Unknown-input pseudo-state observer synthesis for fractional-order systems: A geometric framework

Published Version:
Nozari, H.A., Rostami, S.J.S., Castaldi, P. (2024). Unknown-input pseudo-state observer synthesis for fractional-order systems: A geometric framework. CHAOS, SOLITONS AND FRACTALS, 187, 1-11 [10.1016/j.chaos.2024.115355].

Availability:
This version is available at: https://hdl.handle.net/11585/985534 since: 2024-09-18

Published:
DOI: http://doi.org/10.1016/j.chaos.2024.115355

(Article begins on next page)

Unknown-input pseudo-state observer synthesis for fractional-order systems: *A geometric framework*

Hasan Abbasi Nozari [a], Seyed Jalil Sadati Rostami [a,*], Paolo Castaldi [b]

[a] *Faculty of Electrical and Computer Engineering, Babol Noshirvani University of Technology, Babol, Iran*
[b] *Department of Electrical, Electronic and Information Engineering, University of Bologna, Cesena, Italy*

A R T I C L E I N F O

Keywords:
Conditioned invariant subspace
Unknown-input pseudo-state observation
Fractional-order linear systems
Caputo derivative

A B S T R A C T

This paper explores the development and characterization of the invariant subspaces in geometric control theory, namely conditioned invariant, input-containing, unobservability, and detectability subspaces for fractional-order linear systems, and the requirements for the existence of such subspaces have been established. By utilizing these invariance notions, the pseudo-state observation problem in the presence of unknown-inputs for fractional linear systems with Caputo derivative is formulated and solved within the geometric framework for the first time. Furthermore, it is shown that the obtained solubility conditions for the unknown-input pseudo-state observation problem are not confined by the choice of fractional derivative operator. Finally, the applicability and effectiveness of the theoretical findings are further supported by two numerical simulations, including an application to a vehicle suspension system.

1. Introduction

The theory of fractional calculus has captivated significant interest in applied mathematics, particularly within the fields of systems and control theory. Numerous papers and textbooks have explored this paradigm (see, for example, [1–6] and the references therein). The interesting properties of fractional differential equations include the long-term memory effect and hereditary provided by the fractional integrals by which the fractional derivatives including the Caputo derivative and the Riemann–Liouville (RL) derivative are defined and the non-integer differential order which allows for a greater degree of freedom in accurately describing the real-world systems [1,7]. In recent decades, fractional-order (FO) systems have seen successful applications in diverse areas, e.g., biological systems [8–10], physical systems [11–14], signal processing [15,16], multi-agent systems [17, 18]. Over the past two decades, several concepts and methods from the control theory of integer-order systems have been extended into FO systems. These include robust control design [19], Lyapunov method to stabilize fractional nonlinear systems [20], feedback linearization approach [21], sliding mode control [5], and fuzzy control [22], to name a few.

Although several concepts from the control theory of integer-order systems including stability, controllability, and observability have been generalized into FO systems, some major aspects of the fractional systems theory are yet to be explored. One such area is geometric control theory, which is considered a powerful tool in understanding the system-theoretic properties of dynamical systems and has been so far scarcely addressed for fractional systems. The area of systems and control theory known as the geometric approach was initially developed for integer-order linear systems to address the problems of disturbance decoupling and unknown-input observation [23,24]. The interest in geometric control theory is motivated by its applications to various synthesis problems, including fault detection and identification, H_2 optimal regulation problems [25], and model following problem [26]. The fundamental principles of geometric control have also been applied to different classes of dynamic systems, including nonlinear systems [27], impulsive systems [28], switching linear systems [29], linear parameter-varying systems [30], multidimensional systems [31], to name just a few. The existing literature contains only one seminal work that explores the extension of classical geometric control theory to the fractional-order time-invariant linear (FO-LTI) systems modeled by the Caputo derivative [32]. This pioneering work generalizes foundational notions of $A-$invariance and $(A, B)-$controlled invariance to the fractional-order context. It also addresses disturbance decoupling problems with and without stability. However, to use the geometric approach for solving estimation problems in fractional-order fields, other fundamental structural tool of the geometric control approach including conditioned invariants, input-containing, and detectability subspaces are yet to be developed.

In the literature, there are limited number of research works dedicated to FO observer design for fractional systems in presence of

* Corresponding author.
E-mail addresses: hanozari@nit.ac.ir (H.A. Nozari), j.sadati@nit.ac.ir (S.J.S. Rostami), paolo.castaldi@unibo.it (P. Castaldi).

unknown inputs. However, none of the proposed methods exploit the geometric approach in the design procedure for an unknown-input FO observer. For example, a method of designing FO observers for continuous FO-LTI systems with unknown inputs was proposed in [33], where sufficient conditions for the asymptotical stability of error dynamics are obtained based on the linear matrix inequalities. An unknown-input FO observer based on finite frequency H_{-}/H_{∞} is developed by Li et al. [34] to solve the fault detection problem that unknown input cannot be completely decoupled from the generated indicative residual signals. An unknown-input nonsingular observer for sensor fault detection and estimation in linear FO singular one-sided Lipschitz systems is designed by Komachali and Shafiee [35] and the asymptotic stability the proposed FO observer are guaranteed based on the linear matrix inequalities. In the work by Djeghali et al. [36], the authors developed some notions including observability, left invertibility and normalforms have not been addressed for nonlinear FO systems, by which they could develop a sliding mode observer to estimate the fault signal as an unknown input.

Geometric approach plays a central role in a plethora of fascinating classes of estimation problems including unknown input observation problem with stability (UIOPS). This problem concerns with state estimation when dealing with unknown inputs. Unknown inputs are of significant importance in decentralized control schemes, particularly in cases where single controllers possess partial information, likewise in general control approaches, where they model disturbances [23]. Furthermore, by addressing the feasibility of solutions for the UIOP, we can effectively tackle fault detection and isolation problems [26,37]. The UIOP has been also addressed in geometric sense for various classes of dynamic systems including switching linear systems [38], 2–D Fornasini Marchesini systems [39], linear impulsive systems [40], linear parameter-varying systems [41], linear hybrid systems with state jumps [42], and singular systems [43]. The present work aims to establish the fundamental algebraic and geometric tools, namely conditioned invariants, input-containing, and detectability subspaces, by which the UIOPS will be addressed for FO-LTI systems for the first time. We will demonstrate that the disturbance-to-error nature of the UIOPS ensures our necessary and sufficient conditions hold for any choice of fractional derivatives. This applies to systems where a solution to the associated fractional differential equation exists and is unique. Consequently, the main novelties and contributions of this work are elaborated as follows:

- Several fundamental notions and tools used in the classic geometric control theory of the integer-order systems including conditioned invariant, input-containing, and detectability subspaces are extended to the fractional-order field.
- Several sufficient and necessary conditions for the existence of conditioned invariant subspaces, hinging on the existence of FO quotient observers, are established in the fractional-order context for the first time and can be degenerated into the standard conditions of the integer-order case.
- The UIOP with the requirement of stability is extended to the FO-LTI systems. The newly established conditions are then employed to solve this problem.

The remainder of this paper is organized as follows. Section 2 starts by introducing some preliminaries including useful definitions and lemmas. The main outcomes of our study are outlined in Section 3. In Section 4 , the unknown-input pseudo-state observation problem and its solvability in fractional-order context is discussed. Finally, to demonstrate the validity of the achieved theoretical findings, two numerical examples are provided in Section 5. The concluding remarks are given in Section 6.

2. Preliminaries

2.1. Notations

Throughout the paper, the following notations are exploited. The symbols $\mathbb{C}$ and $\mathbb{N}^0$ denote the sets of complex numbers and natural numbers with zero, respectively. We define $\mathbb{R}_a^+ := \{s \in \mathbb{R} \mid s \geq a\}$, where $\mathbb{R}$ represents the set of real numbers. Slanted capital letters, such as A, are used to symbolize linear maps between vector spaces and matrices. Calligraphic capital letters $(\mathcal{X}, \mathcal{Y}, \mathcal{W})$ denote sets, vector spaces, and subspaces. The image and kernel vector spaces of a linear map A are denoted by $\mathrm{im}(A)$ and $\ker(A)$. Given a linear map $A : \mathcal{X} \to \mathcal{Y}$, a subspace $\mathcal{W} \subseteq \mathcal{X}$ is called an $A-$invariant subspace if $A\mathcal{W} \subseteq \mathcal{W}$, where $A\mathcal{W} := \{Aw \mid w \in \mathcal{W}\}$. The inverse image of $\mathcal{W} \subseteq \mathcal{Y}$ under A is the subspace $A^{-1}\mathcal{W} := \{x \in \mathcal{X} \mid Ax \in \mathcal{W} \subseteq \mathcal{Y}\} \subseteq \mathcal{X}$. We denote by $\mathcal{X}/\mathcal{W}$ as the quotient space of a vector space $\mathcal{X}$ modulo a subspace $\mathcal{W} \subseteq \mathcal{X}$. The orthogonal complement of a given subspace $\mathcal{W}$ of a linear space $\mathcal{X}$ is denoted by $\mathcal{W}^{\perp} := \{x \in \mathcal{X} \mid x^{\mathrm{T}}w = 0 \quad \forall w \in \mathcal{W}\}$. The restriction of the linear map A to an $A-$invariant subspace is denoted by $A \mid \mathcal{W}$. For a square matrix A, the spectrum of eigenvalues of A is denoted by $\mathrm{spec}(A)$. The argument of a complex number $z \in \mathbb{C}$ restricted to the interval $(-\pi, \pi]$ is denoted by $\arg(z)$. The symbol I_n stands for the n-dimensional identity matrix. For the arbitrary $\alpha \in \mathbb{R}_0^+$, the symbols $\lceil \alpha \rceil$ and $\lfloor \alpha \rfloor$ denote the "ceiling" function and the "floor" function, respectively.

2.2. Caputo fractional derivative and its properties

In the following section, some fundamental concepts about FO calculus are recalled, which will be useful in establishing our later results. One is referred to Podlubny [7] and Kilbas [44] for more information.

Definition 1 (*Caputo Derivative [7]*)**.** The Caputo fractional derivative of order $\alpha > 0$ starting from t_0 of a function $f : \mathbb{R}_{t_0}^+ \to \mathbb{R}$ is defined by

$$
{}^C D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_{t_0}^{t} \frac{f^{(\lceil \alpha \rceil)}(\tau)}{(t-\tau)^{\alpha - \lceil \alpha \rceil + 1}} d\tau, \tag{1}
$$

where $\Gamma(.)$ designates the Gamma function. Whenever $t_0 = 0$, we write ${}^C D^{\alpha}$ for notational simplicity.

Definition 2 (*Podlubny [7]*)**.** The fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}_{t_0}^+ \to \mathbb{R}$ starting from $t_0 \in \mathbb{R}_{t_0}^+$ is defined by

$$
I_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{2}
$$

where $m \in \mathbb{N}^0 \backslash \{0\}$ such that $\alpha \in (m-1, m)$.

Definition 3 (*Mittag-Leffler Function [45]*)**.** The Mittag–Leffler function of two parameters α and β which includes a convergent power series is given by

$$
E_{\alpha,\beta}\left(\mu\left(t-t_0\right)^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\left(\mu\left(t-t_0\right)^{\alpha}\right)^k}{\Gamma(\alpha k + \beta)}, \tag{3}
$$

for all $\mu \in \mathbb{C}$ and $\alpha, \beta > 0$.

It is noted that, for the case $\beta = 1$, we simplify the notation by denoting $E_{\alpha,1}(\cdot)$ as $E_{\alpha}(\cdot)$.

Remark 1. Seeking readability, it is assumed that $t_0 = 0$. It is worth pointing out that the results for $t_0 = 0$ can be simply extended to the case $t_0 \neq 0$.

Lemma 1. *Consider the following two-parameter function*

$$
\psi_{\alpha,\beta}(\mu, t) = \sum_{k=0}^{\infty} \frac{\mu^k \, t^{\alpha k + \beta - 1}}{\Gamma(\alpha k + \beta)}, \tag{4}
$$

for all $\mu \in \mathbb{C}$ *and* $\alpha \in (0, 2)$, $\beta > 0$. *Then*

$$
{}^C D^{\alpha} \psi_{\alpha,\beta}(\mu, t) = \mu \, \psi_{\alpha,\beta}(\mu, t). \tag{5}
$$

Proof. The proof is presented in Appendix.

2.3. Fractional-order linear systems

The non-autonomous multivariable commensurate FO-LTI system can be represented by using a pseudo-state space formulation of the following form [46]:

$$\Sigma:\begin{cases}{}^{C}D^{\alpha}x(t)=Ax(t)+Bu(t), & \forall t\geq 0\\ y(t)=Cx(t),\end{cases} \tag{6}$$

(henceforth, the arguments t will often be omitted for the sake of readability) where ${}^{C}D^{\alpha}$ denotes the Caputo derivative operator of commensurate order $\alpha\in(0,2)$ with the initial conditions given in a compact form as $x^{(k)}(0)=x_{0,k}$ for $k\in\{0,\lfloor\alpha\rfloor\}$. For all $t\geq 0$, $x(t)\in\mathcal{X}=\mathbb{R}^{n}$ is the pseudo-state, $u(t)\in\mathcal{U}=\mathbb{R}^{m}$ is the input function, and $y(t)\in\mathcal{Y}=\mathbb{R}^{p}$ is the output of the system. A, B, and C are constant real-valued matrices with appropriate dimensions

It must be noted that the fractional derivative differs from the integer-order derivative in that it is established over a growing finite-interval of time, as opposed to being based on the characterization of the differentiable function in an arbitrarily small ("infinitesimal") neighborhood around the point at which the derivative is calculated. As a result, the future behavior of a fractional system is influenced by the historical past of the vector x. This means that the value of the vector $x(t)$ at a single instant of time does not accurately reflect the complete past history of the system's behavior. Hence, the term "pseudo-state" is used instead of the term "state" for x in fractional systems defined by nonlocal FO operators (see e.g. [19,46,47] for more details).

Remark 2. It is remarked that fractional orders $\alpha\geq 2$ are not taken into account since system (6) is not asymptotically stable or stabilizable for these values of α [7].

Refer now to the following autonomous commensurate FO-LTI system,

$$\begin{cases}{}^{C}D^{\alpha}x(t)=Ax(t), & \forall t\geq 0\\ x^{(k)}(0)=x_{0,k},\end{cases} \tag{7}$$

for $k\in\{0,\lfloor\alpha\rfloor\}$, then the pseudo-state response of the system (6) for $\alpha\in(0,2)$ is given by [7]:

$$x(t)=\sum_{k=0}^{\lfloor\alpha\rfloor}t^{k}\,E_{\alpha,k+1}\left(At^{\alpha}\right)x_{0,k}. \tag{8}$$

Thus, given initial conditions $x^{(k)}(0)=x_{0,k}\in\mathcal{X}$ for $k\in\{0,\lfloor\alpha\rfloor\}$ and an input function $u(t)\in\mathcal{U}$ defined on $[0,t)$, the solution $x(t)$ to non-autonomous FO-LTI system (6) can be stated as follows [7]:

$$x(t)=\sum_{k=0}^{\lfloor\alpha\rfloor}t^{k}E_{\alpha,k+1}\left(At^{\alpha}\right)x_{0,k}+\int_{0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-\tau)^{\alpha}\right)Bu(\tau)d\tau. \tag{9}$$

Thus, the corresponding system output is represented by:

$$y(t)=\sum_{k=0}^{\lfloor\alpha\rfloor}Ct^{k}E_{\alpha,k+1}\left(At^{\alpha}\right)x_{0,k}+\int_{0}^{t}\underbrace{C(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-\tau)^{\alpha}\right)B}_{g_{u,y}(t-\tau)}u(\tau)d\tau, \tag{10}$$

where $g_{u,y}(t):=Ct^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})B$ represents the pseudo-state impulse response of system (6) [48]. Since $y(t)$ depends on t, $x_{0,k}$ and $u(t)$ and also on the choice of α, we will usually exploit the notation $y(t)=y(t,x_{0,k},u)$ for $k\in\{0,\lfloor\alpha\rfloor\}$.

The stability of commensurate FO-LTI system (6) can be analyzed in terms of the eigenvalue locations of the pseudo-state matrix A. This result was investigated for system (6) with $\alpha\in(0,2)$ in [49,50], and can be stated by the subsequent Lemma.

Lemma 2 (*Matignon [49] and Sabatier et al. [50]*). *Consider the commensurate FO-LTI system (6) with $\alpha\in(0,2)$. It is asymptotically stable if and only if*

$$|\arg(\mathrm{spec}(A))|>\frac{\alpha\pi}{2}.$$

The following lemma shows that when the initial pseudo-state $x_{0,k}$ is derived from an $A-$invariant subspace $\mathcal{W}$ for $k\in\{0,\lfloor\alpha\rfloor\}$, the pseudo-state trajectory will remain entirely within $\mathcal{W}$ for all $t>0$. This result is generalized from the integer-order case [51] to FO systems as follows [32].

Lemma 3 (*Padula et al. [32]*). *A subspace $\mathcal{W}\subseteq\mathcal{X}$ is $A-$invariant under system (7) if and only if $A\mathcal{W}\subseteq\mathcal{W}$.*

3. Main results

Consider the following disturbed commensurate FO-LTI system:

$$\begin{cases}{}^{C}D^{\alpha}x(t)=Ax(t)+Ed(t), & \forall t\geq 0\\ y(t)=Cx(t),\end{cases} \tag{11}$$

where $d(t)\in\mathcal{D}=\mathbb{R}^{v}$ represents the unknown disturbance input. For the given initial conditions $x_{0,k}$ for $k\in\{0,\lfloor\alpha\rfloor\}$ and disturbance d, the output of system (11) for $\alpha\in(0,2)$ is given by:

$$y(t)=\sum_{k=0}^{\lfloor\alpha\rfloor}Ct^{k}\,E_{\alpha,k+1}\left(At^{\alpha}\right)x_{0,k}+\int_{0}^{t}\underbrace{C(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-\tau)^{\alpha}\right)E}_{g_{d,y}(t-\tau)}d(\tau)d\tau, \tag{12}$$

where $g_{d,y}(t):=Ct^{\alpha-1}E_{\alpha,\alpha}\left(At^{\alpha}\right)E$ represents the pseudo-state impulse-response between d and y. It is obvious that the disturbance d has no influence on the output y if $g_{d,y}(t)=0$. The following lemma characterizes this result in a geometric sense for the FO systems.

Lemma 4. *Consider the disturbed system (11). Let $\mathcal{E}:=\mathrm{im}(E)=\{x\in\mathcal{X}\mid x=Ed, d\in\mathcal{D}\}$ and $\mathcal{C}:=\ker(C)=\{x\in\mathcal{X}\mid Cx=0\in\mathcal{Y}\}$. Let Then the following statements are equivalent:*

(i) the output y is unaffected by disturbance d, i.e., $g_{d,y}(t)=0$,
(ii) there exists an $A-$invariant subspace $\mathcal{J}$ satisfying $\mathcal{E}\subseteq\mathcal{J}\subseteq\mathcal{C}$

Proof. $(i)\Rightarrow(ii)$. Suppose that $g_{d,y}(t)=0$. Then, by repeatedly applying Lemma 1, it also follows that ${}^{C}D^{(k)\alpha}(g_{d,y}(t)):=\underbrace{{}^{C}D^{\alpha C}D^{\alpha}\ldots{}^{C}D^{\alpha}}_{k\text{-times}}(g_{d,y}(t))$ $=CA^{k}t^{\alpha-1}E_{\alpha,\alpha}\left(At^{\alpha}\right)E=0$ for all $t>0$, which is equivalent to

$$\sum_{k=0}^{\infty}CA^{k}\frac{t^{\alpha(k+1)-1}}{\Gamma(\alpha k+\alpha)}E=0\quad\Leftrightarrow\quad\sum_{k=0}^{\infty}\frac{t^{\alpha(k+1)-1}}{\Gamma(\alpha k+\alpha)}CA^{k}E=0,$$

for all $t>0$. In light of the identity principle applied to the fractional power-series expansion, we get $CA^{k}E=0$ for $k\in\mathbb{N}^{0}$. In the virtue of the Cayley–Hamilton theorem, we have that $CA^{k}E=0$ for $k=0,1,\ldots,n-1$ which can be alternatively formulated in a geometric sense as $A^{k}\mathcal{E}\subseteq\mathcal{C}$ for $k=0,1,\ldots,n-1$. It in turn implies that $\langle A\mid\mathcal{E}\rangle=\sum_{k=0}^{n-1}\mathrm{im}(A^{k}E)\subseteq\mathcal{C}$. The subspace $\mathcal{J}:=\langle A\mid\mathcal{E}\rangle$ has the invariance property and is the minimal $A-$invariant subspace containing $\mathcal{E}$.

$(ii)\Rightarrow(i)$. If $\mathcal{E}\subseteq\mathcal{J}$ and $\mathcal{J}$ is invariant under A, then we have $\mathrm{im}(A^{k}E)\subseteq\mathcal{J}$ for $k\geq 0$. Moreover, if $\mathcal{J}\subseteq\mathcal{C}$, it then follows that $CA^{k}E=0$ for $k\geq 0$. Hence, we have that

$$\sum_{k=0}^{\infty}\frac{t^{\alpha(k+1)-1}}{\Gamma(\alpha k+\alpha)}CA^{k}E=0,\quad\forall t\geq 0.$$

Consequently, we get that $g_{d,y}(t)=0$. This concludes the proof.

3.1. Observability of FO-LTI systems

In what follows, for the sake of completeness, we go over the possibility of reconstruction of the pseudo-state x when the pair (u,y) in system (6) is known.

Definition 4. Consider system (6). Let $T > 0$. The pseudo-states $x^0_{0,k}, x^1_{0,k} \in \mathcal{X}$ for $k \in \{0, \lfloor\alpha\rfloor\}$ are indistinguishable on $[0,T]$ if $y\left(t, x^0_{0,k}, u\right) = y\left(t, x^1_{0,k}, u\right)$ for every $u \in \mathcal{U}$ and $t \in [0,T]$.

Consider first the case $\alpha \in (0,1)$. Two pseudo-states $x^0_{0,0}$ and $x^1_{0,0}$ are indistinguishable if, for any input $u \in \mathcal{U}$, the corresponding outputs are equal. Hence, in order to have $x^0_{0,0}$ and $x^1_{0,0}$ indistinguishable over $[0,T]$, we need that

$$CE_\alpha\left(At^\alpha\right)x^0_{0,0} + \int_0^t g_{u,y}(t-\tau)u(\tau)d\tau = CE_\alpha\left(At^\alpha\right)x^1_{0,0} + \int_0^t g_{u,y}(t-\tau)u(\tau)d\tau,$$

for all $t \in [0,T]$. It is observed that x^0_0 and x^1_0 are indistinguishable if and only if

$$CE_\alpha\left(At^\alpha\right)x^0_{0,0} = CE_\alpha\left(At^\alpha\right)x^1_{0,0}, \quad \forall t \in [0,T]. \tag{13}$$

The relation (13) holds if and only if

$${}^C D^{(k)\alpha}\left\{CE_\alpha\left(At^\alpha\right)(x^0_{0,0} - x^1_{0,0})\right\} = 0, \quad \forall t \in [0,T], \tag{14}$$

and for $k \in \mathbb{N}^0$. Thus, by sequential using of Lemma 1, we get $CA^k E_\alpha(At^\alpha)(x^0_{0,0} - x^1_{0,0}) = 0$ for every $t \in [0,T]$. Take $t = 0$, then we have $CA^k(x^0_{0,0} - x^1_{0,0}) = 0$ for $k \in \mathbb{N}^0$. Let define nonzero vector $q := x^0_{0,0} - x^1_{0,0}$. In light of the Cayley–Hamilton theorem, it then follows that $CA^k q = 0$ for $k = 0, 1, \ldots, n-1$. Hence, we get that

$$q \in \ker\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right). \tag{15}$$

Let us now consider $\alpha \in (1,2)$. The set of pseudo-states $x^0_{0,k}$ and $x^1_{0,k}$ with $k \in \{0, \lfloor\alpha\rfloor\}$ are indistinguishable if, for any input u, the corresponding output values are the same. Hence, we must have that

$$\sum_{k=0}^{\lfloor\alpha\rfloor} Ct^i E_{\alpha,k+1}\left(At^\alpha\right)x^0_{0,k} + \int_0^t g_{u,y}(t-\tau)u(\tau)d\tau = \sum_{k=0}^{\lfloor\alpha\rfloor} Ct^k E_{\alpha,k+1}\left(At^\alpha\right)x^1_{0,k} + \int_0^t g_{u,y}(t-\tau)u(\tau)d\tau, \quad \forall t \in [0,T].$$

Hence, the set of pseudo-states $x^0_{0,k}$ and $x^1_{0,k}$ are indistinguishable if and only if

$$\sum_{k=0}^{\lfloor\alpha\rfloor} Ct^k E_{\alpha,k+1}\left(At^\alpha\right)x^0_{0,k} = \sum_{k=0}^{\lfloor\alpha\rfloor} Ct^k E_{\alpha,k+1}\left(At^\alpha\right)x^1_{0,k}, \quad \forall t \in [0,T]. \tag{16}$$

The relation (16) holds if and only if

$${}^C D^{(k)\alpha}\left\{\sum_{k=0}^{\lfloor\alpha\rfloor} Ct^k E_{\alpha,k+1}\left(At^\alpha\right)\left(x^0_{0,k} - x^1_{0,k}\right)\right\} = 0, \quad \forall t \in [0,T], \tag{17}$$

and for $i \in \mathbb{N}^0$. Hence, by repeatedly using of Lemma 1, we get that

$$CA^k E_\alpha\left(At^\alpha\right)\left(x^0_{0,0} - x^1_{0,0}\right) + Ct\,A^k E_{\alpha,2}\left(At^\alpha\right)\left(x^0_{0,1} - x^1_{0,1}\right) = 0, \quad \forall t \in [0,T],$$

and for $k \in \mathbb{N}^0$. By taking $t = 0$ we thus obtain that $CA^k(x^0_{0,0} - x^1_{0,0}) = 0$ for $k \in \mathbb{N}^0$. Now similar to the case $\alpha \in (0,1)$, defining nonzero vector $q := x^0_{0,0} - x^1_{0,0}$ yields the inclusion in (15).

Analogous to the integer-order context [26], the space of vectors q for which the inclusion in (15) holds is called the complementary controllability subspace (also referred to as unobservability subspace) that is independent of T and can be represented in the geometric sense as the maximum A−invariant subspace contained in $\mathcal{C} = \ker(C)$ denoted by:

$$\mathcal{N} := \langle \mathcal{C} \mid A\rangle = \bigcap_{k=0}^{n-1} \mathcal{C}A^k. \tag{18}$$

As a consequence, also analogous to the integer-order context, the FO-LTI system ruled by (6) is completely observable if and only if the following geometric expression holds:

$\langle \mathcal{C} \mid A\rangle = \{0\}$.

3.2. Conditioned invariant subspaces for FO-LTI systems

A fundamental concept for the geometric design of an unknown-input observer in fractional-order context is that of conditioned invariance. With this motivation, the notion of conditioned invariance (also referred to as (C,A)−invariance) for the FO-LTI systems is introduced and discussed for the first time.

Definition 5. A linear subspace $\mathcal{S}$ of the pseudo-state space $\mathcal{X}$ of system (6) is called conditioned invariant, if there exists an output-injection gain $G \in \mathbb{R}^{n\times p}$ such that $\mathcal{S}$ is A_G−invariant, where $A_G := A + GC$.

The set of all conditioned invariant subspaces will be referred to as $\underline{\mathcal{S}}$. Being G a friend of a conditioned invariant subspace $\mathcal{S} \in \underline{\mathcal{S}}$, then the set of friends is defined as $\underline{G}(\mathcal{S}) := \{G \in \mathbb{R}^{n\times p} : A_G\mathcal{S} \subseteq \mathcal{S}\}$. Consider system (6) with $B = 0$. By applying the output injection $v(t) := Gy(t)$, which is an output-to-pseudostate feedback law, we have

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + v(t) = Ax(t) + Gy(t) = A_G x(t), \\ y(t) = Cx(t), \end{cases} \tag{19}$$

where $v(t)$ serves as a function that directly impacts the pseudo state as an input.

Thus, Definition 5 implies that a subspace $\mathcal{S} \subseteq \mathcal{X}$ is conditioned invariant if for all $x_{0,k} \in \mathcal{S}$ with $k \in \{0, \lfloor\alpha\rfloor\}$ there exists an output-to-pseudostate feedback $Gy(t)$ that maintains $x(t)$ in $\mathcal{S}$, i.e.,

$$x(t) = \sum_{k=0}^{\lfloor\alpha\rfloor} t^k E_{\alpha,k+1}\left(A_G t^\alpha\right)x_{0,k} \in \mathcal{S}. \tag{20}$$

By Definition 5, $\mathcal{S}$ is a (C,A)−invariant subspace (or equivalently, it is invariant with respect to A_G in the view of Lemma 3).

Theorem 1. *A subspace $\mathcal{S}$ of the pseudo-state space $\mathcal{X}$ of system (6) is called conditioned invariant if and only if it satisfies the subspace inclusion:*

$$A(\mathcal{S} \cap \mathcal{C}) \subseteq \mathcal{S}. \tag{21}$$

Proof. ($\Rightarrow$) The proof is in the same line as the corresponding point in [37, Theorem 5.5] and is omitted.

($\Leftarrow$) Suppose that $\mathcal{S}$ is conditioned invariant subspace, thus by definition there exists an output injection map G such that $A_G\mathcal{S} \subseteq \mathcal{S}$. Let $v := \dim(\mathcal{S})$ and $\{x_1, x_2, \ldots, x_l\}$ ($l \le v$) be a basis of $\mathcal{S} \cap \mathcal{C}$. Thus, $A_G x_k \in \mathcal{S}$ for $k = 1, 2, \ldots, l$. But $Cx_k = 0$. Hence, $Ax_k \in \mathcal{S}$ and $A(\mathcal{S} \cap \mathcal{C}) \subseteq \mathcal{S}$.

It turns out that the definitions of (C,A)−invariant subspaces given in this paper are the duals of (A,B)−invariant subspaces developed in fractional-order framework by Padula et al. [32, Theorem 5.2].

In what follows, we characterize the geometric notion of conditioned invariant subspaces for the FO-LTI system (6) with $B = 0$ in terms of the existence of a fractional-order *quotient* observer.

Theorem 2. *Consider system (6) with $B = 0$. A subspace $\mathcal{S} \subseteq \mathcal{X}$ is conditioned invariant if and only if there exists a FO quotient observer ruled by*

$$\Sigma^o : \begin{cases} {}^C D^\alpha w(t) = Nw(t) + My(t), \\ \zeta(t) = w(t), \end{cases} \tag{22}$$

such that for every $x^{(k)}(0)$ and $w^{(k)}(0)$ with $k \in \{0, \lfloor\alpha\rfloor\}$, there holds

$w^{(k)}(0) = Px^{(k)}(0) \Rightarrow w(t) = Px(t), \quad \forall t \ge 0,$

where $P : \mathcal{X} \longrightarrow \mathcal{X}/\mathcal{S}$ is the canonical projection of $\mathcal{S}$.

Proof. (⇒) Suppose that $\mathcal{S}$ is a conditioned invariant subspace. Then, by Definition 5 there exist an output-injection matrix G such that $\mathcal{S}$ is A_G-invariant subspace. Let $P : \mathcal{X} \longrightarrow \mathcal{X}/\mathcal{S}$ be the canonical projection of $\mathcal{S}$. Let us define the matrices K and G as follows:

$$N := \langle A + GC \mid \mathcal{X}/\mathcal{S} \rangle \;\; and \;\; M := -PG,$$

Assume that $w^{(k)}(0) = P\,x^{(k)}(0)$ for each $k \in \{0, \lfloor\alpha\rfloor\}$. Since

$$\begin{aligned} {}^{C}D^{\alpha} P\,x(t) &= PA\,x(t) \\ &= PA\,x(t) + PGC\,x(t) - PGC\,x(t) \\ &= P(A + GC)\,x(t) - PG\,y(t), \\ &= N\,x(t) + M\,y(t), \end{aligned}$$

it follows that $P\,x(t) = w(t)$ for all $t \geq 0$.

(⇐) Let Σ^o ruled by (22) be a FO observer for the system (6) with $B = 0$. For any $x^{(k)}(0)$ with $k \in \{0, \lfloor\alpha\rfloor\}$, the setting $w^{(k)}(0) = P\,x^{(k)}(0) = P\,x_{0,k}$ yields ${}^{C}D^{\alpha}\; w(t)|_{t=0^+} = P\;{}^{C}D^{\alpha}\; x(t)|_{t=0^+}$ from which we get

$$\begin{aligned} {}^{C}D^{\alpha}\; w(t)|_{t=0^+} - P\,{}^{C}D^{\alpha}\; x(t)|_{t=0^+} &= 0 \\ N\,w^{(0)}(0) + MC\,x^{(0)}(0) - PA\,x^{(0)}(0) &= 0 \\ NP\,x^{(0)}(0) + MC\,x^{(0)}(0) - PA\,x^{(0)}(0) &= 0. \end{aligned}$$

Thus, we have that

$$PA = NP + MC.$$

This implies that for any $x(t) \in \mathcal{S} \cap \mathcal{C}$, equivalently, $Px = 0$ and $Cx = 0$, there holds that $PA\,x(t) = 0$ (equivalently, $Ax \in \mathcal{S}$). By virtue of Theorem 1 it follows that $\mathcal{S}$ is conditioned invariant subspace.

Remark 3. It is remarkable that if $B \neq 0$ the result in Theorem 2 still holds by adding $Ru(t)$, $R := PB$ to Eq. (22) as follows: ${}^{C}D^{\alpha}w(t) = Nw(t) + M\,y(t) + Ru(t)$, $\zeta(t) = w(t)$. In this case, if $\mathcal{S}$ is such that $\mathrm{im}(B) \subseteq \mathcal{S}$ then $R = 0$. That is, the FO quotient observer ruled by (22) does not require any knowledge of the input $u(t)$ to generates a perfect estimate up to time t.

The set of all the conditioned invariants containing a given subspace $\mathcal{L} \in \mathcal{X}$ is closed with respect to intersection and it has an infimal element that coincides with their intersection. It can be determined by the following $(C, A)-$invariant subspace algorithm:

$$\begin{cases} \mathcal{S}_0 = \mathcal{L} \\ \mathcal{S}_i = \mathcal{S}_{i-1} + A(\mathcal{S}_{i-1} \cap \mathcal{C}), \quad i \in \mathbb{N}^0 \backslash \{0\} \end{cases} \tag{23}$$

where $\mathcal{S}^* = \lim \mathcal{S}_i$ (as $i \to \infty$). This algorithm is strictly increasingly and will stop after a finite number of steps (at most in $n - 1$ steps). That is, the inclusion chain $\{\mathcal{S}_i\}$ has the form $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_i = \mathcal{S}_{i+1} = \cdots$ that implies $\mathcal{S}^\star = \mathcal{S}_i$ with $i \leq n-1$. This algorithm converges to $\mathcal{S}^*(\mathcal{L})$ as the minimal conditioned invariant subspace which contains $\mathcal{L}$. This infimal element, i.e. $\mathcal{S}^*$, is also referred to as the minimal input-containing conditioned invariant subspace of system (6) and represents the set of initial pseudo states for which there exist pseudo-state trajectories x and control u such that $y = 0$.

In the following, we present an analogous result to Padula et al. [32] for conditioned invariant subspace. We will provide a system-theoretic interpretation for condition invariant subspace $\mathcal{S}$ through output injection, in the case that system is initialized at a generic instant of time $t^\star \geq 0$, assuming that $x(t)$ is equal to zero for all $t < 0$.

Theorem 3. *Consider system (6) with $B = 0$. Let $\mathcal{S}$ be a $(C, A)-$invariant subspace. Given any $t^\star > 0$, $x^{(k)}(t^\star)$ belongs to $\mathcal{S}$ for $k \in \{0, \lfloor\alpha\rfloor\}$, then there exists a control law by output injection $Gy(t)$ such that $x(t) \in \mathcal{S}$ for all $t > t^\star$ if and only if $x(t) \in \mathcal{S}$ for almost every $t \in [0, t^\star]$.*

Proof. The proof is inspired by the corresponding points in [32, Theorem 5.5].

(⇐) Let $G \in \underline{G}(\mathcal{S})$. By applying the output-to-pseudostate feedback law $Gy(t)$ to system (6) and by the linearity of the Caputo derivative, we can write

$$\begin{aligned} {}^{C}D^{\alpha}_{t^\star}\,x(t) &= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_{t^\star}^{t} \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}} d\tau \\ &= (A + GC)x(t) + \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \underbrace{\int_{0}^{t^\star} \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}} d\tau}_{\mathfrak{T}(t)}, \end{aligned} \tag{24}$$

for all $t > t^\star$. Given that $x(t)$ belongs to $\mathcal{S}$ for every $t \leq t^\star$, applying Lemma 5.4 from [32] reveals that $x^{(\lceil\alpha\rceil)}(t)$ also belongs to $\mathcal{S}$ for almost every $t \leq t^\star$, thus implying that $\mathfrak{T}(t) \in \mathcal{S}$ for every $t \geq t^\star$. The relation (24) can be viewed as a set of $\alpha-$order differential equations with the fractional derivative defined at $t_0 = t^\star$ and $x^{(k)}(t_0) \in \mathcal{S}$ for $k \in \{0, \lfloor\alpha\rfloor\}$. Let take the system map A_G such that $A_G\mathcal{S} \subseteq \mathcal{S}$, the system input as $\mathfrak{T}(t)$, and the input map as $\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} I_n$. Since, $\mathfrak{T}(t) \in \mathcal{S}$, as per Lemma 3.4 in [32] concerning system (24), it can be deduced that $x(t)$ belongs $\mathcal{S}$ for all $t > t^\star$.

(⇒) From the linearity of the Caputo derivative we have

$$\begin{aligned} {}^{C}D^{\alpha}x(t) = &\frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \underbrace{\int_{0}^{t^\star} \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}} d\tau}_{\mathfrak{T}_1(t)} \\ &+ \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \underbrace{\int_{t^\star}^{t} \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}} d\tau}_{\mathfrak{T}_2(t)}. \end{aligned} \tag{25}$$

Let $G \in \underline{G}(\mathcal{S})$. By applying the output injection $Gy(t)$ to system (6) and using the property in (25) we have

$$\frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \mathfrak{T}_1(t) = (A + GC)x(t) - \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \mathfrak{T}_2(t). \tag{26}$$

Since, by assumption, $x(t) \in \mathcal{S}$ for all $t > t^\star$, it follows that $A_G x(t) \in \mathcal{S}$ for all $t > t^\star$. Thus, ${}^{C}D^{\alpha}x(t)$ must also lies on $\mathcal{S}$ so that the fractional differential Eq. (26) is satisfied. It follows that $\mathfrak{T}_2(t) \in \mathcal{S}$ for all $t > t^\star$. In order to satisfy the relation (26), we must have also $\mathfrak{T}_1(t) \in \mathcal{S}$ for $t \in [0, t^\star]$, from which, we have $x^{(\lceil\alpha\rceil)}(t) \in \mathcal{S}$ for $t \in [0, t^\star]$. Therefore, by using Lemma 5.4 from [32], it follows that $x(t) \in \mathcal{S}$ for $t \in [0, t^\star]$.

Remark 4. It is remarked that if $B \neq 0$ in (6), the result of Theorem 3 still remains valid if input $u(t)$ is such that $u(t) \in B^{-1}\mathcal{S}$.

Remark 5. It is noteworthy that the feature addressed by Theorem 3 has no analog in the integer-order case, where the system can always evolve on a conditioned invariant from a generic time $t^\star \geq 0$ onward, irrespective of what happens between $t_0 = 0$ and $t^\star \geq 0$.

From the results of Section 3.1, given an input-containing subspace $\mathcal{S}$ and a corresponding output-injection friend $G \in \underline{G}(\mathcal{S})$, we can define the corresponding detectably subspace as $\mathcal{N}_S := \langle \mathcal{S} + \mathcal{C} \mid A_G \rangle$. The eigenvalues of A_G can be decomposed into $\mathrm{spec}\,(A_G \mid \mathcal{S})$ and $\mathrm{spec}\,(A_G \mid \mathcal{X}/\mathcal{S})$. The eigenstructure of $\mathrm{spec}\,(A_G \mid \mathcal{S})$ relative to the mapping A_G restricted to $\mathcal{S}$ can be devided into the two parts: the eigenvalues in $\mathrm{spec}\,(A_G \mid \mathcal{S}/(\mathcal{S} \cap \mathcal{N}))$ which are freely assignable, i.e, depend on $G \in \underline{G}(\mathcal{S})$, and those eigenvalues in $\mathrm{spec}\,(A_G \mid \mathcal{S} \cap \mathcal{N})$ which are fixed for all the choices of G in $\underline{G}(\mathcal{S})$. Similarly, the eigenstructure $\mathrm{spec}\,(A_G \mid \mathcal{X}/\mathcal{N}_S)$ is arbitrarily assignable with a suitable choice of $G \in \underline{G}(\mathcal{S})$, while the eigenvalues in $\mathrm{spec}\,(A_G \mid \mathcal{N}_S/\mathcal{S})$ are fixed and independent of the choice of $G \in \underline{G}(\mathcal{S})$. Thus, by Lemma 2, $\mathcal{S}$ is said to be:

- *inner detectability subspace*, if there exist $G \in \underline{G}(\mathcal{S})$ satisfying $|\arg(\mathrm{spec}\,(A_G \mid \mathcal{S}))| > \frac{\alpha\pi}{2}$, or, equivalently, if $|\arg(\mathrm{spec}\,(A_G \mid \mathcal{S} \cap \mathcal{N}))| > \frac{\alpha\pi}{2}$ for all $\alpha \in (0, 2)$.
- *outer detectability subspace*, if there exist $G \in \underline{G}(\mathcal{S})$ satisfying $\left|\arg\left(\mathrm{spec}\,\left(A_G \mid \mathcal{X}/\mathcal{S}\right)\right)\right| > \frac{\alpha\pi}{2}$, or, equivalently, if $|\arg(\mathrm{spec}\,(A_G \mid \mathcal{N}_S/\mathcal{S}))| > \frac{\alpha\pi}{2}$ for all $\alpha \in (0, 2)$.

We adopt the terminology of 'detectability input-containing subspace' as an alternative designation for an input-containing subspace that satisfies the outer detectability condition. Analogous to the integer-order context, the set of detectability input-containing subspaces has an infimum, which we denote it by $\mathcal{S}^{\star}_{g,\alpha}$. An input-containing subspace $\mathcal{S}$ for which there exists $G \in \underline{G}(\mathcal{S})$ such that $\mathrm{spec}\,(A_G \mid \mathcal{X}/\mathcal{S})$ is freely assignable, is called an unobservability input-containing subspace. The set of unobservability input-containing subspaces is closed with respect to subspace intersection, and thus it contains an infimum denoted by $\mathcal{N}^{\star}$, which satisfies $\mathcal{S}^{\star} \subseteq \mathcal{S}^{\star}_{g,\alpha} \subseteq \mathcal{N}^{\star}$. There holds also that the subspace $\mathcal{N}^{\star}$ is also the unobservability input-containing subspace on $\mathcal{N}_{\mathcal{S}^{\star}}$ (i.e., $\mathcal{N}^{\star} = \mathcal{N}_{\mathcal{S}^{\star}}$. The invariant-zero structure $\mathcal{Z}_I$ of the system (6) is defined as follows:

$$\mathcal{Z}_I := \mathrm{spec}(A_G \mid \mathcal{N}^{\star}/\mathcal{S}^{\star}).$$

Thus, a fractional system is said to be minimum-phase if we have $|\arg(z)| > \alpha\pi/2$ for all $z \in \mathcal{Z}_I$ and $\alpha \in (0, 2)$.

Remark 6. Consider FO-LTI system (6). Let $\mathcal{S}^{\star}_{g,1}$ be the detectability input-containing subspaces of the corresponding integer-order LTI system. It is worth observing that $\mathcal{S}^{\star}_{g,\alpha} \supseteq \mathcal{S}^{\star}_{g,1}$ for $\alpha \in (0, 1)$, whereas $\mathcal{S}^{\star}_{g,\alpha} \subseteq \mathcal{S}^{\star}_{g,1}$ for $\alpha \in (1, 2)$.

4. Unknown-input fractional-order observer design

In this section, we apply the geometric concepts previously established to address UIOPS in the realm of FO systems. Solving this particular problem is crucial for dealing with numerous intricate synthesis problems, such as signal reconstruction, fault detection and isolation, as well as non-interacting control for FO systems. Consider a FO-LTI system Σ governed by the following differential equations:

$$\Sigma^d : \begin{cases} {}^C D^{\alpha} x(t) = A x(t) + E d(t), \\ y(t) = C x(t), \\ z(t) = H x(t), \end{cases} \tag{27}$$

where $d \in \mathcal{D} = \mathbb{R}^{v}$ is an unknown input which is not measurable and is not forced with any specific assumption and limitation (e.g., boundedness of d). The variable z is to be estimated output based on the information collected from the measured output y. Thus, we aim to design a FO-LTI system $\Sigma_{d,o}$ governed by:

$$\Sigma^{d,o} : \begin{cases} {}^C D^{\alpha} w(t) = N w(t) + M y(t), \\ \zeta(t) = L w(t) + K y(t). \end{cases} \tag{28}$$

where $w(t) \in \mathcal{W}$ is the state of the FO-LTI observer $\Sigma^{d,o}$. Let the overall system from the disturbance input d to the output $e := z - \zeta$ is denoted by Σ^e, as depicted in Fig. 1 (i.e., an extension of Σ^d). Then, the FO-LTI system Σ^e is given by

$$\begin{pmatrix} {}^C D^{\alpha} x(t) \\ {}^C D^{\alpha} w(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ MC & N \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} E \\ 0 \end{pmatrix} d(t),$$
$$e(t) = \begin{pmatrix} H - KC & -L \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}. \tag{29}$$

Let $\mathcal{X}^e := \mathcal{X} \oplus \mathcal{W}$ represents the extended pseudostate vector space. We can rewrite (29) in the following form:

$$\Sigma^e : \begin{cases} {}^C D^{\alpha} x^e(t) = A^e x^e(t) + E^e d(t), \\ e(t) = H^e x^e(t). \end{cases} \tag{30}$$

where $x^e := (x, w) \in \mathcal{X}^e$. The maps A^e, E^e, and H^e correspond to matrices given in (29). In this paper, we address the UIOPS by seeking a FO quotient observer $\Sigma^{d,o}$ governed by (28). The objective is for $e(t) := z(t) - \zeta(t)$ to converge to zero when $t \to \infty$ for all initial conditions $x_{0,k}$ with $k \in \{0, \lfloor\alpha\rfloor\}$ and for all disturbance inputs d (i.e., $e(t)$ is solely dependent on $x_{0,k}$ with $k \in \{0, \lfloor\alpha\rfloor\}$ and not on $d(t)$). This can be also viewed as the problem of finding an FO-LTI observer $\Sigma^{d,o}$ such that the fractional-order impulse-response matrix between d and e is equal to zero, i.e.,

$$g_{d,e}(t) := H^e t^{\alpha-1} E_{\alpha,\alpha}(A^e t^{\alpha}) E^e = 0,$$

and the overall FO-LTI system Σ^e is asymptotically stable, or equivalently, such that $|\arg(\mathrm{spec}\,(A^e))| > \alpha\pi/2$. Let $\mathcal{H} := \ker(H)$, $\mathcal{C} := \ker(C)$, and $\mathcal{E} := \mathrm{im}(E)$. The condition for solvability of the UIOPS in the fractional setting is provided by the following theorem.

Theorem 4. *The UIOPS admits solutions if and only if the following subspace inclusion holds:*

$$\mathcal{S}^{\star}_{g,\alpha}(\mathcal{E}) \cap \mathcal{C} \subseteq \mathcal{H}. \tag{31}$$

Proof. ($\Rightarrow$) Let $\underline{S}^e$ and $\underline{S}^e_{g,\alpha}$ denote the sets of all conditioned invariant and detectability subspaces associated with a given pair (H^e, A^e), respectively. Let $e(t) := z(t) - \zeta(t)$ and consider the dynamics of $x^e \in \mathcal{X}^e$ given by (30). Suppose that $g_{d,e}(t) = 0$ which is equivalent to $\langle A^e \mid \mathcal{E}^e \rangle \subseteq \langle \mathcal{H}^e \mid A^e \rangle =: \mathcal{S}^e$. Then, by Lemma 4, there exists an A^e-invariant subspace satisfying $\mathcal{E}^e \subseteq \mathcal{S}^e \subseteq \mathcal{H}^e$. Hence, any A^e-invariant subspace belongs to $\underline{S}^e$ such that $\mathcal{S}^e \in \underline{S}^e$ and $\mathcal{S}^e \cap \mathcal{X} =: \mathcal{S} \in \underline{S}$. It is observed that $\mathcal{E} = \mathcal{E}^e \cap \mathcal{X} \subseteq \mathcal{S} \subseteq \mathcal{H}^e = \ker(H - KC)$ which, in the view of Lemma 5.21 in [37], implies that $\mathcal{S} \cap \mathcal{C} \subseteq \mathcal{H}$. We observe that the spectrum of the dynamics e equals the $\mathrm{spec}\,(A^e \mid \mathcal{X}^e/\mathcal{S}^e)$. Hence, we have $\mathcal{S}^e \subseteq \underline{S}^e_{g,\alpha}$ and thus $\mathcal{S} \subseteq \underline{S}_{g,\alpha}$. Since $\mathcal{E} \subseteq \mathcal{S} \subset \mathcal{S}_{g,\alpha}$ it in turn implies $\mathcal{S}^{\star}_{g,\alpha} \subseteq \mathcal{S}_{g,\alpha}$. Thus, it is obvious that subspace inclusion $\mathcal{S} \cap \mathcal{C} \subseteq \mathcal{H}$ implies (31).

($\Leftarrow$) Let $P : \mathcal{X} \longrightarrow \mathcal{X}/\mathcal{S}$ be the canonical projection. Assume $\mathcal{S} \in \underline{S}_{g,\alpha}(\mathcal{E})$ with $\mathcal{S}^{\star}_{g,\alpha}(\mathcal{E}) \cap \mathcal{C} \subseteq \mathcal{H}$. By this last inclusion there exists mappings L, M such that $Hx(t) = LPx(t) + KCx(t)$. Define $N := A + GC \mid \mathcal{X}/\mathcal{S}$ and $M := -PG$. Let $G \in \underline{G}(\underline{S})$ and consider the FO-LTI observer ${}^C D^{\alpha} w(t) = N w(t) + M y(t)$, $\zeta(t) = L w(t) + K y(t)$, with $\mathcal{W} := \mathcal{X}/\mathcal{S}$. Define the auxiliary variable $e_{aux} := Px - w$, then we have

$$\begin{aligned} {}^C D^{\alpha} e_{aux}(t) &= P\, {}^C D^{\alpha} x(t) - P\, {}^C D^{\alpha} w(t) \\ &= PAx(t) + PEd(t) - Nw(t) - MCx(t) \\ &= P(A + GC)x(t) - Nw(t) \\ &= N e_{aux}(t). \end{aligned}$$

Hence, $e := z - \zeta$ is ruled by ${}^C D^{\alpha} e_{aux}(t) = N e_{aux}(t)$, $e(t) = L e_{aux}(t)$. It is clearly seen that $g_{d,e}(t) = 0$ (i.e., the dynamics $e(t)$ is unaffected by $d(t)$) and the error spectrum is stable for all initial conditions $x^{(k)}(0)$ and $w^{(k)}(0)$ with $k \in \{0, \lfloor\alpha\rfloor\}$.

Remark 7. It is worth noting that UIOP does not depend on the initial conditions, while it only established on the disturbance-to-error (input-to-output) properties of the extended fractional system Σ^e ruled by (30). Let $t^{\star} > t_0 \geq 0$ be a certain instant of time from which the behavior of the fractional system Σ^e is characterized. By using initialization function ψ proposed in [52]:

$$\psi(e, \alpha, 0, t^{\star}, t) \equiv \frac{1}{\Gamma(\alpha)} \int_0^{t^{\star}} (t - \tau)^{\alpha-1} e(\tau) d\tau, \quad t \geq t^{\star}, \tag{32}$$

which serves as a time-varying function incorporated into the nonlocal fractional-order operator to consider the effects of the system's past history, the overall time response of system Σ^e is represented by:

$$\begin{aligned} e(t) = &\int_0^t H^e \tau^{\alpha-1} E_{\alpha,\alpha}(A^e \tau^{\alpha}) E^e d(t - \tau) d\tau \\ &- \int_0^t H^e \tau^{\alpha-1} E_{\alpha,\alpha}(A^e \tau^{\alpha}) \psi(t - \tau) d\tau, \end{aligned} \tag{33}$$

where the second integral term represents the initialization response of the system Σ^e due to the historical past of $e(t)$ between $t_0 = 0$ and $t^{\star}$. It is observed that the effect of disturbance $d(t)$ on output $e(t)$ does not depend on the initialization function ψ. As a result, the solvability condition Theorem 3 remains valid for the fractional differential equation of the form (27) irrespective of which definition for fractional derivative is adopted.

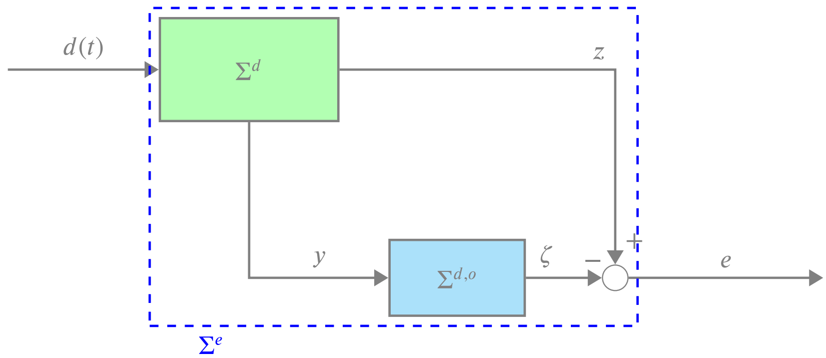

Fig. 1. Schematic diagram of unknown-input pseudo-state fractional observer.

It is worth observing that the condition in Theorem 4 still remains valid for the solvability of UIOP without stability by replacing $\mathcal{S}^\star_{g,\alpha}(\mathcal{E})$ with $\mathcal{S}^\star(\mathcal{E})$ in (31) without the detectability requirement of the pair (H^e, A^e).

5. Numerical examples

Example 1. Let consider the FO-LTI system governed by (27) with the following maps, pseudo-state space $\mathcal{X} = \mathbb{R}^3$, and $\alpha = \frac{1}{3}$:

$$A = \begin{pmatrix} -0.6667 & 1.0000 & 0.6667 \\ -3.0000 & 1.0000 & 2.0000 \\ 1.3333 & 1.0000 & -1.3333 \end{pmatrix},\ E = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$
$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By using the algorithm given in (23), we can find out that

$$\mathcal{S}^\star = \mathcal{S}^\star_{g,\alpha} = \mathcal{N}^\star = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Given that H is the identity matrix, a full-order FO observer is required. It can be easily observed that $\mathcal{S}^\star_{g,\alpha} \cap \mathcal{C} \subseteq \mathcal{H}$ when $\alpha = \frac{1}{3}$. As a result, the condition stated in Theorem 4 is met, and the UIOPS is solvable. Therefore, our goal is to find $G \in \underline{G}(\mathcal{S}^\star_{g,\alpha})$ (i.e., a friend G of $\mathcal{S}^\star_{g,\alpha}$) that ensures the asymptotic stability of the FO dynamics on $\mathcal{X}/\mathcal{S}^\star_{g,\alpha}$. The freely assignable eigenvalues of A_G are chosen to be $\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ with $\alpha = \frac{1}{3}$. By following the procedure described in [26], we obtain the following matrices for the FO quotient observer that solves UIOPS:

$$N = \begin{pmatrix} 10.83 & 13.16 \\ -8.165 & -9.831 \end{pmatrix}, \quad M = \begin{pmatrix} 12.5 & -1 \\ -8.498 & -1 \end{pmatrix},$$
$$L = \begin{pmatrix} -0.6667 & 0.3333 \\ 0.6667 & -0.3333 \\ 0.3333 & -0.6667 \end{pmatrix},\ K = \begin{pmatrix} 0.3333 & -1.11e{-16} \\ -0.3333 & 1 \\ 0.3333 & -8.327e{-17} \end{pmatrix}.$$

The disturbance signal $d(t)$ is represented by a random process, as revealed in Fig. 2 (a). Fig. 2 (b) demonstrates the effectiveness of the designed quotient observer in decoupling the disturbance signal $d(t)$ from the error signal $e(t)$ in the case where Σ^d is initialized at $x^{(0)}(0) = (0, 0, 0.1)^{\mathrm{T}}$ and $\Sigma^{d,o}$ is initialized at $w^{(0)}(0) = (0, 0)^{\mathrm{T}}$. Furthermore, the overall system Σ^e governed by Eq. (30) is indeed asymptotically stabilized.

In Fig. 3, the error signals are depicted in the same configuration as in the previous case. Here, $\alpha = \frac{2}{3}$ leads to two free eigenvalues placed on the boundary of stability region. Conversely, when α is equal to 1, the system Σ^e transforms into an integer-order system that is non-stabilizing. It is noteworthy that when $\alpha = \frac{2}{3}$ and $\alpha = 1$, the structural condition specified in Theorem 4 is still satisfied. Nevertheless, the overall system Σ^e governed by Eq. (30) is not asymptotically stabilized.

Table 1
Suspension model parameters [53].

Parameter description	Value
Sprung mass, m_s	250 kg
Upsprung mass, m_{us}	43.5 kg
Stiffness of car body spring, k_s	249 830 N/m
Damping of car body, b_s	1607 N s/m
Tire stiffness, k_t	191.86
Tire damping, b_t	112.2 N s/m

It should be noted that the solvability of UIOPS in the presence of stable error dynamics does not necessarily mean that any friend of $\mathcal{S}^\star_{g,1}$ and $\mathcal{S}^\star_{g,\frac{2}{3}}$ can be used. However, if stability is not required then every friend of $\mathcal{S}^\star$ solves the UIO problem.

Example 2 (*Application to the Suspension System*)**.** In this example, we frame the task of estimating the unknown road profile and the pseudo-states of a suspension model as an observation problem with unknown inputs and then solve it within the proposed geometric framework. The suspension FO model being examined has with two degrees of freedom and involves a sprung mass that represents the body frame of the vehicle, and an unsprung mass that embodies the wheel and its corresponding components. A spring and a damper are used to establish the connection between these masses. Similarly, the tire, which exhibits viscoelastic characteristics, is also represented by an equivalent spring and damper. The model can be described as an FO-LTI system governed by (27) with the following maps and $\alpha = \frac{1}{2}$ [53]:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k_s}{m_s} & \frac{k_s}{m_s} & -\frac{b_s}{m_s} & \frac{b_s}{m_s} & 0 \\ \frac{k_s}{m_{us}} & -\frac{k_s+k_t}{m_{us}} & \frac{b_s}{m_{us}} & -\frac{b_s+b_t}{m_{us}} & \frac{k_t}{m_{us}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{b_s}{m_{us}} \\ 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and $H = I_5$, where $x = \begin{pmatrix} x_s & x_{us} & {}^{C}D^{\alpha}x_s & {}^{C}D^{\alpha}x_{us} & x_r \end{pmatrix}^{\mathrm{T}}$ is the pseudo-state vector. We consider ${}^{C}D^{\alpha}x_r$ to be an unknown input that is inaccessible for measurement. In order to account for the system's starting position at rest, we establish the initial velocities and displacements as zero. Table 1 lists the vehicle dynamics parameters employed in the simulations.

In order to achieve more realistic riding scenarios, we incorporate two distinct categories of road irregularities: deterministic and random. The deterministic irregular excitation is simulated by implementing two

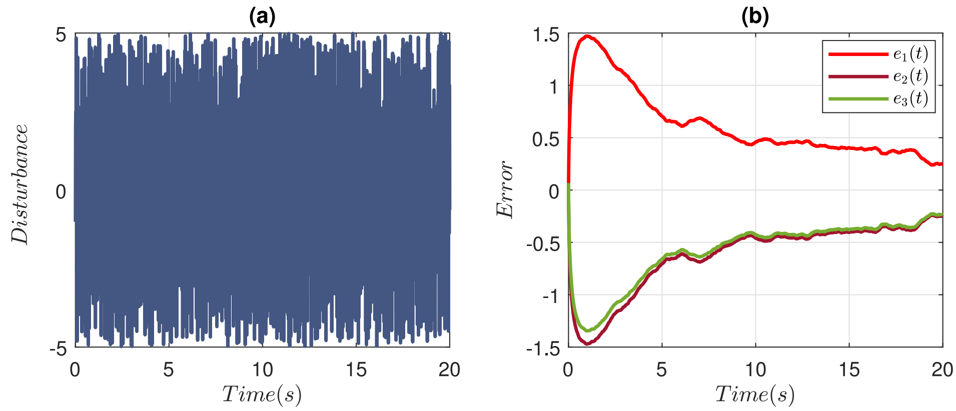

Fig. 2. Unknown-input FO observer performance for $\alpha = \frac{1}{3}$: (a) unknown disturbance $d(t)$ and (b) error $e(t)$.

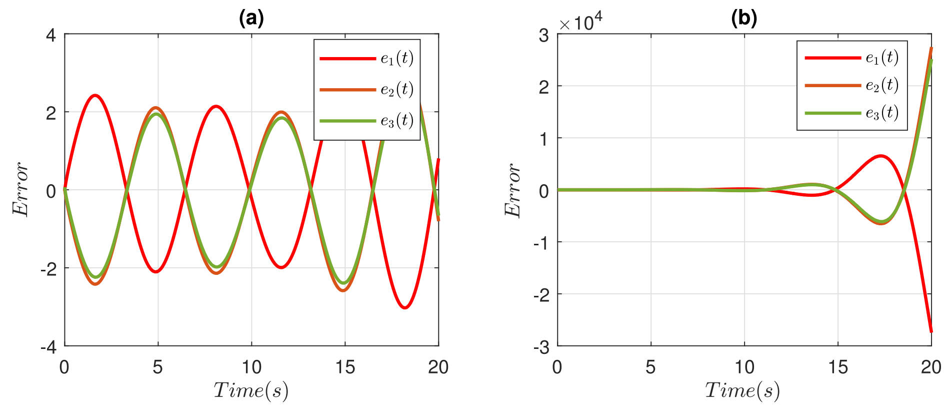

Fig. 3. Unknown-input FO observer performance for (a) $\alpha = \frac{2}{3}$ and (b) $\alpha = 1$.

consecutive bumps on the road surface as follows:

$$x_r(t) := \begin{cases} h\left(1-\cos\left(\frac{2\pi v}{l}t\right)\right) & t_{b_1} \le t \le t_{b_1} + \frac{1}{v}, \\ \frac{h}{2}\left(1-\cos\left(\frac{2\pi v}{l}t\right)\right) & t_{b_2} \le t \le t_{b_2} + \frac{1}{v}, \\ 0 & \text{Otherwise}, \end{cases} \tag{34}$$

where l and h are the length and height of the bump, t_{b_1} and t_{b_2} denote the start times of the first and second bump over a total travel time t. In addition, when conducting robustness evaluation, we take into consideration the random road irregularities profile attributed to class D, commonly known as poor road quality. Assuming a constant vehicle speed v, the integral white noise random pavement is represented by the model described in [54]:

$$x_r(t) := \sqrt{k}\int_0^t w(t)dt, \tag{35}$$

where k denotes the spectral density constant, which is defined as in [54], and $w(t)$ stands for the white noise. It is easy to see that

$$\mathcal{S}^\star = \mathcal{S}^\star_{g,\alpha} = \mathcal{N}^\star = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

By using the GA algorithm [26], the following matrices are obtained for the full-order FO unknown-input observer ruled by (30) which geometrically solves the formulated UIOPS (see Figs. 4 and 5):

$$N = \begin{pmatrix} -11.57 & 6.379e-13 & 1 & -5.532e-16 \\ -0.1921 & 9.271e-14 & -6.004e-16 & -1.27e-16 \\ -29.62 & 999.3 & -6.428 & -3.818e-14 \\ -77.7 & 155.5 & -0.9996 & -0.1194 \end{pmatrix},$$

$$M = \begin{pmatrix} 11.57 & 3.074e-15 \\ 0.1921 & 1 \\ -969.7 & 6.428 \\ -77.7 & 1.066 \end{pmatrix}, \quad K = \begin{pmatrix} 0.5 & 0 \\ -5.553e-17 & 1.17e-18 \\ 2.32e-19 & -5.776e-32 \\ 3.006e-18 & 0.02707 \\ 5.553e-17 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} 0.5 & -2.138e-50 & -2.106e-34 & 5.585e-33 \\ 5.553e-17 & 1 & -7.613e-17 & -5.622e-17 \\ -2.32e-19 & 5.738e-17 & 1 & 4.689e-16 \\ -5.553e-17 & 7.225e-19 & -4.154e-18 & -1.049e-16 \\ -3.006e-18 & 1.363e-16 & -2.534e-16 & 1 \end{pmatrix}.$$

6. Conclusion

The paper provides a detailed characterization of (C, A)−invariant subspaces in fractional-order setting. It has been stressed that the conditioned invariance of a subspace hinges on the existence of a FO quotient observer, which is responsible for rendering an estimate of the pseudostate modulo the specified subspace. Moreover, the unknown-input pseudo-state observation problem with stability has been addressed within the context of fractional systems and solved by using the developed geometric notions. Moving forward, the research aims to tackle the fault detection and identification problem by resorting these geometric tools. This will involve the development of geometric residual filters in the fractional-order setting.

CRediT authorship contribution statement

Hasan Abbasi Nozari: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Seyed Jalil Sadati Rostami:**

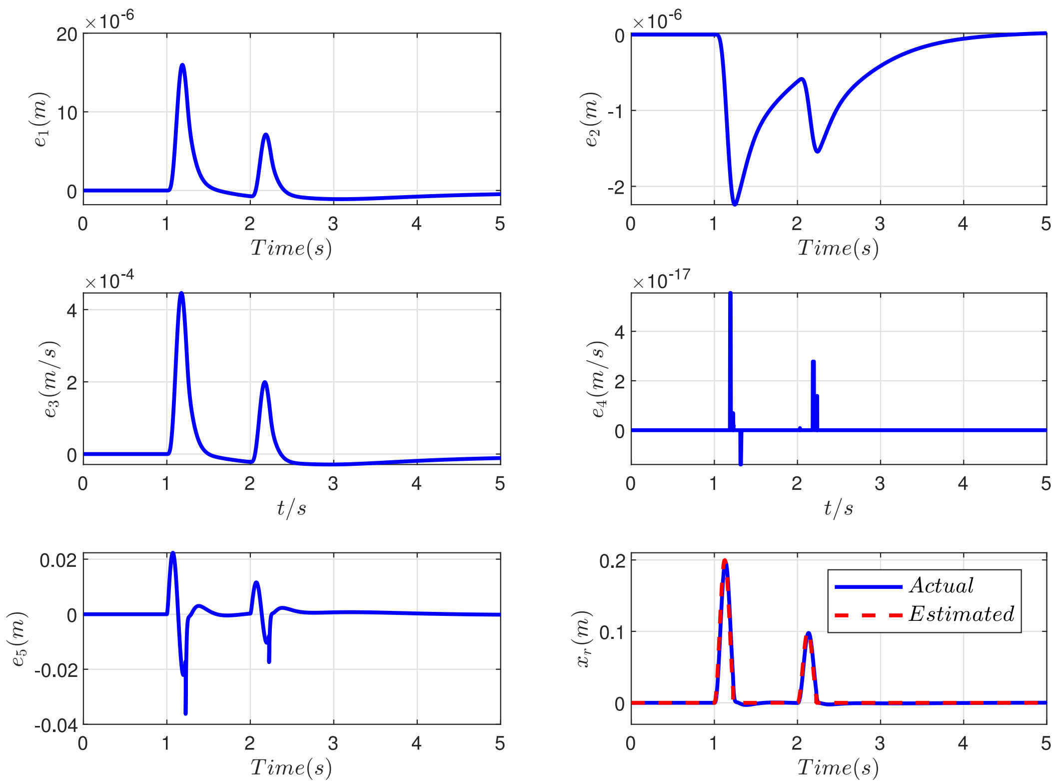

Fig. 4. Unknown-input FO observer performance for deterministic road profile.

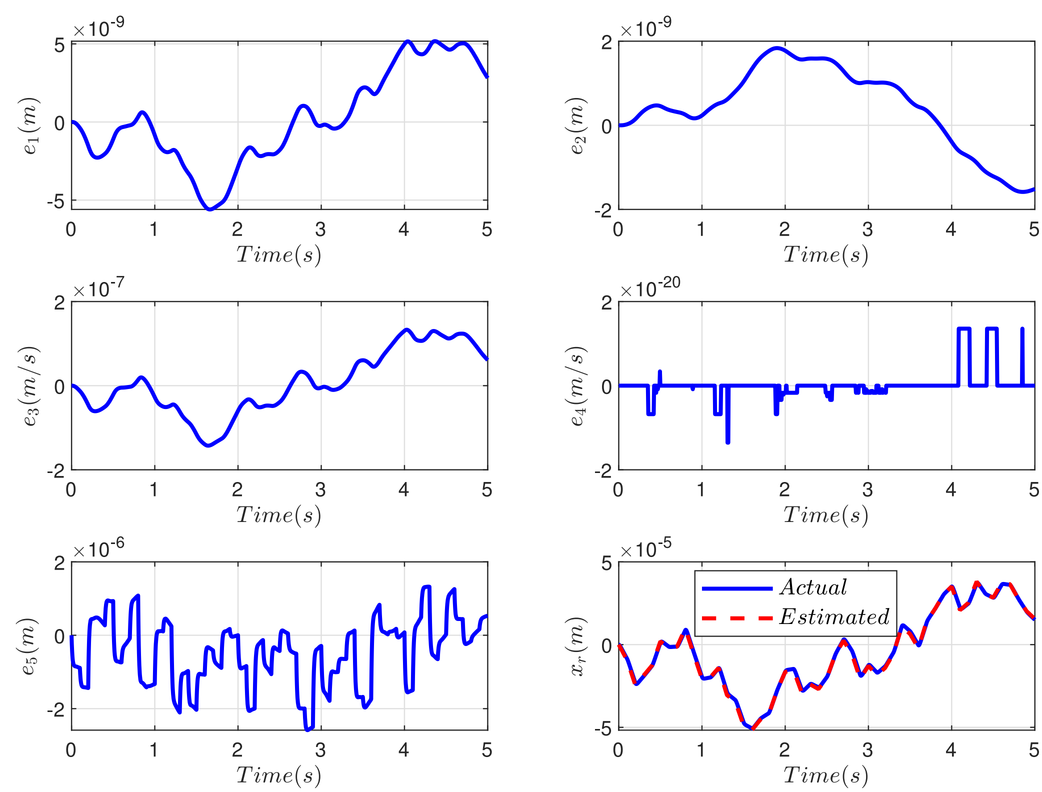

Fig. 5. Unknown-input FO observer performance for random road profile.

Writing – review & editing, Supervision, Resources, Project administration, Methodology, Conceptualization. **Paolo Castaldi:** Writing – review & editing, Supervision, Project administration, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix. Proof of Lemma 1

Proof. It should be emphasized that the upcoming lemma will be necessary to establish the proof of Lemma 1.

Lemma A.1 (*Wei [55]*). *For the Beta function*

$$B(z,w)=\int_0^1 r^{z-1}(1-r)^{w-1}dr,\quad (\mathrm{Re}(z)>0,\ \mathrm{Re}(w)>0),$$

we have

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

Now we continue with the proof of Lemma 1. For $\alpha\in(0,1)$, the proof is given by Wei [55]. Consider now $\alpha\in(1,2)$. We have that

$$\begin{aligned}
{}^{C}D^{\alpha}\psi_{\alpha,\beta}(\mu,t)&=\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{\psi^{(2)}_{\alpha,\beta}(\mu,\tau)}{(t-\tau)^{\alpha-1}}d\tau\\
&=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}\int_0^t\frac{\mu^k\tau^{\alpha k+\beta-3}(\alpha k+\beta-1)(\alpha k+\beta-2)}{(t-\tau)^{\alpha-1}\Gamma(\alpha k+\beta)}d\tau\\
&=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}\int_0^t\frac{\mu^k\tau^{\alpha k+\beta-3}}{(t-\tau)^{\alpha-1}\Gamma(\alpha k+\beta-2)}d\tau\\
&=\frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{\infty}\int_0^t\frac{\mu^{k+1}\tau^{\alpha(k+1)+\beta-3}}{(t-\tau)^{\alpha-1}\Gamma(\alpha(k+1)+\beta-2)}d\tau\\
&=\sum_{k=1}^{\infty}\int_0^1\frac{\mu^{k+1}r^{\alpha(k+1)+\beta-3}t^{\alpha(k+1)+\beta-3}}{\Gamma(2-\alpha)t^{\alpha-1}(1-r)^{\alpha-1}\Gamma(\alpha(k+1)+\beta-2)}t\,dr\\
&=\sum_{k=1}^{\infty}\frac{\mu^{k+1}t^{\alpha k+\beta-1}}{\Gamma(2-\alpha)\Gamma(\alpha(k+1)+\beta-2)}\\
&\quad\times\int_0^1 r^{\alpha(k+1)+\beta-3}(1-r)^{(-\alpha+1)+1-1}dr
\end{aligned}\tag{36}$$

By applying Lemma A.1 to the integral term in (36), we get that

$$\begin{aligned}
{}^{C}D^{\alpha}\psi_{\alpha,\beta}(\mu,t)&=\sum_{k=1}^{\infty}\frac{\mu^{k+1}t^{\alpha k+\beta-1}}{\Gamma(2-\alpha)\Gamma(\alpha(k+1)+\beta-2)}B(\alpha(k+1)+\beta-2,2-\alpha)\\
&=\mu\sum_{k=0}^{\infty}\frac{\mu^k t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}\\
&=\mu\,\psi_{\alpha,\beta}(\mu,t).\quad\square
\end{aligned}$$

References

[1] Sabatier J, Agrawal OP, Machado JT. Advances in fractional calculus. Vol. 4, Springer; 2007.
[2] Balci E, Ozturk I, Kartal S. Dynamical behavior of fractional order tumor model with caputo and conformable fractional derivative. Chaos Solitons Fractals 2019;123:43–51.
[3] Ibrahim R, Altulea D, Elobaid R. Dynamical system of the growth of covid-19 with controller. Adv Differ Equ 2021;9. http://dx.doi.org/10.1186/s13662-020-03168-w.
[4] Kumar M, Ghosh S, Das S. Analytical formulation for power, energy, and efficiency measurement of ultracapacitor using fractional calculus. IEEE Trans Instrum Meas 2019;68:4834–44.
[5] Wang Z, Huang X, Shen H. Control of an uncertain fractional order economic system via adaptive sliding mode. Neurocomputing 2012;83:83–8.
[6] Sheng D, Wei Yand, Cheng S, Shuai J. Adaptive backstepping control for fractional order systems with input saturation. J Franklin Inst 2017;354:2245–68.
[7] Podlubny I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. San Diego, CA: Academic Press; 1999.
[8] Padder A, Almutairi L, Qureshi S, Soomro A, Afroz A, Hincal E, Tassaddiq A. Dynamical analysis of generalized tumor model with caputo fractional-order derivative. Fractal Fract 2023;7(258).
[9] Amilo D, Sadri K, Kaymakamzade B, Hincal E. A mathematical model with fractional-order dynamics for the combined treatment of metastatic colorectal cancer. Commun Nonlinear Sci Numer Simul 2024;130:107756.
[10] Ahmad S, Ullah A, Al-Mdallal QM, Khan H, Shah K, Khan A. Fractional order mathematical modeling of covid-19 transmission. Chaos Solitons Fractals 2020;139:110256.
[11] Gómez-Aguilar J, Rosales-García J, Bernal-Alvarado J, Córdova-Fraga T, Guzmán-Cabrera R. Fractional mechanical oscillators. Rev Mex Fís 2012;58:348–52.
[12] Maity S, Saha M, Saha P, Khanra M. Fractional calculus-based modeling and state-of-charge estimation of supercapacitor. J Energy Storage 2024;81:110317.
[13] You H, Shen Y, Xing H, Yang S. Optimal control and parameters design for the fractional-order vehicle suspension system. J Low Freq Noise Vib Act Control 2018;37:456–67.
[14] Tuwa PRN, Molla T, Noubissie S, Kingni ST, Rajagopal K. Analysis of a quarter car suspension based on a kelvin–voigt viscoelastic model with fractional-order derivative. Int J Non-Linear Mech 2021;137:103818.
[15] Ortigueira MD. Fractional calculus for scientists and engineers. Vol. 84, Springer Science & Business Media; 2011.
[16] Qiao Z, Elhattab A, Shu X, He C. A second-order stochastic resonance method enhanced by fractional-order derivative for mechanical fault detection. Nonlinear Dynam 2021;106:707–23.
[17] Ji Y, Guo Y, Liu Y, Tian Y. Leader-following consensus of fractional-order multi-agent systems via adaptive control. Internat J Adapt Control Signal Process 2020;34:283–97.
[18] Liu B, Su H, Wu L, Li X, Lu X. Fractional-order controllability of multi-agent systems with time-delay. Neurocomputing 2021;424:268–77.
[19] Sabatier J, Lanusse P, Melchior P, Oustaloup A. Fractional order differentiation and robust control design. Intell Syst Control Autom: Sci Eng 2015;77:13–8.
[20] Li Y, Chen Y, Podlubny I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized mittag–leffler stability. Comput Math Appl 2010;59:1810–21.
[21] Kaczorek T. Linearization of fractional nonlinear systems by state-feedbacks. In: Szewczyk R, Zieliński C, Kaliczyńska M, editors. Advances in intelligent systems and computing. Berlin: Springer; 2020, Automation 2020: Towards Industry of the Future Automation.
[22] Liu H, Pan Y, Li S, Chen Y. Adaptive fuzzy backstepping control of fractional-order nonlinear systems. IEEE Trans Syst Man Cybern: Syst 2017;47:2209–17.
[23] Wonham WM, Morse AS. Decoupling and pole assignment in linear multivariable systems: a geometric approach. SIAM J Control 1970;8:1–18.
[24] Basile G, Marro G. Controlled and conditioned invariant subspaces in linear system theory. J Optim Theory Appl 1969;3:306–15.
[25] Ferrante A, Marro G, Ntogramatzidis L. A parametrization of the solutions of the finite-horizon lq problem with general cost and boundary conditions. Automatica 2005;41:1359–66.
[26] Marro G, Zattoni E. Self-bounded controlled invariant subspaces in model following by output feedback: Minimal-order solution for nonminimum-phase systems. J Optim Theory Appl 2005;125:409–29.
[27] Isidori A, Krener A, Gori-Giorgi C, Monaco S. Nonlinear decoupling via feedback: a differential geometric approach. IEEE Trans Autom Control 1981;26:331–45.
[28] Perdon AM, Zattoni E, Conte G. Disturbance decoupling with stability for linear impulsive systems. IFAC-PapersOnLine 49.
[29] Otsuka N. Disturbance decoupling with quadratic stability for switched linear systems. Systems Control Lett 2010;59:349–52.
[30] Stikkel G, Bokor J, Szabo Z. Disturbance decoupling problem with stability for lpv systems. In: 2003 European control conference. ECC, IEEE; 2003, p. 558–63.
[31] Ntogramatzidis L. Structural invariants of two-dimensional systems. SIAM J Control Optim 2012;50:334–56.
[32] Padula F, Ntogramatzidis L, Schmid R, Loxton R. Geometric control and disturbance decoupling for fractional systems. SIAM J Control Optim 2020;58:1403–28.
[33] N'Doye I, Darouach M, Voos H, Zasadzinski M. Design of unknown input fractional-order observers for fractional-order systems. Int J Appl Math Comput Sci 2013;23:491–500.
[34] Li H, Li J, Fan QY. Fault detection of fractional-order system based on finite frequency unknown input observer. IET Control Theory Appl 2024;18:640–8.

[35] Komachali FPourdadashi, Shafiee M. Sensor fault diagnosis in fractional-order singular systems using unknown input observer. Int J Syst Sci 2020;51:116–32.
[36] Djeghali N, Djennoune S, Bettayeb M, Ghanes M, Barbot JP. Observation and sliding mode observer for nonlinear fractional-order system with unknown input. ISA Trans 2016;63:1–10.
[37] Trentelman HL, Stoorvogel AA, Hautus M. Control theory for linear systems. Springer Science & Business Media; 2012.
[38] Conte G, Perdon AM, Zattoni E. A structural approach to unknown inputs observation for switching linear systems. Automatica 2021;129:109572.
[39] Ntogramatzidis L, Cantoni M. Conditioned invariance and unknown-input observation for two-dimensional fornasini-marchesini models. In: 2007 European control conference. ECC, IEEE; 2007, p. 3891–6.
[40] Lawrence DA. Conditioned invariant subspaces for linear impulsive systems. In: 2015 American control conference. ACC, IEEE; 2015, p. 4850–5.
[41] Bokor J, Balas G. Detection filter design for lpv systems—a geometric approach. Automatica 2004;40:511–8.
[42] Conte G, Perdon A, Zattoni E. Unknown input observers for hybrid linear systems with state jumps. IFAC-PapersOnLine 2017;50:6458–64.
[43] Lewis FL. Geometric design techniques for observers in singular systems. Automatica 1990;26:411–5.
[44] Kilbas A. Theory and applications of fractional differential equations.
[45] Haubold HJ, Mathai AM, Saxena RK. Mittag-leffler functions and their applications. J Appl Math 2011;2011.
[46] Sabatier J, Farges C, Trigeassou J. Fractional systems state space description: some wrong ideas and proposed solutions. J Vib Control 2014;339.
[47] Sabatier J, Merveillaut M, Malt R, Oustaloup A. How to impose physically coherent initial conditions to a fractional system. Commun Nonlinear Sci Numer Simul 2010;15:1318–26.
[48] Monje C. Fractional-order systems and controls: fundamentals and applications.
[49] Matignon D. Stability results for fractional differential equations with applications to control processing. In: Computational engineering in systems applications. Citeseer; 1996, p. 963–8.
[50] Sabatier J, Moze M, Farges C. On stability of fractional order systems. In: Third IFAC workshop on fractional differentiation and its applications fDA'. Vol. 08, 2008, p. xxx.
[51] Basile G, Marro G. Controlled and conditioned invariants in linear system theory. Prentice Hall Englewood Cliffs; 1992.
[52] Hartley TT, Lorenzo CF. Dynamics and control of initialized fractional-order systems. Nonlinear Dynam 2002;29:201–33.
[53] Nodeh TF, Mirzaei M, Khosrowjerdi MJ. Simultaneous output selection and observer design for vehicle suspension system with unknown road profile. IEEE Trans Veh Technol 2021;70:4203–11.
[54] Wang J, Qiang B. Road simulation for four-wheel vehicle whole input power spectral density. In: AIP conference proceedings. AIP Publishing LLC; 2017, 020147.
[55] Wei J. The controllability of fractional control systems with control delay. Comput Math Appl 2012;64:3153–9.