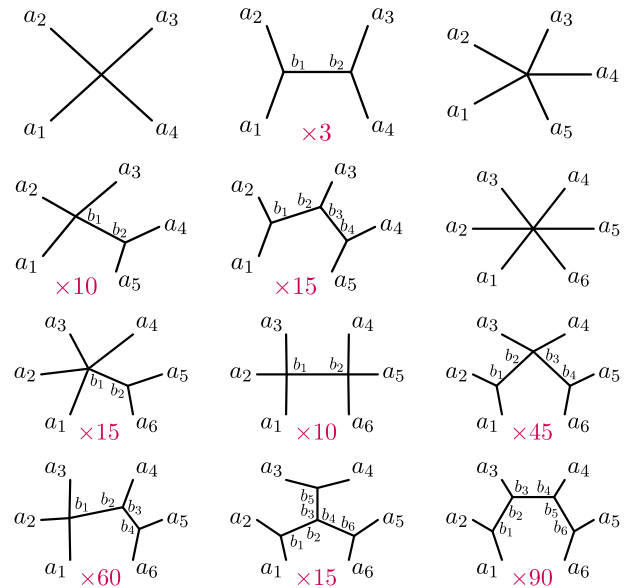


FIG. 2. Diagrams for the construction of on-shell amplitudes \mathcal{A}_n with $n = 4, 5, 6$ via covariant Feynman rules. The number of independent index contractions for each topology is shown in magenta.

$$\mathcal{A}_{a_1 a_2 a_3 a_4} = \mathcal{R}_{a_1 a_2 a_3 a_4} + \frac{1}{4!} \sum_{S_4} [3 \mathcal{R}_{a_1 a_2 b_1} \Delta^{b_1 b_2}(s_{12}) \mathcal{R}_{b_2 a_3 a_4}] \quad (21)$$

$$\begin{aligned} \mathcal{A}_{a_1 a_2 a_3 a_4 a_5} &= \mathcal{R}_{a_1 a_2 a_3 a_4 a_5} + \frac{1}{5!} \sum_{S_5} [10 \mathcal{R}_{a_1 a_2 a_3 b_1} \Delta^{b_1 b_2}(s_{45}) \mathcal{R}_{b_2 a_4 a_5} \\ &\quad + 15 \mathcal{R}_{a_1 a_2 b_1} \Delta^{b_1 b_2}(s_{12}) \mathcal{R}_{a_3 b_2 b_3} \Delta^{b_3 b_4}(s_{45}) \mathcal{R}_{b_4 a_4 a_5}] \quad (22) \end{aligned}$$

The corresponding diagrammatic representations are provided, together with the result for $n = 6$, in Appendix B. Besides combinatorial complexity, there is no conceptual obstacle to extending these formulas to $n \geq 7$.

Our results for $n = 3, 4$ agree with the literature (see, e.g., [12,16,49]) up to the geometrization of the scalar potential. The results for $n \leq 6$ agree with those in [21] in the limit $V \rightarrow 0$ where the two can be compared. To our knowledge, the complete scalar potential contributions for $n \geq 5$ have not been reported before (although formally similar structures appear in 5-point amplitudes for scalar-gauge theories [16]), which highlights the computational gain provided by our methods.

Finally, it is worth highlighting that the expressions of \mathcal{A}_n obtained by gluing covariant Feynman rules make the pole structure of the amplitude more transparent: the poles of \mathcal{A}_n are those of the contributing $\Delta^{ij}(s)$, and any additional s dependence is collected in the contact terms. This is not necessarily the case when one calculates with conventional Feynman rules, see the example in Appendix F.

1. Cancellation of nontensor terms

It can be instructive to examine the cancellation of nontensorial terms for $2 \rightarrow 2$ scattering. The contribution of the nontensor pieces to $\mathcal{A}_{a_1 a_2 a_3 a_4}$ can be obtained replacing $\mathcal{R} \rightarrow F$ in Eq. (21), expanding the Feynman rules via (12) and setting the four external legs on-shell. The last step makes the $N_{a_1 a_2 a_3 a_4}^{(2)}$ terms vanish, leaving

$$\begin{aligned} \mathcal{A}_{a_1 a_2 a_3 a_4}^{NT} &= \bar{N}_{a_1 a_2 a_3 a_4} + \frac{3}{4!} \sum_{S_4} \left[N_{a_1 a_2 b_1}^{(2)} \Delta^{b_1 b_2}(s_{12}) \mathcal{R}_{b_2 a_3 a_4} \right. \\ &\quad \left. + \mathcal{R}_{a_1 a_2 b_1} \Delta^{b_1 b_2}(s_{12}) N_{b_2 a_3 a_4}^{(2)} + N_{a_1 a_2 b_1}^{(2)} \Delta^{b_1 b_2}(s_{12}) N_{b_2 a_3 a_4}^{(2)} \right] \quad (23) \end{aligned}$$

We used that $\bar{N}_{a_1 a_2 a_3} = 0$. This can be evaluated using the expressions for the nontensor components of the 3- and 4-point Feynman rules in the fiber bundle picture

$$N_{a_1 a_2 a_3}^{(2)} = - \sum_{S_3} \Gamma_{a_1 a_2}^{b_1} \Delta_{b_1 a_3}^{-1}(p_3^2) \quad (24)$$

$$\begin{aligned} \bar{N}_{a_1 a_2 a_3 a_4} &= \frac{3i}{4!} \sum_{S_4} \left[\frac{\Lambda^4}{2} R^\mu{}_{b_1 \mu b_2} \Gamma_{a_1 a_2}^{b_1} \Gamma_{a_3 a_4}^{b_2} \right. \\ &\quad \left. + s_{12} \Gamma_{b_1 a_1 a_2} \Gamma_{a_3 a_4}^{b_1} + \Lambda^4 \Gamma_{a_1 a_2}^{b_1} \nabla_{a_3} R^\mu{}_{b_1 \mu a_4} \right] \quad (25) \end{aligned}$$

It is easy to check that, inserting these into (23), together with (16) and (17), yields $\mathcal{A}_4^{NT} = 0$. In particular, the contribution from the first line of Eq. (25) cancels against the two contributions of the form $N \Delta \mathcal{R}$ in (23), while the second line cancels the contribution $\sim N \Delta N$.

2. From trees to loops

Finally, the covariant Feynman rules can also be used to obtain the loop corrections to lower-point vertices. To illustrate this, at 1-point and 2-point we have 1-loop corrections coming from the \mathcal{R}_3 and \mathcal{R}_4 vertices, which are

$$\mathcal{A}_{a_1}^{(1)} = \int \frac{d^d k}{(2\pi)^d} \mathcal{R}_{a_1 b_1 b_2} \Delta^{b_1 b_2}(k^2) \quad (26)$$

$$\begin{aligned} \mathcal{A}_{a_1 a_2}^{(1)} &= \frac{1}{2} \sum_{S_2} \int \frac{d^d k}{(2\pi)^d} [\mathcal{R}_{a_1 a_2 b_1 b_1} \Delta^{b_1 b_2}(k^2) \\ &\quad + \mathcal{R}_{a_1 a_2 b_1} \Delta^{b_1 b_2}(0) \mathcal{R}_{b_2 b_3 b_4} \Delta^{b_3 b_4}(k^2) \\ &\quad + \mathcal{R}_{a_1 b_1 b_2} \mathcal{R}_{a_2 b_3 b_4} \Delta^{b_1 b_3}(k^2) \Delta^{b_2 b_4}((p_1 + k)^2)], \quad (27) \end{aligned}$$

matching the results of [31]. Extending this to higher points is beyond the scope of this paper, and will be treated in future work.

V. CONCLUSIONS AND OUTLOOK

On-shell amplitudes in general scalar theories are built from quantities that are covariant under field redefinitions. This paper demonstrated that, for tree-level calculations, one can obtain the amplitudes by adopting covariant Feynman rules from the outset and substituting these into a closed formula valid for any n -point amplitude. For Lagrangians with generic 0- and 2-derivative interactions we show how these covariant Feynman rules can be efficiently derived from a metric on a fiber bundle, to high order in n .

Major computational obstacles due to the presence of nontensorial terms in canonical Feynman rules are removed. This enables the calculation of amplitudes with many external legs, which we compute up to $n = 10$. The resulting expressions are universal, and can be specialized to a given scalar theory by computing the relevant covariants as functions of its Lagrangian parameters (see Appendixes E and F).

For fixed external states, the covariant Feynman rules are individually invariant, and so can themselves be interpreted as observable quantities associated to fundamental properties of the theory. In particular, two theories are physically equivalent *iff* their covariant Feynman rules, computed in a mass-eigenstate basis, are identical. In this sense, covariant Feynman rules could be used to test the equivalence of a set of theories, without identifying an explicit mapping among them, nor computing a full process. Furthermore, their calculation can be used to cast effective Lagrangians in simpler forms, see, e.g., the example in Appendix F.

These results constitute an important stepping stone toward the geometric interpretation of general EFTs. The next steps forward, that we will pursue in future work, include the extension of our formalism to loop amplitudes, including fermion and gauge fields, and the systematic inclusion of higher-derivative interactions by going from the fiber bundle to its r -jet bundle $J^r(E)$. In that case, the entire EFT Lagrangian with up to $2(r+1)$ derivatives is obtained by pulling back a metric on $J^r(E)$ [38], and so the techniques developed here for efficiently building amplitudes starting from a single metric tensor should for the most part carry over.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: COVARIANT FEYNMAN RULES FROM FIBER BUNDLE GEOMETRY

The ‘‘canonical’’ Feynman rules $F_{a_1\dots a_n}$ are n -point 1PI correlation functions derived from the tree-level effective action. In the fiber bundle formalism, this yields

$$F_{a_1\dots a_n} = i \sum_{S_n} \left[\frac{\Lambda^4}{2} \eta^{\mu\nu} \partial_{a_1} \dots \partial_{a_n} g_{\mu\nu} - \frac{1}{2} p_1 \cdot p_2 \partial_{a_3} \dots \partial_{a_n} g_{a_1 a_2} \right]_{\phi_{\text{cl}}} \quad (\text{A1})$$

Covariant Feynman rules $\mathcal{R}_{a_1\dots a_n}$ can be obtained from here by expanding the metric and retaining only the covariant piece. The symmetries of the Riemann tensor ensure that the two blocks of the metric do not mix.

In particular expressions such as $R_{a_1\nu a_j}^\mu R_{a_k a_l a_r a_s}$ vanish upon symmetrizing the expression in all its indices. Additionally note that $p_i^2 R_{a_i a_j a_k a_l}$ also vanishes after summing over permutations, due to the Bianchi identity. Terms proportional to Γ can safely be dropped as $\Gamma_{a_i\nu}^\mu|_{\text{vac}} = 0$ and there are no 3-index tensors when using the Levi-Civita connection, so $\Gamma_{a_i a_j a_k}$ terms will always cancel. In the derivation of covariant rules we implement substitutions that are equal up to powers of Γ .

To derive the $g_{\mu\nu}$ contributions to the covariant Feynman rules we use a number of simplifying tricks. First, we exchange derivatives of the metric for Christoffel symbols; then, using also Poincaré invariance, we find

$$R_{a_i\nu a_j}^\mu = -\partial_{a_j} \Gamma_{a_i\nu}^\mu + \Gamma_{\nu a_m}^\mu \Gamma_{a_i a_j}^{a_m} - \Gamma_{a_j\rho}^\mu \Gamma_{a_i\nu}^\rho \quad (\text{A2})$$

which allows us to define higher derivatives of $\Gamma_{a_i\nu}^\mu$ recursively. Since the blocks of the metric do not mix and only vertical tensors contribute to the amplitude we find $\partial^n R_{a_i\nu a_j}^\mu \rightarrow \nabla^n R_{a_i\nu a_j}^\mu$ for any n .

Now we turn our attention to the contributions from $g_{a_i a_j}$ to the covariant Feynman rules. The symmetries of the Riemann tensor make the theory behave as though it were massless, in that we can replace $p_i \cdot p_j \rightarrow \frac{1}{2} s_{ij}$. Momentum conservation, $\sum_{i \neq j} s_{ij} = 0$, allows us to eliminate n variables from a set of $\binom{n}{2}$. It is convenient to eliminate the variables s_{in} for $1 \leq i \leq n-1$ as well as the variable s_{n-1n-2} , allowing the substitution

$$R_{a_1 a_3 a_4 a_2} + \Gamma_{q a_1 a_4} \Gamma_{a_2 a_3}^q - \Gamma_{q a_1 a_2} \Gamma_{a_3 a_4}^q \quad (\text{A3})$$

$$\rightarrow \frac{1}{2} (g_{a_1 a_2, a_3 a_4} + g_{a_3 a_4, a_1 a_2} - g_{a_2 a_3, a_1 a_4} - g_{a_1 a_4, a_2 a_3}), \quad (\text{A4})$$

which, in addition to using the Bianchi identity, means the contribution to (A1) from the kinetic term reads

$$\sum_{S_n} \frac{1}{n!} \left(\binom{n}{2} - n \right) s_{12} \partial_{a_3} \dots \partial_{a_{n-2}} \times (R_{a_1 a_{n-1} a_n a_2} + \Gamma_{q a_1 a_n} \Gamma_{a_2 a_{n-1}}^q - \Gamma_{q a_1 a_2} \Gamma_{a_n a_{n-1}}^q). \quad (\text{A5})$$

To go to $n \geq 5$ we need to then systematically replace partial derivatives of the Riemann tensor and Christoffel symbols with tensors. To do so we observe that, somewhat intriguingly, momentum conservation imposes similar conditions on the Christoffel symbols as switching to geodesic coordinates would do. For example, of relevance to $n = 4$ points, we have $\partial_{a_4} \Gamma_{a_1 a_2 a_3} (s_{12} + s_{13} + s_{14}) \rightarrow 0$. By permuting indices, this implies $s_{12} (\partial_{a_4} \Gamma_{a_1 a_2 a_3} + \partial_{a_3} \Gamma_{a_1 a_2 a_4} + \partial_{a_2} \Gamma_{a_1 a_4 a_3}) \rightarrow 0$, enabling us to substitute $s_{12} \partial_{a_2} \Gamma_{a_1 a_3 a_4} \rightarrow -\frac{1}{3} s_{12} (R_{a_1 a_3 a_4 a_2} + R_{a_1 a_4 a_3 a_2})$.

This kind of replacement rule can be generalized to arbitrary n , where it reads

$$\sum_{S_{n-2}} s_{12} \binom{n-2}{2} \partial_{a_n} \dots \partial_{a_2} \Gamma_{a_1 a_3 a_4} \rightarrow - \sum_{S_{n-2}} s_{12} \binom{n-2}{1} \partial_{a_n} \dots \partial_{a_4} \Gamma_{a_1 a_2 a_3}. \quad (\text{A6})$$

Finally, we caution that when going beyond $n \geq 6$ it becomes important to distinguish between Christoffel symbols with the first index raised or lowered

$$s_{12} \partial_{a_6} \partial_{a_5} \partial_{a_2} \Gamma_{a_1 a_3 a_4} \rightarrow -s_{12} \left(\frac{3}{5} \nabla_{a_6} \nabla_{a_5} R_{a_1 a_3 a_4 a_2} + \frac{8}{15} R_{q a_5 a_6 a_1} R^q_{a_3 a_4 a_2} \right), \quad (\text{A7})$$

whereas

$$s_{12} \partial_{a_6} \partial_{a_5} \partial_{a_2} \Gamma_{a_3 a_4}^{a_1} \rightarrow -s_{12} \left(\frac{3}{5} \nabla_{a_6} \nabla_{a_5} R^{a_1}_{a_3 a_4 a_2} - \frac{2}{15} R^{a_1}_{a_3 a_4 q} R^q_{a_5 a_6 a_2} \right). \quad (\text{A8})$$

As advertised, these conditions are identical to those we get from going to Riemann normal coordinates, and formally lead to the same metric expansion as verified by comparing our results with [48] up to $n = 10$. However, it is important to reiterate that these conditions in our case originate from imposing the physical condition of momentum conservation, and that regardless of the coordinate system of choice the covariant Feynman rules are not the same as the ‘‘canonical’’ Feynman rules of the theory. However, they both yield the same results for the S-matrix elements.

The results for $6 \leq n \leq 10$ are

$$\begin{aligned} \mathcal{R}_{a_1 \dots a_6} = & \frac{i}{6!} \sum_{S_6} \left[\frac{\Lambda^4}{2} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} R^{\mu}_{a_5 \mu a_6} - 11 \Lambda^4 R^{\mu}_{a_1 \nu a_2} \nabla_{a_3} \nabla_{a_4} R^{\nu}_{a_5 \mu a_6} - 7 \Lambda^4 \nabla_{a_1} R^{\mu}_{a_2 \nu a_3} \nabla_{a_4} R^{\nu}_{a_5 \mu a_6} \right. \\ & \left. + 8 \Lambda^4 R^{\mu}_{a_1 \nu a_2} R^{\nu}_{a_3 \rho a_4} R^{\rho}_{a_5 \mu a_6} - 9 s_{12} \nabla_{a_5} \nabla_{a_6} R_{a_1 a_3 a_2} - 8 s_{12} R_{a_1 a_3 a_4}{}^{b_1} R_{b_1 a_5 a_6 a_2} \right], \quad (\text{A9}) \end{aligned}$$

$$\begin{aligned} \mathcal{R}_{a_1 \dots a_7} = & \frac{i}{7!} \sum_{S_7} \left[\frac{\Lambda^4}{2} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} R^{\mu}_{a_6 \mu a_7} - 16 \Lambda^4 R^{\mu}_{a_1 \nu a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} R^{\nu}_{a_6 \mu a_7} - 25 \Lambda^4 \nabla_{a_1} R^{\mu}_{a_2 \nu a_3} \nabla_{a_4} \nabla_{a_5} R^{\nu}_{a_6 \mu a_7} \right. \\ & \left. + 64 \Lambda^4 R^{\mu}_{a_1 \nu a_2} R^{\nu}_{a_3 \rho a_4} \nabla_{a_5} R^{\rho}_{a_6 \mu a_7} - 14 s_{12} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R_{a_1 a_4 a_3 a_2} - 56 s_{12} R_{a_1 a_3 a_4}{}^{b_1} \nabla_{a_5} R_{b_1 a_6 a_7 a_2} \right], \quad (\text{A10}) \end{aligned}$$

$$\begin{aligned} \mathcal{R}_{a_1 \dots a_8} = & \frac{i}{8!} \sum_{S_8} \left[\frac{\Lambda^4}{2} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} R^{\mu}_{a_7 \mu a_8} - 22 \Lambda^4 R^{\mu}_{a_1 \nu a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} R^{\nu}_{a_7 \mu a_8} \right. \\ & - 41 \Lambda^4 \nabla_{a_1} R^{\mu}_{a_2 \nu a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} R^{\nu}_{a_7 \mu a_8} - 25 \Lambda^4 \nabla_{a_1} \nabla_{a_2} R^{\mu}_{a_3 \nu a_4} \nabla_{a_5} \nabla_{a_6} R^{\nu}_{a_7 \mu a_8} + 144 \Lambda^4 R^{\mu}_{a_1 \nu a_2} R^{\nu}_{a_3 \rho a_4} \nabla_{a_5} \nabla_{a_6} R^{\rho}_{a_7 \mu a_8} \\ & + 182 \Lambda^4 R^{\mu}_{a_1 \nu a_2} \nabla_{a_3} R^{\nu}_{a_4 \rho a_5} \nabla_{a_6} R^{\rho}_{a_7 \mu a_8} - 32 R^{\mu}_{a_1 \nu a_2} R^{\nu}_{a_3 \rho a_4} R^{\mu}_{a_5 \sigma a_6} R^{\sigma}_{a_7 \rho a_8} - 20 s_{12} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} R_{a_1 a_7 a_8 a_2} \\ & \left. - 136 s_{12} R_{a_1 a_3 a_4 b_1} \nabla_{a_5} \nabla_{a_6} R^{b_1}_{a_7 a_8 a_2} - 110 s_{12} \nabla_{a_3} R_{a_1 a_4 a_5}{}^{b_1} \nabla_{a_6} R_{b_1 a_7 a_8 a_2} - 32 s_{12} R_{a_1 a_3 a_4}{}^{b_1} R_{b_1 a_5 a_6}{}^{b_2} R_{a_2 a_7 a_8 b_2} \right], \quad (\text{A11}) \end{aligned}$$

$$\begin{aligned} \mathcal{R}_{a_1 \dots a_9} = & \frac{i}{9!} \sum_{S_9} \left[\frac{\Lambda^4}{2} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R^{\mu}_{a_8 \mu a_9} - 29 \Lambda^4 R^{\mu}_{a_1 \nu a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R^{\nu}_{a_8 \mu a_9} \right. \\ & - 63 \Lambda^4 \nabla_{a_1} R^{\mu}_{a_2 \nu a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R^{\nu}_{a_8 \mu a_9} - 91 \Lambda^4 \nabla_{a_1} \nabla_{a_2} R^{\mu}_{a_3 \nu a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R^{\nu}_{a_8 \mu a_9} \\ & + 284 \Lambda^4 R^{\mu}_{a_1 \nu a_2} R^{\nu}_{a_3 \rho a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R^{\rho}_{a_8 \mu a_9} + 890 \Lambda^4 R^{\mu}_{a_1 \nu a_2} \nabla_{a_3} R^{\nu}_{a_4 \rho a_5} \nabla_{a_6} \nabla_{a_7} R^{\rho}_{a_8 \mu a_9} \\ & + 192 \Lambda^4 \nabla_{a_1} R^{\mu}_{a_2 \nu a_3} \nabla_{a_4} R^{\nu}_{a_5 \rho a_6} \nabla_{a_7} R^{\rho}_{a_8 \mu a_9} - 464 R^{\mu}_{a_1 \nu a_2} R^{\nu}_{a_3 \rho a_4} R^{\mu}_{a_5 \sigma a_6} \nabla_{a_7} R^{\sigma}_{a_8 \rho a_9} \\ & - 27 s_{12} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R_{a_1 a_8 a_9 a_2} - 276 s_{12} R_{a_1 a_3 a_4}{}^{b_1} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} R_{b_1 a_8 a_9 a_2} \\ & \left. - 594 s_{12} \nabla_{a_3} R_{a_1 a_4 a_5}{}^{b_1} \nabla_{a_6} \nabla_{a_7} R_{b_1 a_8 a_9 a_2} - 432 s_{12} R_{b_1 a_3 a_4 a_1} R^{b_1}_{a_5 a_6 b_2} \nabla_{a_7} R^{b_2}_{a_8 a_9 a_2} \right], \quad (\text{A12}) \end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{a_1 \dots a_{10}} = & \frac{i}{10!} \sum_{S_{10}} \left[\frac{\Lambda^4}{2} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^\mu{}_{a_9 \mu a_{10}} - 37 \Lambda^4 R^\mu{}_{a_1 \nu a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^\nu{}_{a_9 \mu a_{10}} \right. \\
& - 92 \Lambda^4 \nabla_{a_1} R^\mu{}_{a_2 \nu a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^\nu{}_{a_9 \mu a_{10}} - 154 \Lambda^4 \nabla_{a_1} \nabla_{a_2} R^\mu{}_{a_3 \nu a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^\nu{}_{a_9 \mu a_{10}} \\
& - 91 \Lambda^4 \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} R^\mu{}_{a_4 \nu a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^\mu{}_{a_9 \nu a_{10}} + 508 \Lambda^4 R^\mu{}_{a_1 \nu a_2} R^\nu{}_{a_3 \rho a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^\rho{}_{a_9 \mu a_{10}} \\
& + 1918 \Lambda^4 R^\mu{}_{a_1 \nu a_2} \nabla_{a_3} R^\nu{}_{a_4 \rho a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^\rho{}_{a_9 \mu a_{10}} + 1436 \Lambda^4 \nabla_{a_1} R^\mu{}_{a_2 \nu a_3} \nabla_{a_4} R^\nu{}_{a_5 \rho a_6} \nabla_{a_7} \nabla_{a_8} R^\rho{}_{a_9 \mu a_{10}} \\
& + 1178 \Lambda^4 R^\mu{}_{a_1 \nu a_2} \nabla_{a_3} \nabla_{a_4} R^\nu{}_{a_5 \rho a_6} \nabla_{a_7} \nabla_{a_8} R^\rho{}_{a_9 \mu a_{10}} - 1360 \Lambda^4 R^\mu{}_{a_1 \nu a_2} R^\nu{}_{a_3 \rho a_4} R^\rho{}_{a_5 \sigma a_6} \nabla_{a_7} \nabla_{a_8} R^\sigma{}_{a_9 \mu a_{10}} \\
& - 2648 \Lambda^4 R^\mu{}_{a_1 \nu a_2} R^\nu{}_{a_3 \rho a_4} \nabla_{a_5} R^\rho{}_{a_6 \sigma a_7} \nabla_{a_8} R^\sigma{}_{a_9 \mu a_{10}} + 128 \Lambda^4 R^\mu{}_{a_1 \nu a_2} R^\nu{}_{a_3 \rho a_4} R^\rho{}_{a_5 \sigma a_6} R^\sigma{}_{a_7 a_8} R^\alpha{}_{a_9 \mu a_{10}} \\
& - 35 s_{12} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R_{a_1 a_9 a_{10} a_2} - 500 s_{12} R_{a_1 a_3 a_4 b_1} \nabla_{a_5} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^{b_1}{}_{a_9 a_{10} a_2} \\
& - 1330 s_{12} \nabla_{a_3} R_{a_1 a_4 a_5 b_1} \nabla_{a_6} \nabla_{a_7} \nabla_{a_8} R^{b_1}{}_{a_9 a_{10} a_2} - 882 s_{12} \nabla_{a_3} \nabla_{a_4} R_{a_1 a_5 a_6 b_1} \nabla_{a_7} \nabla_{a_8} R^{b_1}{}_{a_9 a_{10} a_2} \\
& - 1328 s_{12} R_{a_1 a_3 a_4 b_1} R^{b_1}{}_{a_5 a_6 b_2} \nabla_{a_7} \nabla_{a_8} R^{b_2}{}_{a_9 a_{10} a_2} - 2120 s_{12} R_{a_1 a_3 a_4 b_1} \nabla_{a_5} R^{b_1}{}_{a_6 a_7 b_2} \nabla_{a_8} R^{b_2}{}_{a_9 a_{10} a_2} \\
& \left. - 128 s_{12} R_{a_1 a_3 a_4 b_1} R^{b_1}{}_{a_5 a_6 b_2} R^{b_2}{}_{a_7 a_8 b_3} R^{b_3}{}_{a_9 a_{10} a_2} \right]. \tag{A13}
\end{aligned}$$

APPENDIX B: TREE-LEVEL ON-SHELL AMPLITUDE WITH $n = 6$

The expression for the $n = 6$ on-shell amplitude as a function of covariant Feynman rules is

$$\begin{aligned}
\mathcal{A}_{a_1 a_2 a_3 a_4 a_5 a_6} = & \mathcal{R}_{a_1 a_2 a_3 a_4 a_5 a_6} + \frac{1}{6!} \sum_{S_6} [15 \mathcal{R}_{a_1 a_2 a_3 a_4 b_1} \Delta^{b_1 b_2}(s_{56}) \mathcal{R}_{b_2 a_5 a_6} + 10 \mathcal{R}_{a_1 a_2 a_3 b_1} \Delta^{b_1 b_2}(s_{123}) \mathcal{R}_{b_2 a_4 a_5 a_6} \\
& + 45 \mathcal{R}_{a_1 a_2 b_1} \Delta^{b_1 b_2}(s_{12}) \mathcal{R}_{a_3 a_4 b_2 b_3} \Delta^{b_3 b_4}(s_{56}) \mathcal{R}_{b_4 a_5 a_6} + 60 \mathcal{R}_{a_1 a_2 a_3 b_1} \Delta^{b_1 b_2}(s_{1234}) \mathcal{R}_{b_2 b_3 a_4} \Delta^{b_3 b_4}(s_{56}) \mathcal{R}_{b_4 a_5 a_6} \\
& + 15 \mathcal{R}_{a_1 a_2 b_1} \Delta^{b_1 b_2}(s_{12}) \mathcal{R}_{b_2 b_3 b_4} \Delta^{b_3 b_5}(s_{34}) \mathcal{R}_{b_5 a_3 a_4} \Delta^{b_4 b_6}(s_{56}) \mathcal{R}_{b_6 a_5 a_6} \\
& + 90 \mathcal{R}_{a_1 a_2 b_1} \Delta^{b_1 b_2}(s_{12}) \mathcal{R}_{b_2 b_3 a_3} \Delta^{b_3 b_4}(s_{123}) \mathcal{R}_{b_4 b_5 a_4} \Delta^{b_5 b_6}(s_{56}) \mathcal{R}_{b_6 a_5 a_6}] \tag{B1}
\end{aligned}$$

Figure 2 shows diagrammatic representations for amplitudes with $4 \leq n \leq 6$.

APPENDIX C: COVARIANCE IN 0-D TOYS

Consider a zero-dimensional quantum field theory (see, e.g., [50]). This drastic assumption brings a number of simplifications—not least rendering kinematics trivial, and reducing functional derivatives to partial derivatives. But it nevertheless preserves the key combinatorial properties of the Feynman graph expansion which guarantee the covariance of amplitudes, which we here illustrate with the 4-point amplitude.

In zero dimensions the scalar field is a map $\phi: \text{pt} \rightarrow \mathcal{M}$, i.e., just an \mathcal{M} -valued variable, no longer a map (or section). This means that a field redefinition $\phi \mapsto \psi(\phi)$ is, in this 0-d setting, *exactly the same* as a diffeomorphism on field space \mathcal{M} . So, amplitudes are precisely tensors on \mathcal{M} . The action is just an ordinary function (not functional) of the field variable ϕ , and correlation functions are just expectation values of operators on the probability distribution defined by $e^{-S(\phi)/\hbar}$, suitably normalized. For instance, we could take $S(\phi) = \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 + \dots$

assuming the highest power in ϕ to be even for convergence of path integrals. The amputated 1PI n -point functions, together with the propagator (even though nothing propagates in zero-dimensions), are now just partial derivatives of the effective action $\Gamma = S + \dots$,

$$\langle \phi_1 \dots \phi_n \rangle_{\text{1PI}} = \frac{i \partial^n \Gamma}{\partial \phi_1 \dots \partial \phi_n}, \quad D_{ij} = - \left(\frac{i \partial^2 \Gamma}{\partial \phi_i \partial \phi_j} \right)^{-1}. \tag{C1}$$

Under a general field redefinition $\phi \mapsto \psi(\phi)$, the variation of the 2-point function is given by the chain rule

$$\frac{\partial^2 \Gamma(\psi)}{\partial \phi \partial \phi} = \frac{\partial^2 \Gamma(\psi)}{\partial \psi \partial \psi} \frac{\partial \psi}{\partial \phi} \frac{\partial \psi}{\partial \phi} + \frac{\partial \Gamma(\psi)}{\partial \psi} \frac{\partial^2 \psi}{\partial \phi^2}, \tag{C2}$$

where we drop indices in the remainder of this section for readability. Imposing the vacuum condition $\partial_\psi \Gamma = 0$ kills the second term, meaning that the 2-point, and hence its inverse (the propagator), transforms as a tensor on \mathcal{M} .

Consider the four-point amplitude. It is

$$i\mathcal{A}_4 = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} = i\Gamma^{(4)} - \Gamma^{(3)} \frac{i}{\Gamma^{(2)}} \Gamma^{(3)}, \quad (\text{C3})$$

where we adopt the notation $f^{(n)}(x) \equiv \frac{\partial^n f(x)}{\partial x^n}$. Note that there is no kinematics so, contrary to our usual diagrammatic thinking, the second diagram does not correspond to a particular channel and is unique. Under $\phi \mapsto \psi(\phi)$, the 4-point contact has variation determined by the Faà di Bruno formula (6), which is just the result of successively applying the chain rule

$$\frac{\partial^4 \Gamma(\psi)}{\partial \phi \partial \phi \partial \phi \partial \phi} = \Gamma^{(4)}(\psi) \left(\frac{\partial \psi}{\partial \phi} \right)^4 + 6\Gamma^{(3)}(\psi) \left(\frac{\partial \psi}{\partial \phi} \right)^2 \frac{\partial^2 \psi}{\partial \phi^2} + \Gamma^{(2)}(\psi) \left(3 \left(\frac{\partial^2 \psi}{\partial \phi^2} \right)^2 + 4 \frac{\partial^3 \psi}{\partial \phi^3} \frac{\partial \psi}{\partial \phi} \right) + \Gamma^{(1)}(\phi) \frac{\partial^4 \psi}{\partial \phi^4} \quad (\text{C4})$$

The last term vanishes due to the vacuum condition $\Gamma^{(1)} = 0$. The penultimate term vanishes only if we impose the on-shell condition $\Gamma^{(2)} = 0$. This leaves only the first two terms. The first one is the transformation of a tensor while the second is not. The leading variation of the second diagram in Eq. (C3), from gluing lower points, precisely cancels the second term in the equation. Using the Faà di Bruno on both 3-point insertions

$$\left(\frac{\partial^2 \Gamma(\psi)}{\partial \phi \partial \phi} \right)^{-1} \left(\frac{\partial^3 \Gamma(\psi)}{\partial \phi \partial \phi \partial \phi} \right)^2 = \frac{1}{\Gamma^{(2)}(\phi)} \left(\frac{\partial \phi}{\partial \psi} \right)^2 \left[\Gamma^{(3)}(\psi) \left(\frac{\partial \psi}{\partial \phi} \right)^3 + 3\Gamma^{(2)}(\psi) \frac{\partial^2 \psi}{\partial \phi^2} \frac{\partial \psi}{\partial \phi} + \Gamma^{(1)}(\phi) \frac{\partial^3 \psi}{\partial \phi^3} \right]^2 \quad (\text{C5})$$

$$= \frac{(\Gamma^{(3)}(\psi))^2}{\Gamma^{(2)}(\psi)} \left(\frac{\partial \psi}{\partial \phi} \right)^4 + 6\Gamma^{(3)}(\psi) \left(\frac{\partial \psi}{\partial \phi} \right)^2 \frac{\partial^2 \psi}{\partial \phi^2} + 9\Gamma^{(2)}(\psi) \left(\frac{\partial^2 \psi}{\partial \phi^2} \right)^2 + \Gamma^{(1)}(\psi) [\dots] \quad (\text{C6})$$

So the glued diagram contains a tensor piece [first in Eq. (C6)] plus a nontensor piece (second term) that precisely matches that coming from the variation of the contact term. The last two terms in (C6) are nontensor pieces that do not cancel, and are only removed by the on-shell and vacuum conditions. The upcoming paper [41] will demonstrate, restoring full d -dimensional kinematics, that similar cancellations occur for all nontensor pieces at n -points, for all finite $n \in \mathbb{Z}_{\geq 2}$. In other words, the expansion of the n -point amplitude in Feynman diagrams coincides precisely with a diagrammatic representation of the Faà di Bruno formula for the transformation of the n th derivative, a correspondence that only breaks down only for the terms linear in $\Gamma^{(1)}$ and $\Gamma^{(2)}$.

APPENDIX D: VERTICAL TENSORS ON FIBER BUNDLES

Given our field space fiber bundle (E, π, Σ) , the quantum field theory amplitudes \mathcal{A}_n are expansions in vertical tensors on that bundle. Here, for completeness, we recap how vertical tensors are constructed. We note that the approach taken in Ref. [37] for distinguishing fiber vs spacetime directions is similar in spirit.

At any point $e \in E$, we can split the tangent space $T_e E$ into vertical and horizontal directions using the projection π (see Fig. 1). The vertical space $V_e E$ is well defined, spanned by the tangent vectors at e that get mapped to

zero under π , $V_e E := \ker(d\pi_e)$, where $d\pi_e: T_e E \rightarrow T_{\pi(e)} \Sigma$ is the differential of the map π at e , which recall is a linear approximation of π , i.e., provides a map $d\pi_e: T_e E \rightarrow T_{\pi(e)} \Sigma$ between tangent spaces. Collecting the $\{V_e E\}$ as fibers over Σ , we form the vertical bundle $VE = \ker(d\pi) \subset TE$, with projection map $\pi_V: VE \rightarrow \Sigma$. From here, we can tensor product (and/or dualize) vertical vectors to build *vertical tensors* T_V .

In order to correctly reproduce the scalar amplitudes using our fiber bundle formalism, we need to build vertical tensors that are Lorentz invariant but which nevertheless end up depending on the $g_{\mu\nu}$ components of the metric, such as that in (11). To form such tensors requires not just the vertical bundle VE but also a horizontal bundle HE . While VE is uniquely defined given π , specifying HE amounts to a *choice* of Ehresmann connection on the bundle, i.e., a complementary horizontal subspace $H_e E$ for every $e \in E$ such that $T_e E = V_e E \oplus H_e E$ (see Fig. 1), and thence a global splitting $TE = VE \oplus HE$. See Fig. 1.

With this splitting, one can then resolve tangent vectors on E into vertical and horizontal directions. Using the connection we can also *lift* horizontal vector fields from the base Σ to HE . Starting from an orthonormal basis $\{e_\mu\}$ of vector fields on Σ , we let $\{\tilde{e}_\mu^H\}$ denote their lifts to HE . We make the convenient choice of Ehresmann connection defined in local coordinates via $\tilde{e}_\mu^H = \partial_\mu$. We can also take the horizontal lift of a covector $e^\mu := g^{\mu\nu} e_\nu$, which we

denote $\tilde{e}^{\mu H}$. Note the covector is first defined by raising with the full bundle metric g .

Equipped with these objects, we can explain how a vertical tensor like (11) in the main text is formally defined. To form \tilde{R} , we start from the Riemann tensor R on E , which is of type (1, 3), and define \tilde{R} by specifying the following action on a pair of vertical vector fields, $\tilde{R}(v_1, v_2) := \pi_V R(\tilde{e}^{\mu H}, v_1, \tilde{e}_\mu^H, v_2)$, where $v_{1,2} \in \Gamma(VE)$. The result is $\tilde{R}_{ij} = R^\mu_{\ i\mu j} = \frac{1}{2\Lambda^4} \text{Tr}[\eta] \partial_i \partial_j V$ as quoted in the main text.

APPENDIX E: EXPLICIT COVARIANT FEYNMAN RULES IN A THEORY OF TWO REAL SCALARS

To better appreciate the physical meaning of the results presented in the main text, it can be useful to evaluate the covariant expressions for a concrete theory. To this end, we consider a theory of two real scalar fields ϕ_1, ϕ_2 . Following Sec. 6.3 of Ref. [38], we parametrize the most general fiber bundle metric through the following Taylor expansions:

$$g_{\mu\nu} = -\frac{1}{2} \eta_{\mu\nu} V(u) \quad (\text{E1})$$

$$V(u) = v_0 + \sum_{pr} \frac{v_{pr}}{2} \frac{u^p u^r}{\Lambda^2} + \sum_{prs} \frac{v_{prs}}{3!} \frac{u^p u^r u^s}{\Lambda^3} + \sum_{prst} \frac{v_{prst}}{4!} \frac{u^p u^r u^s u^t}{\Lambda^4} + \sum_{prstw} \frac{v_{prstw}}{5!} \frac{u^p u^r u^s u^t u^w}{\Lambda^5} + \dots \quad (\text{E2})$$

$$g_{11}(u) = a_0 + \sum_p a_p \frac{u^p}{\Lambda} + \sum_{pr} \frac{a_{pr}}{2} \frac{u^p u^r}{\Lambda^2} + \sum_{prs} \frac{a_{prs}}{3!} \frac{u^p u^r u^s}{\Lambda^3} + \dots \quad (\text{E3})$$

$$g_{22}(u) = b_0 + \sum_p b_p \frac{u^p}{\Lambda} + \sum_{pr} \frac{b_{pr}}{2} \frac{u^p u^r}{\Lambda^2} + \sum_{prs} \frac{b_{prs}}{3!} \frac{u^p u^r u^s}{\Lambda^3} + \dots \quad (\text{E4})$$

$$g_{12}(u) = c_0 + \sum_p c_p \frac{u^p}{\Lambda} + \sum_{pr} \frac{c_{pr}}{2} \frac{u^p u^r}{\Lambda^2} + \sum_{prs} \frac{c_{prs}}{3!} \frac{u^p u^r u^s}{\Lambda^3} + \dots \quad (\text{E5})$$

where the indices p, r, s, t, w take values in $\{1, 2\}$, all a, b, c, v parameters are dimensionless, and the ellipses stand for terms inducing 6- or higher-point interactions. The notation generalizes in an obvious way to terms with arbitrarily many u insertions.

For a simple illustration, we restrict our attention to the particular theory with

$$g_{11}(u) = 1 + a_2 \frac{\phi_2}{\Lambda} \quad g_{22}(u) = 1 \quad g_{12}(u) = 0, \quad (\text{E6})$$

and

$$V(u) = 1 + \frac{v_{11}}{2} \frac{(u^1)^2}{\Lambda^2} + \frac{v_{22}}{2} \frac{(u^2)^2}{\Lambda^2} + \frac{v_{111}}{3!} \frac{(u^1)^3}{\Lambda^3} + \frac{v_{1111}}{4!} \frac{(u^1)^4}{\Lambda^4} + \frac{v_{1112}}{3!} \frac{(u^1)^3 (u^2)}{\Lambda^4} + \frac{v_{1122}}{2!2!} \frac{(u^1)^2 (u^2)^2}{\Lambda^4} \\ + \frac{v_{11111}}{5!} \frac{(u^1)^5}{\Lambda^5} + \frac{v_{11112}}{4!} \frac{(u^1)^4 u^2}{\Lambda^5} + \frac{v_{11122}}{2!3!} \frac{(u^1)^3 (u^2)^2}{\Lambda^5} \quad (\text{E7})$$

which leads to the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 \left[1 + a_2 \frac{\phi_2}{\Lambda} \right] + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{\Lambda^2 v_{11}}{2} \phi_1^2 - \frac{\Lambda^2 v_{22}}{2} \phi_2^2 - \frac{\Lambda v_{111}}{6} \phi_1^3 \\ - \frac{v_{1111}}{4!} \phi_1^4 - \frac{v_{1112}}{3!} \phi_1^3 \phi_2 - \frac{v_{1122}}{2!2!} \phi_1^2 \phi_2^2 - \frac{v_{11111}}{5!} \frac{\phi_1^5}{\Lambda} - \frac{v_{11112}}{4!} \frac{\phi_1^4 \phi_2}{\Lambda} - \frac{v_{11122}}{2!3!} \frac{\phi_1^3 \phi_2^2}{\Lambda}. \quad (\text{E8})$$

In this theory, mixings are absent and the two ϕ_1, ϕ_2 scalars are canonically normalized mass eigenstates, with masses $M_i^2 = v_{ii} \Lambda^2 = -\Lambda^4 \tilde{R}^\mu_{\ i\mu i} / 2$. The propagators [Eq. (16)] therefore take the form

$$\Delta^{ij}(s) = \frac{\delta^{ij}}{\Lambda^2} \frac{i}{s/\Lambda^2 - v_{ii}} \quad (\text{E9})$$

In principle, the vacuum of the theory should be determined by solving $\partial_i V(u) \equiv 0$. For simplicity, we expand around the vacuum solution $u^i = 0$.

1. Covariant Feynman rules

The covariant Feynman rules can be computed by evaluating the relevant covariant tensors appearing in the expressions for $\mathcal{R}_{3,4,5}$ at the vacuum. We performed this operation with the `xTensor` and `xCoba` packages from the `xAct` suite in *Mathematica* [51]. From the Lagrangian in Eq. (E8) we obtain the following nonvanishing rules:

$$\mathcal{R}_{111} = -i\Lambda v_{111} \quad (\text{E10})$$

$$\mathcal{R}_{112} = \frac{i\Lambda}{2} a_2 (2v_{11} - v_{22}) \quad (\text{E11})$$

$$\mathcal{R}_{1111} = -iv_{1111} + ia_2^2 \left(2v_{11} - \frac{3v_{22}}{4} \right) \quad (\text{E12})$$

$$\mathcal{R}_{1112} = -iv_{1112} + \frac{3i}{2} a_2 v_{111} \quad (\text{E13})$$

$$\mathcal{R}_{1122} = -iv_{1122} + \frac{ia_2^2}{4} \left(\frac{s_{12}}{\Lambda^2} - 8v_{11} + 2v_{22} \right) \quad (\text{E14})$$

$$\mathcal{R}_{11111} = -\frac{iv_{11111}}{\Lambda} - \frac{5ia_2}{\Lambda} v_{1112} + \frac{5ia_2^2}{\Lambda} v_{111} \quad (\text{E15})$$

$$\begin{aligned} \mathcal{R}_{11112} = & -\frac{iv_{11112}}{\Lambda} + \frac{ia_2}{\Lambda} (2v_{1111} - 3v_{1122}) \\ & + \frac{ia_2^3}{2\Lambda} (3v_{22} - 10v_{11}) \end{aligned} \quad (\text{E16})$$

$$\mathcal{R}_{11122} = -\frac{iv_{11122}}{\Lambda} + \frac{3ia_2}{\Lambda} v_{1112} - \frac{7ia_2^2}{2\Lambda} v_{111} \quad (\text{E17})$$

$$\mathcal{R}_{11222} = \frac{3ia_2}{\Lambda} v_{1122} - \frac{ia_2^3}{4\Lambda} \left(\frac{2s_{12}}{\Lambda^2} - 22v_{11} + 5v_{22} \right) \quad (\text{E18})$$

In these expressions $s_{ij} = (p_i + p_j)^2$ where p_i is the incoming momentum of the i th Feynman rule leg. For instance, in \mathcal{R}_{1122} , s_{12} is the invariant mass of the two incoming ϕ_1 's.

It is worth noting that these expressions depend on both 2- and 0-derivative interactions, which are associated to a and v parameters respectively. This property follows naturally from these objects being vertical tensors on the fiber bundle E , and it is crucial in order to account properly for the invariance under field redefinitions.

2. Invariance under field redefinitions

By definition, the covariant Feynman rules transform as tensors under diffeomorphism field redefinitions. For fixed external indices their expressions are invariant. This property can be straightforwardly tested in explicit examples. Consider for instance the field redefinition

$$\phi_1 \mapsto \phi_1 + \alpha\phi_1^2, \quad \phi_2 \mapsto \phi_2. \quad (\text{E19})$$

The action of this shift on the Lagrangian, Eq. (E8), is equivalent to a redefinition of the a , v coefficients: the transformed Lagrangian up to 5-point interactions will have the same polynomial form as the initial one, upon replacing

$$v_{111} \rightarrow v_{111} + 6\alpha v_{11} \quad v_{1111} \rightarrow v_{1111} + 12\alpha(v_{111} + \alpha v_{11}) \quad (\text{E20})$$

$$v_{11111} \rightarrow v_{11111} + 20\alpha(v_{1111} + 3\alpha v_{111}) \quad v_{11112} \rightarrow v_{11112} + 12\alpha v_{1112} \quad (\text{E21})$$

$$v_{11122} \rightarrow v_{11122} + 6\alpha v_{1122} \quad (\text{E22})$$

and reintroducing the four terms

$$a_1 \rightarrow 4\alpha \quad a_{11} \rightarrow 8\alpha^2 \quad a_{12} \rightarrow 4\alpha a_2 \quad a_{112} \rightarrow 8\alpha^2 a_2, \quad (\text{E23})$$

that were taken to vanish in Eq. (E8). These terms would have given the following contributions to the covariant Feynman rules:

$$\Delta\mathcal{R}_{111} = \frac{3i\Lambda}{2} a_1 v_{11} \quad (\text{E24})$$

$$\Delta\mathcal{R}_{1111} = 3ia_1 v_{111} - \frac{19i}{4} a_1^2 v_{11} + 2ia_{11} v_{11} \quad (\text{E25})$$

$$\Delta\mathcal{R}_{1112} = \frac{ia_1 a_2}{4} (2v_{22} - 15v_{11}) + \frac{ia_{12}}{2} (3v_{11} - v_{22}) \quad (\text{E26})$$

$$\begin{aligned} \Delta\mathcal{R}_{11111} &= \frac{5ia_1}{\Lambda} \left(v_{1111} - \frac{a_2^2}{4}(13v_{11} - 2v_{22}) - \frac{15}{4}a_{11}v_{11} \right) + \frac{5i}{\Lambda}a_{11}v_{111} + \frac{5i}{4\Lambda}a_1^2(-11v_{111} + 18a_1v_{11}) \\ &\quad + \frac{5i}{4\Lambda}a_{12}a_2(7v_{11} - 2v_{22}) \end{aligned} \quad (\text{E27})$$

$$\begin{aligned} \Delta\mathcal{R}_{11112} &= \frac{3ia_1}{\Lambda}(v_{1112} - 3a_2v_{111}) + \frac{ia_{112}}{2\Lambda}(4v_{11} - v_{22}) + \frac{ia_1^2a_2}{4\Lambda}(76v_{11} - 5v_{22}) + \frac{ia_{11}}{2\Lambda}a_2(v_{22} - 12v_{11}) \\ &\quad + \frac{3ia_{12}}{\Lambda}v_{111} + \frac{ia_1a_{12}}{4\Lambda}(5v_{22} - 38v_{11}) \end{aligned} \quad (\text{E28})$$

$$\Delta\mathcal{R}_{11122} = \frac{3i}{4}a_1(2v_{1122} + 7a_2^2v_{11}) + \frac{ia_2}{2\Lambda}(a_{12} - a_1a_2) \left(\frac{s_{45}}{\Lambda^2} - 15v_{11} + 3v_{22} \right) \quad (\text{E29})$$

One can easily check that applying the replacements (E20)–(E23) onto each $\mathcal{R} + \Delta\mathcal{R}$ gives again the expressions in Eqs. (E10)–(E18), verifying that the covariant Feynman rules are invariant under the field redefinition considered. Importantly, in most cases the invariance is ensured by nontrivial cancellations among shifts of v and a parameters.

APPENDIX F: APPLICATION TO A $O(N)$ -SYMMETRIC THEORY WITH SPONTANEOUS SYMMETRY BREAKING

As a further example, we present an application to a toy model with N scalars ϕ^i , that is reminiscent of the scalar sector of SMEFT/HEFT. The theory respects a global $O(N)$ symmetry, while its vacuum is only symmetric under $O(N-1)$ rotations. We parametrize the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi} \cdot \partial^\mu\vec{\phi} - V(\phi), \quad V(\phi) = -\frac{m^2}{2}\frac{\vec{\phi} \cdot \vec{\phi}}{2} + \lambda\frac{(\vec{\phi} \cdot \vec{\phi})^2}{4}, \quad (\text{F1})$$

such that the vacuum condition is

$$\langle \vec{\phi} \cdot \vec{\phi} \rangle = \frac{m^2}{2\lambda} \equiv v^2, \quad (\text{F2})$$

and we can choose the configuration where the expectation value is only taken by the last component: $\langle \phi^i \rangle = v\delta^{iN}$.

Then one can expand around the vacuum with a SMEFT-like parametrization

$$\mathcal{L}_S = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu\vec{\varphi} \cdot \partial^\mu\vec{\varphi} - V(\varphi, h), \quad (\text{F3})$$

$$V(\varphi, h) = \lambda v^2 h^2 + \lambda v h^3 + \frac{\lambda}{4}(h^2 + \vec{\varphi} \cdot \vec{\varphi})^2 + \lambda v h \vec{\varphi} \cdot \vec{\varphi}, \quad (\text{F4})$$

where we took $\phi^N = v + h$ and relabeled $\phi^i = \varphi^i$ for $i = 1, \dots, N-1$ such that $\vec{\varphi}$ is a $(N-1)$ -vector. As an alternative, one can work with a nonlinear, HEFT-like parametrization, taking

$$\vec{\phi} = (v + \eta) \exp(T^i \omega^i / v) \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \quad (\text{F5})$$

where the dynamical fields are now called $\eta, \vec{\omega}$ with $\vec{\omega}$ a $(N-1)$ -vector, and T^i are the $N \times N$ generators of the preserved $O(N-1)$. The resulting Lagrangian is

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{(v + \eta)^2}{2} \left[\sin^2 \left[\frac{\sqrt{\vec{\omega} \cdot \vec{\omega}}}{v} \right] \frac{\partial_\mu\vec{\omega} \cdot \partial^\mu\vec{\omega}}{\vec{\omega} \cdot \vec{\omega}} + \left(\frac{\vec{\omega} \cdot \vec{\omega}}{v^2} - \sin^2 \left[\frac{\sqrt{\vec{\omega} \cdot \vec{\omega}}}{v} \right] \right) \frac{(\vec{\omega} \cdot \partial^\mu\vec{\omega})^2}{(\vec{\omega} \cdot \vec{\omega})^2} \right] - V(\eta), \quad (\text{F6})$$

$$V(\eta) = \lambda v^2 \eta^2 + \lambda v \eta^3 + \frac{\lambda}{4} \eta^4, \quad (\text{F7})$$

and the two representations are related by the field redefinitions

$$\varphi^i = (v + \eta) \frac{\omega^i}{\sqrt{\vec{\omega} \cdot \vec{\omega}}} \sin \frac{\sqrt{\vec{\omega} \cdot \vec{\omega}}}{v}, \quad \omega^i = v \frac{\varphi^i}{\sqrt{\vec{\varphi} \cdot \vec{\varphi}}} \arctan \frac{\sqrt{\vec{\varphi} \cdot \vec{\varphi}}}{v + h}, \quad (\text{F8})$$

$$h = -v + (v + \eta) \cos \frac{\sqrt{\vec{\omega} \cdot \vec{\omega}}}{v}, \quad \eta = -v + \sqrt{(v + h)^2 + \vec{\varphi} \cdot \vec{\varphi}}. \quad (\text{F9})$$

The h and η (φ^i and ω^i) fields represent the same physical states as, upon Taylor-expanding, they are equal up to nonlinear terms in the fields, e.g., $\eta = h + \mathcal{O}(h^2, |\vec{\varphi}|^2)$ and analogously for the others.

In both notations, the nonvanishing covariant Feynman rules are

$$\Delta^{\varphi_i \varphi_i}(s) = \Delta^{\omega_i \omega_i}(s) = \frac{i}{s}, \quad \Delta^{hh}(s) = \Delta^{\eta\eta}(s) = \frac{i}{s - 2\lambda v^2}, \quad (\text{F10})$$

$$\mathcal{R}_{\varphi_i \varphi_i h} = \mathcal{R}_{\omega_i \omega_i \eta} = -2i\lambda v, \quad \mathcal{R}_{\varphi_i \varphi_i \varphi_j \varphi_j} = \mathcal{R}_{\omega_i \omega_i \omega_j \omega_j} = -2i\lambda(1 + 2\delta_{ij}), \quad (\text{F11})$$

$$\mathcal{R}_{hhh} = \mathcal{R}_{\eta\eta\eta} = -6i\lambda v, \quad \mathcal{R}_{\varphi_i \varphi_i hh} = \mathcal{R}_{\omega_i \omega_i \eta\eta} = -2i\lambda, \quad (\text{F12})$$

$$\mathcal{R}_{hhhh} = \mathcal{R}_{\eta\eta\eta\eta} = -6i\lambda. \quad (\text{F13})$$

The nonvanishing on-shell scattering amplitudes with up to 4 external legs are

$$\mathcal{A}_{\varphi_i \varphi_i \varphi_j \varphi_j} = \mathcal{A}_{\omega_i \omega_i \omega_j \omega_j} = 6\lambda + 4\lambda^2 v^2 \left(\frac{1}{s_{12} - 2\lambda v^2} + \frac{1}{s_{13} - 2\lambda v^2} + \frac{1}{s_{14} - 2\lambda v^2} \right) \quad (\text{F14})$$

$$\mathcal{A}_{\varphi_i \varphi_i \varphi_j \varphi_j} = \mathcal{A}_{\omega_i \omega_i \omega_j \omega_j} = 2\lambda + 4\lambda^2 v^2 \left(\frac{1}{s_{12} - 2\lambda v^2} \right) \quad (\text{F15})$$

$$\mathcal{A}_{\varphi_i \varphi_i hh} = \mathcal{A}_{\omega_i \omega_i \eta\eta} = 2\lambda + 4\lambda^2 v^2 \left(\frac{3}{s_{12} - 2\lambda v^2} + \frac{1}{s_{13}} + \frac{1}{s_{14}} \right) \quad (\text{F16})$$

$$\mathcal{A}_{hhhh} = \mathcal{A}_{\eta\eta\eta\eta} = 6\lambda + 36\lambda^2 v^2 \left(\frac{1}{s_{12} - 2\lambda v^2} + \frac{1}{s_{13} - 2\lambda v^2} + \frac{1}{s_{14} - 2\lambda v^2} \right) \quad (\text{F17})$$

where it is understood that $j \neq i$, plus the 3-point amplitudes $\mathcal{A}_{a_1 a_2 a_3} = \mathcal{R}_{a_1 a_2 a_3}$. The first term in each expression stems from the contact Feynman rule $\mathcal{R}_{a_1 a_2 a_3 a_4}$, while the terms with propagators stem from the gluing of two covariant three-point vertices. Note that, for this simple model, the covariant Feynman rules coincide with the conventional ones when the Lagrangian is cast in the form \mathcal{L}_S .

The conventional Feynman rules for the \mathcal{L}_H Lagrangian are different. It can be instructive to compare the calculation of a scattering amplitude with the conventional and covariant rules: let us take for instance $\omega_1 \omega_1 \rightarrow \eta\eta$. The calculation through conventional diagrams yields

$$\mathcal{A}_{\omega_1 \omega_1 \eta\eta} = \frac{s_{12}}{v^2} + \frac{6\lambda s_{12}}{s_{12} - 2\lambda v^2} + \frac{(s_{13} - 2\lambda v^2)^2}{s_{13} v^2} + \frac{(s_{14} - 2\lambda v^2)^2}{s_{14} v^2}, \quad (\text{F18})$$

where the four terms stem respectively from the contact $\omega_1 \omega_1 \eta\eta$ vertex, the s -channel exchange of an η particle and the t - and u -channel exchanges of an ω_1 particle. This expression is equal to the result in Eq. (F16), obtained with covariant Feynman rules. However, Eq. (F18) contains spurious powers of s_{ij} in the numerators, while Eq. (F16) does not. In this sense, covariant Feynman rules make the pole structure of the amplitude more transparent.

This example demonstrates how fiber bundle geometry allows the computation of covariant Feynman rules for any scalar Lagrangian, including with a nonlinear dependence on the fields. This operation can help unveil the underlying structure of apparently complex theories

and allow them to be cast into simpler forms, without performing explicit field redefinitions. It could also be used as an equivalence test between Lagrangians, as two theories are equivalent *iff* they yield identical covariant Feynman rules.

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