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On the implied volatility skew outside the at-the-money point

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The small-maturity implied volatility of an asset pricing model is fully determined by the asymptotics of traded option prices, and thus model-free expressions are available. We show how by sharpening one such expression it is possible to derive a novel general formula for the leading order of the in-the-money and out-of the money (OTM/ITM) implied volatility skew. We apply this formula to find expressions of the small maturity limiting skew of the Heston stochastic volatility model, of exponential Lévy models and their time changes, as well as that of some recently proposed pricing models with independent log-returns.

Keywords: Implied volatility, out-of-the-money implied volatility skew

JEL Classification: G12, C69

1. Introduction

Market practitioners very often quote option prices in “implied volatilities”, that is the units of normal standard deviations in the Black-Scholes pricing formula needed to obtain the traded option price. It is observed that these values are not constant but vary across strikes and maturities, typically exhibiting for some fixed maturity a U or V shape (“the smile”) whose arms are directed towards the deep in and out-of-the-money (ITM/OTM) strikes, with the left arm typically steeper than the right one, and a minimum located around the at-the-money (ATM) point. Any theoretically consistent option pricing model to be useful in practice should reproduce this stylized fact. Of particular importance for risk-management and trading purposes is being able to reproduce the ATM skew time rate, that is the rate, as maturity tends to zero, of the steepness of the implied volatility in the ATM point. The asymptotic analysis of the implied volatility serves then a twofold purpose. For model selection, the quality of mathematical models can be assessed, among the other factors, based on their ability to reproduce the empirically observed market skew time rate at small maturities. Concerning implementation and practical industrial uses, asymptotic expansions can be used as calibration instruments, since standard numerical methods are typically unstable at short maturities.

Early research was focused on stochastic volatility and jump diffusion models, with particular regard to the Heston (1993) model. See for example Medvedev and Scaillet (2007), Alos et al. (2006), Forde and Jacquier (2009) and Forde et al. (2012). Implied volatility at small maturities for exponential Lévy models were later considered in Roper (2008), Tankov (2011), Figueroa-López and Forde (2012), Gerhold et al. (2016), and Figueroa-López and Olafsson (2016) who also add

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a stochastic volatility component. A detailed analysis of the Lévy framework is also presented Andersen and Lipton (2013), while an alternative view based on time-rescaling of maturities is proposed by Mijatović and Tankov (2016). Model-free properties and bounds are presented in Lee (2005), whereas Gao and Lee (2014) derive model-free implied volatility asymptotics to arbitrary order in various combination of strike/maturity parameters based on saddle point approximations of option prices. Results on the volatility surface asymptotics have been also made available for the now popular strain of research of "rough volatility", which uses asset dynamics with stochastic volatility driven by a fractional Brownian motion with Hurst exponent smaller than 1/2. See for instance Guennoun et al. (2018), Fukasawa (2017), Bayer et al. (2019), Forde and Zhang (2017).

A model-free general formula for the small maturity implied volatility is found in Roper and Rutkowski (2009), one to which, one way or another, all the model-specific literature should reconcile with. Such formula clarifies that the small maturity limit of the implied volatility level, whatever the moneyness regime, only depends on vanilla option prices at short expiries. The derivation is based on the first order expansion of a certain inverse function. As we shall show in this paper, simply considering the second order in such expansion has a profound impact on the formulae. Not only it sharpens, with very little added complexity, the implied volatility asymptotics in Roper and Rutkowski (2009) but also breaks the ground for asymptotic formulae of the skew outside the ATM point, which as far as the authors are aware have not been systematically studied before. As we shall see, in order to obtain the skew approximation, the first order in the implied volatility expansion is insufficient.

2. A refinement

Let us introduce some basic notation. On a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ admitting an equivalente pricing measure $\mathbb{Q}$, so that, in particular, the market admits no arbitrage. We assume that the market risk-free interest rate is zero; this does not lead to loss of generality upon replacing spot prices with forward prices throughout. Let $S = (S_t)_{t\geq 0}$, $S_0 > 0$, be a martingale asset pricing model, and assume that S_t is absolutely continuous. Denote by $C(S_0, K, \tau) = \mathbb{E}[(S_\tau - K)^+]$ the value of call option struck at $K > 0$, expirying at $\tau > 0$ when the underlying spot price is S_0. Then

$$DC(S_0, K, \tau) = \mathbb{Q}(S_\tau \geq K) = -\frac{\partial C}{\partial K}(S_0, K, \tau) \tag{2.1}$$

is the digital call option price of respective parameters. Correspondingly, we denote $P(S_0, K, \tau) = \mathbb{E}[(K - S_\tau)^+]$ the put option value and $DP(S_0, K, \tau) = \mathbb{Q}(S_\tau \leq K) = \frac{\partial P}{\partial K}(S_0, K, \tau)$ the digital put price.

Throughout we indicate with $C_{BS}(S_0, K, \tau, \sigma)$ the Black-Scholes call price; the Black-Scholes implied volatility $\sigma_{BS}(S_0, K, \tau)$ is defined as the unique function such that

$$C_{BS}(S_0, K, \tau, \sigma_{BS}(S_0, K, \tau)) = C(S_0, K, \tau). \tag{2.2}$$

The existence of such function is guaranteed because C is increasing as a function of σ and $C_{BS}(S_0, K, \tau, 0) = (S_0 - K)^+$, $\lim_{\sigma\to\infty} C_{BS}(S_0, K, \tau, \sigma) = S_0$.

Further, let $v(S_0, K, \tau) = \sigma_{BS}(S_0, K, \tau)^2\tau$ be the Black-Scholes total implied variance. In Roper and Rutkowski (2009), Theorem 5.1, the authors find the model free approximation of the implied volatility for $K \neq S_0$. They establish the following asymptotic leading order

$$v(S_0, K, \tau) = \frac{\log(K/S_0)^2}{-2\log O(S_0, K, \tau)} + o(1), \qquad \tau \sim 0, \tag{2.3}$$

where $O(S_0, K, \tau) = C(S_0, K, \tau) - (S_0 - K)^+$ is the value of the vanilla option starting OTM (call if $K > S_0$, put if $K < S_0$).

In the next proposition we strengthen the above OTM/ITM asymptotics.

Proposition 1 *Let S be a martingale underlying asset price process, whose marginal laws are absolutely continuous. For $S_0 \neq K$ we have, as when $\tau \sim 0$, the following asymptotics*

$$v(S_0, K, \tau) = \frac{\log(K/S_0)^2}{-2\Big(\log\left(c_{K,S_0} O(S_0, K, \tau)\right) + \frac{3}{2}\log\left(-\log(c_{K,S_0} O(S_0, K, \tau))\right)\Big)} + o(1) \tag{2.4}$$

or, equivalently,

$$v(S_0, K, \tau) = \frac{\log(K/S_0)^2}{-2\log O(S_0, K, \tau)}\left[1 - \frac{\log\left(c_{K,S_0}(-\log\left(c_{K,S_0} O(S_0, K, \tau)\right))^{3/2}\right)}{\log O(S_0, K, \tau)} + o\left(\frac{1}{\log O(S_0, K, \tau)}\right)\right]. \tag{2.5}$$

Proof. Since S is a martingale, assumptions (2.1) to (2.3) in Roper and Rutkowski (2009) are automatically satisfied (see Tehranchi 2020, Theorem 3.2). Under such assumptions, the authors in the proof of Theorem 5.1 find that

$$F(v(S_0, K, \tau)) = \Phi_{3/2}\left(\frac{1}{G(O(S_0, K, \tau))(1 + o(1))}\right), \qquad \tau \sim 0 \tag{2.6}$$

where F and G are given by

$$F(x) = \frac{\log(K/S_0)^2}{2x}, \qquad G(x) = c_{K,S_0} x, \qquad x > 0 \tag{2.7}$$

with

$$c_{K,S_0} = \frac{4\sqrt{\pi}}{|\log(K/S_0)|\sqrt{KS_0}}. \tag{2.8}$$

and the function Φ_α, $\alpha > 0$, is a type of generalized Lambert function, i.e. the mapping $x \mapsto \Phi_\alpha(x) = y$, consists of the only solution to the equation $y^\alpha e^y = x$. Jeffrey et al. (1995) report the following asymptotic expansion for Φ_α:

$$\Phi_\alpha(x) = \log x - \alpha \log\log x + O\left(\frac{\log\log x}{\log x}\right), \qquad x \to \infty. \tag{2.9}$$

Using expression (2.9) – as opposed to only the leading logarithmic order as in Roper and Rutkowski (2009) – and rearranging the terms in (2.6) we arrive to (2.4). Now letting

$$x(S_0, K, \tau) = \frac{\log\left(c_{K,S_0}(-\log\left(c_{K,S_0} O(S_0, K, \tau)\right))^{3/2}\right)}{\log O(S_0, K, \tau)} \tag{2.10}$$

we can rewrite (2.4) as

$$v(S_0, K, \tau) = \frac{\log(K/S_0)^2}{-2\log O(S_0, K, \tau)}\left(1 + x(S_0, K, \tau)\right)^{-1} + o(1). \tag{2.11}$$

Observe that both $x(S_0, K, \tau) \to 0$ and $x(S_0, K, \tau)^2 \log O(S_0, K, \tau) \to 0$ as $\tau \to 0$, because $(\log(-\log y))^n / \log y \to 0$ for all $n > 0$, when $y \to 0$. Hence using $(1+y)^{-1} \sim 1 - y + O(y^2)$, $y \sim 0$, in (2.11) we obtain (2.5). □

It is apparent that (2.5) sharpens (2.3). Also, as we shall see in the following sections, (2.4) and (2.5) correctly specify to known model-specific second order expansions. Still, for our purposes such expressions are mainly a tool for the skew analysis. Crucially, for large x both the first two orders in (2.9) tend to infinity, whereas the higher ones tend to zero. This means that (2.4) and (2.5) are much better suited for further asymptotic analyses involving the implied variance compared to (2.3), since the latter ignores a subleading diverging term which can be relevant should cancellations arise. As it turns out this is precisely what happens in the OTM and ITM skew derivation of many models, with (2.4), as opposed to (2.3), leading to the correct expression.

3. The skew outside the ATM point

In option markets liquidity at small maturities obviously concentrates around the ATM point. Therefore the deep ITM and OTM skew asymptotic rate is certainly not as important as that of small moneynesses. This may be part of the reason why the literature almost invariably focuses on the ATM skew, which is implicitly considered as a proxy for the small moneyness skew. However, in a way, the question arises of whether this is really the case. In financial data it is commonly found that as maturity gets smaller the ATM skew explodes or shows a discontinuity, implying that even small changes in the underlying value could have a large impact on the skew. This is only worsened by the fact that in principle, different moneynesses can show different skew time rates, and these two effects combined can have profound implications on skew risk management. That is to say that the inherently singular nature of the skew in the ATM point may motivate the introduction of asymptotic formulae for the OTM and ITM cases as well.

We recall that the implied volatility skew is defined as

$$\mathcal{S}(S_0, K, \tau) = \frac{\partial \sigma_{BS}}{\partial K}(S_0, K, \tau). \tag{3.1}$$

Let N and ϕ be respectively the standard normal PDF and CDF and set

$$d(S_0, K, \tau, \sigma) = \frac{\log(K/S_0)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}. \tag{3.2}$$

The Black-Scholes call vega can be taken to be of the form

$$\mathcal{V}(S_0, K, \tau, \sigma) = \frac{\partial C_{BS}}{\partial \sigma}(S_0, K, \tau, \sigma) = K\sqrt{\tau}\phi\left(-d(S_0, K, \tau, \sigma)\right). \tag{3.3}$$

Also, recall the Black-Scholes call-put symmetry relation for zero risk-free rates $C_{BS}(S_0, K, \tau, \sigma) = P_{BS}(K, S_0, \tau, \sigma)$, with Δ_P the Black-Scholes put delta

$$\frac{\partial C_{BS}}{\partial K}(S_0, K, \tau, \sigma) = \Delta_P(K, S_0, \tau, \sigma) = -N(-d(S_0, K, \tau, \sigma)). \tag{3.4}$$

Letting

$$d(S_0, K, \tau) = d(S_0, K, \tau, \sigma_{BS}(S_0, K, \tau)) = \frac{\log(K/S_0)}{\sqrt{v(S_0, K, \tau)}} + \frac{\sqrt{v(S_0, K, \tau)}}{2} \tag{3.5}$$

and using the chain rule, it follows from (3.4), (3.3) and (3.5) that

$$\mathcal{S}(S_0, K, \tau) = \frac{N(-d(S_0, K, \tau)) - DC(S_0, K, \tau)}{K\sqrt{\tau}\phi(-d(S_0, K, \tau))}. \tag{3.6}$$

If the model S is sticky-by-moneyness, that is, if the implied volatility only depends on K/S_0, one writes $\sigma_{BS}(k, \tau)$ and $v(k, \tau)$ for the Black-Scholes implied volatility and total implied variance respectively, and in $k = \log(K/S_0)$ coordinates (3.6) becomes

$$\mathcal{S}^{(k)}(S_0, k, \tau) := \frac{\partial \sigma_{BS}}{\partial k}(k, \tau) = e^k \frac{N(-d(k, \tau)) - DC(S_0, S_0 e^k, \tau)}{\sqrt{\tau}\phi(-d(k, \tau) + \sqrt{v(k, \tau)})} \tag{3.7}$$

where $d(k, \tau) = d(1, e^k, \tau, \sigma_{BS}(k, \tau))$. One can pass from (3.6) to (3.7) using the chain and $\phi(-d(k, \tau)) = \phi(-d(k, \tau) + \sqrt{v(k, \tau)})S_0/K$.

Setting $k = 0$ in equation (3.7) or $K = S_0$ in equation (3.6) determines the ATM implied volatility skew, whose small time behavior is seen to depend on the digital call ATM option price small maturity limit and on the vanilla call ATM price to the second order (e.g. Figueroa-López and Olafsson 2016, Gerhold et al. 2016).

The asymptotics of (3.6) and (3.7) as $\tau \sim 0$ and $K = S_0$ (i.e. $k = 0$) are well-investigated: see the mentioned Figueroa-López and Olafsson (2016), Gerhold et al. (2016), as well as Gatheral (2011). We are here interested to the $K \neq S_0$ ($k \neq 0$) case. We obtain the asymptotic behavior of the OTM/ITM implied volatility skew in the next theorem.

THEOREM 2 *Let S be a martingale underlying asset price process whose marginal laws are absolutely continuous. For $S_0 \neq K$ the implied volatility skew $\mathcal{S}$ of S is given by*

$$\mathcal{S}(S_0, K, \tau) \sim \frac{\operatorname{sgn}\Big(\log(K/S_0)\Big)}{K\sqrt{\tau}\left(-2\log\left(O(S_0, K, \tau)\right)\right)^{1/2}}$$

$$- \log(K/S_0)\frac{DO(S_0, K, \tau)}{\sqrt{\tau}O(S_0, K, \tau)(-2\log(c_{K,S_0}O(S_0, K, \tau)))^{3/2}} \tag{3.8}$$

where $DO(S_0, K, \tau) = DC(S_0, K, \tau)\mathbb{1}_{\{K>S_0\}} + DP(S_0, K, \tau)\mathbb{1}_{\{K<S_0\}}$ is the price of the digital option starting OTM.

Proof. It is convenient to treat the OTM and ITM cases (resp. $K > S_0$ and $K < S_0$) separately. We denote with $\mathcal{S}^+$ and $\mathcal{S}^-$ the implied volatility skew with such implicit conditions.

Let us first introduce the Mills ratios

$$U_+(x) = \frac{N(x)}{\phi(x)}, \qquad U_-(x) = \frac{1 - N(x)}{\phi(x)}, \quad x \in \mathbb{R}. \tag{3.9}$$

We have the asymptotic behavior

$$U_+(x) \sim U_-(-x) = \frac{1}{x} + O(x^{-2}), \qquad x \to \infty. \tag{3.10}$$

Assuming first $K > S_0$ we can rewrite (3.6) as $\mathcal{S}^+(S_0,K,\tau) = \mathcal{S}_1^+(S_0,K,\tau) + \mathcal{S}_2^+(S_0,K,\tau)$ with

$$\mathcal{S}_1^+(S_0,K,\tau) = \frac{U_+(-d(S_0,K,\tau))}{K\sqrt{\tau}}, \qquad \mathcal{S}_2^+(S_0,K,\tau) = -\frac{DC(S_0,K,\tau)}{K\sqrt{\tau}\phi(-d(S_0,K,\tau))}. \tag{3.11}$$

Consider $\mathcal{S}_1^+$. Applying the asymptotics (3.10) to definition (3.5), and in view of (2.3), results in

$$\mathcal{S}_1^+(S_0,K,\tau) = \frac{1}{K\sqrt{\tau}}\frac{1}{d(S_0,K,\tau)} + O(d(S_0,K,\tau)^{-2}) \sim \frac{\sigma_{BS}(S_0,K,\tau)}{K\log(K/S_0)} \sim \frac{1}{K\sqrt{\tau}\,(-2\log C(S_0,K,\tau))^{1/2}}. \tag{3.12}$$

Regarding S_2^+, observe first the following relationship, again obtained applying (2.3),

$$\begin{aligned}\phi(-d(S_0,K,\tau)) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{S_0}{K}}\exp\left(-\frac{1}{2}\left(\frac{\log(K/S_0)^2}{v(S_0,K,\tau)} + \frac{v(S_0,K,\tau)}{4}\right)\right)\\ &\sim \frac{1}{\sqrt{2\pi}}\sqrt{\frac{S_0}{K}}\exp\left(-\frac{\log(K/S_0)^2}{2v(S_0,K,\tau)}\right).\end{aligned} \tag{3.13}$$

Using the full asymptotic expression for v from (2.4) in (3.13) and then replacing in the denominator of $\mathcal{S}_2^+$ we obtain

$$\begin{aligned}&K\sqrt{\tau}\phi(-d(S_0,K,\tau))\\ &\sim \frac{1}{\sqrt{2\pi}}\sqrt{\tau S_0 K}c_{K,S_0}C(S_0,K,\tau)(-\log(c_{K,S_0}C(S_0,K,\tau)))^{3/2}\\ &\sim \frac{\sqrt{8}}{|\log(K/S_0)|}\sqrt{\tau}C(S_0,K,\tau)(-\log(c_{K,S_0}C(S_0,K,\tau)))^{3/2}\end{aligned} \tag{3.14}$$

leading to

$$\mathcal{S}_2^+(S_0,K,\tau) \sim -\frac{\log(K/S_0)DC(S_0,K,\tau)}{\sqrt{8}\sqrt{\tau}C(S_0,K,\tau)(-\log(c_{K,S_0}C(S_0,K,\tau)))^{3/2}}. \tag{3.15}$$

If $K < S_0$ with $\mathcal{S}^- = \mathcal{S}_1^- + \mathcal{S}_2^-$ we use instead

$$\mathcal{S}_1^-(S_0,K,\tau) = -\frac{U_-(-d(S_0,K,\tau))}{K\sqrt{\tau}}, \qquad \mathcal{S}_2^-(S_0,K,\tau) = \frac{DP(S_0,K,\tau)}{K\sqrt{\tau}\phi(-d(K,S_0,\tau))}. \tag{3.16}$$

Again the asymptotics (3.10) together with (2.3) lead immediately to the equivalence

$$\mathcal{S}_1^-(S_0,K,\tau) \sim -\frac{1}{K\sqrt{\tau}\,(-2\log P(S_0,K,\tau))^{1/2}}. \tag{3.17}$$

Concerning $\mathcal{S}_2^-$ we have the identical analysis to that for S_2^+ with the digital put price replacing the digital call price, so that

$$\mathcal{S}_2^-(S_0,K,\tau) = \frac{-\log(K/S_0)DP(S_0,K,\tau)}{\sqrt{8}\sqrt{\tau}P(S_0,K,\tau)(-\log\left(c_{K,S_0}(P(S_0,K,\tau))\right))^{3/2}}. \tag{3.18}$$

Summing up, if $K \neq S_0$, we arrive to the final OTM/ITM small maturity asymptotic formula in the thesis. $\square$

The applicability of (3.8), just as that of (2.4)–(2.5), then hinges on the availability of asymptotic expressions for O and DO. In many popular models, these are available, e.g. using saddlepoint approximations of integral valuation expression, or follow from general probabilistic model properties (see references in the section below). Theorem 2 reinforces the case for explicit models, i.e. option pricing models in which the option prices can be calculated in an analytic form, as they yield exact implied volatility surface asymptotics, which in turn may be used to optimize calibration procedures. These authors advocate one such model in Azzone and Torricelli (2023), which we briefly detail Subsection 3.3 below.

4. Applications

In the following we calculate the refined OTM and ITM asymptotics of the implied volatility in Section 1, together with the implied volatility skew approximation obtained in Section 2 to some classes of models. Namely, we discuss exponential Lévy models (and time changes thereof), the Heston stochastic volatility model, and recently introduced models of exponential additive [1] type. It should become apparent how the improved level formula (2.5) unifies and simplifies the derivation of results already present in the literature, and at the same time rigorously determines the small maturity OTM and ITM skew.

4.1. *Lévy and time changed Lévy exponential models*

Assume $S = (S_0 e^{X_t})_{t\geq 0}$ is a martingale with $X = (X_t)_{t\geq 0}$ a Lévy process with finite first exponential moment (see e.g., Cont and Tankov 2003 for definitions and applications of Lévy processes in finance). The ATM small time skew analysis is carried out among the others in Tankov (2011), Gerhold et al. (2016), Figueroa-López and Forde (2012), Figueroa-López and Olafsson (2016), and Andersen and Lipton (2013), showing that the small ATM skew typically explodes at rate $\tau^{-1/2}$, but slower rates are also possible. See also Bertoin (1996) and Rüschendorf and Woerner (2002) for a full outlook of the relation between probability and Lévy integrals. We obtain the OTM/ITM Lévy exponential models implied variance and skew in the next proposition.

Proposition 3 *Let S be a Lévy exponential model with Lévy measure ν. The $\tau \sim 0$ implied variance and skew for $K \neq S_0$ are given by respectively*

$$v(S_0, K, \tau) = \frac{\log(K/S_0)^2}{-2\log(C_\pm \tau)}\left[1 - \frac{\log\left(c_{K,S_0}(-\log(C_\pm c_{K,S_0}\tau))^{3/2}\right)}{\log(C_\pm \tau)} + o\left(\frac{1}{\log \tau}\right)\right]. \tag{4.1}$$

$$\mathcal{S}^{\pm}(S_0, K, \tau) \sim \frac{\operatorname{sgn}(\log(K/S_0))}{K\sqrt{\tau}\sqrt{-2\log(C_\pm \tau)}} - \log(K/S_0)\frac{c_\pm}{C_\pm\sqrt{\tau}(-2\log(c_{K,S_0}C_\pm \tau))^{3/2}}. \tag{4.2}$$

where

$$C_+ = \int_k^\infty (S_0 e^x - K)\nu(dx), \qquad C_- = \int_{-\infty}^k (K - S_0 e^x)\nu(dx). \tag{4.3}$$

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[1] An additive process is a stochastically continuous process with independent increments

and

$$c_+ = \int_k^\infty \nu(dx), \qquad c_- = \int_{-\infty}^k \nu(dx). \tag{4.4}$$

Proof. It is known by Figueroa-López and Forde (2012), Theorem 2.1, Proposition 2.2 that if $S_0 < K$, $DC(S_0, K, \tau) \sim c_+\tau$, $C(S_0, K, \tau) \sim C_+\tau$ whereas if $S_0 > K$, $DP(S_0, K, \tau) \sim c_-\tau$, $P(S_0, K, \tau) \sim C_-\tau$. We obtain the implied variance using equation (2.5) together with (4.3). We derive the skew using Theorem 2 and substituting in the options prices (4.3) and (4.4). □

Remark 1 Tankov (2011), Proposition 4, gives the leading order of implied variance for small τ.

$$v(S_0, K, \tau) \sim \frac{\log(K/S_0)^2}{-2\log\tau}, \qquad \tau \sim 0. \tag{4.5}$$

Then Proposition 3 evidently sharpens the above. An improvement of (4.5) has also been presented by Figueroa-López and Forde (2012), Theorem 2.3, as

$$v(S_0, K, \tau) = \frac{\log(K/S_0)^2}{-2\log\tau}\left[\left(1 - \frac{\log\left(C_\pm c_{K,S_0}(-\log\tau)^{3/2}\right)}{\log\tau}\right) + o\left(\frac{1}{\log\tau}\right)\right] \tag{4.6}$$

Proceeding as in Proposition 3 but starting from our equation (2.4) instead of (2.5), and using the geometric expansion $(1+x)^{-1} \sim 1 - x$ we would obtain

$$\begin{aligned} v(S_0, K, \tau) &= \frac{\log(K/S_0)^2}{-2\log\tau}\left[\frac{1}{1 + \frac{\log\left(C_\pm c_{K,S_0}\left(-\log(C_\pm c_{K,S_0}\tau)\right)^{3/2}\right)}{\log\tau}} + o\left(\frac{1}{\log\tau}\right)\right] \\ &= \frac{\log(K/S_0)^2}{-2\log\tau}\left[\left(1 - \frac{\log\left(C_\pm c_{K,S_0}\left(-\log(C_\pm c_{K,S_0}\tau)\right)^{3/2}\right)}{\log\tau}\right) + o\left(\frac{1}{\log\tau}\right)\right] \end{aligned} \tag{4.7}$$

which is equivalent to (4.6).

Remark 2 The leading order of the small time skew in Lévy models is provided by the first term in (4.2). It can be noticed that differentiating (4.5) or (4.6) with respect to K, besides not being mathematically justified[1], would not recover the rate in (4.2). This example should make one very wary of naively differentiating maturity expansions in order to get the small time skew, which is not just mathematically unsound but can also easily produce suboptimal or entirely wrong answers.

Let us add that this approach can be straightforwardly extended to some of the so-called time-changed Lévy models, i.e. models of the exponential form $S = (S_t)_{t\geq 0}$, $S_t = S_0 \exp(Y_t)$, $Y = (Y_t)_{t\geq 0}$ given by $Y_t = X_{T_t}$, $X = (X_t)_{t\geq 0}$ being a Lévy process and the "time change" $T = (T_t)_{t\geq 0}$ modeled as $T_t = \int_0^t V_s ds$ for some positive activity rate process $V = (V_t)_{t\geq 0}$ independent of X (see Carr et al. 2003 for details). In such case, and under a stationarity assumption on V, we can follow Figueroa-López and Forde (2012), Corollary 3.3, and deduce that the skew equation follows (4.2) with $\hat{c}_\pm = \mathbb{E}[V_0]c_\pm$, $\hat{C}_\pm = \mathbb{E}[V_0]C_\pm$, replacing respectively $c_\pm$ and $C_\pm$ in (4.2).

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[1] A general sufficient condition for the limit of the derivatives to coincide with the derivative of the limit (even if uniform) if the sequence of the derivatives converges uniformly (e.g. Bartle and Sherbert 2000, Theorem 8.2.3), which cannot be verified here.

4.2. *Heston stochastic volatility model*

The popular Heston (1993) stochastic volatility models postulates that the price process is modeled by $S = (S_t)_{t\geq 0}$, $S_t = S_0 \exp(X_t)$, $S_0 > 0$, where and $X = (X_t)_{t\geq 0}$ is a solution of the system of SDEs coupling a compensated stochastic integral with a mean reverting diffusive variance with square root dynamics, i.e.

$$\begin{cases} dX_t = \sqrt{v_t} dW_t - \dfrac{v_t}{2} dt, & X_0 = 0 \\ dv_t = k(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_t,\ v_0 > 0, \end{cases} \tag{4.8}$$

for Brownian motions $Z = (Z_t)_{t\geq 0}, W = (W_t)_{t\geq 0}$ such that $\langle W, Z\rangle_t = \rho t$, $\rho \in [-1, 1]$, and $k, \theta, \sigma > 0$ obeying certain constraints for a unique strong solution of (4.8) to exist. An explicit expansion for the short maturity ATM implied volatility skew is presented in Forde et al. (2012); see also Lewis (2009), and Forde and Jacquier (2009) for a large deviation approach.

Define the function U as follows

$$U(p) := \exp\left(\frac{k\theta}{\sigma^2}\left((i\rho\sigma - d_0)ip - 2\log\left(\frac{1 - g_0 e^{-id_0 p}}{1 - g_0}\right)\right) + \frac{v_0 e^{-id_0 p}}{(1 - g_0 e^{-id_0 p})\sigma^2}\Big((i\rho\sigma - d_0)ipd_1 \right.$$
$$\left. -(k - d_1)\left(1 - e^{id_0 p}\right) + \frac{(i\rho\sigma - d_0)(1 - e^{-id_0 p})(g_1 - id_1 g_0 p)}{1 - g_0 e^{-id_0 p}}\Big)\right), \tag{4.9}$$

where

$$d_0 := \sigma\sqrt{1 - \rho^2}, d_1 := \frac{2k\rho - \sigma}{2\sqrt{1 - \rho^2}} i, \quad g_0 = \frac{i\rho - \sqrt{1 - \rho^2}}{i\rho + \sqrt{1 - \rho^2}}, \quad g_1 = \frac{2k - \rho\sigma}{\sigma\sqrt{1 - \rho^2}(i\rho + \sqrt{1 - \rho^2})^2}. \tag{4.10}$$

We determine the small maturity implied variance and OTM/ITM skew of the Heston model using our approach.

Proposition 4 *Let $S = (S_t)_{t\geq 0}$, $S_0 > 0$, be the asset pricing model defined by $S_t = S_0 \exp(X_t)$, with $X = (X_t)_{t\geq 0}$ a solution of (4.8). Let $k = \log(K/S_0)$; the $k \neq 0$ small-maturity implied variance and skew are given respectively by*

$$v(k, \tau) \sim \frac{k^2 \tau}{2\Lambda^*(k)}\left(1 + \frac{\tau}{\Lambda^*(k)}\log\left(A(k)c_{K,S_0}(\Lambda^*(k))^{3/2}\right)\right) \tag{4.11}$$

$$\mathcal{S}(k, K, \tau) \sim \frac{\operatorname{sgn}(k)}{K\sqrt{2\Lambda^*(k)}} - \frac{k|p^*(k)|}{K(2\Lambda^*(k))^{3/2}}, \tag{4.12}$$

where Λ^* is the Fenchel-Legendre transform of the model's limiting cumulant generating function $\Lambda(z) = \lim_{\tau\to 0} \tau \log \mathbb{E}[e^{zX_\tau/\tau}]$, p^* the inverse function to Λ' and

$$A(k) = \frac{KU(p^*(k))}{(p^*(k))^2\sqrt{2\pi\Lambda''(p^*(k))}}, \tag{4.13}$$

where U is given in (4.9).

Proof. According to Forde et al. (2012), Theorem 3.1, and combining with put call parity, we have that that as $\tau \sim 0$ the Heston OTM option price $O(k, \tau)$ in log-moneyness k satisfies

$$O(k, \tau) = \exp\left(-\frac{\Lambda^*(k)}{\tau}\right)\left(A(k)\tau^{3/2} + O(\tau^{5/2})\right), \qquad \tau \sim 0, \tag{4.14}$$

Substituting (4.14) in (2.4), using $\log(x^{-1} + a\log x) \sim \log(x^{-1})$, $x \sim 0$, $a \in \mathbb{R}$ we obtain

$$\begin{aligned} v(k,\tau) &\sim -\frac{k^2}{2}\left(-\frac{\Lambda^*(k)}{\tau} + \log\left(A(k)c_{K,S_0}\tau^{3/2}\right) + \frac{3}{2}\log\left(\frac{\Lambda^*(k)}{\tau}\right)\right)^{-1} \\ &\sim \frac{k^2\tau}{2\Lambda^*(k)}\left(1 - \frac{\tau}{\Lambda^*(k)}\log\left(A(k)c_{K,S_0}(\Lambda^*(k))^{3/2}\right)\right)^{-1} \end{aligned} \tag{4.15}$$

and (4.11) follows since $(1-x)^{-1} \sim 1 + x$, $x \sim 0$.

Moving on to the skew analysis, we need to compute the asymptotic of the digital call price. We start from the representations of the digital call option value in Theorem 5.1 of Lee (2004) and proceed as in Forde et al. (2012), Theorem 3.1, for the European option. First, we write the digital call price as the Fourier-Plancharel integral as

$$DC(k,\tau) = \frac{1}{2\pi}\int_{-\infty+i\zeta}^{\infty+i\zeta} e^{iuk}\frac{\psi_\tau(-u)}{iu}du, \tag{4.16}$$

with $\psi_\tau(z)$ the characteristic function of X_τ, and $\zeta > 0$. Second, we make the change of variable $z = u/\tau$, for every τ set $\zeta = p^*(k)\tau$, where $p^*(k)$ is the inverse function of Λ', and use Forde et al. (2012), Lemma 6.1: as $\tau \to 0$, $\psi_\tau(-u/\tau) = U(-iu)\exp\left(\frac{\Lambda(-iu)}{\tau}\right)(1 + O(\tau))$. We can rewrite the digital call price as

$$\begin{aligned} DC(k,\tau) &= \frac{1}{2\pi}\int_{-\infty+ip^*(k)}^{\infty+ip^*(k)} e^{iuk/\tau}\frac{\psi_\tau(-u/\tau)}{iu}du \\ &= \left(\frac{1}{2\pi}\int_{-\infty+ip^*(k)}^{\infty+ip^*(k)} U(-iu)\frac{e^{\frac{F(u)}{\tau}}}{iu}du\right)(1 + O(\tau)), \end{aligned} \tag{4.17}$$

where $F(u) := \Lambda(-iu) + iuk$. Let us point out that $\Lambda(z) = \frac{v_0 z}{\sigma(\sqrt{1-\rho^2}\cot(z\sqrt{1-\rho^2}\sigma/2)-\rho)}$ (see Forde et al. 2012, Equation 2.3). The functions U and F are both analytic in the integration contour. From Lemma 6.4 and Remark 6.5 of Forde et al. (2012), for every $u \neq ip^*(k)$ on the integration domain, $\Re(F(u)) > \Re(F(ip^*(k)))$ and $\Re(F(u)) \to \infty$ as $|u| \to \infty$. Moreover, $F''(ip^*(k)) = \Lambda''(p^*(k)) \neq 0$. Then, we can use the steepest descent/Laplace integral approximation for large $1/\tau$ (see e.g., Olver 1997, Chapter 4, Theorem 7.1) on (4.17) and obtain the following asymptotic expansion for the digital OTM call option

$$DC(k,\tau) = \exp\left(-\frac{\Lambda^*(k)}{\tau}\right)\left(\frac{U(p^*(k))}{|p^*(k)|\sqrt{2\pi\Lambda''(p^*(k))}}\sqrt{\tau} + O(\tau^{3/2})\right), \qquad \tau \sim 0. \tag{4.18}$$

By following the same steps for the digital OTM put option we find out that in such case $DP(k,\tau) \sim DC(k,\tau) \sim DO(k,\tau)$ as $\tau \sim 0$. Finally, (3.8), (4.14), and (4.18) together imply

$$\begin{aligned} \mathcal{S}(k,K,\tau) \sim &\frac{\operatorname{sgn}(k)}{K\sqrt{2\tau}\left(\frac{\Lambda^*(k)}{\tau} - \log(c_{K,S_0}A(k)\tau^{3/2})\right)^{1/2}} \\ &- k\frac{|p^*(k)|}{K\tau^{3/2}\sqrt{8}\left(\frac{\Lambda^*(k)}{\tau} - \log(c_{K,S_0}A(k)\tau^{3/2})\right)^{3/2}} \sim \frac{\operatorname{sgn}(k)}{K\sqrt{2\Lambda^*(k)}} - \frac{k|p^*(k)|}{K(2\Lambda^*(k))^{3/2}}, \end{aligned} \tag{4.19}$$

which is the required limit. □

A simple computation shows that the expression we obtain in (4.11) is consistent with the first two orders of the implied volatility expansion in Forde et al. (2012), Section 4.

Remark 3 The skew expression in (4.12) formally coincides with the expression attained by differentiating the first order term in (4.11) with respect to K, after recalling the key property of the Legendre transform $(\Lambda^*)'(k) = p^*(k)$, and the fact that $\operatorname{sgn}(p^*(k)) = \operatorname{sgn}(k)$ (see Forde et al. 2010, Section 2). Hence, retrospectively, the limiting and differentiation operations commute, although we stress again that in absence of other properties, differentiating the limiting implied variance is not a mathematically justified way of obtaining the limiting skew.

Remark 4 The Heston small time skew limits also illustrate how delicate the skew asymptotic analysis may be. Unlike the case of Lévy models, we have that the two terms in (4.12) are of the same order, in particular constants. Thus had we used here (2.3) instead of (2.5) in the skew formula (3.6) we would have wrongly ended up with only the first of two such terms. As announced, this provides clear motivation for the second-order sharpening of the implied variance operated Proposition 1.

4.3. *Additive models*

The generalized Beta additive (GBA) model has been recently proposed in Azzone and Torricelli (2023), developing and improving on the conjugate power Dagum additive (CPDA) model of Carr and Torricelli (2021). The martingale asset dynamics $S = (S_t)_{t\geq 0}$ are of the familiar exponential form $S_t = S_0 \exp(X_t)$ where $X = (X_t)_{t\geq 0}$ is an additive process, that is a process with independent increments started at 0 such that the law of X_t is generalized logistic type 4 (Balakrishnan 2013). As a consequence, the distribution of S_t is generalized Beta of the second kind. The distinctive feature of the GBA model is that produces fully analytical option prices. In turn this allows complete control of the implied volatility surface by means of appropriate term functions $\alpha, \beta, \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\sigma(0) = 0, \alpha(0), \beta(0) > 0$ satisfying some compatibility conditions. Summarizing Azzone and Torricelli (2023), Proposition 4.4 and Corollary 4.5, an application of (2.5) produces, in log-moneyness k, and as $\tau \sim 0$, the equivalences

$$\sigma_{BS}(k,\tau) \sim \begin{cases} \left(\dfrac{k}{2\beta(0)}\dfrac{\sigma(\tau)}{\tau}\right)^{1/2}\left(1 - \dfrac{3}{2}\dfrac{\log\left(\frac{\sigma(\tau)}{k\beta(0)}\right)}{k\beta(0)}\sigma(\tau)\right)^{1/2}, & \text{if } k > 0, \\[2ex] \left(\dfrac{|k|}{2\alpha(0)}\dfrac{\sigma(\tau)}{\tau}\right)^{1/2}\left(1 - \dfrac{3}{2}\dfrac{\log\left(\frac{\sigma(\tau)}{|k|\alpha(0)}\right)}{|k|\alpha(0)}\sigma(\tau)\right)^{1/2}, & \text{if } k < 0. \end{cases} \tag{4.20}$$

The numerical computations in the next section show that (4.20) performs much better than the corresponding formula attained using the crude (2.3) asymptotics. With regard to the OTM and ITM skew, it is shown in Azzone and Torricelli (2023), Proposition 4.6, through an application of (3.8) that

$$\mathcal{S}(k,K,\tau) \sim \begin{cases} \dfrac{1}{2K\sqrt{2\beta(0)k}}\left(\dfrac{\sigma(\tau)}{\tau}\right)^{1/2}, & \text{if } k > 0 \\[2ex] -\dfrac{1}{2K\sqrt{2\alpha(0)k}}\left(\dfrac{\sigma(\tau)}{\tau}\right)^{1/2}, & \text{if } k < 0 \end{cases} \tag{4.21}$$

for small τ. Furthermore, while proving (4.21) one notices $S_2^{\pm} = -S_1^{\pm}/2$. Exactly as argued in Remark 4 this implies that ignoring the $\log\log$ term in the $\Phi_{3/2}$ function expansion (i.e. using (2.3) instead of (2.4)-(2.5) in the asymptotic skew analysis) would produce the wrong constant in the leading order. Notably, we see from (4.21) that the skew may or may not explode, depending upon the asymptotic behavior of σ near zero. This provides greater flexibility for modelling the volatility surface. See Azzone and Torricelli (2023) for full details.

5. Numerical examples

To test the improved level and skew formulae (2.4)–(2.5) and (3.8), we take into account the Normal inverse Gaussian (NIG, Barndorff-Nielsen 1997) Lévy model, the Heston (1993) model, and the GBA model of Azzone and Torricelli (2023). We numerically compute the smile and skew for each of these models and compare them with the model-specific approximations of the previous section. The numerical skew is obtained by using the central difference approximation of the model implied volatility derivative with step 10^{-4}. The results are visualized in Figures 1 to 3, taking into a account a daily maturity for the Lévy and GBA model and a monthly maturity for the Heston model. The upper panels report the implied volatility, the middle and bottom panel the skew and skew convergence. The second order approximation (2.5) clearly improves on (2.3), consistently with the numerical findings in Figueroa-López and Forde (2012) and Forde et al. (2012). Overall it appears necessary to include the second order to achieve a reliable approximation in all the models accounted and for the parameters chosen. The values of the slopes are quite robust and their quality improves as maturity shortens, as expected. We point out that for the Heston model the implied volatility and skew approximations work very well already for the one-month maturity. In the case of jump processes (such as the NIG and the GBA models) we do need shorter time to maturities, or farther OTM strikes, to achieve the same accuracy in the approximations, possibly due to the interaction of the OTM implied volatility blow up and a finite/vanishing ATM volatility (Carr and Wu 2003, Tankov 2011, Azzone and Torricelli 2023), yielding lack of uniformity in the convergence of levels and slopes.

6. Conclusion

In this paper we have shown that the small-time implied volatility skew away from the ATM point can be obtained in a model-free way from model (or traded) option prices, just as in the ATM case. The key to such derivation is a simple second-order refinement of a known implied volatility asymptotic formula. Since there is a substantial body of literature on the short term asymptotics of vanilla and digital options, small-maturity OTM and ITM skew formulae are at hand for many popular asset pricing models.

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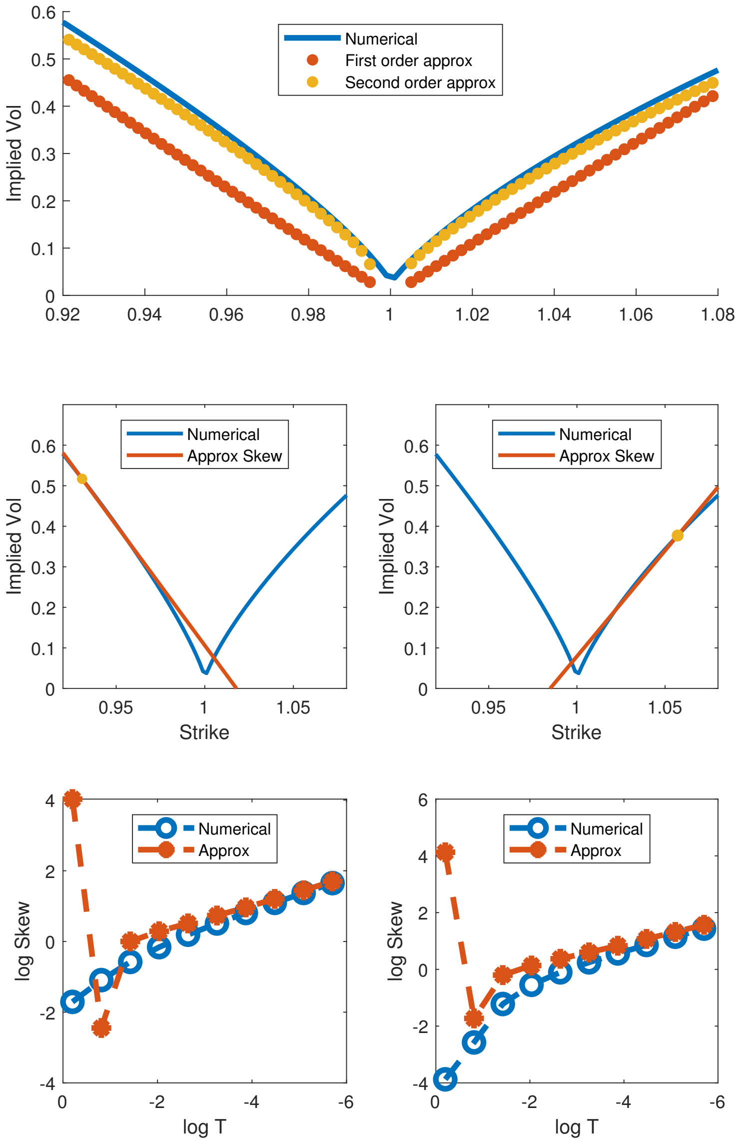

Figure 1. NIG process with parameters $\theta = -0.05$, $\sigma = 0.1$, and $\kappa = 0.5$ (in the parametrization of Cont and Tankov 2003, Table 4.4). Top panel: one-day numerical implied volatility and first and second order approximations. Middle panel: one-day numerical implied volatility and approximated ITM and OTM slopes. Bottom panel: convergence of the ITM and OTM skew in log-log scale.

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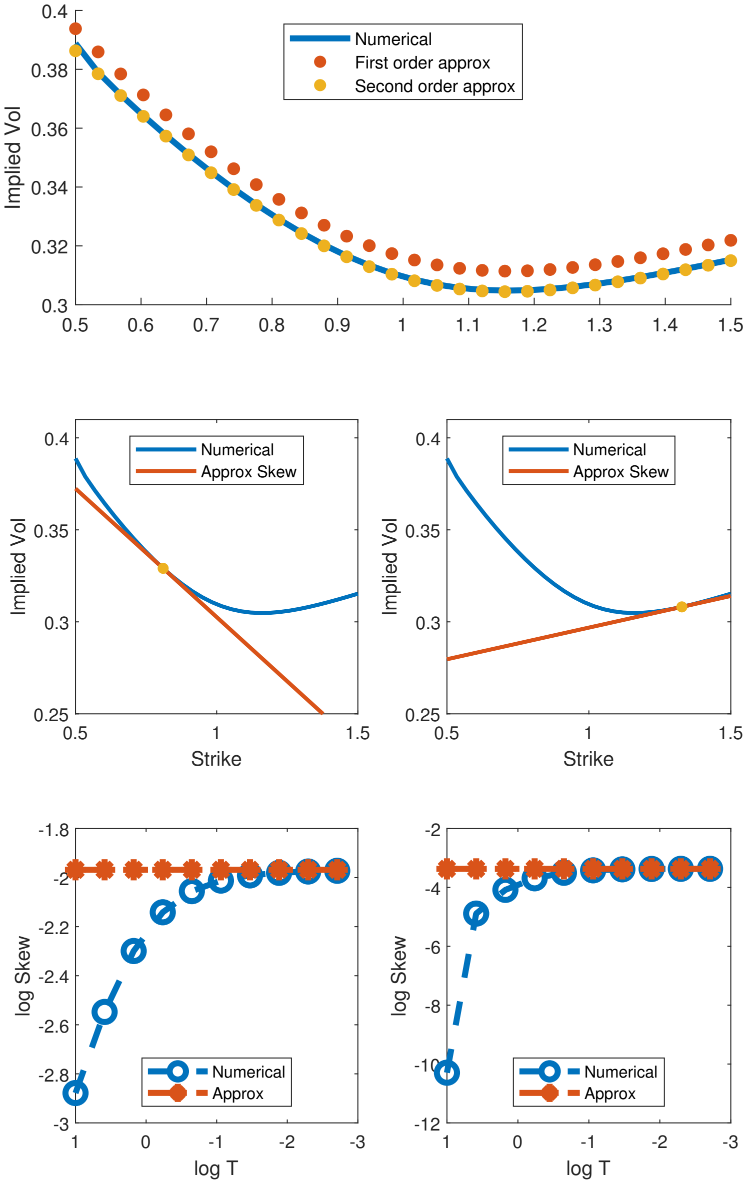

Figure 2. Heston model with parameters $v_0 = 0.2$, $\theta = 0.05$, $\kappa = 4$, $\sigma = 0.2$, $\rho = -0.1$. Top panel: one-month numerical implied volatility and first and second order approximations. Middle panel: one-month numerical implied volatility and approximated ITM and OTM slopes. Bottom panel: convergence of the ITM and OTM skew in log-log scale.

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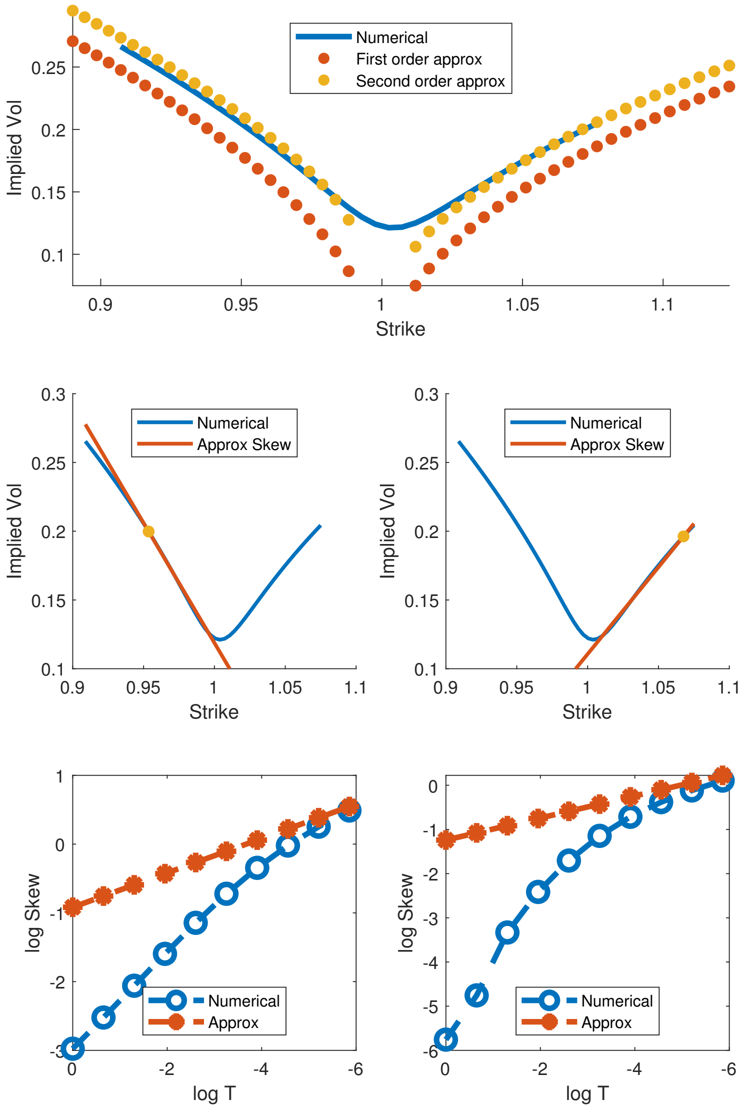

Figure 3. GBA model with $\sigma(\tau) = \sigma_0\sqrt{\tau}$, $\beta(\tau) = \sigma(\tau) + \beta_0$ and $\alpha(\tau) = \alpha_0$, where $\sigma_0 = 0.1$, $\beta_0 = 2$ and $\alpha_0 = 1.5$. Top panel: one-day numerical implied volatility and first and second order approximations. Middle panel: one-day numerical implied volatility and approximated ITM and OTM slopes. Bottom panel: convergence of the ITM and OTM skew in log-log scale.

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