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# Reconstuction of a convolution kernel in an integrodifferential problem with a fractional time derivative

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## 1 Introduction

se1

Problem:

$$\begin{cases} D_t^\alpha u(t) = Au(t) + \int_0^t k(t-s)Bu(s)ds + F(t), & t \in [0, T], \\ u(0) = u_0, \\ \Phi(u(t)) = g(t), & t \in [0, T]. \end{cases} \quad (1.1) \quad \text{eq1.1}$$

Unknowns:  $u, k$ .

Basic assumptions:

(A1)  $X$  is a complex Banach space with norm  $\|\cdot\|$ ,  $\alpha \in (0, 2)$ ,  $D_t^\alpha u$  is the Caputo derivative of  $u$  with respect to  $t$ .

(A2)  $A : D(A) \rightarrow X$  is a linear operator; there exist,  $M, R \in \mathbb{R}^+$ , such that  $\{\lambda \in \mathbb{C} : |\lambda| \geq R, |\text{Arg}(\lambda)| \leq \frac{\alpha\pi}{2}\} \subseteq \rho(A)$ , and, for  $\lambda$  in this set,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M|\lambda|^{-1},$$

$B \in \mathcal{L}(D(A), X)$ .

(A3)  $\Phi \in X'$ .

Notation: if  $\theta \in (0, 2)$ ,

$$D_\phi(A) = \begin{cases} (X, D(A))_{\phi, \infty} & \text{if } \phi \in (0, 1), \\ D(A) & \text{if } \phi = 1, \\ \{x \in D(A) : Ax \in (X, D(A))_{\phi-1, \infty}\} & \text{if } \phi \in (1, 2). \end{cases}$$

The following characterization of  $D_\theta(A)$  ( $0 < \theta < 1$ ) holds (see ....)

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**Theorem 1.1.** Suppose that  $S$  satisfies the condition (A2). Let  $\theta \in (0, 1)$ . Then

$$D_\theta(A) = \{x \in X : \sup_{\xi \geq R} \xi^\theta \|A(\xi - A)^{-1}x\| < \infty\}.$$

An equivalent norm in  $D_\theta(A)$  is

$$\|x\|_\theta := \sup\{\|x\| + \xi^\theta \|A(\xi - A)^{-1}x\| : \xi \geq R\} = \sup\{\|x\| + \xi^\theta \|A(\xi - A)^{-1}x\| : \xi \geq R, \xi \in \mathbb{Q}\}.$$

**Lemma 1.2.** Let  $(\Omega, \mu)$  a measure space and let  $f : \Omega \rightarrow X$  be measurable. Then

- (I) the function  $t \rightarrow \|f(t)\|_\theta$  is measurable ( $\|f(t)\|_\theta = \infty$  if  $f(t) \notin D_\theta(A)$ );  
 (II) if  $\int_\Omega \|f(t)\|_\theta d\mu < \infty$ ,  $\int_\Omega f(t) d\mu \in D_\theta(A)$  and

$$\left\| \int_\Omega f(t) d\mu \right\|_\theta \leq \int_\Omega \|f(t)\|_\theta d\mu$$

*Proof.* (I) It follows from  $\|f(t)\|_\theta = \sup_{\xi \geq R, \xi \in \mathbb{Q}} g_\xi(t)$ , with

$$g_\xi(t) = \|f(t)\| + \xi^\theta \|A(\xi - A)^{-1}f(t)\|.$$

(II) If  $\xi \geq R, \xi \in \mathbb{Q}$ ,

$$\left\| \int_\Omega f(t) d\mu \right\| + \xi^\theta \|A(\xi - A)^{-1} \int_\Omega f(t) d\mu\| \leq \int_\Omega g_\xi(t) d\mu \leq \int_\Omega \|f(t)\|_\theta d\mu.$$

Taking the supremum in  $\xi$ , we obtain the assertion. □

We shall employ the following

**th1.1**

**Theorem 1.3.** Let  $\alpha \in (0, 2)$ . Consider system

$$\begin{cases} D_t^\alpha v(t) = Av(t) + f(t), & t \in [0, T], \\ v^{(k)}(0) = v_k, & k < \alpha, \end{cases} \quad (1.2) \quad \text{eq1.2}$$

supposing that (A1)-(A2) hold; then:

(I) (??) has, at most, one solution, for every  $f \in C([0, T]; X)$ ,  $u_0 \in D(A)$ ,  $u_1 \in X$  in case  $\alpha > 1$  (solution means  $D_t^\alpha v \in C([0, T]; X)$ ,  $v \in C([0, T]; D(A))$ ).

(II) Let  $\theta \in (0, 1)$ ,  $\alpha\theta \neq 1$ . Then necessary and sufficient conditions implying that (??) has a strict solution  $v$  such that  $D_t^\alpha v$  and  $Av$  are bounded with values in  $D_\theta(A)$  are :

$$u_k \in D_{1+\theta-\frac{k}{\alpha}}(A) (k < \alpha), \quad f \in C([0, T]; X) \cap B([0, T]; D_\theta(A)).$$

(III) If  $T_0 \in \mathbb{R}^+$ , there exists  $C(T_0) \in \mathbb{R}^+$  such that, if  $0 < T \leq T_0$ ,

$$\|D_t^\alpha v\|_{B([0, T]; D_\theta(A))} + \|v\|_{B([0, T]; D_{1+\theta}(A))} \leq C(T_0) \left( \sum_{k < \alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|f\|_{B([0, T]; D_\theta(A))} \right).$$

*Proof.* Concerning (I)-(II), see ... . We show (III). We set  $F : [0, T_0] \rightarrow D_\theta(A)$ ,  $F(t) = f(t)$  if  $0 \leq t \leq T$ ,  $F(t) = f(t_0)$  if  $T \leq t \leq T_0$ . Let  $V$  be the solution of

$$\begin{cases} D_t^\alpha V(t) = AV(t) + F(t), & t \in [0, T_0], \\ V^{(k)}(0) = v_k, & k < \alpha. \end{cases}$$

Then  $v = V|_{[0,T]}$ , so that

$$\begin{aligned}
& \|D_t^\alpha v\|_{B([0,T];D_\theta(A))} + \|v\|_{B([0,T];D_\theta(A))} \\
& \leq \|D_t^\alpha V\|_{B([0,T];D_\theta(A))} + \|V\|_{B([0,T];D_\theta(A))} \\
& \leq C(T_0)(\sum_{k<\alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|F\|_{B([0,T_0];D_\theta(A))}) \\
& = C(T_0)(\sum_{k<\alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|f\|_{B([0,T];D_\theta(A))}).
\end{aligned}$$

Moreover, by ...

$$\|v\|_{C^\alpha([0,T];D_\theta(A))} \leq C(\alpha)\|D^\alpha v\|_{B([0,T];D_\theta(A))},$$

and  $D(A) \in J_{1-\theta}(D_\theta(A), D_{1+\theta}(A))$ , so that, if  $0 \leq s < t \leq T$ ,

$$\begin{aligned}
\|v(t) - v(s)\|_{D(A)} & \leq C\|v(t) - v(s)\|_{D_\theta(A)}^\theta \|v(t) - v(s)\|_{D_{1+\theta}(A)}^{1-\theta} \\
& \leq C_1(T_0)(t-s)^{\alpha\theta}(\sum_{k<\alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|f\|_{B([0,T];D_\theta(A))}).
\end{aligned}$$

□

$v$  can be represented in the form

$$v(t) = \sum_{k<\alpha} S_k(t)v_k + \int_0^t T(t-s)f(s)ds, \quad (1.3) \quad \boxed{\text{eq1.3}}$$

with

$$\begin{aligned}
S_k(t) &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1-k} (\lambda^\alpha - A)^{-1} d\lambda, \\
T(t) &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda,
\end{aligned}$$

and  $\Gamma$  describing the boundary of

$$\{\lambda \in \mathbb{C} : |\lambda| \geq R^{\frac{1}{\alpha}}, |\text{Arg}(\lambda)| \leq \frac{\pi}{2} + \epsilon\},$$

with  $\epsilon$  positive suitably small, oriented from  $\infty e^{-i(\frac{\pi}{2}+\epsilon)}$  to  $\infty e^{i(\frac{\pi}{2}+\epsilon)}$

**1e1.2** **Lemma 1.4.** *Suppose that (A1)-(A2) hold. Let  $f_0 \in D_{\theta'}(A)$ , with  $\theta < \theta'$  and let*

$$z(t) = \int_0^t T(t-s)f(s)ds.$$

*Then  $Av \in C^1((0,T];X)$  and  $\|(Av)'(t)\|_{D_\theta(A)} \leq Ct^{\alpha(\theta'-\theta)-1}$ .*

*Proof.* From (??), we have  $z'(t) = T(t)f_0$  and, if  $t \in (0,T]$ ,

$$Az'(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} A(\lambda^\alpha - A)^{-1} f_0 d\lambda.$$

We can assume  $\theta' \in (\theta, 1)$ . So,  $D_{\theta'}(A) = (D_\theta(A), D_{1+\theta}(A))_{\theta'-\theta, \infty}$ . This implies, for,  $|\mu| \geq R$ ,  $|\text{Arg}(\mu)| \leq \frac{\alpha\pi}{2}$ ,

$$\|A(\mu - A)^{-1} f_0\|_{D_\theta(A)} \leq C|\mu|^{\theta-\theta'}.$$

So

$$\begin{aligned}
Az'(t) &= \frac{1}{2\pi i t} \int_{\Gamma'} e^{\lambda t} A(\lambda^\alpha t^{-\alpha} - A)^{-1} d\lambda. \\
\|Az'(t)\|_{D_\theta(A)} &\leq C_0 t^{-1} \int_{\Gamma'} e^{Re(\lambda)t} |\lambda^\alpha t^{-\alpha}|^{\theta-\theta'} |d\lambda| \leq C_1 t^{\alpha(\theta'-\theta)-1}.
\end{aligned}$$

□

le1.4

**Lemma 1.5.** Suppose that (A1)-(A2),  $\alpha \in (1, 2)$ ,  $\theta < \frac{1}{\alpha}$ . Let  $f_0 \in D_{\theta'}(A)$ , with  $\theta' > \theta + 1 - \frac{1}{\alpha}$  and let

$$z(t) = S_1(t)f_0.$$

Then  $Av \in C^1((0, T]; X)$  and  $\|(Av)'(t)\|_{D_{\theta}(A)} \leq Ct^{\alpha(\theta' - \theta - 1)}$ . Consequently, if  $\theta' > \theta + 1 - \frac{1}{\alpha}$ ,

$$\int_0^T \|Az'(t)\|_{\theta} dt < \infty.$$

*Proof.* If  $t > 0$ , we have

$$\begin{aligned} Az'(t) &= \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} A(\lambda^{\alpha} - A)^{-1} f_0 d\lambda \\ &= \frac{1}{2\pi i t^{\alpha}} \int_{\Gamma} e^{\lambda} \lambda^{\alpha-1} \left(\frac{\lambda}{t}\right)^{\alpha} - A)^{-1} f_0 d\lambda \end{aligned}$$

so that

$$\|Az'(t)\|_{\theta} \leq C_0 t^{-\alpha} \int_{\Gamma} e^{Re(\lambda)} |\lambda|^{\alpha-1-\alpha(\theta' - \theta)} t^{\alpha(\theta' - \theta)} \|x\|_{\theta'} |d\lambda| \leq C_1 t^{\alpha(\theta' - \theta - 1)}.$$

□

pr1.3

**Proposition 1.6.** We consider the problem

$$\begin{cases} D_t^{\alpha} u(t) = Au(t) + F(t), & t \in [0, T], \\ u^{(k)}(0) = u_k, \end{cases}$$

with the following conditions:

(a)  $F(t) = G(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} v_{[\alpha]}$ , with  $G \in C^1([0, T]; X)$ ,  $G' \in B([0, T]; D_{\theta}(A))$ ,  $v_{[\alpha]} \in D_{1+\theta-\frac{[\alpha]}{\alpha}}(A)$ ;

(b)  $u_0 \in D_{1+\theta}(A)$ ,  $Au_0 + F(0) \in D_{\theta'}(A)$ , for some  $\theta' > \theta$ .

Then  $u(t) = U(t) + z(t)$ , with:

(I)  $U \in C^1([0, T]; X)$ ,  $v = U'$  solution of

$$\begin{cases} D^{\alpha} v(t) = Av(t) + G'(t), & t \in [0, T], \\ v(0) = v_0; \end{cases} \quad (1.4) \quad \text{eq1.4}$$

(II)  $z$  solution of

$$\begin{cases} D^{\alpha} z(t) = Az(t) + Au_0 + F(0), & t \in [0, T], \\ z(0) = 0. \end{cases} \quad (1.5) \quad \text{eq1.5A}$$

*Proof.* By Theorem ??, (??) has a unique solution  $v$ , with  $D^{\alpha}v, Av \in C([0, T]; X) \cap B([0, T]; D_{\theta}(A))$ . We deduce

$$(1 * D^{\alpha}v)(t) = A(1 * v)(t) + G(t) - G(0), \quad t \in [0, T].$$

We set

$$J_{\alpha}g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds.$$

Then  $D^{\alpha}v = (J_{\alpha})^{-1}(v - v_0)$ . We deduce that

$$J_{\alpha}(1 * D^{\alpha}v) = 1 * J_{\alpha}(D^{\alpha}v) = 1 * (v - v_0) = 1 * v - tv_0$$

and

$$D^{\alpha}(1 * v) = D^{\alpha}(tv_0) + 1 * D^{\alpha}v = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} v_0 + 1 * D^{\alpha}v,$$

$$D^{\alpha}(1 * v) = A(1 * v)(t) + G(t) - G(0) + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} v_0, \quad t \in [0, T].$$

Setting

$$U(t) = (1 * v)(t) + u_0,$$

we deduce

$$D^\alpha U(t) = AU(t) + G(t) - F(0) - Au_0 + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} v_0 = F(t) - F(0) - Au_0, \quad t \in [0, T].$$

The conclusion follows. □

**co1.4** **Corollary 1.7.** *Suppose that (A1)-(A2) hold. Suppose, moreover, that*  
 (a)  $k \in C([0, T])$ ,  
 (b)  $u \in C^1((0, T]; D(A))$ ,  $\|Au'(t)\|_{D_\theta(A)} \leq Ct^{\epsilon-1}$ , for some  $\epsilon \in \mathbb{R}^+$ ;  
 (c)  $u$  is a strict solution to

$$\begin{cases} D^\alpha u(t) = Au(t) + \int_0^t k(t-s)Au(s)ds + F(t), & t \in [0, T], \\ u(0) = u_0, & t \in [0, T], \end{cases}$$

with  $F(t) = G(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} v_0$ ,  $G \in C^1([0, T]; X)$ ,  $G' \in B([0, T]; D_\theta(A))$ ,  $v_0 \in D_{1+\theta}(A)$ ,  $u_0 \in D_{1+\theta}(A)$ ,  $Au_0 + F(0) \in D_{\theta'}(A)$ ,  $\theta' > \theta$ .

Then  $u(t) = U(t) + z(t)$ , with

(I)  $U \in C^1([0, T]; X)$ ,  $v = U'$  solution of

$$\begin{cases} D^\alpha v(t) = Av(t) + G'(t) + k(t)Au_0 + \int_0^t k(t-s)Au'(s)ds, & t \in [0, T], \\ v(0) = v_0; \end{cases} \quad (1.6) \quad \text{eq1.5}$$

(II)  $z$  solution of

$$\begin{cases} D^\alpha z(t) = Az(t) + Au_0 + F(0), & t \in [0, T], \\ z(0) = 0. \end{cases}$$

*Proof.* From the assumptions,

$$(k * Au)(t) = k(t)Au_0 + \int_0^t k(t-s)Au'(s)ds$$

belonging to  $C([0, T]; X) \cap B([0, T]; D_\theta(A))$ . So the conclusion follows from Proposition ?? □

**re1.5** **Remark 1.8.** On account of Lemma ??, (??) can be written also in the form

$$\begin{cases} D^\alpha v(t) = Av(t) + G'(t) + k(t)Au_0 + \int_0^t k(t-s)Av(s)ds + \int_0^t k(t-s)Az'(s)ds, & t \in [0, T], \\ v(0) = v_0. \end{cases} \quad (1.7) \quad \text{eq1.6}$$

We set

$$S(v, k)(t) := (k * A(v + z'))(t). \quad (1.8) \quad \text{eq1.7}$$

**le1.6** **Lemma 1.9.** *Suppose that the assumptions of Corollary ?? are satisfied. Let  $\Phi \in X'$ . We set*

$$h(t) = g(t) - \Phi(z(t)).$$

We suppose  $\Phi(Au_0) \neq 0$  and set

$$\chi := \Phi(Au_0)^{-1}.$$

Then  $h \in C^1([0, 1])$ ,  $D^\alpha h'$  is defined and

$$k(t) = K_0(t) - \chi\Phi(Av(t)) - R(v, k)(t), \quad t \in [0, T], \quad (1.9) \quad \text{eq1.8}$$

with

$$K_0(t) = \chi[D^\alpha h'(t) - \Phi(G'(t))], \quad (1.10)$$

$$R(v, k) = -\chi\{k * \Phi[A(v + z')]\}(t) = -\chi\Phi[S(v, k)(t)]. \quad (1.11) \quad \boxed{\text{eq1.10}}$$

On the other hand, suppose that

$$\Phi(u_0) = h(0), \quad \Phi(v_0) = h'(0). \quad (1.12) \quad \boxed{\text{eq1.11}}$$

Let  $(v, k)$  be a strict solution to (??)-(??), with  $v \in C([0, T]; D(A)) \cap B([0, T]; D_{1+\theta}(A))$  and  $k \in C([0, T])$ . We set

$$u := u_0 + 1 * v + z.$$

Then,  $u \in C^1([0, T]; D(A))$ ,  $\|Au'(t)\|_{D_\theta(A)} \leq Ct^{\alpha(\theta' - \theta) - 1} \forall t \in (0, T]$  and  $(u, k)$  is a solution to (??).

*Proof.* Applying  $\Phi$  to the first equation in (??), we easily deduce (??).

On the other hand, let  $(v, k)$  be a strict solution to (??)-(??), with  $v \in C([0, T]; D(A)) \cap B([0, T]; D_{1+\theta}(A))$  and  $k \in C([0, T])$ . Then

$$k(\cdot)Au_0 + k * A(v + z') = k(\cdot)Au_0 + k * Au'.$$

So, by Corollary ??, the two first conditions in (??) are satisfied.

It remains to show that  $\Phi(u) = g$ . Applying  $\Phi$  to the first equation in (??) and comparing with (??), we deduce

$$D^\alpha(\Phi v)(t) = \Phi(D^\alpha v(t)) = D^\alpha h'(t), \quad t \in [0, T].$$

From (??), we deduce  $\Phi v = h'$  and  $\Phi U = h$ . We deduce that

$$\Phi(u) = \Phi(U) + \Phi(z) = g.$$

□

In conclusion, we are reduced to study the system (??)-(??), which we write in the equivalent form

$$\begin{cases} D^\alpha v(t) = Av(t) + G_1(t) + \Psi(Av(t))Au_0 + S_1(v, k)(t), & t \in [0, T], \\ v(0) = v_0, \\ k(t) = K_0(t) + \Psi(Av(t)) - R(v, k)(t), & t \in [0, T], \end{cases} \quad (1.13) \quad \boxed{\text{eq1.12}}$$

with

$$G_1(t) = G'(t) + K_0(t)Au_0,$$

$$\Psi = -\chi\Phi,$$

$$S_1(v, k)(t) = R(v, k)(t)Au_0 + S(v, k)(t), \quad (1.14) \quad \boxed{\text{eq1.13A}}$$

**1e1.7** **Lemma 1.10.** Suppose that (A1)-(A2) hold. We consider the problem

$$\begin{cases} D^\alpha v(t) = Av(t) + \Psi(Av(t))f_0 + f(t), & t \in [0, T], \\ v(0) = v_0, \end{cases} \quad (1.15) \quad \boxed{\text{eq1.13}}$$

Assume that  $\Psi \in X'$ ,  $f_0 \in D_\theta(A)$ ,  $f \in C([0, T]; X) \cap B([0, T]; D_\theta(A))$ ,  $v_0 \in D_{1+\theta}(A)$ . Then (??) has a unique solution  $v$  in  $C([0, T]; D(A)) \cap B([0, T]; D_{1+\theta}(A))$ . Moreover, If  $T_0 \in \mathbb{R}^+$ , there exists  $C(T_0) \in \mathbb{R}^+$  such that, if  $0 < T \leq T_0$ ,

$$\|v\|_{C^\alpha([0, T]; D_\theta(A))} + \|v\|_{C^{\alpha\theta}([0, T]; D(A))} + \|v\|_{B([0, T]; D_{1+\theta}(A))} \leq C(T_0)(\|v_0\|_{D_{1+\theta}(A)} + \|f\|_{B([0, T]; D_\theta(A))}).$$

*Proof.* We set, for  $0 < \tau \leq T$ ,

$$X_\tau := \{V \in C([0, \tau]; D(A)) : V(0) = v_0\},$$

which is a complete metric space with the distance

$$d(V_1, V_2) := \|V_1 - V_2\|_{C([0, T]; D(A))}. \quad (1.16) \quad \boxed{\text{eq1.14A}}$$

If  $V \in X(\tau)$ , we consider the problem

$$\begin{cases} D^\alpha v(t) = Av(t) + \Psi(AV(t))f_0 + f(t), & t \in [0, \tau], \\ v(0) = v_0, \end{cases} \quad (1.17) \quad \boxed{\text{eq1.14}}$$

which, by Theorem ??, has a unique solution  $v = v(V)$ , belonging to  $B([0, T]; D_{1+\theta}(A))$ , with  $D^\alpha v \in B([0, T]; D_\theta(A))$ . Clearly, the solutions in  $[0, \tau]$  are the fixed points of the mapping  $V \rightarrow v(V)$ . If  $V_1, V_2 \in X_\tau$ , we have, setting  $v_j := v(V_j)$ ,

$$d(v_1, v_2) \leq C(T_0)\tau^{\alpha\theta} \|\Psi(A(V_1 - V_2))\|_{C([0, \tau])} \leq C_1(T_0)\tau^{\alpha\theta} d(V_1, V_2).$$

So, if  $\tau$  is sufficiently small, (??) has a unique solution in  $[0, \tau]$ .

In order to extend it, we show that a solution with the desired regularity  $\tilde{v}$  is given in  $[0, \sigma]$ , with  $\sigma \in (0, T)$ , it can be extended in a unique way to a solution, again with the prescribed regularity, in  $[0, (\sigma + \delta) \wedge T]$ . So we set now, for  $\delta \in (0, T - \sigma]$ ,

$$Y_\delta := \{V \in C([0, \sigma + \delta]; D(A)) : V|_{[0, \sigma]} = \tilde{v}\},$$

again equipped with the distance (??) (replacing  $T$  with  $\sigma + \delta$ ). If  $V \in Y_\delta$ , we consider again the problem (??) in the interval  $[0, \sigma + \delta]$ . Again, by Theorem ?? we have a unique solution  $v = v(V)$ ; by the uniqueness guaranteed by this theorem in  $[0, \sigma]$ , we deduce  $v|_{[0, \sigma]} = \tilde{v}$ , so that  $v \in Y_\delta$ . If  $v_j = v(V_j)$ , with  $V_j \in Y_\delta$ ,  $j = 1, 2$ , we deduce from Theorem ?? (III)

$$d(v_1, v_2) = \|v_1 - v_2\|_{C([0, \sigma + \delta]; D(A))} \leq \delta^{\alpha\theta} \|v_1 - v_2\|_{C^{\alpha\theta}([0, \sigma + \delta]; D(A))} \leq C(T_0)\delta^{\alpha\theta} d(V_1, V_2).$$

Choosing  $\delta$  so small that  $C(T_0)\delta^{\alpha\theta} < 1$  (independently of  $\sigma$ ), we can extend in a unique way the solution to  $[0, \sigma + \delta]$ .

The remaining part of the proof is analogous to that of Theorem ??. □

Now we study problem (??). We indicate with  $V_0$  the solution of the problem

$$\begin{cases} D^\alpha V_0(t) = AV_0(t) + G_1(t) + \Psi(AV_0(t)), & t \in [0, T], \\ v(0) = v_0 \end{cases} \quad (1.18) \quad \boxed{\text{eq1.16}}$$

and set

$$K_1(t) = K_0(t) + \Psi(AV_0(t)), \quad t \in [0, T],$$

Of course, the existence and uniqueness of a solution  $V_0$  in  $B([0, T]; D_\theta(A))$  is guaranteed by Lemma ??. We begin with the existence and, to some extent, uniqueness of a solution in a small interval:

**Lemma 1.11.** *Let  $\delta \in \mathbb{R}^+$ . Then there exists  $\tau(\delta) \in (0, T]$ , such that, if  $0 < \tau \leq \tau(\delta)$  (??) has a unique solution  $(v, k)$  with  $D^\alpha v, Av$  in  $B([0, \delta]; D_\theta(A))$ ,  $k \in C([0, \delta])$  and*

$$\max\{\|v - V_0\|_{B([0, \tau]; D_\theta(A))}, \|k - K_0\|_{C([0, \tau])}\} \leq \delta.$$

*Proof.* We set, for  $\tau \in (0, T]$ ,

$$X_{\delta, \tau} := \{(V, H) \in (C([0, \tau]; D(A)) \cap B([0, \tau]; D_{1+\theta}(A))) \times C([0, \tau]) : \max\{\|v - V_0\|_{B([0, \tau]; D_\theta(A))}, \|H - K_1\|_{C([0, \tau])}\} \leq \delta\},$$

which is a complete metric space with the distance

$$d((V_1, H_1), (V_2, H_2)) = \max\{\|V_1 - V_2\|_{B([0, \tau]; D_{1+\theta}(A))}, \|K_1 - K_2\|_{C([0, \tau])}\}.$$

Given  $(V, H)$  in  $X_{\delta, \tau}$ , we consider the problem

$$\begin{cases} D^\alpha v(t) = Av(t) + G_1(t) + \Psi(Av(t))Au_0 + S_1(V, H)(t), & t \in [0, T], \\ v(0) = v_0, \\ k(t) = K_0(t) + \Psi(Av(t)) - R(V, H)(t), & t \in [0, T], \end{cases} \quad (1.19) \quad \boxed{\text{eq1.17}}$$

By Lemma ??, (??) has a unique solution  $(v, k)$  with the prescribed regularity. Clearly, as usual, solving (??) is equivalent to find a fixed point of  $(V, H) \rightarrow (v, k)$ .

From (??), we get

$$\begin{cases} D^\alpha(v - V_0)(t) = A(v - V_0)(t) + \Psi(A(v - V_0)(t))Au_0 + S_1(V, H)(t), & t \in [0, T], \\ (v - V_0)(0) = 0, \\ k(t) - K_1(t) = \Psi(A(v - V_0)(t)) - R(V, H)(t), & t \in [0, T], \end{cases}$$

so that

$$\|v - V_0\|_{B([0, \tau]; D_{1+\theta}(A))} \leq C(T)\|S_1(V, H)\|_{B([0, \tau]; D_\theta(A))}$$

We estimate  $\|S_1(V, H)\|_{B([0, \tau]; D_\theta(A))}$ . By (??), (??), (??) and Lemma ?? we have

$$\begin{aligned} \|S_1(V, H)\|_{B([0, \tau]; D_\theta(A))} &\leq C_0\|S(V, H)\|_{B([0, \tau]; D_\theta(A))} \leq C_1\|H\|_{C([0, \tau])}(\tau\|V\|_{B([0, \tau]; D_{1-\theta}(A))} + \tau^{\alpha(\theta' - \theta)}) \\ &\leq C_1(\|K_1\|_{C([0, T])} + \delta)[\tau(\|V_0\|_{B([0, \tau]; D_{1-\theta}(A))} + \delta) + \tau^{\alpha(\theta' - \theta)}] := \omega_0(\delta, \tau). \end{aligned}$$

So

$$\|v - V_0\|_{B([0, \tau]; D_{1+\theta}(A))} \leq C(T)\omega_0(\delta, \tau).$$

We have also

$$\|k - K_1\|_{C([0, \tau])} \leq C_1\|v - V_0\|_{B([0, \tau]; D_{1+\theta}(A))} + \|R(V, H)\|_{C([0, \tau])} \leq C_2\omega_0(\delta, \tau).$$

As  $\lim_{\tau \rightarrow 0} \omega_0(\delta, \tau) = 0$ , if  $\tau \leq \tau_0(\delta)$  and  $(V, H) \in X_{\delta, \tau}$ ,  $(v, k) \in X_{\delta, \tau}$ .

Let now  $(V_1, H_1), (V_2, H_2)$  belong to  $X_{\delta, \tau}$ . We indicate with  $(v_j, k_j)$  ( $j = 1, 2$ ) the corresponding solutions of (??). It follows

$$\begin{cases} D^\alpha(v_1 - v_2)(t) = A(v_1 - v_2)(t) + \Psi(A(v_1 - v_2)(t))Au_0 + S_1(V_1, H_1)(t) - S_1(V_2, H_2)(t), & t \in [0, \tau], \\ (v_1 - v_2)(0) = 0, \\ k_1(t) - k_2(t) = \Psi(A(v_1 - v_2)(t)) - (R(V_1, H_1)(t) - R(V_2, H_2)(t)), & t \in [0, \tau]. \end{cases}$$

We have

$$\begin{aligned}
\|v_1 - v_2\|_{B([0,\tau];D_{1+\theta}(A))} &\leq C_0(T)\|S_1(V_1, H_1)(t) - S_1(V_2, H_2)\|_{B([0,\tau];D_\theta(A))} \\
&\leq C_1(T)\|S(V_1, H_1)(t) - S(V_2, H_2)\|_{B([0,\tau];D_\theta(A))} \\
&\leq C_1(T)(\|(H_1 - H_2) * A(V_1 + z')\|_{B([0,\tau];D_\theta(A))} + \|H_2 * A(V_1 - V_2)\|_{B([0,\tau];D_\theta(A))}) \\
&\leq C_2(T)(\|H_1 - H_2\|_{C([0,\tau])}(\tau\|V_0\|_{B([0,T];D_{1+\theta}(A))} + \delta) + \tau^\alpha(\theta' - \theta)) \\
&\quad + \tau(\|K_1\|_{C([0,T])} + \delta)\|V_1 - V_2\|_{B([0,\tau];D_{1+\theta}(A))} \\
&\leq \omega_1(\delta, \tau)d((V_1, H_1), (V_2, H_2)),
\end{aligned}$$

with  $\lim_{\tau \rightarrow 0} \omega_1(\delta, \tau) = 0$ . It follows

$$\begin{aligned}
&\|k_1 - k_2\|_{C([0,\tau])} \\
&\leq C_2(\|v_1 - v_2\|_{B([0,\tau];D_{1+\theta}(A))} + \|R(V_1, H_1) - R(V_2, H_2)\|_{C([0,\tau])}) \\
&\leq C_3(\|v_1 - v_2\|_{B([0,\tau];D_{1+\theta}(A))} + \|S(V_1, H_1)(t) - S(V_2, H_2)\|_{B([0,\tau];D_\theta(A))}) \\
&\leq C_3\omega_1(\delta, \tau)d((V_1, H_1), (V_2, H_2)).
\end{aligned}$$

So the conclusion follows from the contraction mapping theorem.  $\square$

We want to show that, in fact, (??) has a unique global solution. The key step is the following

**Lemma 1.12.** *Suppose that (A1)-(A3) hold. Consider problem (??), with  $G_1 \in C([0, T]; X) \cap B([0, T]; D_\theta(A))$ ,  $u_0, v_0 \in D_{1+\theta}(A)$ . Let  $0 < \tau_0 \leq \tau_1 < \min\{2\tau_0, T\}$  and let  $(V, K)$  be a solution in  $[0, \tau_1]$ , with  $V \in B([0, \tau_1]; D_{1+\theta}(A))$ ,  $K \in C([0, \tau_1])$ . Then there exists  $\delta$  positive, independent of  $\tau_1$ , such that (??) has a unique solution  $(v, k)$  in  $[0, (\tau_1 + \delta) \wedge 2\tau_0 \wedge T]$  with  $v \in B([0, (\tau_1 + \delta) \wedge 2\tau_0 \wedge T]; D_{1+\theta}(A))$ ,  $k \in C([0, (\tau_1 + \delta) \wedge 2\tau_0 \wedge T])$  and coinciding with  $(V, K)$  in  $[0, \tau_1]$ .*

*Proof.* Let  $\delta \in \mathbb{R}^+$ . We set

$$\tau(\delta) := (\tau_1 + \delta) \wedge (2\tau_0) \wedge T$$

and

$$\begin{aligned}
X_\delta &:= \{(W, H) \in (C([0, \tau(\delta)]; X) \cap B([0, \tau(\delta)]; D_{1+\theta}(A))) \times C([0, \tau(\delta)]) \\
&\quad : W|_{[0, \tau_1]} = V, H|_{[0, \tau_1]} = K\}.
\end{aligned}$$

For  $(W, H) \in X_\delta$ , we consider the problem

$$\begin{cases}
D^\alpha v(t) = Av(t) + G_1(t) + \Psi(Av(t))Au_0 + S_1(W, H)(t), & t \in [0, (\tau_1 + \delta) \wedge 2\tau_0], \\
v(0) = v_0, \\
k(t) = K_0(t) + \Psi(Av(t)) - R(V, H)(t), & t \in [0, (\tau_1 + \delta) \wedge 2\tau_0],
\end{cases} \tag{1.20} \quad \boxed{\text{eq1.19}}$$

For any  $(W, H) \in X_\delta$ , (??) has a unique solution  $(v, k)$  with  $v \in B([0, \tau(\delta)]; D_{1+\theta}(A))$ ,  $k \in C([0, \tau(\delta)])$ . We observe that, by the uniqueness of the solution of (??),  $v|_{[0, \tau_1]} = V$  and  $k|_{[0, \tau_1]} = K$ . We deduce that  $(v, k) \in X_\delta$ , which we equip with the usual distance

$$d((V_1, H_1), (V_2, H_2)) = \max\{\|V_1 - V_2\|_{B([0, \tau(\delta)]; D_{1+\theta}(A))}, \|H_1 - H_2\|_{C([0, \tau(\delta)])}\}.$$

Now we look for conditions ensuring that the mapping  $(W, H) \rightarrow (v, k)$  is a contraction in  $X_\delta$ . As usual, we get

$$d((v_1, k_1), (v_2, k_2)) \leq C(T)\|S(V_1, H_1) - S(V_2, H_2)\|_{B([0, \tau(\delta)]; D_\theta(A))}.$$

Let  $\tau_1 \leq t \leq \tau(\delta)$ . Then

$$\|S(V_1, H_1)(t) - S(V_2, H_2)(t)\|_{D_\theta(A)}$$

$$\leq \left\| \int_0^t (H_1(t-s) - H_2(t-s))A(V_1(s) + z'(s))ds \right\|_{D_\theta(A)} + \left\| \int_0^t H_2(t-s)(A(V_1(s) - V_2(s)))ds \right\|_{D_\theta(A)}.$$

We set  $\tilde{v} := V_{|[0, \tau_0]}$ ,  $\tilde{h} := H_{|[0, \tau_0]}$ . Then we have, on account of  $t - \tau_1 \leq \tau_0$ ,

$$\int_0^t (H_1(t-s) - H_2(t-s))A(V_1(s) + z'(s))ds = \int_0^{t-\tau_1} (H_1(t-s) - H_2(t-s))A(\tilde{v}(s) + z'(s))ds,$$

so that

$$\left\| \int_0^t (H_1(t-s) - H_2(t-s))A(V_1(s) + z'(s))ds \right\|_{D_\theta(A)} \leq \|H_1 - H_2\|_{C([0, \tau(\delta)])} (\|\tilde{v}\|_{B([0, \tau_0]; D_{1+\theta}(A))} \delta + C_0 \delta^{\alpha(\theta' - \theta)}).$$

Analogously,

$$\begin{aligned} \left\| \int_0^t H_2(t-s)(A(V_1(s) - V_2(s)))ds \right\|_{D_\theta(A)} &= \left\| \int_{\tau_1}^t \tilde{h}(t-s)(A(V_1(s) - V_2(s)))ds \right\|_{D_\theta(A)} \\ &\leq \delta \max(|\tilde{h}|) \|V_1 - V_2\|_{B([0, \tau(\delta)]; D_{1+\theta}(A))} \end{aligned}$$

We deduce that

$$\|v_1 - v_2\|_{B([0, \tau(\delta)]; D_{1+\theta}(A))} \leq \omega_0(\delta) d((V_1, H_1), (V_2, H_2)),$$

with  $\lim_{\delta \rightarrow 0} \omega_0(\delta) = 0$ . We observe that  $\omega(\delta)$  does not depend on  $\tau_1$ . We have also

$$\|k_1 - k_2\|_{C([0, \tau(\delta)])} \leq \|\Psi(A(V_1 - V_2))\|_{C([0, \tau(\delta)])} + \|R(V_1, H_1) - R(V_2, H_2)\|_{C([0, \tau(\delta)])} \leq \omega_1(\delta) d((V_1, H_1), (V_2, H_2)),$$

with  $\lim_{\delta \rightarrow 0} \omega_1(\delta) = 0$ , and the conclusion follows.  $\square$

Now we are able to prove the main result of the paper:

**th1.10**

**Theorem 1.13.** *Suppose that (A1)-(A3). Consider problem ??, with  $u, k$  unknown. Assume that the following further conditions are fulfilled:*

- (a)  $\alpha \in (0, 1]$ ;
- (b)  $F(t) = G(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} v_0$ , with  $G \in C^1([0, T]; X)$ ,  $G' \in B([0, T]; D_\theta(A))$ ,  $\theta \in (0, 1)$ ,  $v_0 \in D_{1+\theta}(A)$ ;
- (c)  $u_0 \in D_{1+\theta}(A)$ ;
- (d)  $Au_0 + F(0) \in D_{\theta'}(A)$ , with  $\theta < \theta'$ ;
- (e)  $\Phi \in X'$ ;
- (f) if  $z$  is the solution of (??) and  $h(t) = g(t) - \Phi(z(t))$ ,  $D^{1+\alpha}h \in C([0, T])$ ,  $h(0) = \Phi(u_0)$ ,  $h'(0) = \Phi(v_0)$ ;
- (g)  $\Phi(Au_0) \neq 0$ .

Then (??) has a unique solution  $(u, k)$  such that  $u - z \in C^1([0, T]; D(A))$ ,  $(u - z)' \in B([0, T]; D_\theta(A))$ ,  $k \in C([0, T])$ .

*Proof.* If  $(u, k)$  is a solution with the required properties,  $k * Au \in C^1([0, T]; X)$  and  $(k * Au)' \in B([0, T]; D_\theta(A))$ . So, by Corollary ??,  $u = U + z$ , with  $v = U'$  solution of (??), or, equivalently (??).

On the other hand, if  $v$  is a solution of (??),  $u := U + z$ , with  $U := u_0 + 1 * v$ , satisfies the two first equations in (??). From (??) we have also  $\Phi(U) = h$  and  $\Phi(D^\alpha v) = D^{1+\alpha}h$ . Applying  $\Phi$  to the first equation in (??), on account of (g), we deduce (??).

.....

$\square$

[?] Problem of determination from final data (not convolution kernels).

Paper [?] Reconstruction of a kernel  $m$  such that  $k = a + m$ , applicable in case  $\alpha \leq 1$ . Even in this case needed not so much regularity, but also more compatibility conditions than here.

[?] Determination of order of derivation  $\alpha$  and coefficient of the second order space derivative  $\alpha \in (0, 1)$ . Hilbert space setting. The operator  $A$  with conditions on the spectrum which are satisfied by a positive self-adjoint compact operator. Assumptions on the Fourier coefficients on the data.

Determination of source term: [?],

## References

- ChNaYaYa1** [1] J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, "Uniqueness in an inverse problem for a one-dimensional fractional diffusion equations", *Inverse Problems* **25**(2009), 16 pp..
- FeKa1** [2] P. Feng, E.T. Karimov, "Inverse source problems for time fractional mixed parabolic-hyperbolic-type equations", *Inverse Ill-Posed Problems* **23** (2015), 339-353.
- Ja2** [3] J. Janno, "Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time fractional diffusion equation", *Electronic Journal of Differential Equations*, **199**(2016), 1-28.
- Ja1** [4] J. Janno, "Determination of time-dependent sources and parameters of nonlocal diffusion and wave equations from final data", *Frac. Calc. Appl. Anal.* **23** (2020), 1678-1701.
- JaKa1** [5] J. Janno, K. Kasemets, "Identification of a kernel in an evolutionary integral occurring in subdiffusion", *J. Inverse Ill Posed Problems*, bf 25 (2017), 777-798.
- KiJa1** [6] N. Kinash, J. Janno, "Inverse problems for a generalized subdiffusion equation with final overdetermination", *Math. Model. Anal.* **24**(2019), 236-262.