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A Convergent Semi-Lagrangian Scheme for the Game ∞-Laplacian

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Abstract
We propose a new semi-Lagrangian scheme for the game ∞-Laplacian. We demonstrate the convergence of the scheme to the viscosity solution of the given problem, showing its consistency, monotonicity, and stability. The proof of this result is established following the Barles-Souganidis analysis. This analysis assumes convergence at the boundary in a strong sense and is applied to our proposed scheme, augmented with an artificial viscosity term.

Keywords Game ∞-Laplacian · Semi-Lagrangian scheme · Convergence analysis · Viscosity solutions

Mathematics Subject Classification 65N06 · 35J25 · 35D40 · 65N12

1 Introduction

We focus on the numerical investigation of the following Dirichlet problem associated with the game ∞-Laplacian:

$$\begin{cases} -\Delta_\infty^G u(x) = f(x) & \text{for } x \in \mathcal{O} \subset \Omega, \\ u(x) = F(x) & \text{for } x \in \overline{\Omega} \setminus \mathcal{O}, \end{cases} \tag{1}$$

where $\mathcal{O}$ and Ω are bounded open domains in $\mathbb{R}^2$ with $\partial\Omega \cap \partial\mathcal{O} = \emptyset$, $F : \overline{\Omega}\setminus\mathcal{O} \to \mathbb{R}$ and $f : \mathcal{O} \to \mathbb{R}$ are continuous functions, and, additionally, f is either identically equal to zero or never zero (cf. [17]). The function $u : \overline{\Omega} \to \mathbb{R}$ is the unknown in this problem, which is given at the boundary in a strong sense, i.e. $u(x) = F(x)$ on $\overline{\Omega} \setminus \mathcal{O}$.

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The game ∞-Laplacian operator, introduced in [16] to model a stochastic game called tug-of-war with noise, is defined as

$$\Delta_\infty^G u(x) := |Du(x)|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{2}$$

where $Du(x)$ denotes the gradient of $u(x)$. At points where $|Du(x)| \neq 0$, the game ∞-Laplacian can be viewed as the second derivative in the direction of $Du(x)$, given by

$$\Delta_\infty^G u(x) = \sigma(Du(x))^T D^2u(x)\,\sigma(Du(x)), \tag{3}$$

where $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\sigma(p) := \frac{1}{|p|} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{4}$$

and $D^2u(x)$ denotes the Hessian matrix.

A generalization of problem (1) has been explored in [5] as an image inpainting model. Indeed, the game ∞-Laplacian can be regarded as a specific case of the game p-Laplacian. More generally, the game ∞-Laplacian operator, expressed as the second derivative in the direction of the gradient, has also found applications in edge detection (see [6, 14] for references).

Regarding the numerical approximation of the game ∞-Laplacian operator, several convergent numerical schemes have been proposed (e.g. in [9, 14, 15]). These methods are, on the one hand, based on stencils composed of a large number of grid nodes, on the other hand, the convergence analysis, as in [9], is limited to a semi-discrete scheme, specifically in the case where $f = 0$.

In this work, our focus is on a theoretical investigation of the numerical scheme presented in [5], considering only the game ∞-Laplacian operator. Our aim is to establish a convergence result for a novel semi-Lagrangian scheme, employing the Barles-Souganidis analysis [3]. In our case, we demonstrate both the consistency and monotonicity of a fully discrete scheme, along with proving its convergence, in particular for the case where $f = 0$. In addition, we make use of a smaller stencil than those adopted in the works mentioned above.

Our study can be easily extended to the game p-Laplacian, as defined in [5], along the same line of the following argument.

1.1 Preliminary Results

We recall here the definition of viscosity sub- and super-solutions in $\mathcal{O}$ for our problem (1). The interested reader can refer to the works [2, 7, 11–13] for more details on the related theory.

Definition 1 Let us consider a bounded open domain $\mathcal{O} \subset \mathbb{R}^2$. Let $f : \mathcal{O} \to \mathbb{R}$ be a continuous function. We say that an upper semicontinuous (usc) (respectively, lower semicontinuous (lsc)) function $u : \mathcal{O} \to \mathbb{R}$ is a viscosity subsolution (respectively, supersolution) of

$$-\Delta_\infty^G u(x) = f(x) \quad x \in \mathcal{O}, \tag{5}$$

if for any $\phi \in C^2(\mathcal{O})$ such that $u(x) - \phi(x)$ has a local maximum (local minimum) at $x \in \mathcal{O}$, the following inequalities hold:

(i) if $D\phi(x) \neq 0, \quad -\Delta_\infty^G \phi(x) \leq f(x) \quad (-\Delta_\infty^G \phi(x) \geq f(x)),$

(ii) if $D\phi(x) = 0$ and $\lambda_1 \leq \lambda_2$, where λ_1, λ_2 denote the eigenvalues of the Hessian matrix $D^2\phi(x)$, then:

$$-\lambda_2 \leq f(x) \qquad (-\lambda_1 \geq f(x)).$$

A function u is a viscosity solution of (5) if u is both a subsolution and a supersolution in $\mathcal{O}$.

For the existence and uniqueness results of problem (1) with $f = 0$, we refer interested readers to [2, 11, 13] and the references therein.

Finally, we recall the following *comparison principle* (Cf. [10, Theorem 3.11]).

Theorem 1 *If $u : \mathcal{O} \to \mathbb{R}$ is a usc subsolution of* (5) *and $v : \mathcal{O} \to \mathbb{R}$ is a lsc supersolution of* (5) *with $f = 0$, then*

$$\sup_{x \in \mathcal{O}} (u - v)(x) = \sup_{x \in \partial\mathcal{O}} (u - v)(x).$$

2 Construction of a Semi-Lagrangian Scheme for the Game ∞-Laplacian

We derive a semi-Lagrangian scheme for the problem (1) by approximating the second order operator with a directional finite difference, supposing $Du \neq 0$. Hence, let us introduce a discretization parameter Δ and let us consider the following approximation:

$$\sigma(Du(x))^T D^2u(x)\, \sigma(Du(x)) \approx \frac{1}{\Delta^2} \big(u(x + \Delta\sigma(Du(x))) + u(x - \Delta\sigma(Du(x))) - 2u(x)\big). \tag{6}$$

From now on, we suppose Ω a square domain, i.e. $\Omega = (0, L) \times (0, L)$. Given $M \in \mathbb{N}$, let us define a space discretization parameter $h := L/M$ and the multi-index set $\mathcal{J} := \{0, \dots M\}^2$. We introduce a uniform grid

$$\mathcal{G}_h := \{x_j = (j_1 h, j_2 h) \mid j = (j_1, j_2) \in \mathcal{J}\},$$

and the sets of indexes

$$Q := \{j \in \mathcal{J} \text{ such that } x_j \in \mathcal{O}\}, \quad B := \{j \in \mathcal{J} \text{ such that } x_j \in \overline{\Omega} \setminus \mathcal{O}\}.$$

Let us define

$$\sigma_\varepsilon(p) := \sigma(p + \bar{\varepsilon}), \tag{7}$$

where σ is defined as in (4), and $\bar{\varepsilon} := \varepsilon(1, 1)^T$ with $\varepsilon > 0$ being a positive parameter. This definition ensures that the map σ_ε is Lipschitz continuous with a constant $L_{\sigma_\varepsilon} = O(\varepsilon^{-1})$ (cf. [8]). Moreover, let us denote the central finite difference approximation of Du at the node x_j with step h as $D_j[u] := (D_{1,j}[u], D_{2,j}[u])^\top$, where

$$D_{1,j}[u] := \frac{u_{j_1+1, j_2} - u_{j_1-1, j_2}}{2h}, \quad D_{2,j}[u] := \frac{u_{j_1, j_2+1} - u_{j_1, j_2-1}}{2h}.$$

We denote by $B(\mathcal{G}_h)$ the space of functions defined on $\mathcal{G}_h$. For a given function $f \in B(\mathcal{G}_h)$, we represent the value of f at the node x_j as f_j.

Following [4, 8], we propose the following approximation scheme for (1):

Find $u_{\varepsilon,\rho} \in B(\mathcal{G}_h)$ solution to

$$G_\varepsilon(\rho, x_j, u_j, u) = 0, \tag{8}$$

where

$$G_\varepsilon(\rho, x_j, u_j, u) := \begin{cases} -S_\varepsilon(\rho, x_j, u_j, u) - f_j & \text{if } j \in Q, \\ u(x_j) - F(x_j) & \text{if } j \in B, \end{cases} \tag{9}$$

with $\rho := (h, \Delta)$ representing a vector of discretization parameters. The map $S_\varepsilon : [0, 1) \times [0, 1) \times \mathcal{G}_h \times \mathbb{R} \times B(\mathcal{G}_h) \to \mathbb{R}$ is defined as

$$S_\varepsilon(\rho, x_j, u_j, u) := \frac{1}{\Delta^2}\left(I[u](x_j + \Delta\sigma_\varepsilon(D_j[u])) + I[u](x_j - \Delta\sigma_\varepsilon(D_j[u])) - 2u_j\right), \tag{10}$$

where $I[u](x)$ denotes the bi-linear Lagrange interpolation of a given function $u \in B(\mathcal{G}_h)$.

Remark 1 For Δ small enough, the scheme (10) is well-defined, since the discrete characteristics $x_j \pm \Delta\sigma_\varepsilon(D_j[u])$ never exit the domain $\overline{\Omega}$.

3 Convergence Analysis

We prove a convergence result by showing that the scheme (8) is consistent and, thanks to the addition of an artificial viscosity term, it is also monotone and stable.

We denote by ϕ a generic test function, $\phi \in C^\infty$, and $\widehat{\phi} := (\phi_j)$, where $\phi_j := \phi(x_j)$.

3.1 Consistency

Let us recall the definition of consistency, as introduced in [3, 14].

Definition 2 For any $x \in \mathcal{O}$, let us consider a sequence of discretization parameters $\rho_m = (h_m, \Delta_m)$, a sequence of nodes $(x_{j_m}) \in \mathcal{G}_{h_m}$ such that, as $m \to \infty$:

$$\rho_m \to (0, 0), \quad \varepsilon_m \to 0, \quad \text{and} \quad x_{j_m} \to x. \tag{11}$$

Additionally, let $\phi \in C^\infty(\mathcal{O})$ and define $\widehat{\phi}_m := (\phi(x_{j_m}))_{x_j \in \mathcal{G}_h}$. The scheme (8) is said to be consistent if

$$-\lim_{m\to\infty} S_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) - f(x_{j_m}) = -\Delta_\infty^G \phi(x) - f(x), \tag{12}$$

for $D\phi(x) \neq 0$, and

$$\begin{aligned} -\lambda_2(D^2\phi(x)) - f(x) &\le -\limsup_{m\to\infty} S_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) - f(x_{j_m}) \\ &\le -\liminf_{m\to\infty} S_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) - f(x_{j_m}) \\ &\le -\lambda_1(D^2\phi(x)) - f(x), \end{aligned} \tag{13}$$

for $D\phi(x) = 0$.

We can now state and prove the following theorem.

Theorem 2 *Assume that there exist four constants* $C_1, C_2, r, q > 0$*, independent of* ϕ *and* h*, such that*

$$\|I[\widehat{\phi}](\cdot) - \phi(\cdot)\|_\infty \le C_1 h^r, \tag{14}$$

$$|D_j[\widehat{\phi}] - D\phi(x_j)| \leq C_2 h^q, \quad \textit{for any } j \in Q, \tag{15}$$

and that

$$h = \Delta^\alpha, \quad \varepsilon = \Delta^\beta, \quad \textit{where} \quad \alpha > \max\left\{\frac{2}{r}, \frac{1+\beta}{q}\right\}, \; \beta > 0. \tag{16}$$

Then, as $\Delta \to 0$*, the scheme* (8) *satisfies the consistency conditions* (12)–(13).

Proof From now on, we neglect the subscript m and we denote $D\phi_j = D\phi(x_j)$ and $D^2\phi_j = D^2\phi(x_j)$ for a better readability. To check consistency, let us consider the following cases.

$\underline{Case 1 : D\phi(x) \neq 0, x \in \mathcal{O}}$. For m big enough, we can also suppose that $D\phi_j \neq 0$. In this first case, consistency analysis may be carried out by essentially the same arguments in [4]. By using assumptions (14), (15), the Lipschitzianity of σ_ε, and a third order Taylor expansion of $\phi(x_j + \Delta\sigma_\varepsilon(D\phi_j))$ and $\phi(x_j - \Delta\sigma_\varepsilon(D\phi_j))$ with respect to the node x_j, we get, for $j \in Q$,

$$S_\varepsilon(\rho, x_j, \phi_j, \widehat{\phi}) = \sigma_\varepsilon(D\phi_j)^T D^2\phi_j \, \sigma_\varepsilon(D\phi_j) + O\left(\frac{h^r}{\Delta^2}\right) + O\left(\frac{h^q}{\varepsilon\Delta}\right) + O\left(\Delta^2\right). \tag{17}$$

Choosing the parameters h, ε as in (16), $S_\varepsilon(\rho, x_j, \phi_j, \widehat{\phi})$ verifies the condition (12). As a consequence, also the scheme $G_\varepsilon(\rho, x_j, u_j, u)$ defined in (8) verifies (12).

$\underline{Case 2 : D\phi(x) = 0, x \in \mathcal{O}.}$

Using assumptions (14) and a third Taylor expansion of $\phi(x_j + \Delta\sigma_\varepsilon(D_j[w]))$ and $\phi(x_j - \Delta\sigma_\varepsilon(D_j[w]))$ with respect to the node x_j we get, for $j \in Q$,

$$S_\varepsilon(\rho, x_j, \phi_j, \widehat{\phi}) = \sigma_\varepsilon(D_j[\widehat{\phi}])^T D^2\phi_j \, \sigma_\varepsilon(D_j[\widehat{\phi}]) + O\left(\frac{h^r}{\Delta^2}\right) + O\left(\Delta^2\right). \tag{18}$$

Since in this case $\sigma_\varepsilon(D_j[\widehat{\phi}])$ is singular in the limit for $(\varepsilon, \rho) \to (0, 0, 0)$, S_ε can be bounded only by its lim inf and lim sup. Using the following eigenvalue properties:

$$\lambda_1(D^2\phi(x)) = \min_{\theta\in\mathbb{R}^2, |\theta|=1} \theta^T D^2\phi(x)\,\theta, \quad \lambda_2(D^2\phi(x)) = \max_{\theta\in\mathbb{R}^2, |\theta|=1} \theta^T D^2\phi(x)\,\theta,$$

and, by assuming (16), we obtain:

$$\begin{aligned}\lambda_1(D^2\phi(x)) &\leq \liminf_{m\to\infty} S_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) \\ &\leq \limsup_{m\to\infty} S_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) \leq \lambda_2(D^2\phi(x)).\end{aligned} \tag{19}$$

From (19), we can easily derive

$$\begin{aligned}-\lambda_2(D^2\phi(x)) - f(x) &\leq \liminf_{m\to\infty} G_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) \\ &= -\limsup_{m\to\infty} S_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) - f(x_{j_m}) \\ &\leq -\liminf_{m\to\infty} S_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) - f(x_{j_m}) \\ &= \limsup_{m\to\infty} G_{\varepsilon_m}(\rho_m, x_{j_m}, \phi_{j_m}, \widehat{\phi}_m) \\ &\leq -\lambda_1(D^2\phi(x)) - f(x),\end{aligned}$$

Under assumption (16) the equality $h^r = o(\Delta^2)$ holds. Then, condition (13) is satisfied by $S_\varepsilon(\rho, x_j, \phi_j, \widehat{\phi})$, hence by the scheme $G_\varepsilon(\rho, x_j, u_j, u)$ defined in (8).

□

Remark 2 This setup allows for various options. For instance, one could consider backward, forward, or higher-order finite difference approximations of the gradient, along with higher-order or weighted essentially non-oscillatory interpolation for reconstructing u along the characteristics. However, our primary focus is on developing a convergence theory for the scheme in two specific scenarios: the centered finite difference scheme for the gradient, as detailed in Sect. 2, which implies $q = 2$, and linear or bilinear interpolation for $I[\cdot]$, resulting in $r = 2$. Although it would be interesting to explore higher-order schemes, and the consistency result formulated in Sect. 3.1 would still be valid, the artificial viscosity term may not be sufficient to ensure monotonicity in general and, hence, a more in-depth study would be required to guarantee convergence.

3.2 Monotonicity

Let L_ϕ denote the Lipschitz constant of the function $\phi \in C^\infty(\Omega)$.

To establish the monotonicity of (8), we follow the idea in [8] and introduce an artificial vanishing viscosity term. The resulting scheme is given by

$$\widetilde{G}_\varepsilon(\rho, x_j, u_j, u) = 0, \tag{20}$$

where

$$\widetilde{G}_\varepsilon(\rho, x_j, u_j, u) := \begin{cases} -\widetilde{S}_\varepsilon(\rho, x_j, u_j, u) - f_j & \text{if } j \in Q, \\ u(x_j) - F(x_j) & \text{if } j \in B. \end{cases} \tag{21}$$

Here

$$\widetilde{S}_\varepsilon(\rho, x_j, u_j, u) := S_\varepsilon(\rho, x_j, u_j, u) + V_\varepsilon(\rho, x_j, u_j, u), \tag{22}$$

and

$$V_\varepsilon(\rho, x_j, u_j, u) := \frac{2W_\varepsilon h}{\Delta} \sum_{i \in \mathcal{D}(j)} \frac{u_i - 4u_j}{h^2},$$

where W_ε is a positive constant, and for $j \in Q$, $\mathcal{D}(j)$ is the set of indices of the nodes defined as

$$\mathcal{D}(j) := \{(j_1 + 1, j_2), (j_1 - 1, j_2), (j_1, j_2 + 1), (j_1, j_2 - 1)\}.$$

The new scheme $\widetilde{G}_\varepsilon$ preserves the consistency property of the original scheme G_ε, as shown in the next proposition.

Proposition 3 *Let us assume* (14), (15), and (16). *Additionally, let us suppose that the following condition holds:*

$$\frac{W_\varepsilon h}{\Delta} = o(1). \tag{23}$$

Then, the scheme $\widetilde{G}_\varepsilon$ satisfies the consistency condition (12)–(13).

Proof Since by definition $V_\varepsilon(\rho, x_j, \phi_j, \widehat{\phi}) \to 0$ as $(\varepsilon, \rho) \to (0, 0, 0)$, the thesis follows straight. □

Let us recall the generalized monotonicity property (see [4]).

Definition 3 The scheme (20) is said to be monotone if it satisfies the following conditions:

$$\text{if } u \leq \widehat{\phi} \quad \text{then} \quad \widetilde{G}_\varepsilon(\rho, x_j, t, u) \geq \widetilde{G}_\varepsilon(\rho, x_j, t, \widehat{\phi}) + o(1), \tag{24}$$

$$\text{if } \widehat{\phi} \leq u \quad \text{then} \quad \widetilde{G}_\varepsilon(\rho, x_j, t, \widehat{\phi}) \geq \widetilde{G}_\varepsilon(\rho, x_j, t, u) + o(1), \tag{25}$$

for any $u \in B(\mathcal{G}_h)$, $\phi \in C^\infty(\Omega)$, $\rho > (0,0)$, $\varepsilon > 0$, $x_j \in \mathcal{G}_h$, and $t \in \mathbb{R}$.

The *monotonicity* property of the scheme (20) follows from the next theorem.

Theorem 4 *Let us suppose*

$$W_\varepsilon > \frac{CL_\phi}{\varepsilon}, \tag{26}$$

and

$$h = \Delta^\alpha \quad \textit{with} \quad \alpha > \frac{2}{r} \tag{27}$$

for $C > 0$, and $\Delta \to 0$. Then, the scheme (20) *is monotone, in the sense that it satisfies* (24)–(25).

Proof In order to prove that $\widetilde{G}_\varepsilon$ is monotone, it is enough to prove that $\widetilde{S}_\varepsilon$ is monotone, i.e.,

$$\text{if} \quad u \leq \widehat{\phi} \quad \text{then} \quad \widetilde{S}_\varepsilon(\rho, x_j, t, u) \leq \widetilde{S}_\varepsilon(\rho, x_j, t, \widehat{\phi}) + o(1), \tag{28}$$

$$\text{if} \quad \widehat{\phi} \leq u \quad \text{then} \quad \widetilde{S}_\varepsilon(\rho, x_j, t, \widehat{\phi}) \leq \widetilde{S}_\varepsilon(\rho, x_j, t, u) + o(1), \tag{29}$$

for all $\rho > (0,0)$, $\varepsilon > 0$, $j \in Q$, and $t \in \mathbb{R}$. We suppose $u_j \leq \phi_j$ for all $j \in \mathcal{J}$.
In the following, we will use the shorthand notation $w(a \pm b) = w(a+b) + w(a-b)$. The forthcoming inequality holds

$$\begin{aligned}
\widetilde{S}_\varepsilon(\rho, x_j, t, u) \leq{}& \frac{1}{\Delta^2}\left(I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[u])) - 2t\right) + \frac{2W_\varepsilon h}{\Delta} \sum_{i \in \mathcal{D}(j)} \frac{u_i - 4t}{h^2} \\
={}& \frac{1}{\Delta^2}\left(I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[\widehat{\phi}])) - 2t\right) + \frac{2W_\varepsilon h}{\Delta} \sum_{i \in \mathcal{D}(j)} \frac{\phi_i - 4t}{h^2} \\
&+ \frac{1}{\Delta^2}\left(I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[u])) - I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[\widehat{\phi}]))\right) \\
&+ \frac{2W_\varepsilon h}{\Delta} \sum_{i \in \mathcal{D}(j)} \frac{u_i - \phi_i}{h^2}.
\end{aligned} \tag{30}$$

Let us prove that

$$\frac{1}{\Delta^2}\left(I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[u])) - I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[\widehat{\phi}]))\right) + \frac{2W_\varepsilon h}{\Delta} \sum_{i \in \mathcal{D}(j)} \frac{u_i - \phi_i}{h^2} \leq o(1). \tag{31}$$

Recalling that L_{σ_ε} is the Lipschitz constant of $\sigma_\varepsilon(p)$, defined in (7), we have $L_{\sigma_\varepsilon} \leq \frac{1}{C\varepsilon}$, $C > 0$. Using assumption (14), and the fact that $\phi \in C^\infty$, we have that

$$\begin{aligned}
&\frac{1}{\Delta^2}\left(I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[u])) - I[\widehat{\phi}](x_j \pm \Delta\sigma_\varepsilon(D_j[\widehat{\phi}]))\right) + \frac{2W_\varepsilon h}{\Delta} \sum_{i \in \mathcal{D}(j)} \frac{u_i - \phi_i}{h^2} \\
&= \frac{1}{\Delta^2}\left(\phi(x_j \pm \Delta\sigma_\varepsilon(D_j[u])) - \phi(x_j \pm \Delta\sigma_\varepsilon(D_j[\widehat{\phi}]))\right) + \frac{2W_\varepsilon h}{\Delta} \sum_{i \in \mathcal{D}(j)} \frac{u_i - \phi_i}{h^2} + O\left(\frac{h^r}{\Delta^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2}{\Delta^2} L_\phi \Delta L_{\sigma_\varepsilon} |D_j[u] - D_j[\widehat{\phi}]| + \frac{2W_\varepsilon h}{\Delta} \sum_{i\in\mathcal{D}(j)} \frac{u_i-\phi_i}{h^2} + O\left(\frac{h^r}{\Delta^2}\right)\\
&\le \frac{2L_\phi L_{\sigma_\varepsilon}}{\Delta} \max_{i\in\mathcal{D}(j)} \left(\frac{\phi_i-u_i}{h}\right) + \frac{2W_\varepsilon h}{\Delta} \sum_{i\in\mathcal{D}(j)} \frac{u_i-\phi_i}{h^2} + O\left(\frac{h^r}{\Delta^2}\right).
\end{aligned}
$$

Then, by (26), we get (31). Now, since (27) implies $h^r = o(\Delta^2)$, using (31) in (30) we get (28).

Analogously, supposing $\widehat{\phi} \le u$, it is possible to prove (29). □

Remark 3 If the function $F : \overline{\Omega}\backslash\mathcal{O} \to \mathbb{R}$ is Lipschitz, the solution of problem (1) inherits a Lipschitz constant denoted as L_u (see [10, Corollary 3.14], [1, 19]). This enables the application of the monotonicity result using test functions with a slightly higher, yet uniformly bounded, Lipschitz constant, such as $2L_u$ (see [4, Remark 3.3]).

3.3 Convergence

Let $\widetilde{u}_{\varepsilon,\rho} \in B(\mathcal{G}_h)$ represent the solution to (20). Our goal is to establish the existence of solutions and demonstrate their convergence towards the solution described by (1).

Theorem 5 *Assume that $F : \overline{\Omega}\backslash\mathcal{O} \to \mathbb{R}$ is a continuous function, then there exists a bounded solution $\widetilde{u}_{\varepsilon,\rho} \in B(\mathcal{G}_h)$ to (20), such that*

$$\max_{j\in\mathcal{J}} |\widetilde{u}_{\varepsilon,\rho}(x_j)| \le \max_{j\in B} |F(x_j)|.$$

uniformly with respect to ρ and ε.

Proof Existence is established using the Leray-Schauder fixed-point theorem (see [18]).

Let us define a map $T : B(\mathcal{G}_h) \to B(\mathcal{G}_h)$ as

$$
T(u) := \begin{cases} \left(\frac{2}{\Delta^2} + \frac{8W_\varepsilon}{\Delta h}\right)^{-1} \left(\frac{1}{\Delta^2} I[u](x_j \pm \Delta\sigma_\varepsilon(D_j[u])) + \frac{2W_\varepsilon \sum_{i\in\mathcal{D}(j)} u_i}{\Delta h}\right) & j \in Q,\\ F(x_j) & j \in B, \end{cases}
$$

and let $r := \max_{j\in B} |F(x_j)|$.

The map T from $B_r := \{u \in B(\mathcal{G}_h) : \max_j |u_j| \le r\}$ to $B(\mathcal{G}_h)$ is continuous since composition of continuous functions. Assuming that $u \in \partial B_r$, then there exists at least one index i_0 such that $|u_{i_0}| = r$. Using the monotonicity of the interpolation, i.e. $|I[u](z)| \le \max_j |u_j| \le r$, for any $u \in B_r$ and $z \in \Omega$, we obtain the following:

$$
(T(u))_j \le \begin{cases} \left(\frac{2}{\Delta^2} + \frac{8W_\varepsilon}{\Delta h}\right)^{-1} \left(\frac{2r}{\Delta^2} + \frac{2W_\varepsilon \sum_{i\in\mathcal{D}(j)} r}{\Delta h}\right) & j \in Q,\\ r & j \in B. \end{cases}
$$

Observing that in the first inequality the right-term is equal to r, we get

$$(T(u))_j \le r, \quad \text{for any} \quad j \in \mathcal{J}.$$

This implies that, if $u_{i_0} = r$, then $(T(u))_{i_0} \ne \lambda u_{i_0}$ for all $\lambda > 1$. Analogously, if $u_{i_0} = -r$, then $(T(u))_{i_0} \ge -r$.

In conclusion, $T(u) \ne \lambda u$ for all $\lambda > 1$ and $u \in \partial B_r$. By the Leray-Schauder fixed-point theorem, T admits a fixed point in B_r. □

Let $\rho_m = (h_m, \Delta_m), \varepsilon_m$ be a sequence of discretization parameters, such that $\rho_m \to (0,0)$, $\varepsilon_m \to 0$ as $m \to \infty$. For any $x \in \overline{\Omega}$, let us define

$$u^*(x) := \limsup_{\substack{m\to\infty,\\ \mathcal{G}_h \ni y_m \to x}} \widetilde{u}_{\varepsilon_m,\rho_m}(y_m), \quad u_*(x :) = \liminf_{\substack{m\to\infty,\\ \mathcal{G}_h \ni y_m \to x}} \widetilde{u}_{\varepsilon_m,\rho_m}(y_m).$$

From the convergence analysis of [3], we get the following result.

Theorem 6 *Let* $F : \overline{\Omega} \setminus \mathcal{O} \to \mathbb{R}$ *be uniformly Lipschitz. Let us assume* (14), (15), (16), (26) *and* $\frac{h}{\varepsilon} = o(\Delta)$. *Then* u^* *and* u_* *are respectively a viscosity subsolution and a viscosity supersolution of*

$$-\Delta_\infty^G u(x) = f \quad \text{for } x \in \mathcal{O}.$$

Proof Let us observe that assumption $\frac{h}{\varepsilon} = o(\Delta)$ implies (23), hypothesis required by Proposition 3. Since also Theorems 4 and 5 hold true, we can follow the proof of Theorem 2.1 in [3] to obtain the desired result. □

Corollary 1 *Let the assumptions of Theorem 6 hold true. If, furthermore,* $f \equiv 0$ *and* $u^* = u_* = F$ *on* $\overline{\Omega}\setminus\mathcal{O}$, *then* $\widetilde{u}_{\varepsilon,\rho} \in B(\mathcal{G}_h)$, *solution to* (20), *converges locally uniformly to the unique continuous viscosity solution of* (1).

Proof Since $f \equiv 0$ and F is uniformly Lipschitz, problem (1) admits a unique bounded viscosity solution $u \in W^{1,\infty}(\overline{\Omega})$ (see [11, 12, 17]). Using Theorem 6 and Theorem 2.1 in [3], we get the thesis. □

Remark 4 Let us assume that the conditions (14) and (15) are verified with $r = 2$ and $q = 2$, choosing for instance respectively a bilinear interpolation operator and a central finite difference approximation of the gradient. Now, assuming $\beta = \frac{1}{4}$, assumptions (16) and (23) are verified when $\alpha > \frac{5}{4}$.

Remark 5 It is important to note that the assumptions in Corollary 1, related to the convergence to the boundary data F, have not been verified in this study. Additional investigation are required to justify the convergence in a strong sense to the specified boundary values.

4 Conclusions

In this work, we have proposed a new numerical scheme for a Dirichlet problem associated with the game ∞-Laplacian, based on a semi-Lagrangian approximation. We have demonstrated that the new scheme converges to the viscosity solution of the problem. Specifically, we have proved that the scheme is consistent, and, thanks to the addition of an artificial viscosity term, we ensure it is also stable and monotone. The proof of the convergence result follows the approach developed by Barles-Souganidis analysis. With respect to other numerical method for the game ∞-Laplacian, the scheme here proposed is fully discrete and make use of a smaller stencil. In the future, an extension of a similar study on the game p-Laplacian could be conducted, by observing that this operator can be expressed as a convex combination of the ∞-Laplacian and 1-Laplacian.

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Data Availability No Data associated in the manuscript.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

Ethical Approval No human or animal participants were involved in this study; hence, ethical approval was not applicable.

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