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On the spectral zeta function of second order semiregular non-commutative harmonic oscillators

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ABSTRACT

In this paper we give a meromorphic continuation of the spectral zeta function for second order *semiregular* Non-Commutative Harmonic Oscillators (NCHO). By "semiregular systems" we mean systems with terms with degree of homogeneity scaling by 1 in their asymptotic expansion. As an application of our results, we first compute the meromorphic continuation of the Jaynes-Cummings (JC) model spectral zeta function. Then we compute the spectral zeta function of the JC generalization to a 3-level atom in a cavity. For both of them we show that it has only one pole in 1.

1. Introduction

One of the most important observables of the spectrum of an elliptic operator is the spectral zeta function. For a complex Hilbert space H and a densely defined linear operator $P : H \to H$, we denote the set of the eigenvalues (repeated by multiplicity) of P by $\operatorname{Spec} P$. When $\operatorname{Spec} P$ is discrete we can define the spectral zeta function of P as

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$$\zeta_P(s) := \sum_{\lambda \in \operatorname{Spec} P} \lambda^{-s},$$

for any given complex number s for which it makes sense. In particular, if P is an elliptic, selfadjoint and positive global pseudodifferential operator of order $\mu > 0$ on $\mathbb{R}^n$, then $s \mapsto \zeta_P(s)$ is holomorphic for $\operatorname{Re} s > 2n/\mu$ since the defining series is absolutely convergent (see Corollary 4.4.4. in [19]). For instance, if we denote by $P = \frac{x^2 - \partial_x^2}{2}$ the harmonic oscillator defined as the maximal operator in $L^2(\mathbb{R})$, then $\operatorname{Spec} P = \{k + 1/2;\ k \in \mathbb{Z}_+\}$ with multiplicity 1, and

$$\zeta_P(s) = \sum_{k \geq 0} (k + 1/2)^{-s} = (2^s - 1)\zeta(s),$$

where $\zeta(s)$ denotes the Riemann zeta function. Note that ζ_P is holomorphic for $\operatorname{Re}(s) > 1$, and has a meromorphic continuation to the whole complex plane. Furthermore, ζ_P has the only pole at $s = 1$, and we have $\zeta_P(s) = 0$ for $s = -2k$, $k \in \mathbb{Z}_+$ which are, thus, called trivial zeros. Moreover, the spectral zeta function entangles information about the spectrum of P in its analytical properties. For instance, the residues of the zeta function at its poles give the coefficients of the Weyl Law for P by the Ikehara Tauberian theorem (see Section 14 of Shubin [28]. See also Proposition (IV.6) in [7] and the references in Ivrii [13]).

The notion of spectral zeta function was introduced for the first time for the Laplacian on a two-dimensional Euclidean domains Ω by Carleman [4] who studied the Dirichlet-type series

$$\sum_{\lambda_j \in \operatorname{Spec} \Delta} \frac{\phi_{\lambda_j}(x_1)\phi_{\lambda_j}(x_2)}{\lambda_j^s},\ x_1, x_2 \in \Omega \tag{1}$$

where ϕ_{λ_j} is the eigenfunction of Δ associated to the eigenvalue λ_j. Later, in the case of a bounded Euclidean domain V of arbitrary dimension N, Minakshisundaram [16] showed through a method different from Carleman's that (1) is an entire function of s with zeros at negative integers and that

$$\sum_{\lambda_j \in \operatorname{Spec} \Delta} \frac{\phi_{\lambda_j}(x_1)^2}{\lambda_j^s}$$

can be continued as a meromorphic function of s with a unique simple pole at $N/2$ and negative integer zeros. Next, the analytic continuation of the spectral zeta function was studied by Minakshisundaram and Pleijel [17] for the Laplacian on a general compact manifold by a method that is a generalization of Carleman's. Seeley [26] studied the spectral zeta function of an elliptic ψ do on a compact manifold without boundary through the trace of complex powers of ψ dos, furthermore giving the value of the zeta function at 0.

Many different techniques have been used to obtain properties of the spectral zeta function. Duistermaat and Guillemin [5] (see also, [6] and the references in Hormander [10]) studied systematically the spectral zeta function of ψ dos on compact boundery-less manifolds basing their approach on the construction of a parametrix for the wave equation. Robert [23] (see also Aramaki [1]) extended meromorphically the spectral zeta function of an elliptic ψ does on $\mathbb{R}^n$ to the whole complex plane with simple poles that he computed along with the corresponding residues. He generalized to the global setting the techniques by Seeley to construct the parametrix of the resolvent by complex powers.

An import distinction to show the relevance of the results in this paper is the one between *regular* and *semiregular* symbols. Since the natural homogeneity of the Poisson bracket of homogeneous symbols is the sum of the orders minus 2, it is natural in the global calculus to call "regular" those symbols whose asymptotic expansion is made of homogeneous symbols for which the j-th term has order $\mu - 2j$ where μ is the order of the principal term. We will call "semiregular" those symbols whose j-th term in the asymptotic expansion has order $\mu - j$. This is indeed parallel to the use of "semiregular" appearing in the paper by Boutet De Monvel [2] on the hypoellipticity of the $\overline{\partial}$ operator. For a semiregular system of order μ we will call *semiprincipal symbol* the term of degree $\mu - 1$ in the asymptotic expansion of its symbol while we call *principal symbol* the one of order μ.

Moreover, following the discussion by [19], [20], [21], and [22], we call second order regular Non-Commutative Harmonic Oscillators (NCHOs) the class of the *regular* global partial differential systems of second order with polynomial coefficients. From now on we will omit the expression "second order" since all the NCHOs considered will be of second order.

Ichinose and Wakayama [12] obtained a meromorphic continuation of the spectral zeta function of a subclass of regular NCHOs and determined some of its special values. In addition, they showed that such a spectral zeta function has only a simple pole at 1 and that the sequence of its trivial zeros coincides with the one of the Riemann zeta function, the non-positive even integers. Their approach is based on the Mellin transform of the heat-semigroup of the operator in the approximation given by a parametrix which they computed *directly*, without using the one for the resolvent, obtaining its asymptotic expansion (see (15) and (16) in their paper). Later, Parmeggiani [19] generalized that approach to obtain the meromorphic continuation of the spectral zeta function of all the regular NCHOs. Nevertheless, while gaining in generality unfortunately his result did not explicitly locate the trivial zeros of the continuation of the spectral zeta function as could Ichinose and Wakayama.

Ichinose and Wakayama's and Parmeggiani's papers deal with *regular* systems. Regarding the *semiregular* systems, Sugiyama explored in [29] the Hurwitz-type spectral zeta function for the quantum Rabi model (describing the interaction of light and matter of a two-level atom coupled with a single quantized photon of the electromagnetic field, see the seminal papers [24] and [25] by Rabi, see also [3] by Braak).

In this paper we study the properties of the spectral zeta function associated with a positive elliptic *semiregular positive partial differential systems* with polynomial coefficients, including *also* models of semiregular NCHOs in the class of the Semiregular Metric Globally Elliptic Systems (SMGES) as those introduced in Section 3 of Malagutti and Parmeggiani [15]. The class of the SMGES is given by those matrix-valued symbols with a scalar principal part (that is the "metric" part) and a smoothly diagonalizable semiprincipal part. This class contains models relevant to Quantum Optics, such as the Jaynes-Cummings model (which describes the interaction between an atom and the electromagnetic field in a cavity and can be derived as an approximation of Rabi's by rotating waves approximation provided that the coupling strength is sufficiently weak, see [27] and the seminal paper [14] by Jaynes and Cummings). Here we follow the construction of the zeta function provided by Ichinose and Wakayama, in analogy to the approach by Parmeggiani in Theorem 7.2.1 of [19].

We will prove a result about the continuation of the spectral zeta function $\zeta_{A^{\mathrm{w}}}$ which turns out to be a meromorphic function whose poles are real and accumulate at $-\infty$. Namely, we will give the continuation as a linear combination of the meromorphic functions $s \mapsto \frac{1}{s-(n-j)+h/2}$, $j \geq 0$ and $h = 0$, 1, modulo a function that is holomorphic on a complex half-plan. Notice that even if in principle our extension can have poles in all the negative semi-integers, unlike the results in [12], [19] and [29] where the poles are all positive, we prove in this paper that only the positive integers are poles for the spectral zeta function of a differential operator with polynomial coefficients. The meromorphic continuation is obtained by following the approach of Theorem 7.2.1 in [19] where the parametrix approximation $U_A(t)$ of the heat-semigroup $e^{-tA^{\mathrm{w}}}$ is used. More precisely, by the Mellin transform we can write $\zeta_{A^{\mathrm{w}}}$ as $s \mapsto \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt$ for $\mathrm{Re} s > 2n/2 = n$ and, at this point, the asymptotic expansion $\sum_{j\geq 0} b_{-j}(t)$ (in the sense of Remark 6.1.5 at p. 83 of [19]) of $U_A(t)$ with $t \in \overline{\mathbb{R}}_+$ becomes crucial. In fact, the approximation of $s \mapsto \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt$ by $s \mapsto \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr}\, U_A(t)\, dt$ leads to the study of integrals of the form

$$(2\pi)^{-n} \int_{\mathbb{R}^{2n}} \chi(X) \mathsf{Tr}\left(b_{-2j-h}(t, X)\right) dX,\ j \in \mathbb{N},\ h = 0, 1, \tag{2}$$

where χ is a chosen excision function and Tr is the classical matrix trace. In fact the computation of (2) will give the coefficients of the linear combination of the aforementioned meromorphic functions. These coefficients will contribute to determine the residues and zeros of the spectral zeta function. Now one needs to go through a Taylor expansion argument as the time variable $t \to 0+$ of the terms arising from the study of $\mathrm{Tr}\, e^{-tA^{\mathrm{w}}} - \sum_{j=0}^{\nu} \sum_{h=0}^{1} \mathrm{Tr}\, B_{-2j-h}(t)$ (where B_{-k} with principal symbol b_{-k}). (The behavior of $e^{-tA^{\mathrm{w}}}$ as $t \to +\infty$ does not affect the result.)

This is a delicate argument since the behavior of the coefficients of the linear combination of the above meromorphic functions must be controlled as $t \to 0+$.

The plan of the paper is the following. First of all, the notation adopted will be introduced in Section 2 along with the parabolic ψ differential calculus needed to define the heat-semigroup parametrix which will be constructed directly in Section 3 by computing the terms of its asymptotic expansion through the solution of eikonal and transport equations. After that, in Section 4, we will control the behavior of the coefficients. We will give the proof of our theorem in Section 5. Actually, in Section 5 we will also obtain a meromorphic continuation for the Hurwitz spectral zeta function $\zeta_{A^{\mathrm{w}}+\tau I}$ for all $\tau \geq 0$. Next, in Section 6 by using our results in this paper we will compute the meromorphic continuation of the spectral zeta function for the Hamiltonians of Jaynes-Cummings and its generalization to a 3-level atom in one cavity. For these Hamiltonians we will show that the meromorphic continuation has only a simple pole at $s = 1$ and no other (even if, recall, the general formula allows all the negative semi-integer as poles). Finally, in Section 7 we prove that the spectral zeta function of a differential operator with polynomial coefficients does not have semi-integer poles as a corollary of the previous results of this paper.

2. Parabolic calculus

In this section, similarly to what is done by Parenti and Parmeggiani in [18] (see also Section 6.1 of [19]), we will introduce a class of symbols suitable for the construction of a pseudodifferential approximation of $e^{-tA^{\mathrm{w}}}$. Let us recall $\overline{\mathbb{R}}_+ = [0, +\infty)$. We will be using the following notation for the Hörmander metric and admissible weight (see Hörmander [11]): with $X = (x, \xi)$, $Y = (y, \eta)$ etc. belonging to the phase-space $\mathbb{R}^n \times \mathbb{R}^n$, and $m(X) := \langle X \rangle = (1 + |X|^2)^{1/2}$ the usual "Japanese bracket", we consider the Hörmander metric $g_X = |dX|^2/m(X)^2$. Then m is an admissible function (and so is m^μ for any given $\mu \in \mathbb{R}$), and we may exploit the full power of the Weyl-Hörmander pseudodifferential calculus.

Definition 2.1. Let M_N denote the algebra of $N \times N$ complex matrices. A symbol $a \in S(m^\mu, g; \mathsf{M}_N)$ is said to be **semiregular** (see Remark 3.2.4 of [19]) and we write $a \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$ if it possesses an asymptotic expansion $\sum_{j \geq 0} a_{\mu-j}$ in isotropic (i.e. positively homogenous and smooth outside the origin) terms $a_{\mu-j}$ positively homogeneous of degree $\mu - j$.

Moreover, we define also a class of symbol depending on the time variable t.

Definition 2.2. Let $r \in \mathbb{R}$ and $m(X) := \langle X \rangle$ for all $X \in \mathbb{R}^{2n}$ where $\langle X \rangle := \sqrt{1 + |X|}$. By $S(\mu, r)$ we denote the set of all smooth maps $b : \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathsf{M}_N$ satisfying the following estimates: for any given $\alpha \in \mathbb{Z}_+^{2n}$ and any given p, $j \in \mathbb{Z}_+$ there exists $C > 0$ such that

$$\sup \left| t^p (\frac{d}{dt})^j \partial_X^\alpha b(t,X) \right| \leq C m(X)^{r-|a|+(j-p)\mu}. \tag{3}$$

For $b \in S(\mu, r)$ we then consider the pseudodifferential operator

$$b^{\mathrm{w}}(t,x,D)u(x) = (2\pi)^{-n} \iint e^{i(x-y,\xi)} b(t, \frac{x+y}{2}, \xi) u(y) dy d\xi,\; u \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N),$$

and we shall say that $B \in \mathrm{OP}S(\mu, r)$ if $B = b^{\mathrm{w}}(t,x,D) + R$, where R is smoothing. In this setting, a smoothing operator R is any **continuous** map

$$R : \mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \mathscr{S}(\overline{\mathbb{R}}_+; \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)).$$

Then we introduce the "classical operators": in this case the key is to take in account the correct homogeneity properties. The basic example to keep in mind is the matrix $e^{-ta_\mu(x,\xi)}$.

Definition 2.3. We say that the operator $B \in \mathrm{OP}S(\mu, r)$, $B = b^{\mathrm{w}} + R$ is **classical**, and write $B \in \mathrm{OP}S_{\mathrm{cl}}(\mu, r)$, if there exists a sequence of functions $b_{r-2j} = b_{r-2j}(t, X),\ j \geq 0, t \geq 0$ and $X \neq 0$, such that:

1. One has the homogeneity

$$b_{r-2j}(t, \tau X) = \tau^{r-2j} b_{r-2j}(\tau^\mu t, X),\ \forall \tau > 0,\ \forall j \geq 0; \tag{4}$$

2. The function

$$\mathbb{R}^{2n} \setminus \{0\} \ni X \longmapsto b_{r-2j}(\cdot, X) \in \mathscr{S}(\overline{\mathbb{R}}_+, \mathsf{M}_N),$$

is smooth for all $j \geq 0$;

3. For all $\nu \geq 1$

$$b(t,X) - \sum_{j=0}^{\nu=1} \chi(X) b_{r-2j}(t,X) \in S(\mu, r - 2\nu), \tag{5}$$

where χ is an excision function.

Remark 2.4. We call $b_r = \sigma_r(B)$ the **principal symbol** of B.

Remark 2.5. Semi-regular classical symbols are defined accordingly, considering also terms with odd degree of homogeneity in the expansion formula (5), and the class of pseudodifferential operators associated to them is denoted by $\mathrm{OP}S_{\mathrm{sreg}}(\mu, r)$.

3. Parametrix of the heat-semigroup

In this section we will construct the parametrix of the heat-semigroup of a semiregular positive elliptic pseudodifferential operator.

Lemma 3.1. *Let $A = A^*$, with $A \sim \sum_{j\geq 0} a_{2-j} \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, be an elliptic second order system such that $A^{\mathrm{w}} > 0$. Then, there exists $U_A \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, 0)$ such that*

$$\frac{d}{dt}U_A + A^{\mathrm{w}}U_A : \mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N) \to \mathscr{S}(\overline{\mathbb{R}}_+; \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N))$$

is smoothing, and

$$U_A|_{t=0} - I_N : \mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N) \to \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$$

is smoothing. Moreover, the principal symbol of U_A is

$$\overline{\mathbb{R}}_+ \times \big(\mathbb{R}^{2n} \setminus \{0\}\big) \ni (t, X) \mapsto e^{-ta_\mu(X)}.$$

Proof. We will prove the lemma by constructing the terms of the expansion of the symbol of U_A. In fact, we determine those terms by solving a sequence of transport equations.

Let

$$\overline{\mathbb{R}}_+ \times \big(\mathbb{R}^{2n} \setminus \{0\}\big) \ni (t, X) \mapsto b_0(t, X) := e^{-ta_\mu(X)},$$

and let $B_0 \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, 0)$ with principal symbol given by b_0. Hence, by Lemma 6.1.3 at p. 81 of [19] we have that $\frac{d}{dt}B_0 + A^{\mathrm{w}}B_0 \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, \mu - 1)$ with principal symbol $r_{\mu-1} := a_{\mu-1}b_0$. Moreover, $B_0|_{t=0} - I_N$ is a pseudodifferential system with symbol in $S_{\mathrm{sreg}}(m^{-1}, g; \mathsf{M}_N)$ and we denote its principal symbol by p_{-1}.

Next, we look for a symbol $b_{-1}(t, X)$, positively homogeneous of degree -1 (in the sense of (4)), such that

$$\begin{cases} \frac{d}{dt}b_{-1} + a_\mu b_{-1} = -r_{\mu-1}, \\ b_{-1}|_{t=0} = -p_{-1}. \end{cases} \tag{6}$$

The solution of (6),

$$b_{-1}(t, X) := -e^{-ta_\mu(X)}p_{-1}(X) - \int_0^t e^{-(t-t')a_\mu(X)}r_{\mu-1}(t', X)\, dt',$$

is easily seen to be smooth and have the required homogeneity properties since

$$
\begin{aligned}
b_{-1}(t,\tau X) &= -\, e^{-ta_\mu(\tau X)}p_{-1}(\tau X) - \int_0^t e^{-(t-t')a_\mu(\tau X)}r_{\mu-1}(t',\tau X)\,dt' \\
&= -\, e^{-\tau^\mu t a_\mu(X)}\tau^{-1}p_{-1}(X) - \int_0^t e^{-\tau^\mu(t-t')a_\mu(X)}\tau^{\mu-1}r_{\mu-1}(\tau^\mu t', X)\,dt' \\
&= \tau^{-1}\left(-e^{-\tau^\mu t a_\mu(X)}p_{-1}(X) - \int_0^{\tau^\mu t} e^{-(\tau^\mu t-t')a_\mu(X)}r_{\mu-1}(t', X)\,dt'\right) \\
&= \tau^{-1}b_{-1}(\tau^\mu t, X),
\end{aligned}
$$

where the last equality follows from the change of variable $t \to \tau^{-\mu}t$ in the integral. Taking $B_{-1} \in \mathrm{OP}S_{\mathrm{sreg}}(\mu,-1)$ with principal symbol given by b_{-1} gives

$$
\frac{d}{dt}(B_0 + B_{-1}) + A^{\mathrm{w}}(B_0 + B_{-1}) \in \mathrm{OP}S_{\mathrm{sreg}}(\mu,\mu-2).
$$

Moreover, $(B_0 + B_{-1})|_{t=0} - I_N$ is a pseudodifferential system with symbol in $S_{\mathrm{sreg}}(m^{-2}, g; \mathsf{M}_N)$ and we denote its principal symbol by p_{-2}.

Iterating the above procedure gives a formal series

$$
\sum_{k\geq 0} B_{-k},\ B_{-k} \in \mathrm{OP}S_{\mathrm{sreg}}(\mu,-k).
$$

Hence, there exists an operator $U_A \in \mathrm{OP}S_{\mathrm{sreg}}(\mu,0)$ for which

$$
U_A - \sum_{k=0}^{\nu-1} B_{-k} \in \mathrm{OP}S(\mu,-\nu),\ \forall \nu \geq 1,
$$

by an adaptation of Proposition 3.2.15 at p. 32 of [19] and therefore we obtain the required parametrix. $\square$

Remark 3.2. In the applications of Lemma 3.1 we shall always consider a parametrix approximation of $e^{-tA^{\mathrm{w}}}$ where $b_{-j}|_{t=0} = 0$ for $j \geq 1$,

$$
B_{-j} := (\chi b_{-j})^{\mathrm{w}}(t, x, D_x),
$$

for all $t \in \overline{\mathbb{R}}_+$, where χ is a chosen excision function. Hence, consider the symbol $c_A(t,X)$ of $U_A(t)$, i.e. $U_A(t) = c_A^{\mathrm{w}}(t,x,D)$, given by

$$
c_A(t,X) = \sum_{j\geq 0} \chi_j(X) b_{-j}(t,X), \tag{7}
$$

where $\chi_0(X) := \chi(X)$ and $\chi_j(X) := \chi(X/R_j)$, $j \geq 1$, with $1 \leq R_j \nearrow +\infty$, as $j \to +\infty$, sufficiently fast (for instance, see the proof of Proposition 3.2.15 at p. 32 of [19]). Thus, the series (7) is locally finite in X and, hence, $c_A(t, \cdot) \in C^\infty$ for all $t \in \overline{\mathbb{R}}_+$. Hence, $c_A - \sum_{j=0}^{\nu-1} \chi_j b_{-j} \in S(\mu, -\nu)$, $\forall \nu \geq 1$ and we shall write $c_A \sim \sum_{j\geq 0} b_{-j}$.

Therefore, if $U_A := c_A^{\mathrm{w}}$ then $U_A - \sum_{k=0}^{\nu-1} B_{-k} \in \mathrm{OP}S(\mu, -\nu)$, $\forall \nu \geq 1$ and we shall write $U_A \sim \sum_{j\geq 0} B_{-j}$.

4. Vanishing property

Let A^{w} be as in the previous section. In this section we prove the technical proposition that we need to control the behavior of the b_{-j} constructed in Lemma 3.1 as $t \to 0+$, that is, its vanishing property, for a class of positive and self-adjoint elliptic *differential* systems with symbol in $S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$. Hence, we will suppose the symbol of A^{w} to be $a_2 + a_1 + a_0$ where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0,\, 1,\, 2$.

Proposition 4.1. *Let $A = a_2 + a_1 + a_0$ be an elliptic of second order where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0, 1, 2$, let $A^{\mathrm{w}} > 0$, and let U_A be the heat-semigroup $e^{-tA^{\mathrm{w}}}$ parametrix constructed by Lemma 3.1. Then, denoting again by $\sum_{j\geq 0} B_{-j}$ the expansion of U_A constructed in the proof of Lemma 3.1 and by b_{-j} the principal symbol of B_{-j}, we have for all $j \geq 0$ and $h = 0,\, 1$*

$$b_{-2j-h}(t, \omega) = O(t^{j+h}),\ t \to 0+,$$

and for all $\alpha,\, \beta \in \mathbb{Z}_+^n$, with $|\alpha| = 2k + 1$, $k \geq 0$ and $|\beta| \leq 1$ we have:

$$\partial_X^{\alpha+\beta} b_{-2j-h}(t, \omega) = O(t^{j+k+h|\beta|+1}),\ t \to 0+,$$

where the constants in $O(\cdot)$ do not depend on $\omega \in \mathbb{S}^{2n-1}$.

Proof. We prove this theorem by induction taking into account the definition of the terms b_{-j} and making straightforward computations.

We won't be writing the dependence on ω, and we will write $b_{-j}^{(\ell)}$ for a generic $\partial_X^\alpha b_{-j}$ with $|\alpha| = \ell$.

First of all, we remind that given two pseudodifferential operators with symbol a and b, then by the composition law for pseudodifferential operators (see, for instance, formula (3.3) at p. 19 of [19]) $a^{\mathrm{w}} b^{\mathrm{w}}$ has symbol

$$a\#b \sim ab + \sum_{j\geq 1} \frac{1}{j!}\left(\frac{-i}{2}\right)^j \{a,b\}_{(j)},$$

where $\{\cdot,\cdot\}_{(1)} = \{\cdot,\cdot\}$ is the Poisson bracket.

The terms r_{2-j}, $j \geq 1$ obtained in the proof of Lemma 3.1 are

$$\begin{aligned} r_{2-j} =& a_0 b_{-(j-2)} + a_1 b_{-(j-1)} + \frac{1}{2}\left(\frac{-i}{2}\right)^2 \{a_2, b_{-(j-4)}\}_{(2)} \\ & - \frac{i}{2}\{a_2, b_{-(j-2)}\} - \frac{i}{2}\{a_1, b_{-(j-3)}\},\ j \geq 0, \end{aligned} \tag{8}$$

where we set $b_k \equiv 0$ for all $k = 1, \ldots, 4$ and we recall that a_0 is a *constant* $N \times N$ Hermitian matrix. Therefore, by the construction in the proof of Lemma 3.1,

$$\begin{cases} b_0(t,X) = e^{-ta_2(X)}, \\ b_{-j}(t,X) = -\int_0^t e^{-(t-t')a_2} r_{2-j}(t',X)\,dt',\ j \geq 1. \end{cases} \tag{9}$$

In fact, $p_{-j} = 0$ for any $j \geq 1$ under our hypotheses.

Denote by $E(a_2^{(2)}, b_{-j}^{(2)})$, resp. $E(a_2^{(1)}, b_{-j}^{(1)})$, a generic expression obtained by taking the (matrix) product of derivatives of order 2, resp. order 1, of a_2 with derivatives of order 2, resp. order 1, of b_{-j}. Hence for all $j \geq 0$,

$$\{a_2, b_{-j}\} = E(a_2^{(1)}, b_{-j}^{(1)}),\ \text{ and }\ \{a_2, b_{-j}\}_{(2)} = E(a_2^{(2)}, b_{-j}^{(2)}).$$

Therefore, in $\{a_2, b_{-j}\}_{(2)} = E(a_2^{(2)}, b_{-j}^{(2)})$ we have a *constant* coefficient matrix (given by partial derivatives of order 2 of a_2) times partial derivatives of order 2 of b_{-j}.

We proceed by induction. We start with the case $j = 0$ and $h = 0$. In this case b_0 is the solution of

$$\begin{cases} \partial_t b_0 + a_2 b_0 = 0, \\ b_0|_{t=0} = I_N, \end{cases} \tag{10}$$

whence $b_0(t) = O(1)$ as $t \to 0+$.

Next, by induction on ℓ we show that $b_0^{(\ell)}$ has the claimed property.

For $\ell = 1$ we take a 1st-order partial derivative with respect to X of (10) and have

$$\begin{cases} \partial_t b_0^{(1)} + a_2 b_0^{(1)} = -a_2^{(1)} b_0, \\ b_0^{(1)}|_{t=0} = 0, \end{cases}$$

whence

$$b_0^{(1)}(t) = -\int_0^t e^{-(t-t')a_2} a_2^{(1)} b_0(t')\,dt' = O(t),\ t \to 0+. \tag{11}$$

For $\ell = 2$ we take a 1st-order partial derivative with respect to X of (11) and have

$$\begin{aligned} b_0^{(2)}(t) = &- \int_0^t (e^{-(t-t')a_2})^{(1)} a_2^{(1)} b_0(t')\, dt' - \int_0^t e^{-(t-t')a_2} a_2^{(2)} b_0(t')\, dt' \\ &- \int_0^t e^{-(t-t')a_2} a_2^{(1)} b_0(t')^{(1)}\, dt' \\ =& O(t^2) + O(t) + O(t^2) = O(t),\ t \to 0+. \end{aligned}$$

Next, suppose $b_0^{(2k-1+\ell)}(t) = O(t^k)$ as $t \to 0+$, for $\ell = 0, 1$ and $k \geq 0$. We want to prove that $b_0^{(2k+1+\ell)}(t) = O(t^{k+1})$, as $t \to 0+$, for $\ell = 0, 1$. Using (10) and taking a $2k+1$-st partial derivative with respect to X we obtain (recall that $a_2^{(p)} = 0$ for all $p \geq 3$ since a_2 has polynomial of degree 2 entries)

$$\begin{cases} \partial_t b_0^{(2k+1)} + a_2 b_0^{(2k+1)} = -a_2^{(1)} b_0^{(2k)} - a_2^{(2)} b_0^{(2k-1)} = O(t^k) + O(t^k) = O(t^k), \\ b_0^{(2k+1)}|_{t=0} = 0, \end{cases}$$

whence $b_0^{(2k+1)}(t) = O(t^{k+1})$ as $t \to 0+$. Then, as before,

$$\begin{aligned} b_0^{(2k+2)}(t) = &- \int_0^t (e^{-(t-t')a_2})^{(1)} (a_2^{(1)} b_0^{(2k)}(t')) + a_2^{(2)} b_0^{(2k-1)}(t'))\, dt' \\ &- \int_0^t e^{-(t-t')a_2} \partial_X (a_2^{(1)} b_0^{(2k)}(t')) + a_2^{(2)} b_0^{(2k-1)}(t'))\, dt' \\ =& O(t^{k+2}) + O(t^{k+1}) = O(t^{k+1}),\ t \to 0+. \end{aligned}$$

Hence, the result is proved for b_0.

Next, we prove the result for the case $j = 0$ and $h = 1$. In this case by (9)

$$\begin{aligned} b_{-1}(t) = &- \int_0^t e^{-(t-t')a_2} r_{2-1}(t')\, dt' \\ =& - \int_0^t e^{-(t-t')a_2} a_1 \underbrace{b_0(t')}_{=O(1),\, t' \to 0+} dt' \\ =& O(t),\ t \to 0+. \end{aligned} \tag{12}$$

By taking the derivative in X of (12)

$$\begin{aligned}
b_{-1}^{(1)}(t) = &- \int_0^t (e^{-(t-t')a_2})^{(1)} a_1 b_0(t')\, dt' \\
&- \int_0^t e^{-(t-t')a_2} a_1^{(1)} b_0(t')\, dt' \\
&- \int_0^t e^{-(t-t')a_2} a_1 \underbrace{b_0^{(1)}(t')}_{=O(t'),\, t'\to 0+} dt' \\
=& O(t^2) + O(t) + O(t^2) = O(t),\ t \to 0+,
\end{aligned}$$

and by taking another derivative in X we obtain that $b_{-1}^{(2)}(t) = O(t^2)$ (recall that $a_1^{(p)} = 0$ for all $p \geq 2$ since a_1 has polynomial of degree 1 entries).

Next, suppose $b_{-1}^{(2k-1+\ell)}(t) = O(t^{k+\ell})$, as $t \to 0+$, for $\ell = 0, 1$ and $k \geq 0$. We want to prove that $b_{-1}^{(2k+1+\ell)}(t) = O(t^{k+\ell+1})$, as $t \to 0+$, for $\ell = 0$, 1. First of all, we notice that, by (9), b_{-1} is the solution of the Cauchy problem

$$\begin{cases} \partial_t b_{-1} + a_2 b_{-1} = -r_1 = -a_1 b_0, \\ b_{-1}|_{t=0} = 0. \end{cases} \tag{13}$$

By using (13) and taking a $2k+1$-st partial derivative with respect to X

$$\begin{cases} \partial_t b_{-1}^{(2k+1)} + a_2 b_{-1}^{(2k+1)} &= -a_2^{(1)} b_{-1}^{(2k)} - a_2^{(2)} b_{-1}^{(2k-1)} - a_1^{(1)} b_0^{(2k)} \\ &= O(t^{k+1}) + O(t^k) + O(t^k), \\ b_{-1}^{(2k+1)}|_{t=0} = 0, \end{cases}$$

whence $b_{-1}^{(2k+1)}(t) = O(t^{k+1})$ as $t \to 0+$. Then, as before,

$$\begin{aligned}
b_{-1}^{(2k+2)}(t) = &- \int_0^t (e^{-(t-t')a_2})^{(1)} (a_2^{(1)} b_{-1}^{(2k)}(t') + a_2^{(2)} b_{-1}^{(2k-1)}(t') + a_1^{(1)} b_0^{(2k)}(t'))\, dt' \\
&- \int_0^t e^{-(t-t')a_2} \partial_X (a_2^{(1)} b_{-1}^{(2k)}(t') + a_2^{(2)} b_{-1}^{(2k-1)}(t') + a_1^{(1)} b_0^{(2k)}(t'))\, dt' \\
=& O(t^{k+2}) + O(t^{k+2}) = O(t^{k+2}),\ t \to 0+.
\end{aligned}$$

Hence, the result has been proved for b_{-1}.

Next, suppose, by induction, that for all $\ell = 0, 1$, all $h = 0, 1$ and all $j' \leq j$

$$b_{-2j'-h} = O(t^{j'+h}),\ b_{-2j'-h}^{(2k+1+\ell)} = O(t^{j'+k+h\ell+1}), \quad t \to 0+.$$

We want to prove $b_{-2(j+1)} = O(t^{j+1})$ and $b^{(2k+1+\ell)}_{-2(j+1)} = O(t^{j+1+k+1})$ for $\ell = 0, 1$, as $t \to 0+$, that is, the case $h = 0$ (after that, we will prove that $b_{-2(j+1)-1} = O(t^{j+1+1})$ and $b^{(2k+1+\ell)}_{-2(j+1)-1} = O(t^{j+1+k+\ell+1})$ for $t \to 0+$, i.e. the case $h = 1$). To do it, we have to examine $r_{2-2(j+1)}$ (see (9)). In the first place we have from (8)

$$\begin{aligned} r_{2-2(j+1)} =& a_0 b_{-2j} + a_1 b_{-2j-1} + \frac{1}{2}\left(\frac{-i}{2}\right)^2 \{a_2, b_{-2(j-1)}\}_{(2)} - \frac{i}{2}\{a_2, b_{-2j}\} \\ & - \frac{i}{2}\{a_1, b_{-(2j-1)}\} \\ =& O(t^j) + O(t^{j+1}) + O(t^{j-1+1}) + O(t^{j+1}) + O(t^{j-1+1}) \\ =& O(t^j), \quad t \to 0+. \end{aligned}$$

Consider next, keeping into account that $a^{(p)}_q = 0$ for all $p \geq q+1$ since a_q has polynomial of degree $q = 1$, 2 entries,

$$\begin{aligned} r^{(2k+1)}_{2-2(j+1)} =& a_0 b^{(2k+1)}_{-2j} + a_1 b^{(2k+1)}_{-2j-1} + E(a^{(1)}_1, b^{(2k)}_{-2j-1}) + E(a^{(2)}_2, b^{(2k+3)}_{-2(j-1)}) + E(a^{(1)}_2, b^{(2k+2)}_{-2j}) \\ & + E(a^{(2)}_2, b^{(2k+1)}_{-2j}) + E(a^{(1)}_1, b^{(2k+2)}_{-(2j-1)}) \\ =& O(t^{j+k+1}) + O(t^{j+k+1}) + O(t^{j+k-1+1+1}) + O(t^{j-1+k+1+1}) + O(t^{j+k+1}) \\ & + O(t^{j+k+1}) + O(t^{j-1+k+1+1}) \\ =& O(t^{j+k+1}), \quad t \to 0+. \end{aligned}$$

Taking an extra derivative, one immediately sees also that

$$r^{(2k+2)}_{2-2(j+1)} = O(t^{j+k+1}), \quad t \to 0+.$$

Hence, for all $\ell = 0, 1$ and for $k \geq -1$

$$r^{(2k+1+\ell)}_{2-2(j+1)} = O(t^{j+k+1}), \quad t \to 0+$$

(when $k = -1$ we take $\ell = 1$). Since $b_{-2(j+1)}$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t b_{-2(j+1)} + a_2 b_{-2(j+1)} = -r_{2-2(j+1)}, \\ b_{-2(j+1)}|_{t=0} = 0, \end{cases} \tag{14}$$

we obtain $b_{-2(j+1)}(t) = O(t^{j+1})$ as $t \to 0+$. As before, taking one partial derivative with respect to X yields

$$\begin{cases} \partial_t b^{(1)}_{-2(j+1)} + a_2 b^{(1)}_{-2(j+1)} = -a^{(1)}_2 b_{-2(j+1)} - r^{(1)}_{2-2(j+1)} = O(t^{j+1}) + O(t^{j+1}), \\ b^{(1)}_{-2(j+1)}|_{t=0} = 0, \end{cases}$$

whence it follows that $b^{(1)}_{-2(j+1)}(t) = O(t^{j+2})$, and, taking an extra derivative, also that, as $t \to 0+$,

$$b^{(2)}_{-2(j+1)}(t) = -\partial_X \left(\int_0^t e^{-(t-t')a_2} \left(a_2^{(1)} b_{-2(j+1)} + r^{(1)}_{2-2(j+1)} \right) dt' \right) = O(t^{j+2}).$$

Supposing then by induction the estimates up to order $2k-1$ and using

$$\begin{cases} \partial_t b^{(2k+1)}_{-2(j+1)} + a_2 b^{(2k+1)}_{-2(j+1)} &= -E(a_2^{(1)}, b^{(2k)}_{-2(j+1)}) - E(a_2^{(2)}, b^{(2k-1)}_{-2(j+1)}) - r^{(2k+1)}_{2-2(j+1)} \\ &= O(t^{j+1+k-1+1}) + O(t^{j+1+k-1+1}) + O(t^{j+k+1}), \\ b^{(2k+1)}_{-1}|_{t=0} = 0, \end{cases}$$

we obtain $b^{(2k+1)}_{-2(j+1)}(t) = O(t^{j+1+k+1})$, as $t \to 0+$, and using

$$b^{(2k+2)}_{-2(j+1)}(t) = -\partial_X \left(\int_0^t e^{-(t-t')a_2} \left(E(a_2^{(1)}, b^{(2k)}_{-2(j+1)}) + E(a_2^{(2)}, b^{(2k-1)}_{-2(j+1)}) + r^{(2k+1)}_{-2j} \right) dt' \right),$$

also that

$$b^{(2k+2)}_{-2(j+1)}(t) = O(t^{j+1+k+1}),\ t \to 0+,$$

which proves the result for the case $h = 0$.

Now, to complete the proof of this proposition we need to prove the result for the case $h = 1$, that is, for all $\ell = 0, 1$

$$b_{-2(j+1)-1} = O(t^{j+2}),\ b^{(2k+1+\ell)}_{-2(j+1)-1} = O(t^{j+1+k+\ell+1}),\ t \to 0+.$$

To do it, we have to examine $r_{2-2(j+1)-1}$ and its derivatives, that is,

$$\begin{aligned} r_{2-2(j+1)-1} =& a_0 b_{-2j-1} + a_1 b_{-2(j+1)} + \frac{1}{2}\left(\frac{-i}{2}\right)^2 \{a_2, b_{-2(j-1)-1}\}_{(2)} - \frac{i}{2}\{a_2, b_{-2j-1}\} \\ & - \frac{i}{2}\{a_1, b_{-2j}\} \\ =& O(t^{j+1}) + O(t^{j+1}) + O(t^{j-1+1+1}) + O(t^{j+1}) + O(t^{j+1}) \\ =& O(t^{j+1}),\ t \to 0+, \end{aligned}$$

and

$$\begin{aligned} r^{(2k+1)}_{2-2(j+1)-1} =& a_0 b^{(2k+1)}_{-2j-1} + a_1 b^{(2k+1)}_{-2(j+1)} + E(a_1^{(1)}, b^{(2k)}_{-2(j+1)}) + E(a_2^{(2)}, b^{(2k+3)}_{-2(j-1)-1}) \\ & + E(a_2^{(1)}, b^{(2k+2)}_{-2j-1}) + E(a_2^{(2)}, b^{(2k+1)}_{-2j-1}) + E(a_1^{(1)}, b^{(2k+2)}_{-2j}) \end{aligned}$$

$$
\begin{aligned}
&= O(t^{j+k+1}) + O(t^{j+1+k+1}) + O(t^{j+1+k-1+1}) + O(t^{j-1+k+1+1}) \\
&\quad + O(t^{j+k+1+1}) + O(t^{j+k+1}) + O(t^{j+k+1}) \\
&= O(t^{j+k+1}),\ t \to 0+.
\end{aligned}
$$

Taking an extra derivative, one immediately sees also that

$$r^{(2k+2)}_{2-2(j+1)-1} = O(t^{j+k+2}),\ t \to 0+.$$

Hence, for all $\ell = 0, 1$ and for all $k \geq -1$

$$r^{(2k+1+\ell)}_{2-2(j+1)-1} = O(t^{j+k+\ell+1}),\ t \to 0+.$$

(again, when $k = -1$ we take $\ell = 1$). Since $b_{-2(j+1)-1}$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t b_{-2(j+1)-1} + a_2 b_{-2(j+1)-1} = -r_{2-2(j+1)-1}, \\ b_{-2(j+1)-1}|_{t=0} = 0, \end{cases} \tag{15}$$

we obtain $b_{-2(j+1)-1}(t) = O(t^{j+1+1})$ as $t \to 0+$. As before, taking one partial derivative with respect to X yields

$$\begin{cases} \partial_t b^{(1)}_{-2(j+1)-1} + a_2 b^{(1)}_{-2(j+1)-1} = -a^{(1)}_2 b_{-2(j+1)-1} - r^{(1)}_{2-2(j+1)-1} = O(t^{j+2}) + O(t^{j+1}), \\ b^{(1)}_{-2(j+1)-1}|_{t=0} = 0, \end{cases}$$

whence it follows $b^{(1)}_{-2(j+1)-1}(t) = O(t^{j+2})$, and, taking an extra derivative, also that, as $t \to 0+$,

$$b^{(2)}_{-2(j+1)-1}(t) = -\partial_X \left(\int_0^t e^{-(t-t')a_2} \left(a^{(1)}_2 b_{-2(j+1)-1} + r^{(1)}_{2-2(j+1)} \right) dt' \right) = O(t^{j+3}).$$

Supposing then by induction the estimates up to order $2k$ and making use of

$$\begin{cases} \partial_t b^{(2k+1)}_{-2(j+1)-1} + a_2 b^{(2k+1)}_{-2(j+1)-1} \\ \quad = -E(a^{(1)}_2, b^{(2k)}_{-2(j+1)-1}) - E(a^{(2)}_2, b^{(2k-1)}_{-2(j+1)-1}) - r^{(2k+1)}_{2-2(j+1)-1} \\ \quad = O(t^{j+1+k-1+1+1}) + O(t^{j+1+k-1+1+1}) + O(t^{j+1+k+1}), \\ b^{(2k+1)}_{-1}|_{t=0} = 0, \end{cases}$$

we obtain $b^{(2k+1)}_{-2(j+1)-1}(t) = O(t^{j+k+2})$, as $t \to 0+$, and using

$$b^{(2k+2)}_{-2(j+1)-1}(t) = -\,\partial_X\left(\int_0^t e^{-(t-t')a_2}\left(E(a_2^{(1)}, b^{(2k)}_{-2(j+1)-1}) + E(a_2^{(2)}, b^{(2k-1)}_{-2(j+1)-1})\right)dt'\right)$$
$$-\,\partial_X\left(\int_0^t e^{-(t-t')a_2}\left(r^{(2k+1)}_{2-2(j+1)-1}\right)dt'\right),$$

also that

$$b^{(2k+2)}_{-2(j+1)-1}(t) = O(t^{j+1+k+1+1}),\ t \to 0+,$$

which proves the proposition. □

5. Meromorphic continuation of $\zeta_{A^{\mathrm{w}}}$

Let A^{w} be as in the Section 3. In this section we will use the parametrix approximation of the heat-semigroup construct in Lemma 3.1 to prove the result about the continuation of the spectral zeta function of the class of positive and self-adjoint elliptic operators A^{w} satisfying the hypotheses of Proposition 4.1. Namely, $\zeta_{A^{\mathrm{w}}}$ can be rewritten modulo a term holomorphic on a half plane of $\mathbb{C}$ as a linear complex combination of meromorphic functions. Moreover, we will give explicit formulas for the coefficients of this linear combination.

Theorem 5.1. *Let $A = a_2 + a_1 + a_0$ be an elliptic system of second order where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0, 1, 2$. Moreover, suppose $A^{\mathrm{w}} > 0$.*

There exist constants $c_{-2j-h,n}$ with $0 \le j \le n-1$, $h = 0, 1$, and constants $c_{-2j-1,n}$, C_{-2j} with $j \ge n$, such that, for any given integer $\nu \in \mathbb{Z}_+$ with $\nu \ge n$,

$$\zeta_{A^{\mathrm{w}}}(s) = \frac{1}{\Gamma(s)}\left[\left(\sum_{h=0}^{1}\sum_{j=0}^{n-1}\frac{c_{-2j-h,n}}{s-(n-j)+h/2}\right) + \left(\sum_{j=n}^{\nu}\frac{c_{-2j-1,n}}{s-(n-j)+1/2}\right) + \left(\sum_{j=n}^{\nu}\frac{C_{-2j}}{s-(n-j)}\right) + H_\nu(s)\right] \tag{16}$$

where $\Gamma(s)$ is the Euler gamma function, and H_ν is holomorphic in the region $\mathrm{Re}\, s > (n-\nu)-1$. *Consequently, the spectral zeta function $\zeta_{A^{\mathrm{w}}}$ is meromorphic in the whole complex plane $\mathbb{C}$ with at most simple poles at $s = n,\ n-1,\ n-2,\ \dots,\ 1$ and $s = n - \frac{1}{2}$, $n - \frac{3}{2}$, $n - \frac{5}{2}$, …. One has*

$$c_{-2j-h,n} = (2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\left(b_{-2j-h}(\rho^2,\omega)\right)\rho^{2(n-j)-1-h}\,d\omega\,d\rho, \tag{17}$$

where $0 \le j \le n-1$, $h = 0, 1$ *or* $j \ge n$, $h = 1$. *In (17) the* b_{-2j-h} *are the terms in the symbol of the parametrix* $U_A \in \mathrm{OP}S_{\mathrm{sreg}}(2,0)$ *constructed in the proof of Lemma 3.1 and Remark 3.2,*

$$U_A \sim \sum_{j\ge 0} B_{-j}.$$

Proof. The proof follows they idea to make use of the asymptotic expansion given by Lemma 3.1 to obtain an asymptotic expansion for the continuation of $\zeta_{A^{\mathrm{w}}}$. To do that we write $\zeta_{A^{\mathrm{w}}}$ by the Mellin transform which gives it in terms of the heat-semigroup of A^{w}. Now the semi-group can be approximated via Lemma 3.1. Hence, we compute an approximation of $\zeta_{A^{\mathrm{w}}}$ whose asymptotic terms, given by integrals, are the $\frac{c_{-2j-h,n}}{s-(n-j)+h/2}$ in (16), obtained by a Taylor expansion argument. Actually, we need the integrals defining the $c_{-2j-h,n}$ to converge. That is why we use Proposition 4.1 to have a control on the vanishing of the asymptotic terms of the parametrix of the heat-semigroup as $t \to 0+$. Finally, we take into account the residuals given by the approximations made and we sum their contributes. Namely, we notice that they do not affect the values of the $c_{-2j-h,n}$ for $j \le n-1$ and those for $h = 1$ if $j \ge n$.

By the properties of the heat semi-group $0 \le t \to e^{-tA^{\mathrm{w}}}$ we may use the Mellin transform and write

$$(A^{\mathrm{w}})^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-tA^{\mathrm{w}}}\, dt, \ \mathrm{Re} s > 2n/2 = n,$$

so that

$$s \mapsto \zeta_{A^{\mathrm{w}}}(s) = \mathrm{Tr}(A^{\mathrm{w}})^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt.$$

Let hence $U_A \sim \sum_{j\ge 0} B_{-j} \in \mathrm{OP}S_{\mathrm{sreg}}(2,0)$ be the parametrix approximation of $e^{-tA^{\mathrm{w}}}$ constructed in Lemma 3.1. We write

$$\zeta_{A^{\mathrm{w}}}(s) = \frac{1}{\Gamma(s)} \left(\int_0^1 + \int_1^{+\infty} \right) t^{s-1} \mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt =: Z_0(s) + Z_\infty(s).$$

In the first place we claim that $Z_\infty(s)$ is holomorphic in $\mathbb{C}$. In fact, on the one hand, since $t \mapsto \mathrm{Tr} R(t)$ is rapidly decreasing for $t \to +\infty$ (where $R(t) := e^{-tA^{\mathrm{w}}} - U_A(t)$), we have that for all $p \in \mathbb{N}$ and for all $t \ge 1$

$$|\mathrm{Tr}\, R(t)| \lesssim t^{-p}.$$

On the other, given any $\nu \geq 0$ and any symbol $b \in S(2, -2\nu)$, we have (by definition of the class $S(\mu, \nu)$ at p. 79 of [19]) that for all $t \geq 1$ and all $p \in \mathbb{N}$

$$\begin{aligned}\left|(2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathsf{Tr}\, b(t, X)\, dX\right| &= \left|(2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\, b(t, \rho\omega)\rho^{2n-1}\, d\omega\, d\rho\right| \\ &= t^{-p} \left|(2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} t^p \mathsf{Tr}\, b(t, \rho\omega)\rho^{2n-1}\, d\omega\, d\rho\right| \\ &\lesssim t^{-p} \int_0^{+\infty} \frac{\rho^{2n+1}}{(1+\rho)^{2\nu+2p}}\, d\rho \\ &\lesssim t^{-p}.\end{aligned}$$

(Here, we use the polar coordinates $0 \neq X = |X|\dfrac{X}{|X|}$ with $\rho \in \mathbb{R}_+$, $\omega \in \mathbb{S}^{2n-1}$, and $d\omega$ is the induced Riemann measure on $\mathbb{S}^{2n-1}$.) It thus follows that for all $p \in \mathbb{N}$ and for all $t \geq 1$

$$|\mathrm{Tr}\, U_A(t)| \lesssim t^{-p}.$$

In conclusion, since

$$\mathrm{Tr}\, e^{-tA^{\mathrm{w}}} = \mathrm{Tr}\, U_A(t) + \mathrm{Tr}\, R(t),$$

for every $p \geq 1$ there exists $C_p > 0$ such that

$$|\mathrm{Tr}\, e^{-tA^{\mathrm{w}}}| \leq C_p t^{-p}, \ \forall t \geq 1,$$

which proves the claim, since the term $1/\Gamma(s)$ is already holomorphic in $\mathbb{C}$. Therefore the crux of the matter lies in the study of the function $Z_0(s)$. To study it we need a better understanding of the terms $\mathrm{Tr}\, B_{-2j-h}$, $j \geq 0$, $h = 0, 1$. Hence, we recall that by the homogeneity of the b_{-2j-h}, for $t > 0$, $j \geq 0$, and $h = 0, 1$

$$\begin{aligned}\mathrm{Tr}\, B_{-2j-h}(t) &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \chi(X) \mathsf{Tr}\left(b_{-2j-h}(t, X)\right)\, dX \\ &= (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \chi(\rho\omega) \mathsf{Tr}\left(b_{-2j-h}(t, \rho\omega)\right) \rho^{2n-1}\, d\omega\, d\rho \\ &= (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \chi(\rho\omega) \mathsf{Tr}\left(b_{-2j-h}(\rho^2 t, \omega)\right) \rho^{2(n-j)-1-h}\, d\omega\, d\rho.\end{aligned}$$

We consider

$$c_{-2j-h,n} := (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\left(b_{-2j-h}(\rho^2, \omega)\right) \rho^{2(n-j)-1-h}\, d\omega\, d\rho.$$

We claim that

$$|c_{-2j-h,n}| < +\infty,\ \forall j \in \mathbb{Z}_+,\ h = 0, 1.$$

In fact, the integral is convergent at $\rho = +\infty$ for all j since $\mathsf{Tr}\,(b_{-2j-h}(\cdot, \omega))$ is a Schwartz function, it is clearly convergent at $\rho = 0$ for $0 \le 2j + h \le 2n - 1$, and finally it is convergent at $\rho = 0$ also when $2j + h \ge 2n$, for the singularity at 0 of the factor $\rho^{2(n-j)-1-h}$ is compensated by $\mathsf{Tr}\,(b_{-2j-h}(t, \omega)) = O(t^{j+h})$ as $t \to 0+$.

We define now the function

$$f_{-2j-h}(t) := -(2\pi)^{-n} \int_0^{1} \int_{\mathbb{S}^{2n-1}} (1 - \chi(\rho\omega))\mathsf{Tr}\,(b_{-2j-h}(t, \rho\omega))\, \rho^{2n-1}\, d\omega\, d\rho,\ j \in \mathbb{Z}_+.$$

Then $f_{-j'} \in C^\infty([0, +\infty); \mathbb{C})$, for all $j' \in \mathbb{Z}_+$, and by Proposition 4.1

$$f_{-2j-h}(t) = O(t^{j+h}),\ t \to 0+. \tag{18}$$

It follows that

$$\mathrm{Tr}\, B_{-2j-h}(t) = c_{-2j-h,n} t^{-(n-j)+h/2} + f_{-2j-h}(t) = c_{-2j-h,n} t^{-(n-j)+h/2} + O(t^{j+h}), \tag{19}$$

as $t \to 0+$, for all $j \ge 0$, $h = 0, 1$, and that (by the proof of Proposition 3.2.15 at p. 32 of [19] adapted to the present setting),

$$\mathrm{Tr}\; U_A(t) - \sum_{h=0}^{1} \sum_{j=0}^{\nu} \mathrm{Tr}\; B_{-2j-h}(t) =: \mathrm{Tr}\; R_{2\nu+2}(t) = O(t^{\nu+1}),\ t \to 0+,$$

$\forall \nu \in \mathbb{Z}_+$, $h = 0, 1$.
However, the information contained in (19) alone is not yet sufficient to obtain the continuation of $\zeta_{A^{\mathrm{w}}}$, and we need a better control of f_{-2j-h}. Notice that for all $j, k \in \mathbb{Z}_+$, denoting $\partial_t^k f_{-2j-h}(t)$ by $f^{(k)}_{-2j-h}(t)$,

$$f^{(k)}_{-2j-h}(t) = -(2\pi)^{-n} \int_0^{1} \int_{\mathbb{S}^{2n-1}} (1 - \chi(\rho\omega))\mathsf{Tr}\left(\partial_t^k b_{-2j-h}(t, \rho\omega)\right) \rho^{2n-1}\, d\omega\, d\rho,$$

so that $f^{(k)}_{-2j-h}(0)$ is finite and can be computed through (8), and through the differential equations (10), (13), (14), and (15) used to construct the b_{-2j-h}. Note, in particular, that

$$f^{(k)}_0(0) = (-1)^{k+1}(2\pi)^{-n}\int_0^1\int_{\mathbb{S}^{2n-1}}(1-\chi(\rho\omega))\mathsf{Tr}\left(a_2(\rho\omega)^k\right)\rho^{2n-1}\,d\omega\,d\rho.$$

We next apply Lemma 7.2.3 at p. 99 of [19] to the functions f_{-2j-h}, so that for any given $\nu \in \mathbb{Z}_+$ we may write, by (18),

$$F_{-2j-h}(s) := \int_0^1 t^{s-1}f_{-2j-h}(t)\,dt = \sum_{k=0}^{\nu}\frac{f^{(j+h+k)}_{-2j-h}(0)}{(j+h+k)!}\frac{1}{s+j+h+k} + F_{-2j-h,\nu}(s),$$

where $F_{-2j-h,\nu}$ is holomorphic for $\mathrm{Re}s > -j-h-\nu-1$.

Using this in (19) we have that for each $j \geq 0$, $h = 0, 1$, for any given $\nu \in \mathbb{Z}_+$,

$$\begin{aligned} s \mapsto \int_0^1 t^{s-1}\mathrm{Tr}\,B_{-2j-h}(t)\,dt =& \frac{c_{-2j-h,n}}{s-(n-j)+h/2} \\ &+\left(\sum_{k=0}^{\nu}\frac{f^{(j+h+k)}_{-2j-h}(0)}{(j+h+k)!}\frac{1}{s+j+h+k}\right) + F_{-2j-h,\nu}(s), \end{aligned}$$

where $F_{-2j-h,\nu}$ is holomorphic for $\mathrm{Re}s > -j-h-\nu-1$.

Analogously, since $0 \leq t \mapsto \mathrm{Tr}\,R(t) \in \mathscr{S}(\overline{\mathbb{R}}_+;\mathbb{C})$ and by Lemma 7.2.3 at p. 99 of [19] we also have, with $f_R(t) := \mathrm{Tr}\,R(t)$, that for any given $\nu \in \mathbb{Z}_+$

$$\int_0^1 t^{s-1}f_R(t)\,dt = \sum_{k=0}^{\nu}\frac{f^{(k)}_R(0)}{k!}\frac{1}{s+k} + F_{R,\nu}(s),$$

where $F_{R,\nu}$ is holomorphic for $\mathrm{Re}s > -\nu-1$.

We therefore obtain that for any given $\nu \in \mathbb{Z}_+$.

$$\begin{aligned} Z_0(s) =& \frac{1}{\Gamma(s)}\left[\left(\sum_{h=0}^{1}\sum_{j=0}^{\nu}\int_0^1 t^{s-1}\mathrm{Tr}\,B_{-2j-h}(t)\,dt\right)\right. \\ &\left. + \int_0^1 t^{s-1}\mathrm{Tr}\,R_{2\nu+2}(t)\,dt + \int_0^1 t^{s-1}\mathrm{Tr}\,R(t)\,dt\right] \end{aligned}$$

Since the function $s \mapsto \int_0^1 t^{s-1}\mathrm{Tr}R_{2\nu+2}(t)\,dt =: F_{2\nu+2}(s)$ is holomorphic for $\mathrm{Re}s > -\nu-1$, we thus obtain that, for any given $\nu \in \mathbb{Z}_+$ with $\nu \geq n$,

$$
\begin{aligned}
Z_0(s) = \frac{1}{\Gamma(s)} \Bigg[& \sum_{h=0}^{1}\sum_{j=0}^{\nu} \frac{c_{-2j-h,n}}{s-(n-j)+h/2} + \left(\sum_{h=0}^{1}\sum_{j,k=0}^{\nu} \frac{f_{-2j-h}^{(j+h+k)}(0)}{(j+h+k)!}\frac{1}{s+j+h+k} \right) \\
& + \sum_{k=0}^{\nu} \frac{f_R^{(k)}(0)}{k!}\frac{1}{s+k} + \left(\sum_{h=0}^{1}\sum_{j=0}^{\nu} F_{-2j-h,\nu}(s) \right) + F_{R,\nu}(s) + F_{2\nu+2}(s) \Bigg] \\
= \frac{1}{\Gamma(s)} \Bigg[& \left(\sum_{h=0}^{1}\sum_{j=0}^{n-1} \frac{c_{-2j-h,n}}{s-(n-j)+h/2} \right) + \left(\sum_{j=n}^{\nu} \frac{c_{-2j-1,n}}{s-(n-j)+1/2} \right) \\
& + \left(\sum_{j=n}^{\nu} \frac{C_{-2j}}{s-(n-j)} \right) + \tilde{H}_\nu(s) \Bigg],
\end{aligned}
$$

with $s \mapsto \tilde{H}_\nu(s)$ holomorphic for $\operatorname{Re} s > (n-\nu)-1$. Since the function $1/\Gamma(s)$ is holomorphic in $\mathbb{C}$ and has zeros at the non-positive integers $-k$, $k \in \mathbb{Z}_+$, this proves the theorem. $\square$

Remark 5.2. An interesting problem can be to use the asymptotics for resolvent expansions and trace regularizations by [8] and [9].

Theorem 5.1 has the following corollary for the Hurwitz-type spectral zeta function of A^{w}.

Corollary 5.3. *Let $A = a_2 + a_1 + a_0$ be an elliptic system of second order where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0, 1, 2$. Moreover, suppose $A^{\mathrm{w}} > 0$.*

For all $\tau > 0$ there exist constants $c_{-2j-h,n}(\tau)$ with $0 \le j \le n-1$, $h = 0, 1$, and constants $c_{-2j-1,n}(\tau)$, $C_{-2j}(\tau)$ with $j \ge n$, such that, for any given integer $\nu \in \mathbb{Z}_+$ with $\nu \ge n$,

$$
\begin{aligned}
\zeta_{A^{\mathrm{w}}+\tau I}(s) = \frac{1}{\Gamma(s)} \Bigg[& \left(\sum_{h=0}^{1}\sum_{j=0}^{n-1} \frac{c_{-2j-h,n}(\tau)}{s-(n-j)+h/2} \right) + \left(\sum_{j=n}^{\nu} \frac{c_{-2j-1,n}(\tau)}{s-(n-j)+1/2} \right) \\
& + \left(\sum_{j=n}^{\nu} \frac{C_{-2j}(\tau)}{s-(n-j)} \right) + H_\nu(\tau)(s) \Bigg]
\end{aligned}
\tag{20}
$$

where $\Gamma(s)$ is the Euler gamma function, and $s \mapsto H_\nu(\tau)(s)$ is holomorphic in the region $\operatorname{Re} s > (n-\nu)-1$*. Consequently, the spectral zeta function $\zeta_{A^{\mathrm{w}}+\tau I}$ is meromorphic in the whole complex plane $\mathbb{C}$ with at most simple poles at $s = n, n-1, n-2, \ldots, 1$ and $s = n-\frac{1}{2}, n-\frac{3}{2}, n-\frac{5}{2}, \ldots$. One has*

$$c_{-2j-h,n}(\tau) =$$

$$(2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\left(b_{-2j-h}(\rho^2,\omega)\right)\rho^{2(n-j)-1-h}\,d\omega d\rho \tag{21}$$

$$-\tau(2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}}\int_0^{\rho^2} e^{-(\rho^2-t')a_2}\mathsf{Tr}(b_{2-2j-h}(t',\omega))\rho^{2(n-j)-1-h}\,dt'd\omega d\rho$$

where $0\le j\le n-1$, $h=0,1$ or $j\ge n$, $h=1$. In (21) the b_{-2j-h} are the terms in the symbol of the parametrix $U_A\in \mathrm{OP}S_{\mathrm{sreg}}(2,0)$ constructed in the proof of Lemma 3.1 and Remark 3.2,

$$U_A\sim\sum_{j\ge 0}B_{-j},$$

where we set $b_k\equiv 0$ for all $k=1,\,2$.

Proof. The proof follows from the demonstration of Theorem 5.1. In fact, we use of the equations in the proof of Lemma 3.1 and (9) to link the asymptotic expansion of the parametrix of the heat semi-group of $A^{\mathrm{w}}+\tau I$ to the one of A^{w}. Let b_j, r_{2-j} be the terms constructed in the proof of Lemma 3.1 (see also (8) and (9)) for A^{w} and $b_{-j}(\tau)$, $r_{2-j}(\tau)$ those for $A^{\mathrm{w}}+\tau I$. Then,

$$\begin{cases} b_0(\tau)(t,X)=e^{-ta_2(X)},\\ b_{-1}(\tau)(t,X)=\int_0^t e^{-(t-t')a_2}r_{2-j}(t',X)\,dt',\\ b_{-j}(\tau)(t,X)=-\int_0^t e^{-(t-t')a_2}r_{2-j}(t',X)\,dt'-\tau\int_0^t e^{-(t-t')a_2}b_{2-j}(t',X)\,dt',\ j\ge 2,\end{cases} \tag{22}$$

since for all $j\ge 2$

$$\begin{cases}\frac{d}{dt}b_{-j}(\tau)+a_2b_{-j}(\tau)=-r_{2-j}(\tau)=-r_{2-j}-\tau b_{2-j},\\ b_{-j}(\tau)|_{t=0}=0.\end{cases}$$

Now, we apply Theorem 5.1 to $\zeta_{A^{\mathrm{w}}+\tau I}$, obtaining (20) with coefficients

$$c_{-2j-h,n}(\tau)=(2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}}\mathsf{Tr}\left(b_{-2j-h}(\tau)(\rho^2,\omega)\right)\rho^{2(n-j)-1-h}\,d\omega\,d\rho. \tag{23}$$

Actually, substituting in (23) the expressions for $b_{-j}(\tau)$ given by (22), we obtain (21) which completes the proof. □

6. Examples

6.1. The meromorphic continuation of Jayne-Cumming model spectral zeta function ($n = 1$, $N = 2$)

The Jaynes-Cumming (JC) model is the model of a two-level atom in one cavity, given by the 2×2 system in one real variable $x \in \mathbb{R}$ (see 3.1 in [15])

$$A^{\mathrm{w}}(x, D) = \alpha p_2^{\mathrm{w}}(x, D) I_2 + \beta\Big(\boldsymbol{\sigma}_+ \psi^{\mathrm{w}}(x, D)^* + \boldsymbol{\sigma}_- \psi^{\mathrm{w}}(x, D)\Big) + \gamma \boldsymbol{\sigma}_3, \ \alpha > 0, \beta, \gamma \in \mathbb{R},$$

where $\psi(x, D) := \frac{x+\partial_x}{\sqrt{2}}$, $\boldsymbol{\sigma}_\pm := \frac{1}{2}(\boldsymbol{\sigma}_1 \pm i\boldsymbol{\sigma}_2)$ with $\boldsymbol{\sigma}_j$, $j = 0, \ldots, 3$, the Pauli-matrices, i.e.

$$\boldsymbol{\sigma}_0 := I_2, \quad \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and the atom levels are given by $\pm\gamma$.

To apply Theorem 5.1 we need to compute the terms b_{-j} of the asymptotic expansion of the semi-group parametrix construct in Lemma 3.1. First of all, if A is the Hamiltonian of the JC model and, in the notations of the previous sections, $A = a_2 + a_1 + a_0$, then

$$a_1 a_0 = -a_0 a_1, \ a_0^2 = I_2, \ \text{and} \ a_1^2 = p_2, \tag{24}$$

where p_2 is the harmonic oscillator symbol. Hence, the product of any number of factors equal to a_1 or a_0 can be rewritten as the multiple (by a function in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$) of a_1, a_0, $a_0 a_1$ or I_2 by using iteratively the identities (24). This fact motivates the following definition.

Definition 6.1. Given a linear combination of products of any number of a_0 and a_1, we say that it is written in *irreducible form* if it is a linear combination of a_1, a_0, $a_0 a_1$ and I_2 with coefficients in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$.

We are going to prove a lemma determining the structure of the b_j as linear combination with coefficients in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$ of a_1, a_0, $a_0 a_1$ and I_2.

Lemma 6.2. *Let $A = a_2 + a_1 + a_0$ be the Hamiltonian of the JC model with a_j homogeneous of degree j. Then, the b_{-j} can be written in irreducible form. Moreover,*

$$j \ \textit{odd} \ \Rightarrow \ \textit{the coefficients of } a_0, I_2 \textit{ in the irreducible form of } b_{-j} \textit{ are } 0, \tag{25}$$

$$j \ \textit{even} \ \Rightarrow \ \textit{the coefficients of } a_1, a_0 a_1 \textit{ in the irreducible form of } b_{-j} \textit{ are } 0. \tag{26}$$

Proof. The proof is by induction, follows the construction of the parametrix in Lemma 3.1 and here we will use the same notations of that lemma. First of all,

$$b_0(t,X) = e^{-tp_2(X)} I_2, \quad b_{-1}(t,X) = -te^{-tp_2(X)} a_1$$

(see also (9)). Hence, b_0 and b_{-1} are already written in irreducible form and satisfy (25) and (26). Now, we suppose that for all $j' \leq 2j-1$ ($j \geq 2$) the thesis is verified and we want to prove the result for b_{2j} and b_{2j+1}. By the construction in Lemma 3.1 and since A is a differential operator (that is, its expansion contains only terms with degree of homogeneity ≥ 0),

$$\begin{cases} \frac{d}{dt} b_{-2j} + p_2 b_{-2j} = -a_0 b_{2-2j} - a_1 b_{1-2j}, \\ b_{-2j}|_{t=0} = 0. \end{cases}$$

Hence, since by inductive hypothesis

$$\begin{aligned} a_0 b_{2-2j} + a_1 b_{1-2j} =& a_0(f_1 a_0 + f_2 I_2) + a_1(g_1 a_1 + g_2 a_0 a_1) \\ =& f_1 a_0^2 + f_2 a_0 + g_1 a_1^2 + g_2 a_1 a_0 a_1 \\ =& f_1 I_2 + f_2 a_0 - g_1 p_2 I_2 - g_2 p_2 a_0, \end{aligned}$$

where the third equality follows from (24) and where the f_j and g_j are function in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$. Hence, the claim is verified for b_{-2j}. Repeating the argument for b_{-2j-1}, we have

$$\begin{cases} \frac{d}{dt} b_{-2j-1} + p_2 b_{-2j-1} = -a_0 b_{2-2j-1} - a_1 b_{1-2j-1}, \\ b_{-2j-1}|_{t=0} = 0, \end{cases}$$

which, since

$$\begin{aligned} a_0 b_{1-2j} + a_1 b_{-2j} =& a_0(\tilde{f}_1 a_1 + \tilde{f}_2 a_0 a_1) + a_1(\tilde{g}_1 a_0 + \tilde{g}_2 I_2) \\ =& \tilde{f}_1 a_0 a_1 + \tilde{f}_2 a_0^2 a_1 + \tilde{g}_1 a_1 a_0 + \tilde{g}_2 a_1 \\ =& \tilde{f}_1 a_0 a_1 + \tilde{f}_2 a_1 - \tilde{g}_1 a_0 a_1 + \tilde{g}_2 a_1, \end{aligned}$$

shows that the claim is verified also for b_{-2j-1} and completes the proof. □

Remark 6.3. By Lemma 6.2 we have that $\mathsf{Tr}\,(b_{-2j-1}) = 0$ since it is a linear combination of matrices with zeros on the principal diagonal. Hence, by (25) and (17) we have that

$$c_{-2j-1,1} = (2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} \mathsf{Tr}\left(b_{-2j-1}(\rho^2, \omega)\right) \rho^{-2j} \, d\omega \, d\rho = 0, \; j \geq 0,$$

and

$$\begin{aligned}c_{0,1} =&(2\pi)^{-1}\int_0^{+\infty}\int_0^{2\pi}\mathsf{Tr}\left(b_0(\rho^2,\omega)\right)\rho\,d\omega\,d\rho\\ =&2(2\pi)^{-1}\int_0^{+\infty}\int_0^{2\pi}e^{-\rho^2/2}\rho\,d\omega\,d\rho\\ =&2\int_0^{+\infty}e^{-\rho^2/2}\rho\,d\rho=2.\end{aligned}$$

Therefore, if A is the JC Hamiltonian, by (16) the spectral zeta function associated to A^{w} is

$$\zeta_{A^{\mathrm{w}}}(s)=\frac{1}{\Gamma(s)}\left[\frac{2}{s-1}+\left(\sum_{j=1}^{\nu}\frac{C_{-2j}}{s-(1-j)}\right)+H_\nu(s)\right],$$

where $\nu\geq 1$, H_ν is holomorphic in the region $\mathrm{Re}s>-\nu$ and the $c_{-2,1}$, C_{-2j} has been defined in Theorem 5.1. Consequently, the spectral zeta function $\zeta_{A^{\mathrm{w}}}$ is meromorphic in the whole complex plane $\mathbb{C}$ with a simple pole at $s=1$. Thus, $\zeta_{A^{\mathrm{w}}}$ has a meromorphic continuation to $\mathbb{C}$.

6.2. The JC-model for one atom with 3-level and one cavity-mode in the so called Ξ-configuration

This generalization of the JC model (that we will denote by 3-Ξ-JCM) describes a 3-level atom in one cavity, given by the 3×3 system in one real variable $x\in\mathbb{R}^2$ (see 3.2 in [15]). In this configuration every level of energy can interact only with the ones near to it, that is the electron can absorb (or emit) a photon moving from the jth level of energy to the $j+1$st (or from the $j+1$st level of energy to the jth) for $j=1,2$. That is mathematically represented by the following Hamiltonian operator. For $\alpha>0$, $\beta_1,\beta_2\in\mathbb{R}\setminus\{0\}$, $\gamma_1,\gamma_2,\gamma_3\in\mathbb{R}$ with $\gamma_1<\gamma_2<\gamma_3$,

$$\begin{aligned}A^{\mathrm{w}}(x,D)=&\alpha p_2^{\mathrm{w}}(x,D)I_3+\frac{1}{2}\sum_{k=1}^{2}\beta_k\left(\psi^{\mathrm{w}}(x,D)^*E_{k,k+1}+\psi^{\mathrm{w}}(x,D)E_{k+1,k}\right)\\ &+\sum_{k=1}^{3}\gamma_kE_{kk},\end{aligned}$$

with

$$E_{jk}:=e_k^*\otimes e_j,\quad 1\leq j,k\leq 3$$

forming the basis of the 3×3 complex matrices, where E_{jk} acts on $\mathbb{C}^3$ as

$$E_{jk}w = \langle w, e_k \rangle e_j, \quad w \in \mathbb{C}^3,$$

and $\psi(x, D) := \frac{x+\partial_x}{\sqrt{2}}$.

Lemma 6.4. *Let $A = a_2+a_1+a_0$ be the Hamiltonian of the 3-Ξ-JCM with a_j homogeneous of degree j. Then,*

$$j \text{ odd } \Rightarrow \text{the principal and secondary diagonal entries of } b_{-j} \text{ are } 0, \tag{27}$$

$$j \text{ even } \Rightarrow \text{the subdiagonal and superdiagonal entries of } b_{-j} \text{ are } 0. \tag{28}$$

Proof. Again the proof is by induction, follows the construction of the parametrix in Lemma 3.1 and here we will use the same notations of that lemma. First of all,

$$b_0(t, X) = e^{-tp_2(X)} I_2, \quad b_{-1}(t, X) = -te^{-tp_2(X)} a_1.$$

Hence, b_0 and b_{-1} satisfy (27) and (28). Now, we suppose that for all $j' \leq 2j-1$ ($j \geq 2$) the thesis is verified and we want to prove the result for b_{2j} and b_{2j+1}. By the construction in Lemma 3.1 and since A is a differential operator

$$\begin{cases} \frac{d}{dt} b_{-2j} + p_2 b_{-2j} = -a_0 b_{2-2j} - a_1 b_{1-2j}, \\ b_{-2j}|_{t=0} = 0. \end{cases}$$

Therefore, by inductive hypothesis $a_0 b_{2-2j}$ subdiagonal and superdiagonal entries are 0 since a_0 is a diagonal matrix. Moreover, $a_1 b_{1-2j}$ subdiagonal and superdiagonal entries are 0 since the principal and secondary diagonal entries of b_{1-2j} are 0. Hence, the claim is verified for b_{-2j}. Repeating the argument for b_{-2j-1}, we have

$$\begin{cases} \frac{d}{dt} b_{-2j-1} + p_2 b_{-2j-1} = -a_0 b_{2-2j-1} - a_1 b_{1-2j-1}, \\ b_{-2j-1}|_{t=0} = 0. \end{cases}$$

Thus, by inductive hypothesis $a_0 b_{1-2j}$ has principal and secondary diagonal entries that are 0 since a_0 is diagonal. Moreover, $a_1 b_{-2j}$ principal and secondary diagonal entries are 0 since b_{-2j} diagonal entries are 0. Hence, the claim is verified also for b_{-2j-1}. $\square$

Remark 6.5. By Lemma 6.4 we have that $\mathsf{Tr}\,(b_{-2j-1}) = 0$ since b_{-2j-1} principal diagonal entries are 0. Hence, by (27) and (28) we have that

$$c_{-2j-1,1} = (2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} \mathsf{Tr}\left(b_{-2j-1}(\rho^2, \omega)\right) \rho^{-2j} \, d\omega \, d\rho = 0, \; j \geq 0,$$

and

$$\begin{aligned} c_{0,1} =&(2\pi)^{-1}\int_0^{+\infty}\int_0^{2\pi}\operatorname{Tr}\left(b_0(\rho^2,\omega)\right)\rho\, d\omega\, d\rho\\ =&3(2\pi)^{-1}\int_0^{+\infty}\int_0^{2\pi} e^{-\rho^2/2}\rho\, d\omega\, d\rho\\ =&3\int_0^{+\infty} e^{-\rho^2/2}\rho\, d\rho = 3. \end{aligned}$$

Therefore, if A is the 3-Ξ-JCM Hamiltonian, by (16) the spectral zeta function associated to A^{w} is

$$\zeta_{A^{\mathrm{w}}}(s) = \frac{1}{\Gamma(s)}\left[\frac{3}{s-1} + \left(\sum_{j=1}^{\nu}\frac{C_{-2j}}{s-(1-j)}\right) + H_\nu(s)\right],$$

where $\nu \geq 1$, H_ν is holomorphic in the region Re$s > -\nu$ and the $c_{-2,1}$, C_{-2j} has been defined in Theorem 5.1. Consequently, the spectral zeta function $\zeta_{A^{\mathrm{w}}}$ is meromorphic in the whole complex plane $\mathbb{C}$ with a simple pole at $s = 1$. Thus, $\zeta_{A^{\mathrm{w}}}$ has a meromorphic continuation to $\mathbb{C}$.

7. Non-existence of semi-integer poles

In this section we prove, as a corollary of Theorem 5.1, that for a differential operator of order 2 with polynomial coefficients there are no semi-integer poles.

Corollary 7.1. *Let $A = a_2 + a_1 + a_0$ be an elliptic system of second order where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0, 1, 2$. Moreover, suppose $A^{\mathrm{w}} > 0$. Then, $c_{-2j-1,n} = 0$ in (16) for all $j \geq 0$.*

Proof. We prove this corollary by showing that for $j \geq 0$ the b_{-2j-1} (respectively, b_{-2j}) in the construction of the parametrix in Lemma 3.1 are odd (respectively, even) functions in the phase variable X by an induction argument on j. In fact, this completes the proof since in (16) $\operatorname{Tr}(b_{-2j-1})$ is again an odd function of X and in the integral over $\mathbb{S}^{2n-1}$ is 0. First of all,

$$b_0(t, X) = e^{-tp_2(X)}I_2, \quad b_{-1}(t, X) = -te^{-tp_2(X)}a_1$$

(see also (9)). Hence, b_0 and b_{-1} are, respectively, an even and odd function of X since a_1 is an homogeneous polynomial of degree -1. Now, we suppose that for all $j' \leq 2j-1$ ($j \geq 1$) the thesis is verified and we want to prove the result for b_{2j} and b_{2j+1}. By the construction in Lemma 3.1 and since A is a differential operator (that is, its expansion contains only terms with degree of homogeneity ≥ 0),

$$\begin{cases} \frac{d}{dt} b_{-2j} + p_2 b_{-2j} = -a_0 b_{2-2j} - a_1 b_{1-2j}, \\ b_{-2j}|_{t=0} = 0. \end{cases}$$

Hence, by the inductive hypothesis b_{2-2j} and b_{1-2j} are, respectively, an even and odd function of X. Therefore, the claim is verified for b_{-2j}. Repeating the argument for b_{-2j-1}, we have

$$\begin{cases} \frac{d}{dt} b_{-2j-1} + p_2 b_{-2j-1} = -a_0 b_{2-2j-1} - a_1 b_{1-2j-1}, \\ b_{-2j-1}|_{t=0} = 0, \end{cases}$$

which completes the proof since by inductive hypothesis b_{1-2j-1} and b_{2-2j-1} are, respectively, an even and odd function of X. □

Declaration of competing interest

The author declares that there is no competing interest.

Data availability

No data was used for the research described in the article.

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