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# ON SPECTRAL PROPERTIES OF CERTAIN SEMIREGULAR SYSTEMS OF PDES WITH POLYNOMIAL COEFFICIENTS

ALBERTO PARMEGGIANI

*To Jorge Hounie*

ABSTRACT. This paper is a survey of some recent results obtained with Marcello Malagutti, concerning asymptotic properties and semi-clustering properties of certain systems of PDEs with polynomial coefficients that contain, as particular examples, some models coming from Quantum Optics.

## 1. INTRODUCTION

In this paper, I will be giving a survey of some recent joint work [12, 13] with Marcello Malagutti, concerning two basic properties in the spectral theory of systems of PDEs with polynomial coefficients: the asymptotics of the Weyl counting function in a two-term form and in a refined three-term form, and the quasi-clustering property (a weak form of clustering property, that is, a concentration of the spectrum near an arithmetic sequence).

Recall that if  $-\infty < \lambda_1 \leq \lambda_2 \leq \dots \lambda_j \leq \dots \rightarrow +\infty$  is the sequence of the eigenvalues repeated (according to multiplicity) of an elliptic (bounded from below) operator  $A$ , then the Weyl counting function is given by

$$N_A(\lambda) = \sum_{\lambda_j \leq \lambda} 1.$$

It is well-known that understanding the asymptotics as  $\lambda \rightarrow +\infty$  of  $N_A$  is a problem of fundamental importance (also related to Number Theory). Several important contributions were given by Weyl (1911), Hörmander (in his paper on sharp remainders by using wave equation methods), Seeley, Ivrii, Shubin, Duistermaat and Guillemin, Melrose, Safarov and Vassiliev for *scalar* elliptic operators on a compact (boundaryless) manifold (see [3, 11, 20, 22] and the references in [10] and [7]). In the case of *scalar* globally elliptic operators (see, e.g., [7, 22], also in the semiclassical setting) important contributions were given by Hörmander, Shubin, Ivrii, Chazarain, Helffer and Robert (see [22, 6, 7, 11]) and recently by Doll, Gannot and Wunsch in [4] and Doll and Zelditch in [5] in refining, under suitable geometric conditions, the asymptotics of  $N_A$  for semiregular operators (see Definition 2.1 below) that was obtained by Helffer and Robert in [6]. (See also Coriasco and Doll [1] for an extension of [4] to the SG-calculus.) Note that the important, and wonderful, point is that all the asymptotics of the Weyl function are described in terms of dynamical symplectic invariants of the symbol.

For *systems* very little is known. It was Ivrii (see [11]) who started studying the asymptotics for systems on compact manifolds, and myself (see [14, 15, 16, 17]) in the case of

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globally elliptic systems with polynomial coefficients in the class of Non-Commutative Harmonic Oscillators (NCHOs), introduced by M. Wakayama and myself some time ago (see [18, 19]). Here, I will explain the approach in [12] that provides the Weyl asymptotics in case the system with polynomial coefficients is of the *semiregular* type, under further suitable conditions that include in the considered class also important models coming from Quantum Optics, namely the Jaynes-Cumming system (that describes the interaction of photons with an atom with a fixed number of energy levels in a cavity; see [21]). An important feature of the class considered here is that the system is semiregular with scalar principal part and smoothly diagonalizable semiprincipal part (the SMGES class; see Section 2). The Weyl asymptotics I obtain here are the analogues for semiregular systems of those of Helffer and Robert [6] in the scalar case and may be refined, under suitable conditions, to those obtained by Doll, Gannot and Wunsch [4] (and by Doll and Zelditch [5]).

As for the “quasi-clustering”, it was Weinstein [23] (in the case of a scalar Schrödinger operator on a compact boundaryless manifold) and Colin de Verdiere [2] (in the case of a scalar elliptic operator on a compact boundaryless manifold; see also Duistermaat and Guillemin [3]) who proved that when the bicharacteristics of the principal symbol are all periodic with period independent of the energy, then one has that the spectrum of the operator clusters around an arithmetic sequence, determined in terms of invariants of the symbol, in the sense that it is contained in the union of intervals centered in that sequence with vanishing diameters as the centers go to infinity. The arithmetic sequence is such that eventually the intervals become disjoint. The *quasi-clustering* property, weaker than the clustering, does not require this latter property, but that the number of intersections of the intervals are bounded by a universal constant. For systems, the situation is vastly unknown, apart from some results I obtained for regular NCHOs (see [14, 15, 16, 17]).

For systems of the SMGES class, the quasi-clustering property of the spectrum may be viewed as a more general property than the refined spectral asymptotics. In fact, the former holds for systems (as we shall see below) for which the eigenvalues of the semiprincipal part are functions of combinations with positive coefficients of the harmonic oscillators in each symplectic plane  $\mathbb{R}_{(x_j, \xi_j)}^2$  inside  $\mathbb{R}^{2n}$ , which gives, for some choices of the coefficients, an obstruction to the refined Weyl-law.

Here, I will describe the approach in [13] that yields quasi-clustering properties of the spectrum for operators in the SMGES class. A delicate point in the approach is to exploit the fact that these systems may be smoothly diagonalized (modulo smoothing operators) through almost unitary diagonalizers (i.e., unitary modulo smoothing operators). Having almost unitary diagonalizers for controlling the spectrum of the system in terms of that of the diagonalized one is not enough, and we have to deform the diagonalizer (or its adjoint, depending on the index of its principal symbol) into a genuine isometry in the global pseudo-differential calculus we are using (i.e. that of the quantum harmonic oscillator; see Section 2) but in certain cases this requires augmenting the system and hence throwing in new centers (explicitly determined) that give a little noise in the description of the spectrum that, however, do not effectively change the quasi-clustering property.

I end this Introduction by giving the plan of the paper. In Section 2, I recall what a semiregular symbol is and introduce the class of systems of the kind SMGES, along with two basic examples. In Section 3, I give the statement of a fundamental step in the study of spectral properties of these system: the diagonalizability modulo smoothing in terms of  $\psi$ -dos that are unitary up to smoothing. Then, in Section 4, I give an idea of the proof of the Weyl

asymptotics in both the case of the two-term and three-term asymptotics, showing also an example (the Jaynes-Cummings system) for which the refined asymptotics does not hold and one, a perturbation of a Laplacian of a connection, for which it does. In Section 5, I survey the proof of the quasi-clustering property, passing through the ‘‘unitarization’’ theorem, and giving, in the end, the quasi-clustering property in the case of the Jaynes-Cummings system. In the final section, I will illustrate two open problems related to this class of systems.

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## 2. THE CLASS SMGES

I will be considering pseudodifferential operators that are Weyl quantizations of symbols in the Weyl-Hörmander classes: for  $\mu \in \mathbb{R}$  and  $X = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n = T^*\mathbb{R}^n$ , consider  $\langle X \rangle = (1 + |X|^2)^{1/2}$  and write

$$S^\mu := S(\langle X \rangle^\mu, |dX|^2 / \langle X \rangle^2; M_N)$$

for the class of *global symbols* of order  $\mu$ , where  $M_N$  denotes the algebra of  $N \times N$  complex valued matrices. In other words,  $a \in S^\mu$  iff

$$|\partial_X^\alpha a(X)|_{M_N} \lesssim \langle X \rangle^{\mu - |\alpha|}, \quad \forall \alpha \in \mathbb{Z}_+^{2n}.$$

We denote by  $a^w(x, D) \in \Psi^\mu$  the corresponding pseudodifferential operator obtained by Weyl-quantization,

$$a^w(x, D)u(x) = (2\pi)^{-n} \iint e^{i\langle x-y, \xi \rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N).$$

Throughout the paper I will write

$$p_2(X) = |X|^2/2, \quad \psi_j(X) = \frac{x_j + i\xi_j}{\sqrt{2}}, \quad 1 \leq j \leq n,$$

so that  $p_2^w(x, D) \in \Psi^2$  is the usual  $n$ -dimensional quantum harmonic oscillator, and  $\psi_j^w(x, D) \in \Psi^1$  is the annihilation operator in the  $j$ th variables, with  $\psi^w(x, D)^* = \bar{\psi}_j^w(x, D)$  the corresponding creation operator,  $j = 1, \dots, n$ .

**Definition 2.1.** We say that  $a \in S^\mu$  is a *regular symbol*, and write  $a \in S_{\text{reg}}^\mu$  when there is  $(a_{\mu-2j})_{j \geq 0} \subset C^\infty(\mathbb{R}^{2n}; M_N)$ , where  $\mathbb{R}^{2n} = \mathbb{R}^{2n} \setminus \{0\}$ , such that  $a \sim \sum_{j \geq 0} a_{\mu-2j}$ , that is, ( $\chi$  being an excision function)

$$a - \chi \sum_{j=0}^N a_{\mu-2j} \in S^{\mu-2(N+1)}, \quad \forall N \in \mathbb{Z}_+.$$

I say that  $a \in S^\mu$  is *semiregular*, and write  $a \in S_{\text{sreg}}^\mu$ , when  $a = a^0 + a^1$  with  $a^k \in S_{\text{reg}}^{\mu-k}$  for  $k = 0, 1$ . The corresponding classes of pseudodifferential operators are  $\Psi_{\text{reg}}^\mu$  and  $\Psi_{\text{sreg}}^\mu$ , respectively.

The terms

$$a_\mu^0, \quad a_{\mu-1}^1, \quad a_{\mu-2}^0$$

are respectively called **principal**, **semiprincipal** and **subprincipal** symbols, and they are symplectically invariant.

We finally say that  $a^w \in \Psi_{\text{sreg}}^\mu$  is **globally elliptic** when

$$|\det a_\mu^0(X)| \approx |X|^{\mu N}, \quad \forall X \neq 0.$$

I may now introduce the class of  $N \times N$  symbols we consider here, the SMGES class, which stands for *Semiregular Metric Globally Elliptic System*, “metric” coming from the differential geometric terminology for which a metric operator is one which has a scalar principal symbol, given by a (smooth) Riemannian metric on the manifold on which one is working.

**Definition 2.2.** I say that  $a \in S_{\text{sreg}}^\mu$  is SMGES when

$$a(X) = a(X)^* = p_\mu(X)I_N + a_{\mu-1}(X) + a_{\mu-2}(X) + S_{\text{sreg}}^{\mu-2}, \quad X \neq 0,$$

where  $p_\mu(X) \approx |X|^\mu$ , and the semiprincipal part  $a_{\mu-1}$  is such that there exists  $e_0 \in C^\infty(\mathbb{R}^{2n}; \mathbb{M}_N)$ , unitary and positively homogeneous of degree 0 such that

$$e_0^* a_{\mu-1} e_0 = \text{diag}(\lambda_{\mu-1,j} I_{N_j}; 1 \leq j \leq r),$$

where  $N = N_1 + N_2 + \dots + N_r$  and  $\lambda_{\mu-1,j} \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$  are positively homogeneous of degree  $\mu - 1$  and such that

$$j < k \implies \lambda_{\mu-1,j}(X) < \lambda_{\mu-1,k}(X), \quad \forall X \neq 0.$$

In particular, the eigenvalues  $\lambda_{\mu-1,j}$  of  $a_{\mu-1}$  have constant multiplicity.

Given  $a \in S^\mu$ ,  $\mu > 0$ , I will think of  $a^w(x, D)$  also as the unbounded operator  $A: D(A) = B^\mu \subset L^2 \rightarrow L^2$  represented by  $a^w(x, D)$  on  $\mathcal{S}$ . Hence, capital letters will be used to indicate abstract operators between Hilbert spaces.

**2.1. Examples.** I next give two examples of such a class of systems (see [12]). The first example is the Jaynes-Cummings model [21] which describes the interaction of a two-level atom with a photon. Consider the Pauli matrices

$$\sigma_0 = I_2, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Set  $\sigma_\pm := (\sigma_1 \pm i\sigma_2)/2$ . For  $n = 1$  the Jaynes-Cummings operator is

$$a_{JC}^w(x, D) = p_2^w(x, D)I_2 + \alpha \left( \sigma_- \psi^w(x, D)^* + \sigma_+ \psi^w(x, D) \right) + a_0 \in \Psi_{\text{sreg}}^2,$$

where  $a_0 = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$ , with  $\alpha, \gamma_1, \gamma_2 \in \mathbb{R}$ . In this case

$$a_1(X) = \alpha \begin{bmatrix} 0 & \psi(X) \\ \overline{\psi(X)} & 0 \end{bmatrix},$$

with eigenvalues

$$\lambda_{1,\pm}(X) = \pm |\alpha| |\psi(X)| / \sqrt{2}.$$

As we will see below,  $a_{JC}^w$  is actually a Laplacian of a connection on the trivial bundle  $\mathbb{R}^n \times \mathbb{C}^N \rightarrow \mathbb{R}^n$  (in general possessing a nonzero curvature).

The second example is a Laplacian associated with a connection over  $\mathbb{R}^n$ . If  $\star$  denotes the usual Hodge-star operator in  $\mathbb{R}^n$  let  $\Omega^k(\mathbb{R}^n)$  be the space of smooth  $k$ -differential forms on  $\mathbb{R}^n$  and let  $d_k$  be the exterior differential on  $\Omega^k(\mathbb{R}^n)$ . Then  $d_k^* = (-1)^{nk+1} \star d_k \star$  is the adjoint operator on  $\Omega^{k+1}(\mathbb{R}^n)$  forms. I next take

$$D_k = \frac{1}{\sqrt{2}}(d_k + \sum_{j=1}^n x_j dx_j \wedge): \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n),$$

$$D_k^* = \frac{1}{\sqrt{2}}(d_k^* + \sum_{j=1}^n x_j \mathbf{i}_{\partial/\partial x_j}): \Omega^{k+1}(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n),$$

where  $\mathbf{i}_{\partial/\partial x_j}$  denotes the contraction in the  $j$ th direction. Let  $N = n + 1$  and let  $E_{jk}$  be the  $N \times N$  matrix that acts on the vectors  $\{e_1, \dots, e_N\}$  of the canonical basis of  $\mathbb{C}^N$  as  $E_{jk}w = \langle w, e_k \rangle e_j$ ,  $1 \leq j, k \leq N$ . For  $\alpha_1, \dots, \alpha_n \in \mathbb{R} \setminus \{0\}$ , consider

$$\mathbf{D}_k = D_k \otimes I_N + \sum_{j=1}^n \alpha_j (dx_j \wedge) \otimes E_{j,j+1}: \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N).$$

Hence

$$\mathbf{D}_k^* = D_k^* \otimes I_N + \sum_{j=1}^n \alpha_j \mathbf{i}_{\partial/\partial x_j} \otimes E_{j+1,j}: \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

In this case the curvature of the connection is

$$F_{\mathbf{D}} = \sum_{j=1}^{n-1} \alpha_j \alpha_{j+2} (dx_j \wedge dx_{j+1}) \otimes E_{j,j+2} \in \Omega^2(\mathbb{R}^n; \mathbf{M}_N).$$

We may consider the associated Laplacian

$$(2.1) \quad \square_k^{(N)} = \mathbf{D}_k^* \mathbf{D}_k + \mathbf{D}_{k-1} \mathbf{D}_{k-1}^*: \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N),$$

which is in  $\Psi_{\text{reg}}^2$  with principal part  $p_2^w(x, D) \mathbf{1}_k \otimes I_N$ , semiprincipal part

$$\sum_{j=1}^n \alpha_j \left( \psi_j^w(x, D)^* \mathbf{1}_k \otimes E_{j,j+1} + \psi_j^w(x, D) \mathbf{1}_k \otimes E_{j+1,j} \right),$$

and subprincipal part

$$\left(k - \frac{n}{2}\right) \mathbf{1}_k \otimes I_N + \sum_{j=1}^n \alpha_j^2 \mathbf{1}_k \otimes E_{j+1,j+1} + \sum_{j=1}^n \alpha_j^2 dx_j \wedge \mathbf{i}_{\partial/\partial x_j} \mathbf{1}_k \otimes (E_{j,j} - E_{j+1,j+1}),$$

where  $\mathbf{1}_k$  denotes the identity on  $\Omega^k(\mathbb{R}^n)$ .

Therefore  $a_{JC}^w$  is related to  $\square_0^{(2)}$  ( $n = 1, N = 2$ ) and  $\square_k^{(N)}$  is a generalization. Both operators ( $\square_0^{(2)}$  and  $\square_k^{(N)}$ ) are SMGES, for they have *semiprincipal part which is smoothly diagonalizable* (see [12]).

### 3. A FUNDAMENTAL PROPERTY: DIAGONALIZABILITY

A fundamental step in the study of the Weyl asymptotics and the possible quasi-clustering is to pass from the given system to a system in diagonal form (in the quasi-clustering setting, we will also need to make quasi-isometries into true isometries in order to control the spectrum of the considered system in terms of that of its diagonalization). For the class SMGES introduced above we have the following result [12].

**Theorem 3.1.** *Let  $\mu > 0$ ,  $a = a^* \in S_{\text{sreg}}^\mu$  with  $a \sim \sum_{j \geq 0} a_{\mu-j}$ , such that  $a_\mu = p_\mu I_N$  with  $p_\mu \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$  positively homogeneous of degree  $\mu$ . Suppose there is  $e_0 \in C^\infty(\mathbb{R}^{2n}; M_N)$  unitary and positively homogeneous of degree 0 such that*

$$e_0^* a_{\mu-1} e_0 = \left[ \begin{array}{c|c} \Lambda_{\mu-1,1} & 0 \\ \hline 0 & \Lambda_{\mu-1,2} \end{array} \right], \quad X \neq 0,$$

where  $N = N_1 + N_2$  and  $\Lambda_{\mu-1,j} \in C^\infty(\mathbb{R}^{2n}; M_{N_j})$  is positively homogeneous of degree  $\mu - 1$  such that

$$\text{Spec}(\Lambda_{\mu-1,1}(X)) \cap \text{Spec}(\Lambda_{\mu-1,2}(X)) = \emptyset, \quad \forall X \neq 0.$$

Then there exists  $e \in S_{\text{sreg}}^0$  with principal symbol  $e_0$  such that

$$e^w(x, D) e^w(x, D)^* - I \in \Psi^{-\infty}, \quad e^w(x, D)^* e^w(x, D) - I \in \Psi^{-\infty},$$

and

$$e^w(x, D)^* a^w(x, D) e^w(x, D) = b^w(x, D) + \Psi^{-\infty},$$

where

$$b \sim \sum_{j \geq 0} \left[ \begin{array}{c|c} b_{\mu-j,1} & 0 \\ \hline 0 & b_{\mu-j,2} \end{array} \right] \in S_{\text{sreg}}^\mu,$$

and

$$b_\mu = a_\mu, \quad b_{\mu-1} = \left[ \begin{array}{c|c} \Lambda_{\mu-1,1} & 0 \\ \hline 0 & \Lambda_{\mu-1,2} \end{array} \right].$$

Hence in the case of an SMGES, we have, by iterating the above result with  $N = N_1 + \dots + N_r$ , that

$$b_\mu(X) = p_\mu(X) I_N, \quad b_{\mu-1}(X) = \text{diag}(\lambda_{\mu-1,j}(X) I_{N_j}; 1 \leq j \leq r).$$

As for the subprincipal part we consider

$$\pi_j: \mathbb{C}^N \longrightarrow \mathbb{C}^N = \bigoplus_{j=1}^r \mathbb{C}^{N_j},$$

the  $j$ th orthogonal projection onto the  $j$ th  $\mathbb{C}^{N_j} \subset \mathbb{C}^N$ . Then the subprincipal symbol of the diagonalized operator is given by (see [12])

$$b_{\mu-2}(X) = \text{diag}(b_{\mu-2}^{(j)}(X); 1 \leq j \leq r), \quad X \neq 0,$$

where

$$b_{\mu-2}^{(j)} = \pi_j e_0^* A_{\mu-2} e_0 \pi_j^* - i \pi_j e_0^* \{p_\mu, e_0\} \pi_j^*, \quad j = 1, \dots, r.$$

Of course, one has (see [12]) transformation formulas for the subprincipal symbol in terms of the principal symbol  $e_0$  of the diagonalizer.

There is also a formula for the semi-subprincipal term (the term of order  $\mu - 3$ ) along with its transformation rules with respect to  $e_0$ , but we won't be needing it in the sequel.

#### 4. ASYMPTOTICS FOR THE WEYL SPECTRAL COUNTING FUNCTION

We are given  $a^w \in \Psi_{\text{sreg}}^2$  in the class SMGES of order  $\mu = 2$  with principal symbol  $p_2$  (*the standard harmonic oscillator*), realized as an unbounded operator  $A$  in  $L^2$  with domain the Shubin Sobolev space  $B^2$ , which is therefore semibounded. It is thus a globally elliptic operator and has a discrete spectrum

$$-\infty < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow +\infty,$$

where the eigenvalues are repeated according to multiplicity. As recalled in the Introduction, the Weyl function associated with  $A$  is defined by

$$N_A(\lambda) = \sum_{\lambda_j \leq \lambda} 1.$$

Such function has a well-known crucial role in spectral theory and in many related problems. At first, we observe that

$$(4.2) \quad \mathcal{F}_{\lambda \rightarrow t} \left( \sum_{j \geq 1} \delta(\lambda - \lambda_j) \right) = \sum_{j \geq 1} e^{-it\lambda_j}.$$

The latter belongs to  $\mathcal{S}'(\mathbb{R})$ . In fact, for all  $\phi \in \mathcal{S}(\mathbb{R})$  we may consider, using the Schrödinger evolution equation associated with  $a^w$  and integration by parts,

$$U_\phi = \int \phi(t) e^{-ita^w} dt = \int (-i)^k (\partial_t^k \phi)(t) (a^w)^{-k} e^{-ita^w} dt,$$

which is therefore, for sufficiently large  $k$ , a trace-class operator. Hence the operator-trace  $\text{Tr}_{\text{op}}(e^{-ita^w}): \mathcal{S}(\mathbb{R}) \ni \phi \mapsto \text{Tr}_\Delta \text{Tr}(U_\phi) \in \mathbb{C}$  is a tempered distribution ( $\text{Tr}_{\text{op}}$  is the restriction  $\text{Tr}_\Delta$  on the diagonal  $\Delta$  of  $\mathbb{R}^n \times \mathbb{R}^n$ , of the matrix-trace  $\text{Tr}$  of the Schwartz kernel of  $U_\phi$ ). Therefore the right-hand side of (4.2) is a temperate distribution and so is the argument of the Fourier transform in the left-hand side, which is therefore the derivative with respect to  $\lambda$  of  $N_A$ . We thus study the group  $t \mapsto e^{-ita^w}$  and for that we use, following Chazarain and Helffer-Robert, an FIO (Fourier integral operator) parametrix of it. The point is therefore to obtain the parametrix (using the WKB method) in a sufficiently friendly class to allow a control on compositions, which are needed because we have to pass through the diagonalization procedure. The class of FIOs introduced by Helffer and Robert in [6] is not sufficiently friendly, and here the approach of Doll, Gannot and Wunsch [4] is crucial. We consider therefore a parametrix of the Schrödinger group  $t \mapsto e^{-itb^w}$  of the diagonalized system which is given by the composition of the evolution  $t \mapsto e^{-itp_2^w}$ , that has a nice representation by Mehler's formula and is the Weyl-quantization of an exponential of a quadratic form (see Hörmander [8]), with the parametrix of the first-order time-dependent operator

$$t \mapsto P(t) := e^{itp_2^w} (B^w - p_2^w I_N) e^{-itp_2^w},$$

which can be easily computed by Egorov's theorem and the fact that the characteristics in the WKB expansion exist for all times. We therefore get, recalling that the blocks of the principal symbol of  $b^w$  are given by  $\Lambda_{1,j} = \lambda_{1,j} I_{N_j}$ ,  $1 \leq j \leq r$ , that the operator

$$t \mapsto F(t) := \text{diag} \left( (e^{i\phi_{j,1}(t)} \alpha_j(t))^w; 1 \leq j \leq r \right),$$

where

$$\phi_{j,1}(t, X) = - \int_0^t (\lambda_{1,j} \circ \exp s H_{p_2})(X) ds,$$

$H_{p_2}$  being the Hamilton vector-field associated with  $p_2$ , and where  $\alpha_j$  is an  $N_j \times N_j$  matrix with  $\alpha_j \in C^\infty(\mathbb{R}_t; \mathcal{S}_{\text{sreg}}^0)$ , is a parametrix for  $P(t)$ ,

$$(i\partial_t - P(t))F(t) \in C^\infty(\mathbb{R}_t; \mathcal{L}(\mathcal{S}', \mathcal{S}) \otimes \mathbb{M}_N), \quad F(0) = I_N + \text{smoothing}.$$

We therefore have the following result [12].

**Theorem 4.1.** *Let  $\varepsilon > 0$  be sufficiently small so that  $(2k \pm 1)\pi \notin (2k\pi - \varepsilon, 2k\pi + \varepsilon) := I_{k,\varepsilon}$ , for all  $k \in \mathbb{Z}$ . Then for  $j = 1, \dots, r$  there is  $\tilde{\alpha}_j \in C^\infty(I_{k,\varepsilon}; \mathcal{S}^0)$  such that, with*

$$\phi_2(t, X) = -2 \tan(t/2) p_2(X), \quad \tilde{\phi}_{j,1}(t, X) = \phi_{j,1}(t, X - 2 \tan(t/2) JX)$$

(where  $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ ) and  $U_0(t) = \cos(t/2)^{-n} (e^{-\phi_2(t)})^w$  we have that

$$U_B(t) = U_0(t)F(t) = \text{diag} \left( (e^{i\phi_2(t) + i\tilde{\phi}_{j,1}(t)}) \tilde{\alpha}_j(t) \right)^w; \quad 1 \leq j \leq r$$

is a parametrix of  $e^{-itB^w}$ . Hence,

$$I_{k,\varepsilon} \ni t \mapsto U_A(t) := E^w U_B(t) (E^w)^*$$

is a parametrix for  $t \mapsto e^{-itA^w}$ .

Once this is established, one can therefore follow the classical construction of Hörmander [10] to obtain the two-term asymptotics that extends that of Helffer and Robert [6] to the vector-valued case.

**Theorem 4.2.** *Let  $A = a^w \in \Psi_{\text{sreg}}^2$  where  $a$  is an SMGES. Let  $\rho \in \mathcal{S}(\mathbb{R})$  be such that  $\hat{\rho} \in C_c^\infty(-\varepsilon, \varepsilon)$  and  $\hat{\rho} \equiv 1$  near 0. Then ( $ds$  being the Riemannian hypersurface measure)*

$$(4.3) \quad (\mathbf{N}_A \star \rho)(\lambda) = (2\pi)^{-n} \left( \sum_{j=1}^r N_j \int_{p_2 + \lambda_{1,j} \leq \lambda} dX - \int_{p_2 = \lambda} \text{Tr}(a_0) \frac{ds}{|\nabla_X p_2|} \right) + O(\lambda^{n-3/2})$$

as  $\lambda \rightarrow +\infty$ , where  $\text{Tr}$  denotes the matrix-trace. Hence,

$$(4.4) \quad \mathbf{N}_A(\lambda) = (2\pi)^{-n} \left( \left( N \int_{p_2 \leq 1} dX \right) \lambda^n - \left( \int_{p_2 = 1} \text{Tr}(a_1) \frac{ds}{|\nabla_X p_2|} \right) \lambda^{n-1/2} \right) + O(\lambda^{n-1})$$

as  $\lambda \rightarrow +\infty$ .

We have (4.4) from (4.3) by using a Taylor expansion as  $\lambda \rightarrow +\infty$  of the volume of the set  $\{X; p_2 + \lambda_{1,j} \leq \lambda\}$  and using the polynomial growth of  $\mathbf{N}_A(\lambda)$  (which follows from the global ellipticity and the minimax principle).

Using the improvement that Doll, Gannot and Wunsch have given in the scalar semiregular case, we may refine the above theorem. We need to impose a condition on the eigenvalues of the semiprincipal part: *Suppose that for  $j = 1, \dots, r$  the sets*

$$(4.5) \quad Z_j := \left\{ \omega \in \mathbb{S}^{2n-1}; \partial_\omega^\alpha \int_0^{2\pi} (\lambda_{1,j} \circ \exp t H_{p_2})(\omega) dt = 0, \forall |\alpha| \geq 1 \right\}$$

have measure zero. (I have denoted by  $\partial_\omega^\alpha$  the iterated tangential derivatives to  $\mathbb{S}^{2n-1}$ .) We refer to this condition as **Condition DGW**.

That condition allows one to use Stein's theorem on the control of the oscillatory integrals that enter the description of  $N_A$ . Since from Theorem 4.2 it follows in particular that  $N'_A \star \rho = O(\lambda^{n-1})$ ,  $\lambda \rightarrow +\infty$ , one is in a position to use the following Tauberian theorem [20].

**Theorem 4.3.** *With  $\rho$  as in Theorem 4.2, if there is  $\gamma \in \mathbb{R}$  such that  $(N'_A \star \rho)(\lambda) = O(\lambda^\gamma)$  and  $(N'_A \star \chi)(\lambda) = o(\lambda^\gamma)$  for all  $\chi \in \mathcal{S}(\mathbb{R})$  such that  $\hat{\chi} \in C_c^\infty(0, +\infty)$  then*

$$N_A(\lambda) = (N_A \star \rho)(\lambda) + o(\lambda^\gamma), \quad \lambda \rightarrow +\infty.$$

One then has the following refined Weyl-asymptotics.

**Theorem 4.4.** *If Condition **DGW** is satisfied then*

$$N_A(\lambda) = (2\pi)^{-n} \left( \sum_{j=1}^r N_j \int_{p_2 + \lambda_{1,j} \leq \lambda} dX - \int_{p_2 = \lambda} \text{Tr}(a_0) \frac{ds}{|\nabla_X p_2|} \right) + o(\lambda^{n-1})$$

as  $\lambda \rightarrow +\infty$ . In particular,

$$\begin{aligned} N_A(\lambda) = (2\pi)^{-n} \left( N\lambda^n \int_{p_2 \leq 1} dX - \lambda^{n-1/2} \int_{p_2=1} \text{Tr}(a_0) \frac{ds}{|\nabla_X p_2|} \right. \\ \left. + \lambda^{n-1} \int_{p_2=1} \left( \frac{n}{2} \text{Tr}(a_1^2) - \text{Tr}(a_0) \right) \frac{ds}{|\nabla_X p_2|} \right) + o(\lambda^{n-1}) \end{aligned}$$

as  $\lambda \rightarrow +\infty$ .

Notice that Condition **DGW** cannot hold when the semiprincipal part has the same symmetries of the principal part, for instance when the eigenvalues of the semiprincipal part are functions of  $|X|$ . In fact, in such a case the average of the eigenvalues along the bicharacteristics of the principal symbol is a constant, and hence the angular derivatives are identically zero. And this is indeed the case of  $A_{JC}$ . In fact, the eigenvalues of the semiprincipal part are  $\lambda_{1,\pm}(X) = \pm|\alpha||X|/\sqrt{2}$ , whence the sets  $Z_\pm$  have full measure. In this case we have only

$$N_{A_{JC}}(\lambda) = (2\pi)^{-1} 2\lambda \int_{p_2 \leq 1} dX + O(1) = 2\lambda + O(1), \quad \lambda \rightarrow +\infty.$$

**4.1. A deformation of  $\square_0^{(3)}$  which has a refined asymptotics.** In the next example, I will break the symmetry of the eigenvalues of the semiprincipal part of  $\square_0^{(3)}$  (see (2.1)), one of which is 0, and construct a perturbation for which Condition **DGW** is satisfied.

Recalling that  $\alpha_1, \alpha_2 \neq 0$ , consider  $\square_0^{(3)} : \mathcal{S}(\mathbb{R}^2; \mathbb{C}^3) \rightarrow \mathcal{S}(\mathbb{R}^2; \mathbb{C}^3)$ . The eigenvalues of the semiprincipal part are

$$\lambda_0(X) = 0, \quad \lambda_{1,\pm}(X) = \pm \underbrace{(\alpha_1^2 |\psi_1(X)|^2 + \alpha_2^2 |\psi_2(X)|^2)^{1/2}}_{=:\alpha\psi(X)},$$

with constant multiplicity 1. Here the problem is generated by  $Z_0$  which has full measure (the same occurs for  $Z_\pm$  when  $|\alpha_1| = |\alpha_2|$ ). Consider then, for the parameter  $\mu \in (0, 1)$  the new eigenvalue  $\lambda_\mu(X) := \mu\lambda_+(X) + (1-\mu)\lambda_-(X)$ . Notice that when  $\mu = 1/2$  then  $\lambda_\mu(X) \equiv 0$ . So we suppose that  $\mu \neq 1/2$  and that  $|\alpha_1| \neq |\alpha_2|$ , whence (as one may easily check) Condition **DGW** holds. Then we consider the SMGES

$$a(X) = p_2(X)I_N + a_{1,\mu}(X) + a_0(X),$$

where, putting  $f_j = \alpha_j \psi_j / |\alpha \psi|$ ,

$$a_{1,\mu}(X) = \begin{bmatrix} \lambda_\mu(X) |f_2(X)|^2 & \lambda_+(X) \overline{f_1(X)} & -\lambda_\mu(X) \overline{f_1(X) f_2(X)} \\ \lambda_+(X) f_1(X) & 0 & \lambda_+(X) f_2(X) \\ -\lambda_\mu(X) f_1(X) f_2(X) & \lambda_+(X) f_2(X) & \lambda_\mu(X) |f_1(X)|^2 \end{bmatrix},$$

and

$$a_0(X) = -I_3 + \sum_{j=1}^2 \alpha_j^2 E_{j+1,j+1} + \sum_{j=1}^2 (E_{j,j} - E_{j+1,j+1}).$$

Then  $\text{Tr}(a_0) = \alpha_1^2 + \alpha_2^2 - 3$  and we have the following refined Weyl-law

$$\begin{aligned} N_A(\lambda) &= (2\pi)^{-2} \left( 3|\mathbb{S}^3| \lambda^2 - \frac{1}{2} \lambda^{3/2} \int_{|X|=2} \lambda_\mu ds \right. \\ &\quad \left. + \frac{1}{2} \lambda \int_{|X|=2} (2\lambda_+^2 + \lambda_\mu^2 - \alpha_1^2 - \alpha_2^2 + 3) ds \right) + o(\lambda) \end{aligned}$$

as  $\lambda \rightarrow +\infty$ .

## 5. QUASI-CLUSTERING

In this section we study *quasi-clustering properties* of SMGES, that is the presence of the spectrum of the system located in the union of a sequence of intervals, whose diameters go to zero as the centers go to infinity, with universally bounded number of overlappings. Note, as already mentioned, that a *clustering property*, which is stronger, is the presence of the spectrum in the union of a sequence of intervals of the above kind that become disjoint for sufficiently large centers. Looking for quasi-clustering is quite natural to do for a system for two main reasons. On the one hand, the principal symbol of a second order SMGES may have bicharacteristics that are all periodic with period independent of the energy. On the other, thinking of a diagonal system, the spectrum is contained in the union of that of the diagonal components, so that, even if each component clusters, we do have overlapping intervals, of course with an a priori bounded number of overlappings. In the case of our class, the systems having as a principal symbol the harmonic oscillator  $p_2(X) = |X|/2$ , we are exactly in the position of having all the bicharacteristics periodic of the same period. In this case it is crucial that the principal symbol be of order 2 (see, e.g., [14], Section 11.1).

As already mentioned in the Introduction, quasi-clustering here will be considered for SMGES for which the semiprincipal part has eigenvalues that are functions of

$$(5.6) \quad p_{2,\alpha}(X) = \sum_{j=1}^n \alpha_j (x_j^2 + \xi_j^2) / 2,$$

with  $\alpha_1, \dots, \alpha_n > 0$ . For certain values of the  $\alpha$ s (e.g., when they are all equal, a case which occurs for  $A_{JC}$ ) the condition for the three-term Weyl asymptotics (i.e. the sets  $Z_j$  in Condition **DGW** (4.5) should have measure zero) does not hold, and that happens for instance when the level sets of  $p_2$  are also level sets of  $p_{2,\alpha}$ . Hence, in this case, we may think of the quasi-clustering property as more general than having the refined Weyl formula.

Once more, a fundamental ingredient to approach the problem is the use of the diagonalization Theorem 3.1, but that turns out to be not enough, for the fact that  $(e^w)^* a^w e^w = b^w + \Psi^{-\infty}$  with  $e^w$  unitary up to smoothing operators is not sufficiently strong to control the spectrum of  $a^w$  in terms of that of  $b^w$  (using the minimax). The way out of this is to correct  $(e^w)^*$  into a true  $L^2$ -isometry (i.e.  $e^w (e^w)^* = I$ ), and that depends on the nonnegativity of

the index of  $e^w$ . When the index of  $e^w$  is negative, one has to ‘‘augment’’ the considered system into a new one which is still diagonalizable, with diagonalizer which now has the right index, and whose diagonalized operator has two diagonal blocks: one is given by  $b^w$  and the other is given by a system of which we know exactly the spectrum. Hence, we obtain a quasi-clustering in which the only addition is a sequence of new centers that, however, are exactly known.

I next explain the main steps in achieving all that.

Recall that we say that a linear, bounded operator  $T : H \rightarrow H$  ( $H$  a separable Hilbert space) is said to be a *quasi-isometry* if  $T^*T = I + F_1$  and  $TT^* = I + F_2$  where  $F_1, F_2 : H \rightarrow H$  are compact.

We have the following ‘‘isometrization’’ theorem [13].

**Theorem 5.1.** *Suppose  $T \in \Psi_{\text{sreg}}^0$  is such that  $T$  is a quasi-isometry, with  $F_j \in \Psi^{-m_j}$ ,  $m_j > 0$ ,  $j = 1, 2$ . Then:*

- (i) *When  $\text{ind}(T) \leq 0$ , there is  $R_1 \in \Psi^{-m_1}$  such that  $T + R_1$  is an isometry, i.e. one has  $(T + R_1)^*(T + R_1) = I$ ;*
- (ii) *When  $\text{ind}(T) \geq 0$ , there is  $R_2 \in \Psi^{-m_2}$  such that  $T^* + R_2$  is an isometry, i.e. one has  $(T^* + R_2)^*(T^* + R_2) = I$ .*

*When  $F_1$ , resp.  $F_2$ , is smoothing then so is  $R_1$ , resp.  $R_2$ .*

Once we have that theorem, we may use the minimax to relate the spectrum of  $A$  to that of  $B$ , where  $A$  and  $B$  are the realizations, as unbounded selfadjoint operators with domains the Shubin Sobolev space of order 2, of  $a^w$  and  $b^w$ , respectively. In fact, if  $E^*$  is the *isometrization* of  $(e^w)^*$ , which is therefore (by virtue of its pseudodifferential nature) continuous on any given Shubin Sobolev space into itself, supposing  $A > 0$  (which we may assume without loss of generality by the Sharp-Gårding inequality) and that  $EE^* = I$ , one has

$$(5.7) \quad \text{Spec}(A) = \text{Spec}(E^*AE) \setminus \{0\}.$$

Recall that  $E^*AE = B + \text{compact operator}$ . When the eigenvalues  $\lambda_{1,h}$  of the semiprincipal symbol  $a_1$  are functions of  $p_{2,\alpha}$  (see (5.6)), i.e.

$$(5.8) \quad \lambda_{1,h}(X) = p_{1/2}^{(h)}(p_{2,\alpha}(X)), \quad X \neq 0, \quad h = 1, \dots, r$$

(with  $p_{1/2}^{(h)}$  smooth and positively homogeneous of degree  $1/2$  in  $\mathbb{R}^{2n}$ ), we have to look at the bicharacteristics of  $p_{2,\alpha}$ . Note that the smooth diagonalizability of  $a_1$  yields then that a sufficient condition for having (5.8) is that the characteristic polynomial  $\lambda \mapsto \det(\lambda - a_1)$  has coefficients that are smooth functions of  $p_{2,\alpha}$ .

When the constants  $\alpha_1, \dots, \alpha_n > 0$  are such that for positive coprime integers  $m_1, \dots, m_n$  one has

$$(5.9) \quad m_j/\alpha_j = q, \quad \forall j = 1, \dots, n,$$

and for a diagonalizer  $e_0$  of the semiprincipal symbol we have that the resulting subprincipal symbol  $b_0$  of the diagonalization  $B$  is constant on the Hamilton flows of both  $p_2$  and  $p_{2,\alpha}$ , then one may find an orthonormal basis of  $L^2(\mathbb{R}^n; \mathbb{C}^N)$ , made of Schwartz functions such that the eigenspaces of  $P_2$  are invariant for  $P_{2,\alpha}$  and such that the eigenspaces of  $P_2 + P_{2,\alpha}$  are left invariant by  $B_0$  whence one obtains the following theorem.

**Theorem 5.2.** *Let  $0 < a^w \in \Psi_{\text{sreg}}^2$  be an SMGES. Suppose that the coefficients of the characteristic polynomial  $\lambda \mapsto \det(\lambda - a_1(X))$  be smooth functions of  $p_{2,\alpha}$  (see (5.8)) and that*

for a diagonalizer  $e_0$  the subprincipal symbol  $b_0$  be constant along both the bicharacteristic flows of  $p_2$  and  $p_{2,\alpha}$ , where the  $\alpha_j$ s are such that (5.9) holds. Then there are  $M > 0$  and  $c > 0$  such that

$$\text{Spec}(A) \subset \bigcup_{h=1}^{r+1} S_h,$$

where

$$S_h = \bigcup_{k \geq 0} \bigcup_{\gamma \in \mathbb{Z}_+^n} \bigcup_{j=1}^{N_h} \left( c(k, \gamma, j) + \left[ -\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}} \right] \right),$$

and where for the centers  $c(k, \gamma, j)$  we have

$$c(k, \gamma, j) = k + n/2 + p_{1/2}^{(h)} \left( \sum_{j=1}^n \alpha_j (\gamma_j + 1/2) \right) + \mu_{\gamma, j}^{(h)}.$$

One has that  $N_{r+1} = N$ ,  $p_{1/2}^{(r+1)} = 0$ ,  $\mu_{\gamma, j}^{(r+1)} = c$  and for  $h = 1, \dots, r$  the  $\mu_{\gamma, j}^{(h)}$ s are the eigenvalues of the block  $B_0^{(h)}$  of the subprincipal operator  $B_0$  restricted to  $\text{Ker}(P_2 \otimes I_{N_h} - (k+n/2))$ .

When  $\text{ind}(e_0) \geq 0$  we may take  $S_{r+1} = \emptyset$ .

For  $h = 1, \dots, r$ , one has that  $|\mu_{\gamma, j}^{(h)}| \leq \|B_0^{(h)}\|_{L^2 \rightarrow L^2}$ , for all  $\gamma \in \mathbb{Z}_+^n$  and all  $j = 1, \dots, N_h$ . Hence they are uniformly bounded by  $\|B_0\|_{L^2 \rightarrow L^2}$ .

Notice also that the set  $S_h$  is the contribution to the spectrum related to the block  $B_h$  of the diagonalization of  $A$ ,  $h = 1, \dots, r$ , and that the presence of  $S_{r+1}$  and  $c \in \mathbb{R}$  is due to the case when  $\text{ind}(e_0) \geq 0$ . In fact, in such a case one augments the system into a  $2N \times 2N$  system of the form

$$\tilde{A} = \left[ \begin{array}{c|c} A & 0_N \\ \hline 0_N & P_2 I_N + P_0 \end{array} \right],$$

where the symbol  $p_0$  of  $P_0$  is chosen as

$$p_0 = -e_0^* \left( e_{-2} e_0^* p_2 + p_2 e_0 e_{-2}^* - \frac{i}{2} (e_0 \{p_2, e_0^*\} + \{e_0, p_2 e_0^*\}) + e_{-1} p_2 e_{-1}^* \right) e_0 + c I_N,$$

where  $c > 0$  is a constant such that  $P_0 > 0$  (the expression of  $p_0$  comes from the diagonalization). Hence

$$\tilde{E}^* \tilde{A} \tilde{E} = \left[ \begin{array}{c|c} B + \Psi^{-\infty} & 0_N \\ \hline 0_N & (P_2 + c) I_N + R \end{array} \right]$$

where  $\tilde{e} := \left[ \begin{array}{c|c} e & 0_N \\ \hline 0_N & e^* \end{array} \right]$  ( $e$  and  $e^*$  being the total symbols of  $E$  and  $E^*$  respectively), and  $R \in \Psi_{\text{sreg}}^{-1}$ . In this case the index of  $\tilde{E}$  is zero, and the subprincipal symbol of  $E(P_2 I_N + P_0)E^*$  is  $c I_N$  by the definition of  $p_0$ . One therefore may proceed by exploiting the hypotheses on the bicharacteristic flows and constancy of  $b_0$  along the bicharacteristics of  $p_2$  and of  $p_{2,\alpha}$ , that grant the eigenspaces invariance and simultaneous diagonalization that lead to the result.

As a matter of example, let us see what Theorem 5.2 says about the Jaynes-Cummings system  $A_{JC}$  (recalling that  $\psi(X) = (x + i\xi)/\sqrt{2}$ )

$$a_{JC}(X) = p_2(X) I_2 + \alpha \left[ \begin{array}{c|c} 0 & \overline{\psi(X)} \\ \hline \psi(X) & 0 \end{array} \right] + \left[ \begin{array}{c|c} \gamma_1 & 0 \\ \hline 0 & \gamma_2 \end{array} \right], \quad X \in \mathbb{R}^2 \setminus \{0\}.$$

In this case the eigenvalues of the semiprincipal symbol are given by

$$\lambda_{\pm}(X) = \pm|\alpha||X|/\sqrt{2},$$

the diagonalizer  $e_0$  is given by

$$e_0(X) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ \psi(X)/|\psi(X)| & -\psi(X)/|\psi(X)| \end{bmatrix}, \quad X \neq 0,$$

and subprincipal term  $b_0$  is given by

$$b_0(X) = \frac{\gamma_1 + \gamma_2 - 1}{2} I_2.$$

Note that  $p_{2,\alpha} = |\alpha|p_2$  and that  $b_0$  is constant on the bicharacteristics of  $p_2$  (and hence of  $p_{2,\alpha}$ ). Using the index formula from [9] one computes  $\text{ind}(e_0) = -1$ , so that we have to augment the system into a  $4 \times 4$  one. Theorem 5.2 then gives that

$$\text{Spec}(A_{JC}) \subset S_0 \cup \bigcup_{\pm} S_{\pm},$$

where the structure of  $S_{\pm}$  and  $S_0$  is as follows. Let, for  $k \in \mathbb{Z}_+$ ,

$$c_{\pm}(k) = k + 1/2 \pm |\alpha|\sqrt{k + 1/2} + (\gamma_1 + \gamma_2 - 1)/2, \quad \mu_k^{(\pm)} = (\gamma_1 + \gamma_2 - 1)/2,$$

and

$$c_0(k) = k + 1/2 + c,$$

where  $c > 0$  is so large to ensure the positivity of the operator  $P_0$ . Then

$$S_{\pm} = \bigcup_{k \geq 0} \left( c_{\pm}(k) + \left[ -\frac{M}{\sqrt{k + 1/2}}, \frac{M}{\sqrt{k + 1/2}} \right] \right),$$

$$S_0 = \bigcup_{k \geq 0} \left( c_0(k) + \left[ -\frac{M}{\sqrt{k + 1/2}}, \frac{M}{\sqrt{k + 1/2}} \right] \right).$$

## 6. PROBLEMS

I end this survey by considering two open problems.

- As mentioned in Section 3, one has the expression of the semi-subprincipal term of the diagonalized operator, along with its transformation rules with respect of the principal symbol of the diagonalizer. Is it possible to obtain a four-term asymptotics of  $N_A$  in terms of the principal, semiprincipal, subprincipal and semi-subprincipal terms?
- In the quasi-clustering setting, what happens in case the bicharacteristic flows of the principal part  $p_2$  and that of  $p_{2,\alpha}$  do not commute? And how about the requirement on the subprincipal part of the *diagonalized* operator, having fixed the gauge  $e_0$ , to be constant on the bicharacteristics? Can it be relaxed? And is it possible to understand the constancy condition in terms of the initial symbol, without having to fix a diagonalizer? Note that the Jaynes-Cummings system satisfies the constancy of  $a_0$  and  $b_0$  on the bicharacteristics, that are commuting since  $p_2$  and  $p_{2,\alpha}$  are in this case proportional.

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