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A Natural Deduction Calculus for S4.2

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Abstract We propose a natural deduction calculus for the modal logic **S4.2**. The system is designed to match as much as possible the structure and the properties of the standard system of natural deduction for first-order classical logic, exploiting the formal analogy between modalities and quantifiers. The system is proved sound and complete w.r.t. the standard Hilbert-style formulation of **S4.2**. Normalization and its consequences are obtained in a natural way, with proofs that closely follow the analogous ones for first-order logic.

1 Introduction

Proof theory offers various approaches to modal logic, which differ considerably in how much additional technology they incorporate into the "basic formats" (such as natural deduction, sequent calculi, tableaux, etc.) to deal with modalities.

Perhaps the most successful approach is that based on labelled deductive systems [34, 39, 29]. These systems are capable of handling a very large set of modal logics, and allow the establishment of most proof-theoretic properties.

Other approaches for modal proof theory are those on based Hyper-Sequents [1], indexed tableaux à la Fitting [11, 12], and 2-Sequents/Linear Nested Sequents [26, 27, 18, 19], and their extensions both in sequent and natural deduction settings.

In this paper, we focus on a variation of the systems introduced in [24, 16]. The central concept is a geometric extension of the conventional natural deduction for first-order logic to handle modalities similarly to quantifiers. While our method shares similarities with labelled deductive systems (detailed in [24]), our goal is not to explicitly embed the accessibility relation into the formal systems with specific deductive rules. Instead, we aim to preserve the structure of the fundamental formulation of first-order logic in the style of Prawitz. In our previous research [24, 16], we examined systems ranging from **K** to **S5** (**K, KT, KD, K4, KT4, KD4**, and **S5**), excluding **S4.2**.

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S4.2 lies between **S4** and **S5** and provides a robust treatment of knowledge and belief concepts with appropriate properties, as advocated in previous work [35, 8, 20, 30]. Moreover, within the context of potential infinity, modal logic **S4.2** sheds light on convergent expansions of infinite collections [6, 21]. Beyond epistemology, **S4.2** finds applications in several domains. For example, it is identified as the temporal logic of Minkowski spacetime [15]. In set theory, **S4.2** is known as the modal logic of forcing: if Zermelo-Fraenkel set theory with the axiom of choice (ZFC) is consistent, then the ZFC-provable principles of forcing correspond precisely to those in modal theory **S4.2** [17]. Finally, in the study of abelian groups, **S4.2** emerges as the modal logic of abelian groups with the accessibility relation defined by subgroup isomorphism [5].

Overall, the distinctive properties and applications of **S4.2** across different disciplines highlight the importance of this logic and thus the interest in developing a distinctive proof-theoretic approach for it.

The aim of this paper is therefore to fill the gap of our previous investigation and to tackle **S4.2**. To do so, we need to make substantial modifications to the technical approach we have used for other logics, while maintaining the philosophy of avoiding an explicit accessibility relation in the rules of the deductive system, in line with [24].

Clearly, we are not the only ones to study the proof theory of S4.2; other interesting approaches, which differ significantly from our proposal, are [36] and [9] for sequent calculi, and [4, 39] for natural deduction.

Structure of the paper

- In **Subsection 1.1** we introduce the reader in a semi-formal manner to the underlying ideas of the deductive system that will be introduced and rigorously examined in the following sections.
- In **Section 2** we give the basic definitions of **S4.2** and of position fomulas.
- In **Section 3**, we present our natural deduction system for **S4.2**.
- In **Section 4**, we provide a semantic interpretation of our position formulas. We exploit the fact that **S4.2** is complete for frames whose accessibility relation is a semi-lattice with a minimum, as a byproduct of the result of Shehtmann [33] and Goldblatt [15].
- **Sections 5** and **6** establish the main proof-theoretic results of our system, with proofs that follow Prawitz's proof-theoretical analysis of classical first-order natural deduction.

1.1 Modal interaction and positions: a gentle introduction In natural deduction systems for classical first-order logic, the introduction rule for the universal quantifier is usually written as:

$$\frac{A(x)}{\forall x.A(x)}\ \forall I$$

with the well-known constraint: *the variable* x *must not occur free in any of the undischarged assumptions on which the deduction of* $A(x)$ *depends.* Let us introduce a notion of *interaction* between formulas, by saying that *a formula* A *interacts with a formula* B *with respect to variable* x, if x occurs free in both A and B. We may now restate the constraint for $\forall I$ as: $A(x)$ should not interact (with respect to variable x) with any of the undischarged assumptions on which the deduction of $A(x)$ depends.

Turning now to modalities, the $\Box$ operator can be interpreted as "for any world", and is thus a form of universal quantification. Therefore, we seek an introduction rule of the form:

$$\frac{A}{\Box A}\ \Box I$$

with the constraint that A does not interact with any of the undischarged assumptions on which the deduction of A depends. The definition of the relation of *modal interaction* is now crucial, as argued in [2]. To achieve this, we add new information to a propositional modal formula: a *position* s, i.e. a finite set of uninterpreted tokens. We denote the formula A at position s as A^s and refer to it as a "position formula", which will be abbreviated as "formula" henceforth. The introduction of positions allows for a precise definition of modal interaction between formulas (and thus the formulation of the constraint for the $\Box$-introduction rule.)

Intuitively, a token can be thought of as a point in a space, and also a set of token $\{x_1, \ldots, x_k\}$ denotes in a functional way a unique point in the space.

This idea is visually represented in Figure 1.

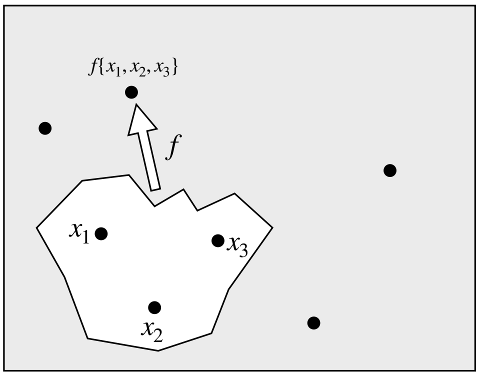

Figure 1: Graphical representation of tokens and positions as points in a space

For the sake of the present paper, this general picture is specialized by taking a semilattice with the minimum as a space, and the supremum operation as the function that returns a point from a set of points, as shown in Figure 2.

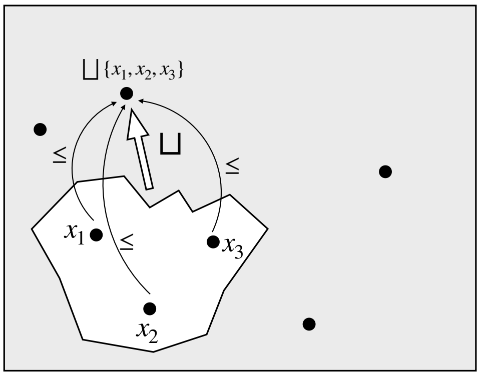

Figure 2: A graphical representation of tokens in a semilattice, where each token is represented as a point in the space. The supremum function is used to determine a unique position in the space from a set of points.

In the syntax that will be used, a set of token $\{x_1, \ldots, x_n\}$ is represented by $x_1, \ldots, x_n$, by dropping the parenthesis, and the union operation is simply denoted by juxtaposition (if s and t are set of token, s, t denotes $s \cup t$, and in particular s, x denotes $s \cup \{x\}$).

Tokens are treated as variables, and modalities operate on tokens in much the same way as quantifiers operate on term variables, taking into account the notion of modal interaction defined via positions.

In functional terms, a position formula $A^{s,x}$ (where $s = x_1,\dots,x_k$ and $x \notin s$) holds at a point $p = s,x$ that functionally depends on the tokens $x_1,\dots,x_k$ and x. Assuming now that x is generic, we can quantify over x, obtaining a formula that holds in a point that depends only on $x_1,\dots,x_k$.

Using this intuition and following the analogy with the universal quantifier, we formulate the introduction rule ($\Box$I) as follows

$$\dfrac{\begin{matrix}\Pi\\ A^{s,x}\end{matrix}}{\Box A^{s}}\ \Box\mathrm{I}$$

with the constraints: $x \notin s$ and x does not occur in (the position of) any undischarged assumption of Π—that is, A does not modally interact with any undischarged assumptions.

Now, let us analyze the dual rule of $\Box$ elimination. Once again, we aim for an elimination rule as similar as possible to the first-order elimination rule for $\forall$:

$$\dfrac{\forall x.A(x)}{A(v)}\ \forall\mathrm{E}$$

with the usual constraint that the term v is free for variable x in $A(x)$.

Intuitively, after the removing of $\Box$ from $\Box A^{s}$, the formula A holds in any position "reachable" from position s, i.e. a position that depends on $s \cup t$ for an arbitrary t. Therefore we define the rule

$$\dfrac{\Box A^{s}}{\Box A^{s,t}}\ \Box\mathrm{E}$$

For the modal operator $\Diamond$ we follow similar ideas and we reason analogously with the first-order existential quantifier $\exists$.

Here is an example of the position-based natural deduction proof of axiom T:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Box A^{\varnothing}]^1}{A^{x,y}}\ \Box\mathrm{E}}{\Box A^{x}}\ \Box\mathrm{I}}{\Box\Box A^{\varnothing}}\ \Box\mathrm{I}}{\Box A \to \Box\Box A^{\varnothing}}\ \to\mathrm{I}^1$$

We conclude the section with a final, important observation.

Remark 1 Once a Kripke model has been fixed, we should *not* assume that a position $x_1,...,x_n$ represents a path that passes through the "tokens/nodes" $x_1,...,x_n$. In fact, if we assume that the accessibility relation is at least a preorder (which is necessary since we are modeling a logic that extends **S4**), it is possible that the worlds corresponding to $x_1,...,x_n$ are incomparable, meaning that they are not in a relation (of preorder) with each other. Therefore, the systems (for logics different from **S4.2**) proposed by Fitting [10, 11], or those of the authors of this article [24, 25], that treat each position as an abstraction of the concept of path, are not directly comparable to the approach of the present paper.

2 The Modal System S4.2

As mentioned in the Introduction, formula occurrences will be labeled with positions—sets of uninterpreted tokens. We assume to have a denumerable collection $\mathcal{T} = \{x_0, x_1, \dots\}$ the elements of which are called *tokens*; x, y, z, possibly indexed, are meta-variables for tokens.

We call *position* each finite (possibly empty) set of tokens. The meta-variables s, t, u possibly indexed, range on the set $\mathcal{P}$ of positions.

We use simple juxtaposition (with an infix comma) for set-theoretic union:

- a finite set of tokens $\{x_1,...,x_n\}$ is represented by $x_1,...,x_n$ (we omit the outer parentheses, but remember that this is not an ordered list—the order of the tokens is irrelevant.)
- With s,t we denote $s \cup t$ and with s,x we denote $s \cup \{x\}$.
- $\varnothing$ denotes the empty set, and therefore $s,\varnothing = s$ and $\varnothing, s = s$.

The language $\mathcal{L}$ of **S4.2** contains the following symbols:

– a countably infinite set of *propositional symbols*, $\mathsf{AT} = \{p_0, p_1, \ldots\}$;
– the *propositional connectives* $\vee, \wedge, \rightarrow, \perp$;
– the *modal operators* $\Box, \Diamond$;
– the *auxiliary symbols* (and).

We define modal formulas as usual; *position-formulas* are obtained by labeling modal formulas with positions.

Definition 2 The set $\mathfrak{mf}$ of *modal formulas* of $\mathcal{L}$ is the least set that contains the propositional symbols and is closed under the propositional connectives and the modal operators. A formula is atomic if it is a propositional symbol, or the connective $\perp$; $\neg A$ is a shorthand for $A \rightarrow \perp$.

Definition 3 A *position-formula* (briefly *p-formula*) is an expression of the form A^s, where A is a modal formula and $s \in \mathcal{P}$. $\mathfrak{I}$ is the set of position-formulas.

In the following, we will sometime write $A * B^s$ for $(A * B)^s$, with $* \in \{\vee, \wedge, \rightarrow\}$.

Given a set/sequence Γ of p-formulas, and a token x, with a little abuse of language we write $x \in \Gamma$ ($x \notin \Gamma$, respectively) to denote that Γ contains (does not contain) a p-formula A^s with $x \in s$ ($x \notin s$).

The notion of subformula is standard. Observe that the subformulas of $\Box A^s$ (or of $\Diamond A^s$) contain all the "instances" of A^s at any position t, like in the first-order case for the subformulas of $\forall x.A(x)$.

Definition 4 (Modal Subformulas) Given a *p-formula* A^s, we recursively define the set $\mathcal{S}(A^s)$ of its subformulas as follows:

- $\mathcal{S}(A^s) = \{A^s\}$ is A is atomic;
- $\mathcal{S}(A \vee B^s) = \mathcal{S}(A^s) \cup \mathcal{S}(B^s) \cup \{A \vee B^s\}$;
- $\mathcal{S}(A \wedge B^s) = \mathcal{S}(A^s) \cup \mathcal{S}(B^s) \cup \{A \wedge B^s\}$;
- $\mathcal{S}(A \rightarrow B^s) = \mathcal{S}(A^s) \cup \mathcal{S}(B^s) \cup \{A \rightarrow B^s\}$;
- $\mathcal{S}(\Box A^s) = \mathcal{S}(A^{s,t}) \cup \{\Box A^s\}$ for any $t \in \mathcal{P}$;
- $\mathcal{S}(\Diamond A^s) = \mathcal{S}(A^{s,t}) \cup \{\Diamond A^s\}$ for any $t \in \mathcal{P}$.

Please note that this definition of subformulas is reminiscent of the concept of subformulas a là Gentzen.

2.1 The Hilbert-style formulation of S4.2 As is standard for all presentations in Hilbert-style of classical systems, we are given the axioms for the language with $\neg, \rightarrow$, and $\Box$, taking into account the following definitions:

1. $\neg A = A \rightarrow \perp$;
2. $A \wedge B = \neg(A \rightarrow \neg B)$;
3. $A \vee B = \neg(\neg A \wedge \neg B)$;
4. $\Diamond A = \neg \Box \neg A$.

We briefly recall the axiomatic ("Hilbert-style") presentation of the logic **S4.2**.

Definition 5 **S4.2** is the smallest set X of formulas that contains all instances of the following schemas:

P1: $A \rightarrow (B \rightarrow A)$
P2: $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

P3: $((\neg B \to \neg A) \to ((\neg B \to A) \to B))$
K: $\Box(A \to B) \to (\Box A \to \Box B)$
T: $\Box A \to A$
4: $\Box A \to \Box\Box A$
C: $\Diamond\Box A \to \Box\Diamond A$

and is closed under the following:

MP: if $A, A \to B \in X$ then $B \in X$;
NEC: if $A \in X$ then $\Box A \in X$.

We write $\vdash_H A$ for $A \in$ **S4.2**. Observe that Axiom **P3** guarantees that the underlying propositional logic is *classical*.

3 Natural Deduction for S4.2

In this section we present the natural deduction system $NK_{4.2}$. The set of derivations from a set Γ of assumptions is defined as the least set that contains Γ and is closed under application of the following rules (where, as usual, a formula in square brackets represents a discharged assumption). Introduction and elimination rules for connectives $\wedge$, $\to$ and the introduction rule for $\vee$ apply between p-formulas with the same position, under some constraints specified below. The $\vee$ elimination rule may act and the rules for $\perp$ may be applied also on p-formulas at different positions.

The notion of discharging of assumptions is standard, and in particular we assume the partial discharging policy, i.e. it is possible to discharge an arbitrary set of occurrences of an assumption, possibly empty. As usual we use square parenthesis to denote discharged assumptions.

Logical rules

$$\dfrac{\begin{matrix}\vdots \\ A^s\end{matrix} \quad \begin{matrix}\vdots \\ B^s\end{matrix}}{A \wedge B^s}\,(\wedge I) \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B^s\end{matrix}}{A^s}\,(\wedge_1 E) \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B^s\end{matrix}}{B^s}\,(\wedge_2 E)$$

$$\dfrac{\begin{matrix}\vdots \\ A^s\end{matrix}}{A \vee B^s}\,(\vee_1 I) \qquad \dfrac{\begin{matrix}\vdots \\ B^s\end{matrix}}{A \vee B^s}\,(\vee_2 I) \qquad \dfrac{\begin{matrix}\vdots \\ A \vee B^s\end{matrix} \quad \begin{matrix}[A^s] \\ \vdots \\ C^t\end{matrix} \quad \begin{matrix}[B^s] \\ \vdots \\ C^t\end{matrix}}{C^t}\,(\vee E)$$

$$\dfrac{\begin{matrix}[A^s] \\ \vdots \\ B^s\end{matrix}}{A \to B^s}\,(\to I) \qquad \dfrac{\begin{matrix}\vdots \\ A \to B^s\end{matrix} \quad \begin{matrix}\vdots \\ A^s\end{matrix}}{B^s}\,(\to E) \qquad \dfrac{\begin{matrix}[\neg A^s] \\ \vdots \\ \perp^t\end{matrix}}{A^s}\,(\perp_c)$$

Modal rules change the position of the p-formula.

$$\dfrac{\begin{matrix}\vdots \\ A^{s,x}\end{matrix}}{\Box A^s}\,(\Box I) \qquad \dfrac{\begin{matrix}\vdots \\ \Box A^s\end{matrix}}{A^{s,t}}\,(\Box E)$$

In the rule $\Box I$, one has $x \notin s$ and $x \notin \Gamma$, where Γ is the set of (open) assumptions on which A^s depends. The token x of $\Box I$ is the *proper token* of the rule.

$$\dfrac{\begin{array}{c}\vdots\\ A^{s,t}\end{array}}{\Diamond A^{s}}\ (\Diamond I) \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \Diamond A^{s}\end{array} \qquad \begin{array}{c}[A^{s,x}]\\ \vdots\\ C^{t}\end{array}}{C^{t}}\ (\Diamond E)$$

In the rule $\Diamond E$, one has $x \notin s$, $x \notin t$ and $x \notin \Gamma$, where Γ is the set of (open) assumptions on which C^{t} depends, with the exception of the discharged assumptions $A^{s,x}$. The token x of $\Diamond E$ is the *proper token* of the rule. Observe that the positions s or T may be $\varnothing$ in any p-formula in those rules (but x is instead a token and must be present.)

Remark 6 At the risk of seeming repetitive, despite the linear typographic notation, we remind that positions are sets, not ordered sequences, and therefore the order in which tokens are listed is irrelevant. For example, the following is a correct derivation of $\wedge I$.

$$\dfrac{\begin{array}{c}\vdots\\ A^{x,y,z}\end{array} \qquad \begin{array}{c}\vdots\\ B^{z,x,y}\end{array}}{A \wedge B^{y,x,z}}\ (\wedge I)$$

The meticulous reader will see positions as equivalence classes of sequences without repetitions modulo the permutation relation and will check that any rule or manipulation we introduce does not depend on the particular representation of a position.

Notation 7

- *With $\Gamma \vdash A^{s}$ we mean that there is a derivation with conclusion A^{s} the non-discharged assumptions of which belong to Γ.*
- *With $\mathcal{H}[\Pi]$ we denote the set of non-discharged assumptions of Π.*

By position renaming we can prove the following (see [38, Vol. 2, pag. 529] for the analogous statement for proper *variables*):

Proposition 8 *Let $\Gamma \vdash A^{s}$. Then there exists a deduction of A^{s} from Γ such that*

1. *each proper token is the proper token of exactly one instance of $\Box I$ or rule $\Diamond E$;*
2. *the proper token of any instance of the $\Box I$ rule occurs only in the sub-derivation above that instance of the rule;*
3. *the proper token of any instance of rule $\Diamond E$ occurs only in the sub-derivation above the minor premiss of that instance of the rule.*

Definition 9 (Token condition) A deduction satisfying the conditions 8.(1)–(3) is said to satisfy the *proper token condition*.

By Proposition 8 we can always assume that, by a suitable renaming of proper tokens, all deductions satisfy the position condition.

Notation 10 *If Π is a deduction, we denote with $\Pi[x/s]$ the tree obtained by replacing each token x in Π with the position s.*

Remark 11 Under reasonable assumptions, this operation of token substitution $\Pi[x/s]$ preserves the token condition. Indeed, if:

1. Π is a deduction satisfying the token condition,
2. x, y are not proper tokens of Π,

then $\Pi[x/\{y\}]$ is a deduction satisfying the token condition.

Finally, we want to make sense of the operation $\Pi[x/s]$ even when the conditions of Remark 11 are not satisfied. Notice that if Π is a deduction satisfying the token condition, we can replace any proper token in Π by a fresh token, to obtain a deduction Π' of the same formula from the same assumptions, and such that x and y satisfy all the conditions of Remark 11. Hence we define $\Pi[x/s]$ as this $\Pi'[x/s]$. In the sequel we will implicitly assume that by $\Pi[x/s]$ we actually mean $\Pi'[x/s]$, for some Π' as above.

3.1 Weak completeness We show that $\mathsf{NK}_{4.2}$ proves, at any position, all the theorems of **S4.2** (weak completeness.) To simplify writing we will sometimes use the derived rule

$$\dfrac{\begin{array}{c}[A^s]\\ \vdots\\ \bot^t\end{array}}{\neg A^s}\,(\rightarrow I)$$

when $s \neq t$, which is a shorthand for

$$\dfrac{\dfrac{\begin{array}{c}[A^s]\\ \vdots\\ \bot^t\end{array}}{\bot^s}\,\bot_c}{\neg A^s}\rightarrow I$$

Remembering that $\neg A$ is defined as $A \rightarrow \bot$.

Since in the formulation of S4.2 we have the abbreviations:

1. $A \wedge B = \neg(A \rightarrow \neg B)$;
2. $A \vee B = \neg(\neg A \wedge \neg B)$;
3. $\Diamond A = \neg\Box\neg A$;

we must prove that:

Proposition 12 *The following formulas are derivable in* $\mathsf{NK}_{4.2}$

1. $A \wedge B \leftrightarrow \neg(A \rightarrow \neg B)^{\varnothing}$;
2. $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)^{\varnothing}$;
3. $\Diamond A \leftrightarrow \neg\Box\neg A^{\varnothing}$;

Proof The formulas $(1)-(2)$ are obtained exactly as in standard propositional natural deduction. Regarding (3), we have

$$\dfrac{\dfrac{[\Diamond A^{\varnothing}]^3 \quad \dfrac{\dfrac{[A^x]^2 \quad \dfrac{[\Box\neg A^{\varnothing}]^1}{\neg A^x}\,\Box E}{\bot^x}\rightarrow E}{\neg\Box\neg A^{\varnothing}}\rightarrow I^{(1)}}{\neg\Box\neg A^{\varnothing}}\,\Diamond E^{(2)}}{\Diamond A \rightarrow \neg\Box\neg A^{\varnothing}}\rightarrow I^{(3)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\Diamond A^{\varnothing}]^2 \quad \dfrac{[A^x]^1}{\Diamond A^{\varnothing}}\,\Diamond I}{\bot^{\varnothing}}\rightarrow E}{\neg A^x}\rightarrow I^{(1)}}{\Box\neg A^{\varnothing}}\,\Box I \quad [\neg\Box\neg A^{\varnothing}]^3}{\bot^{\varnothing}}\rightarrow E}{\Diamond A^{\varnothing}}\,\bot_c^{(2)}}{\neg\Box\neg A \rightarrow \Diamond A^{\varnothing}}\rightarrow I^{(3)}$$

□

Proposition 13

Axioms.:

The following formulas are derivable in $\mathsf{NK}_{4.2}$

1. $A \to (B \to A)^{\varnothing}$;
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))^{\varnothing}$;
3. $((\neg B \to \neg A) \to ((\neg B \to A) \to B))^{\varnothing}$;
4. $\Box(A \to B) \to (\Box A \to \Box B)^{\varnothing}$;
5. $\Box A \to A^{\varnothing}$;
6. $\Box A \to \Diamond A^{\varnothing}$;
7. $\Box A \to \Box\Box A^{\varnothing}$;
8. $\Diamond\Box A \to \Box\Diamond A^{\varnothing}$.

Rules.:

- *if* $\vdash A^{\varnothing}$, *then also* $\vdash \Box A^{\varnothing}$ *(closure under* ***NEC****);*
- *if* $\vdash A^{\varnothing}$ *and* $\vdash A \to B^{\varnothing}$, *then also* $\vdash B^{\varnothing}$ *(closure under* ***MP****).*

Proof

Axioms:

The formulas $(1)-(3)$ are obtained exactly as in standard propositional natural deduction. We give cases $(4)-(8)$.

4):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Box A^{\varnothing}]^1}{A^x}\,\Box E \qquad \dfrac{[\Box(A \to B)^{\varnothing}]^2}{A \to B^x}\,\Box E}{B^x} \to E}{\Box B^{\varnothing}}\,\Box I}{\Box A \to \Box B^{\varnothing}} \to I^{(1)}}{\Box(A \to B) \to (\Box A \to \Box B)^{\varnothing}} \to I^{(2)}$$

5):

$$\dfrac{\dfrac{[\Box A^{\varnothing}]^1}{A^{\varnothing}}\,\Box E}{\Box A \to A^{\varnothing}} \to I^{(1)}$$

6):

$$\dfrac{\dfrac{\dfrac{[\Box A^{\varnothing}]^1}{A^x}\,\Box E}{\Diamond A^{\varnothing}}\,\Diamond I}{\Box A \to \Diamond A^{\varnothing}} \to I^{(1)}$$

7):

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Box A^{\varnothing}]^1}{A^{x,y}}\,\Box E}{\Box A^x}\,\Box I}{\Box\Box A^{\varnothing}}\,\Box I}{\Box A \to \Box\Box A^{\varnothing}} \to I^{(1)}$$

8):

$$
\begin{prooftree}
\AxiomC{$[\Diamond\Box A^{\varnothing}]^2$}
\AxiomC{$[\Box A^{x}]^1$}
\RightLabel{$\Box E$}
\UnaryInfC{$A^{x,y}$}
\RightLabel{$\Diamond I$}
\UnaryInfC{$\Diamond A^{y}$}
\RightLabel{$\Box I$}
\UnaryInfC{$\Box\Diamond A^{\varnothing}$}
\RightLabel{$\Diamond E^{(1)}$}
\BinaryInfC{$\Box\Diamond A^{\varnothing}$}
\RightLabel{$\to I^{(2)}$}
\UnaryInfC{$\Diamond\Box A \to \Box\Diamond A^{\varnothing}$}
\end{prooftree}
$$

Rules.:

For **NEC**, let us suppose that $\vdash A^{\varnothing}$, i.e., we have a proof Π with conclusion $A^{\varnothing}$. Pick a fresh token z, and substitute each position T in Π with T, z. We obtain in this way a proof Π' of A^{z}. Apply now $\Box I$ with proper token z, to obtain a proof of $\vdash \Box A^{\varnothing}$. Closure under **MP** is trivially ensured by rule $(\to E)$.

$\Box$

Theorem 14 (Weak Completeness) *For each position* s, *if* $\vdash_H A$, *then* $\vdash A^{s}$.

4 Semantics and Soundness for $\mathrm{NK}_{4.2}$

As an instrument to obtain the converse of Theorem 14, we give in this section a semantical interpretation of $\mathrm{NK}_{4.2}$, showing—as it is certainly manifest to the reader at this point—that we may interpret positions as ranging over worlds of Kripke models. We start by recalling the standard Kripke semantics for **S4.2**.

4.1 Classical semantics of S4.2 A *frame* is a pair $\langle W, R\rangle$ s.t. W is a non empty set of so-called possible worlds and $R \subseteq W \times W$ is a relation (possibly empty), called accessibility relation. A *model* is a triple $\mathcal{W} = \langle W, R, V\rangle$ s.t. $\langle W, R\rangle$ is a frame and $V : W \to 2^{AT}$ is an evaluation function for propositional symbols.

Definition 15 (Satisfiability) The satisfiability relation $\models$ between a model $\mathcal{W}$, a possible word w ad a formula A is inductively defined in the following way:

- $\mathcal{W}, w \not\models \bot$
- $\mathcal{W}, w \models p \Leftrightarrow p \in V(w)$ for $p \in AT$
- $\mathcal{W}, w \models A \to B \Leftrightarrow \mathcal{W}, w \models A \Rightarrow \mathcal{W}, w \models B$
- $\mathcal{W}, w \models A \vee B \Leftrightarrow \mathcal{W}, w \models A$ or $\mathcal{W}, w \models B$
- $\mathcal{W}, w \models A \wedge B \Leftrightarrow \mathcal{W}, w \models A$ and $\mathcal{W}, w \models B$
- $\mathcal{W}, w \models \Box A \Leftrightarrow \forall v \in W (wRv \Rightarrow \mathcal{W}, v \models A)$
- $\mathcal{W}, w \models \Diamond A \Leftrightarrow \exists v \in W : (wRv$ and $\mathcal{W}, v \models A)$.

Notation 16

- *We write* $\mathcal{W} \models A$ *when for all* $w \in W$ *we have* $\mathcal{W}, w \models A$; $\models A$ *denotes that* $\mathcal{W} \models A$ *for each model* $\mathcal{W}$.
- *Let* $\mathcal{F} = \langle W, R\rangle$ *be a frame, we write* $\mathcal{F} \models A$ *to denote that* $\forall V : W \to 2^{AT}, \langle W, R, V\rangle \models A$.

Definition 17 Let $\mathfrak{C}$ be a class of frames, we say that a modal logic L is characterized (i.e. it is sound and complete) by $\mathcal{C}$ if

$$\vdash A \Leftrightarrow \forall \mathcal{F} \in \mathfrak{C}, \mathcal{F} \models A.$$

Like many modal logics, **S4.2** is characterized by a certain class of frames, that of *direct pre-orders* [8].

Definition 18 A frame $\langle \mathcal{W}, R\rangle$ is a *direct pre-order* iff

1. R is reflexive and transitive;
2. $\forall u, v, w \in \mathcal{W} : uRv \,\&\, uRw \Rightarrow \exists z \in \mathcal{W} : vRz \,\&\, wRz$.

Theorem 19 *S4.2 is characterized by the class of direct pre-orders.*

4.1.1 S4.2 and semi-lattices If we now try to assign semantics to positional formulas using Theorem 19, we run into a non-trivial problem. We have to decide which point in the Kripke model could be uniquely associated with a set of tokens $\{x_1, \ldots, x_n\}$. The standard, naive choice would be to take one of the upper bounds of the worlds associated with each token, but this choice would not be unique, and in a direct pre-order the supremum of a finite set of elements might not exist. However, we can use some results of Goldblatt and Shetmann, which imply that **S4.2** is also characterized by a class of ordered structures much smaller than direct pre-orders, that of semilattices with a minimum element, where the problem disappears.

With *semi-lattice with minimum* we mean a structure $\langle A, \leq, \mathbf{m}\rangle$ where:

1. $\langle A, \leq\rangle$ is a partial order;
2. $\forall x, y \in A$ there exists the least-upper bound $x \sqcup y$;
3. $\mathbf{m} \in A$ is the minumum.

Let $\mathfrak{L}$ be the class of semi-lattices with minumum. Let $\mathfrak{R}_0 = \langle \mathbb{R}_0, \leq, 0\rangle$ be the set of non negative real numbers with the standard total order: $\mathfrak{R}_0$ is a semi-lattice with 0 as minumum. Let us now consider the nonnegative quadrant $\mathfrak{R}_0^2 = \langle \mathbb{R}_0^2, \sqsubseteq\rangle$ where $\sqsubseteq \subseteq \mathbb{R}_0^2$ is the partial order relation defined as

$$(x, y) \sqsubseteq (x', y') \Leftrightarrow x \leq x' \,\&\, y \leq y'$$

$\mathfrak{R}_0^2$ is a semi-lattice with $(0, 0)$ as minimum. The following difficult result is independently due to Goldblatt [15] and Shethmann [33].

Theorem 20 (Shehtmann, Goldblatt) *For each modal formula A*

$$\vdash_H A \;\Leftrightarrow\; \mathfrak{R}_0^2 \models A,$$

where $\vdash_H A$ denotes $A \in$ **S4.2** (i.e. A is derivable in the Hilbert-style presentation, see Section 2.1).

Since $\mathfrak{R}_0^2$ is a semi-lattice, from Theorem 20 we obtain easily:

Theorem 21 *S4.2 is characterized by (that is, it is sound and complete with respect to) the class $\mathfrak{L}$.*

Proof *Soundness:* Since each semi-lattice (with minumum) is a direct pre-order, the result follows from the soundness of **S4.2** w.r.to direct pre-orders. *Completeness:* Let us suppose that $\forall \mathcal{F} \in \mathfrak{L}, \mathcal{F} \models A$. Then $\mathfrak{R}_0^2 \models A$, and therefore $\vdash_H A$, by Theorem 20. □

4.2 The semantics of $\mathrm{NK}_{4.2}$ and its soundness In this section we give a semantic interpretation of position formulas, by proving that each theorem of our system is valid under the semantics. This step is preparatory for obtaining Theorem 31, which shows that all the theorems proved by the system $\mathrm{NK}_{4.2}$ at the empty position are also theorems of **S4.2**.

Let us denote with $\langle \mathcal{W}, \leq, \mathbf{m}\rangle$ a generic semi-lattice with minumum $\mathbf{m}$. Moreover let us denote with $\sqcup : \mathcal{W}^2 \longrightarrow \mathcal{W}$ the *lub* operator (extended in a natural way to finite subsets of $\mathcal{W}$).

Definition 22 (Semi-lattice models and interpretations)

1. A semi-lattice Kripke model is a 4-ple $\mathfrak{A} = \langle \mathcal{W}, \leq, \mathbf{m}, V\rangle$ where $\langle \mathcal{W}, \leq, \mathbf{m}\rangle$ is a semi-lattice with minumum $\mathbf{m}$ and $V : \mathcal{W} \longrightarrow 2^{AT}$ is a function that maps worlds in sets of atomic formulas.
2. An interpretation is a pair $\langle \mathfrak{A}, \rho_{\mathfrak{A}}\rangle$ s.t.
   (a) $\mathfrak{A} = \langle \mathcal{W}, \leq, \mathbf{m}, V\rangle$ is a semi-lattice model;

(b) $\rho_{\mathfrak{A}} : \mathcal{T} \to \mathcal{W}$ is a function that maps tokens into worlds. We extend $\rho_{\mathfrak{A}}$ to finite set of tokens in the following way (observe that the empty position is interpreted on the minimum of the semi-lattice):
  (i) $\rho_{\mathfrak{A}}(\varnothing) = \mathbf{m}$;
  (ii) $\rho_{\mathfrak{A}}(x_1, \ldots, x_n) = \bigsqcup\{\rho_{\mathfrak{A}}(x_1), \ldots, \rho_{\mathfrak{A}}(x_n)\}$

In the following, with $|\mathfrak{A}|$ we denote the set of possible worlds of $\mathfrak{A}$, $\mathcal{W}$. The satisfaction relation $\Vdash$ for position-formulas on an interpretation is now defined as:

$$\rho_{\mathfrak{A}} \Vdash A^s \Leftrightarrow \mathfrak{A}, \rho_{\mathfrak{A}}(s) \models A.$$

Relation $\Vdash$ is extended in the trivial way to sequences Γ of p-formulas. Finally, *logical consequence* is defined as:

$$\Gamma \Vdash A^s \Leftrightarrow \forall \mathfrak{A}. \forall \rho_{\mathfrak{A}}. (\mathfrak{A}, \rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \mathfrak{A}, \rho_{\mathfrak{A}} \Vdash A^s).$$

4.2.1 Soundness As a first step towards soundness we prove that $\models$ and $\Vdash$ are "interchangeable". The following is trivial for *propositional* p-formulas.

Proposition 23 *For any interpretation $\mathfrak{A}$:*

- $\rho_{\mathfrak{A}} \nVdash \bot^s$
- $\rho_{\mathfrak{A}} \Vdash p^s \Leftrightarrow p \in V(w)$ *for any* $p \in AT$
- $\rho_{\mathfrak{A}} \Vdash A \to B^s \Leftrightarrow \rho_{\mathfrak{A}} \Vdash A^s \Rightarrow \rho_{\mathfrak{A}} \Vdash B^s$
- $\rho_{\mathfrak{A}} \Vdash A \vee B^s \Leftrightarrow \rho_{\mathfrak{A}} \Vdash A^s$ *or* $\rho_{\mathfrak{A}} \Vdash B^s$
- $\rho_{\mathfrak{A}} \Vdash A^s \wedge B^s \Leftrightarrow \rho_{\mathfrak{A}} \Vdash A^s$ *and* $\rho_{\mathfrak{A}} \Vdash B^s$.

To extend the proposition to modal p-formulas, we first observe this simple fact about lattices.

Fact 24 *Given a semi-lattice $\langle A, \leq, \boldsymbol{m} \rangle$ and $a \in A$ we have:*

$$\{w \in A : w \geq a\} = \{w \in A : w = a \sqcup u, u \in A\}$$

To improve readability, in the following proofs all the quantifications should be understood relative to the set of worlds of the interpretation/model under consideration. With a little abuse of notation, if $\rho_{\mathfrak{A}}$ is an interpretation, with $\rho_{\mathfrak{A}}[x/w]$ we denote the interpretation obtained by mapping the token x to the world w.

Lemma 25 *Let s be a position and let x be a token s.t. $x \notin s$.*

$$\rho_{\mathfrak{A}} \Vdash \Box A^s \Leftrightarrow \forall w \in |\mathfrak{A}|, \rho_{\mathfrak{A}}[x/w] \Vdash A^{s,x}$$

Proof $\rho_{\mathfrak{A}} \Vdash \Box A^s$
$\Leftrightarrow$
$\mathfrak{A}, \bigsqcup \rho_{\mathfrak{A}}(s) \Vdash \Box A$
$\Leftrightarrow$
$\forall w \geq \bigsqcup \rho_{\mathfrak{A}}(s) : \quad \mathfrak{A}, w \Vdash A$
$\Leftrightarrow$ by fact 24
$\forall u, \mathfrak{A} : \quad \bigsqcup \rho_{\mathfrak{A}}(s) \sqcup u \Vdash A$
$\Leftrightarrow$
$\forall u, \mathfrak{A} : \quad \bigsqcup \rho_{\mathfrak{A}}[x/u](s, x) \Vdash A$
$\Leftrightarrow$
$\forall u : \quad \rho_{\mathfrak{A}}[x/u] \Vdash A^{s,x}$ □

By contraposition we have thus:

Corollary 26 *Let s be a position and let x be a token s.t. $x \notin s$.*

$$\rho_{\mathfrak{A}} \Vdash \Diamond A^s \Leftrightarrow \exists w \in |\mathfrak{A}|, \rho_{\mathfrak{A}}[x/w] \Vdash A^{s,x}$$

The following is the analogue to the first-oder substitution lemma [31]:

Lemma 27 *Let* s, t *be positions and let* x *be a token s.t.* $s \cap t = \varnothing$ *and* $x \notin s$*:*

$$\rho_{\mathfrak{A}} \Vdash A^{s,t} \Leftrightarrow \rho_{\mathfrak{A}}[x/\rho_{\mathfrak{A}}(t)] \Vdash A^{s,x}$$

Proof $\rho_{\mathfrak{A}} \Vdash A^{s,t}$
$\Leftrightarrow$
$\mathfrak{A}, \rho_{\mathfrak{A}}(s,t) \Vdash A$
$\Leftrightarrow$
$\mathfrak{A}, \rho_{\mathfrak{A}}[x/\rho_{\mathfrak{A}}(t)](s,x) \Vdash A$
$\Leftrightarrow$
$\rho_{\mathfrak{A}}[x/\rho_{\mathfrak{A}}(t)] \Vdash A^{s,x}$. □

We can now state the properties of logical consequence w.r.t. modal operators.

Lemma 28

1. $x \notin s, x \notin \Gamma, \Gamma \Vdash A^{s,x} \Rightarrow \Gamma \Vdash \Box A^{s}$;
2. $s \cap t = \varnothing$, $\Gamma \Vdash \Box A^{s} \Rightarrow \Gamma \Vdash A^{s,T}$;
3. $s \cap t = \varnothing$, $\Gamma \Vdash A^{s,T} \Rightarrow \Gamma \Vdash \Diamond A^{s}$;
4. $x \notin s,t,x \notin \Gamma, \Gamma \Vdash \Diamond A^{s}, \Gamma, A^{s,x} \Vdash B^{T} \Rightarrow \Gamma \Vdash B^{T}$.

Proof Simple. We give only the first two cases for $\Box$, the other two cases of $\Diamond$ being handled in a symmetric way. The reader should note that the proof mimic the analogous standard proof for first order quantifiers.

1. Let $x \notin s$ and $x \notin \Gamma$.
   $\Gamma \Vdash A^{s,x}$
   $\Leftrightarrow$
   $\forall \mathfrak{A}, \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \rho_{\mathfrak{A}} \Vdash A^{s,x})$
   $\Leftrightarrow$
   $\forall \mathfrak{A}, \forall w \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}}[x/w] \Vdash \Gamma \Rightarrow \rho_{\mathfrak{A}}[x/w] \Vdash A^{s,x})$
   $\Rightarrow$ since $x \notin \Gamma$
   $\forall \mathfrak{A}, \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \forall w, \rho_{\mathfrak{A}}[x/w] \Vdash A^{s,x})$
   $\Leftrightarrow$ by Lemma 25 since $x \notin s$
   $\forall \mathfrak{A}, \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \rho_{\mathfrak{A}} \Vdash \Box A^{s})$
   $\Leftrightarrow$
   $\Gamma \Vdash \Box A^{s}$.
2. let $s \cap t = \varnothing$.
   $\Gamma \Vdash \Box A^{s}$
   $\Leftrightarrow$
   $\forall \mathfrak{A}, \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \rho_{\mathfrak{A}} \Vdash \Box A^{s})$
   $\Leftrightarrow$ x fresh and by lemma 25
   $\forall \mathfrak{A}, \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \forall w, \rho_{\mathfrak{A}}[x/w] \Vdash A^{s,x})$
   $\Rightarrow$
   $\forall \mathfrak{A}, \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \rho_{\mathfrak{A}}[x/\rho_{\mathfrak{A}}(t)] \Vdash A^{s,x}))$
   $\Leftrightarrow$ by Lemma 27
   $\forall \mathfrak{A}, \forall \rho_{\mathfrak{A}}, (\rho_{\mathfrak{A}} \Vdash \Gamma \Rightarrow \rho_{\mathfrak{A}} \Vdash A^{s,t}))$
   $\Leftrightarrow$
   $\Gamma \Vdash A^{s,t}$

□

We can therefore state the following easy lemma whose proof is a simple induction on the height of Π (i.e. the height of Π when seen as a tree). The propositional cases are handled as for classical propositional logic. The modal cases are a direct consequence of Lemma 28.

Lemma 29 *For each A^s and for each derivation Π of A^s, it holds that $\mathcal{H}[\Pi] \Vdash A^s$.*

As an immediate corollary we have:

Theorem 30 (Soundness)

$$\Gamma \vdash A^s \Rightarrow \Gamma \Vdash A^s$$

And finally:

Theorem 31

$$\vdash A^{\varnothing} \Rightarrow \vdash_H A$$

Proof If $\vdash A^{\varnothing}$, by soundness, for all interpretations $\langle \mathfrak{A}, \rho_{\mathfrak{A}} \rangle$, it holds that $\rho_{\mathfrak{A}} \Vdash A^{\varnothing}$. Therefore, for all interpretations $\mathfrak{A}$, $\rho_{\mathfrak{A}}(\varnothing) \Vdash A$. Or, since $\rho_{\mathfrak{A}}(\varnothing)$ is the minumum of the semilattice:

$$\vdash A^{\varnothing} \Rightarrow \forall \langle \mathcal{W}, \leq, \mathbf{m}, V \rangle,\ \langle \mathcal{W}, \leq, \mathbf{m}, V \rangle, \mathbf{m} \models A. \tag{1}$$

For the sake of contradiction, let us now suppose that there exists a semi-lattice model $\mathfrak{A} = \langle \mathcal{W}, \leq, \mathbf{m}, V \rangle$ and a world $w_0 \in \mathcal{W}$, $w_0 \neq \mathbf{m}$, for which $\mathfrak{A}, w_0 \not\models A$. Consider then the semi-lattice model
$\mathfrak{A}_0 = \langle \mathcal{W}_0, \leq_{w_0}, \mathbf{w}_0, V_{w_0} \rangle$, where $\mathcal{W}_0 = \{x \in \mathcal{W} : x \geq w_0\}$ and $\leq_{w_0}$ and V_{w_0} are the restrictions of $<$ and V, respectively, to $\mathcal{W}_0$. By assumption, $\mathfrak{A}_0, \mathbf{w}_0 \not\models A$, which contradicts (1) since we have a model where A is not true in the minimum element. Summarizing, if $\vdash A^{\varnothing}$, then $\mathfrak{A} \models A$ *for all semi-lattice models* $\mathfrak{A}$. Theorem 21 gives the conclusion.

□

5 Normalization and its Consequences

In this section we define a notion of reduction between proofs and show that any proof can be reduced to a normal form in a finite number of steps. We then derive the main proof-theoretic properties of the system from the normalization result. As we noted in the Introduction, we make a point of following the argument and the proofs in Prawitz's [31] for first order logic[1], in order to substantiate our claim that position-formulas allow for a treatment *as close as possible to that of first-order logic.*

To be brief, and following Prawitz, we restrict our analysis of classical **S4.2** to the negative fragment $\bot, \Box, \rightarrow$[2] in order to obtain a normalization theorem with a meaningful subformula property and consequently a purely syntactical consistency proof of the full natural deduction system, since $\vee$ and $\Diamond$ can be trivially defined.

It would be easy to extend the definitions and results to the whole language, once again following closely those of standard first-order logic.

We write

$$\begin{matrix} B^t \\ \Pi \\ A^s \end{matrix}$$

to say that Π is a deduction of A^s having some (possibly zero) occurrences of formula B^t among its assumptions, and we write

$$\begin{matrix} \Pi \\ A^s\, {}^{R} \end{matrix}$$

to say that Π is a deduction of formula A^s whose last rule is R.

It is easy to show that one can assume that the conclusion of the $\bot_c$ rule is always atomic. To show this fact let us repeatedly apply the following transformations [31, pp. 39][3]:

$$\begin{array}{c}[\neg(A\to B)^s]\\ \Pi\\ \dfrac{\bot^t}{A\to B^s}\end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\dfrac{[A^s]^3\,[A\to B^s]^1}{B^s}\qquad[\neg B^s]^2}{\bot^t}}{\neg(A\to B)^s}\to I^{(1)}\\ \Pi\\ \dfrac{\dfrac{\bot^t}{B^s}\,\bot_c^{(2)}}{A\to B^s}\to I^{(3)}\end{array}$$

$$\begin{array}{c}[\neg(\Box A)^s]\\ \Pi\\ \dfrac{\bot^t}{\Box A^s}\end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\dfrac{[\Box A^s]^1}{A^{s,x}}\qquad[\neg A^{s,x}]^2}{\bot^{s,x}}}{\neg(\Box A)^s}\to I^{(1)}\\ \Pi\\ \dfrac{\dfrac{\bot^t}{A^{s,x}}\,\bot_c^{(2)}}{\Box A^s}\end{array}$$

where x is a fresh token.

In order to define the notion of deduction in *normal form*, we first introduce the notions of *contractions*, *reduction steps*, and *reduction sequence* (see, e.g., [13].)

5.1 Proper contractions and reducibility The relation $\rhd$ of *proper contractibility* between deductions is defined as follows.

$$\dfrac{\dfrac{\begin{array}{c}[A^s]\\ \Pi_1\\ B^s\end{array}}{A\to B^s}\qquad\begin{array}{c}\Pi_2\\ A^s\end{array}}{B^s} \quad\rhd\quad \begin{array}{c}\Pi_2\\ A^s\\ \Pi_1\\ B^s\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\Pi\\ A^{s,x}\end{array}}{\Box A^s}}{A^{s,t}} \quad\rhd\quad \begin{array}{c}\Pi[x/t]\\ A^{s,t}\end{array}$$

Remark 32 It is easy to verify that contractions transform deductions into deductions (modulo some token renaming).

Definition 33 (Reducibility between deductions)

1. The relation $\succ$ of *immediate reducibility* between deductions is the "context closure" of $\rhd$, defined as follows: $\Pi_1 \succ \Pi_2$ if and only if there exist deductions Π_3 and Π_4 such that $\Pi_3 \rhd \Pi_4$ and Π_2 is obtained by replacing Π_3 with Π_4 in Π_1.
2. The relation $\overset{*}{\succ}$ of *reducibility* is the transitive and reflexive closure of $\succ$.

5.2 Normalization

Definition 34 (Normal forms and normalizable deductions)

1. A deduction Π is in *normal form* if there is no deduction Π' such that $\Pi \succ \Pi'$;
2. A deduction Π is *normalizable* if there is a deduction Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and Π' is in normal form.

The proof of the normalization theorem is standard (see e.g. [32, 37, 14]). Nevertheless, we present a comprehensive exposition of all the pertinent details. We introduce the notions of the *degree* of a formula and the notion of *maximum formula*.

Definition 35 (Degree of a formula)

1. The *degree* $\deg(A)$ of a modal formula A is recursively defined as:
   (a) $\deg(p) = 0$ if p is a proposition symbol;
   (b) $\deg(\neg A) = \deg(\Box A) = \deg(\Diamond A) = \deg(A) + 1$;
   (c) $\deg(A \wedge B) = \deg(A \vee B) = \deg(A \to B) = \max\{\deg(A), \deg(B)\} + 1$.
2. The *degree* $\deg(A^s)$ of formula A^s is just $\deg(A)$.

Definition 36 (Maximum formula)

1. A *maximum formula* in a derivation Π is a formula A^x which is conclusion of an introduction rule $I*$ of a connective $*$, and principal premiss of an elimination rule $E*$ of the same connective.
2. A maximum formula A^s in Π is of *highest degree* if
$$\deg(A^s) = \max\{\deg(B^t) : B^t \text{ is a maximum formula in } \Pi\}.$$

Let $C[\Pi]$ be the set of maximum formulas of Π. For the normalization theorem we will use the lexicographic ordering between pairs of natural numbers[4].

Theorem 37 (Weak normalization) *For each derivation Π there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and Π' is in normal form.*

Proof The proof is on well ordering induction on pairs (d, n) of natural numbers. We associate to each derivation Π a pair (called *rank*) $\#[\Pi] = (d, n)$ s.t.

- $d = max\{\deg(A^s) : A^s \in C[\Pi]\}$;
- $n =$ number of maximum formulas of degree d.

We then prove the following **claim**:
$$\#[\Pi] > (0,0) \Rightarrow \exists \Pi' (\Pi \overset{*}{\succ} \Pi' \ \&\ \#[\Pi'] < \#[\Pi]).$$

1. Let us suppose that $\#[\Pi] > (0,0)$;
2. pick a maximum formula of highest degree A^s in Π s.t. the sub-derivation Π^* ending with A^x does not contain any other maximum formula of the same degree. Let us call *good* such a maximum formula;
3. perform the relevant contraction.

The resulting derivation Π' has a smaller rank compared to Π, meaning $\#[\Pi'] < \#[\Pi]$. In fact, no contraction of a good maximum formula of rank d can generate other maximum formulas of degree greater than or equal to d. So either d strictly decreases or n. More precisely we have two cases to be examinated:

1. $\#[\Pi] = (d, n)$ with $n > 1$. In this case the contraction of the good maximum formula could produce other maximum formulas but with degree lower than d, therefore for the resulting derivation Π' we have $\#[\Pi'] = (d, n-1) < (d, n)$;
2. $\#[\Pi] = (d, 1)$. Obviously the resulting derivation cannot have no more maximum formulas of degree d but possibly lower. We have two subcases:
   (a) the resulting derivation Π' has no more maximum formulas (we have only one maximum formula in Π and the reduction does not generate other maximum formulas), namely it has rank $(0,0)$ and we have finished;
   otherwise
   (b) the reduction may generate other maximum formulas, in addition to those already present, but all these maximum formulas has degree $d^o < d$. Let d' the maximal degree of the maximum formulas , we have therefore that $\#[\Pi'] = (d', k) < (d, n)$.

Using the **claim**, since the lexicographic order is well founded, for each derivation Π there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and $\#[\Pi'] = (0,0)$, so we have proved the theorem. □

6 Consequences of Normalization

In this section we prove a Consistency Theorem, as a main consequence of normalization. We strictly follow Prawitz. Proofs of main results are easy variations of the ones in [31] for classical first order logic.

Definition 38 (Thread) A finite sequence $({A_i}^{s_i})_{i \leq m}$ of formulas in a deduction is a *thread* if:

1. for all $i < m$, ${A_i}^{s_i}$ is immediately above ${A_{i+1}}^{s_{i+1}}$;
2. ${A_m}^{s_m}$ is the end-formula of the deduction;
3. ${A_0}^{s_0}$ is an assumption (either discharged or undischarged);

A *branch* is an initial segment of a thread satisfying some properties:

Definition 39 (Branch) By a branch in a deduction we understand an initial segment ${A_1}^{s_1} \dots {A_i}^{s_n}$ of a thread ${A_1}^{s_1} \dots {A_n}^{s_n} {A_{n+1}}^{s_{n+1}} \dots {A_m}^{s_m}$ in a deduction such that ${A_n}^{s_n}$ is either

1. the first formula occurrence in the thread that is the minor premiss of an application of $\rightarrow \mathrm{E}$ rule
2. the last formula occurrence in the thread (the end-formula of the deduction) if there is no such minor premiss in the thread.

By a *main branch* we mean a branch that is also a thread and contains no minor premiss of $\rightarrow \mathrm{E}$ rule.

Branches in normal deductions have a nice structure. It is easy to prove the following proposition:

Proposition 40 (Structure of branches in normal deduction) *Let Π be a normal deduction and let $b = {A_1}^{s_1} \dots {A_i}^{s_n}$ a branch in Π. Then there exists a modal formula occurrence ${A_i}^{s_i}$, called the* minimum formula *in b, that splits b in two (possibly empty) parts with the following properties:*

1. *each ${A_j}^{s_j}$, $j < i$, is a major premiss on an $\rightarrow \mathrm{E}$ rule or the premiss of a $\Box\mathrm{E}$ rule and contains ${A_{j+1}}^{s_{j+1}}$ as a subformula; we call the sequence $({A_j}^{s_j})_{j \leq i}$ an* elimination sequence*;*
2. *each ${A_j}^{s_j}$, $i < j$ and $j \neq n$ is a premiss of an $\rightarrow \mathrm{I}$ rule or of a $\Box\mathrm{I}$ rule and is a subformula of ${A_{j+1}}^{s_{j+1}}$; we call the sequence $({A_j}^{s_j})_{i \leq j}$ an* introduction sequence*;*
3. *the last formula ${A_n}^{s_n}$, provided $n \neq i$, is premiss of an $\rightarrow \mathrm{I}$ rule or premiss of an $\Box\mathrm{I}$ rule or the $\perp_c$ rule.*

Proof The formula occurrences in b that are major premisses of an $\rightarrow \mathrm{E}$ rule precede all the formula occurrences that are premisses of a $\rightarrow \mathrm{I}$ rule or a $\Box\mathrm{I}$ rule or a $\perp_c$ rule. Otherwise, there is a first formula occurrence in b, call it ${A_k}^{s_k}$ that is a major premiss of an $\rightarrow \mathrm{E}$ rule but succeeds a premiss of an $\rightarrow \mathrm{I}$ rule or a $\Box\mathrm{I}$ rule or a $\perp_c$ rule: ${A_k}^{s_k}$ would be a maximum formula formula, and this contradicts the assumption that Π is normal. The formula occurrences in b that are major premisses of an $\Box\mathrm{E}$ rule precede all the formula occurrences that are premisses of an $\Box\mathrm{I}$ rule or a $\perp_c$ rule. Otherwise we can reason as in the previous case.

Let ${A_l}^{s_l}$ be the first formula occurrence in b that is premiss of an $\rightarrow \mathrm{I}$ rule or premiss of the $\Box\mathrm{I}$ rule or the $\perp_c$ rule, or let ${A_l}^{s_l} = {A_n}^{s_n}$: by definition, ${A_l}^{s_l}$ is a minumum formula and satisfies clauses 1) and 3). Moreover, every ${A_j}^{s_j}$ such that $j < i < n$ is a premiss of an $\rightarrow \mathrm{I}$ rule or of an $\Box\mathrm{I}$ rule or of the $\perp_c$ rule. We can exclude the latter possibility, since a premiss of the $\perp_c$ rule is an occurrence of the formula $\perp$ and can be a consequence of an $\rightarrow \mathrm{E}$ rule only. Hence ${A_l}^{s_l}$ satisfies also clause 2).

□

In order to prove the following, desirable Corollary, we need to define a notion *order between branches* in a deduction Π. A main branch (i.e. a branch that ends with the end-formula of the sequence) has order 0. A branch ending with the minor premiss of an application of the $\rightarrow \mathrm{E}$ rule has order $n+1$ if the major premiss of the same application of the $\rightarrow \mathrm{E}$ rule belongs to a branch of order n.

Corollary 41 (Subformula property) *Given a normal deduction Π of a formula A^s from a set Γ of assumptions, we call Δ the set of all formula occurrences in Π except for assumptions discharged by a $\bot_c$ rule and for occurrences of $\bot$ that stand below such assumptions. Then each modal formula occurrence ${A_i}^{s_i}$ in Δ is either a subformula of A^s or a formula in Γ.*

Proof It is enough to prove that the subformula property holds for all formula occurrences ${A_i}^{s_i}$ in a branch of order k, on the assumption that the property holds for all formulas in branches of order $h < k$. Let be Π a normal deduction of the formula A from assumptions Γ. Consider a branch $\mathbf{b} = {A_1}^{s_1} \ldots {A_n}^{s_n}$ and its minimum formula ${A_m}^{s_m}$. The property clearly holds for ${A_n}^{s_n}$: in fact, ${A_n}^{s_n}$ is either the conclusion A^s or the minor premiss of an application of $\rightarrow$ E rule with the major premiss of the shape $A_l \rightarrow {B_l}^{s_l}$ which belongs to a branch of order $h = k-1 < k$. By Proposition 40, the Corollary holds for all ${A_j}^{s_j}$ such that $m < j < n$.

If ${A_1}^{s_1}$ has not been discharged by an application of a $\bot_c$ rule, then either ${A_1}^{s_1} \in \Gamma$ (and we can conclude) or ${A_1}^{s_1}$ is discharged by an application of $\rightarrow$ I. In the latter case, the consequence of the application of $\rightarrow$ I has the form $A_1 \rightarrow {A_l}^{s_1}$ and then belongs either to the introduction sequence of the branch $\mathbf{b}$ or to some branches of order less than k. So ${A_1}^{s_1}$ is a subformula of the described kind, and we can conclude again by Proposition 40 (the results holds for all ${A_j}^{s_j}$, $j < m$).

Finally, if ${A_1}^{s_1}$ has been discharged by an application of the $\bot_c$ rule, we have three cases: i) either ${A_1}^{s_1}$ is a premiss of an introduction rule ($\rightarrow$ I or $\Box$I). then ${A_1}^{s_1} = {A_m}^{s_m}$; ii) or ${A_1}^{s_1}$ is a major premiss of an $\rightarrow$ E rule and so ${A_2}^{s_2} = {A_m}^{s_m}$ is an occurrence of $\bot^t$; iii) or ${A_1}^{s_1}$ is a minor premiss of an $\rightarrow$ E rule and so ${A_1}^{s_1} = {A_n}^{s_n}$. Hence the proof is complete also in the latter three cases $\Box$

As an immediate consequence we have the following *Consistency Theorem*:

Theorem 42 (Consistency) *For each position* s, $\nvdash \bot^s$.

7 Related Work and Conclusions

The system presented here is a natural offspring of the natural deduction systems presented in [24, 25, 16], where the reader can find a detailed comparison of our position-based approach with the more semantically oriented labeled (or indexed) systems by Negri [29], Simpson [34] and Viganò [39].

Masini started to use positions in his 2-Sequents [26, 27], and subsequently with the collaboration of Martini in [23, 22]. At that time the restricted notion of positions he used (only natural numbers) did not allow proper treatment of larger classes of modal logics. Some years ago Lellmann et al. reformulated 2-Sequents as Linear Nested Sequents [18], a restricted form of Brünnler's modal nested sequents [7]. Nested sequents, in turn, "are essentially the same thing" as Fitting's prefixed tableaus [10, 11] "in the same way that classical tableau proofs and classical sequent proofs are" [12, p. 292]. We refer the reader to [25, Sect. 5] for a detailed description of how position-based systems relate to prefixed tableaus and to Mints' indexed systems [28], of course in the context of sequent calculi, since all these formalisms are (or can be seen as) generalizations of sequent calculi. These formalisms all share the same approach—limit the "semantical" decoration to the minimum, preserving the basic structure of the proof-theory of first-order logic.

In all these systems, however, positions (or labels, indexes, etc. according to the terminology of the specific approach) can be understood semantically as paths in the nodes of a suitable Kripke model. As we noted in Remark 1, this is not the case for the system of this paper, which has to use a novel notion of position in order to tackle **S4.2**'s peculiarities.

Further work

Intuitionism: We are planning a study of intuitionistic **S4.2**. From a syntactic point of view, the intuitionistic system is obtained naturally, by removing the rule $\perp_c$ and including the intuitionistic rule

$$\frac{\perp^s}{A^t}\perp_i.$$

The technical aspect of this work will be challenging as we intend to tackle strong normalisation and confluence. We also want to extend the Curry-Howard correspondence to the resulting system. The intuitionistic system could be used in the (difficult) study of the relations between such an intuitionistic **S4.2** and the two main constructive set theories **ZFI** and **CZF**.

Completeness and refutation: We have proved the completeness of our system in a "minimalist" way, showing that we derive all the theorems of the Hilbert-style system. We could, of course, look for a *direct proof of completeness*, without detours into other systems. One possible approach would be to use our system of natural deduction to "disprove" a formula, that is, to extract countermodels of underivable formulas.

To obtain such a standalone completeness, we plan to move to a sequent calculus.

We can first transform the rules of natural deduction into sequent rules, following the well-known transformations for first-order logic, at the cost of introducing some cuts. For example the left/right rules for $\Box$ could become:

$$\frac{\Gamma, A^{s,t} \vdash \Delta}{\Gamma, \Box A^s \vdash \Delta}(\Box\vdash) \qquad\qquad \frac{\Gamma, \vdash A^{s,x}, \Delta}{\Gamma, \vdash \Box A^s \Delta}(\vdash\Box)$$

where in $\vdash\Box$ the token x does not occur in s, Γ, Δ.

Then we have to prove a cut elimination theorem (which does not come for free from normalization) and address completeness and the concept of disproof using a Schütte-like technique adapted to the finite case (see e.g. [13]). This corresponds to constructing a potentially infinite tree which, if finite, provides a derivation, otherwise has an infinite branch which *should* allow us to construct a countermodel. We think this is a difficult task (hence the "should"). That is, the infinite branch allow us to construct a syntactic structure, but it is possible that such a structure is not automatically a Kripke model corresponding to a lattice with a minimum.

A similar scenario occurs, for example, in [3], where a proposal for a multimodal temporal logic was made. Unfortunately, the existence of an infinite branch, if it occurs, allows the construction of a first-order syntactic structure, that cannot be used as a model for the logic under study. To finalize the completeness proof, it was essential to show that the first-order syntactic structure is isomorphic (using the back-and-forth technique) to a Kripke model of the modal logic.

Notes

1. Not for modal logics.

2. Prawitz examines the functionally complete fragment $\perp, \forall, \rightarrow$

3. Unless necessary, we will sometimes omit the names of the rules and the labels of the assumptions that have been removed.

4. $(n,m) < (p,q)$ if either $n < p$ or ($n = p$ and $m < q$)

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