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COMPUTABLE TOPOLOGICAL ABELIAN GROUPS

MARTINO LUPINI, ALEXANDER MELNIKOV, AND ANDRE NIES

ABSTRACT. We study the algorithmic content of Pontryagin - van Kampen duality. We prove that the process of dualization is computable in the important cases of compact and locally compact totally disconnected Polish abelian groups. The applications of our main results include solutions to questions of Kihara and Ng about presentations of connected Polish spaces, and an unexpected arithmetical characterisation of direct products of solenoid groups among all Polish groups.

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1. INTRODUCTION

1.1. Overview. We study Polish groups by combining ideas from computability theory with tools of abstract harmonic analysis and algebraic topology.

The celebrated Pontryagin - van Kampen duality essentially reduces the study of compact abelian groups to the theory of discrete abelian groups. For instance, for connected compact abelian G and H , we have that G is topologically isomorphic to H if and only if their discrete duals \widehat{G} and \widehat{H} are isomorphic. So the expectation might be that *deciding* the topological isomorphism problem for compact abelian groups can be reduced to solving this problem for the dual groups, which are discrete (a context in which much is known [39, 32]). Unfortunately, one fundamental issue with Pontryagin - van Kampen duality in the literature is that calculating duals usually involves non-constructive considerations, all proofs tend to be non-algorithmic, and even the definition of the dual of a group (1.3) appears to be completely non-effective in an algorithmic sense.

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We use tools of effective algebra [1, 14] and computable analysis [47, 56] to resolve this issue. More specifically, using a wide variety of techniques we prove that *Pontryagin - van Kampen duality is computable in the case of connected compact Polish groups*. Our second main result establishes *a computable version of the duality for totally disconnected locally compact abelian groups*. The methods we use are new and promise a lot more. For instance, we give the first example of an effective version of Čech cohomology from algebraic topology, and extend a result in effective algebra due to Dobritsa [7] to the setting of procountable groups.

We apply these results and techniques to give unexpectedly low (arithmetical) estimates for the isomorphism problem for natural subclasses of compact connected abelian groups, perhaps most notably for direct products of solenoid groups. Interestingly, with some extra work our techniques can be used to solve several open problems in computable topology seemingly unrelated to Polish groups. For instance, we give the first known example of a Δ_2^0 -metrized connected compact Polish space not homeomorphic to any computably metrized Polish space; note that we do not restrict ourselves to Polish groups here. This answers a question recently raised by Kihara and also independently posed in [24]. See §1.6.2 for further open questions and their solutions.

To formally state and discuss our results we need some background. The rest of the introduction will proceed as follows. In Subsection 1.2 below we give a general introduction to computable mathematics to motivate our investigations. Then in Subsection 1.3 we briefly discuss Pontryagin - van Kampen duality and what is known about its algorithmic content. Subsections 1.4 and 1.5 contain the main results about compact and totally disconnected locally compact groups, respectively. In Subsection 1.6 we state and discuss the above-mentioned corollaries of our results. Finally, in Subsection 1.7 we briefly discuss the duals of t.d.l.c. groups.

1.2. Computable mathematics. Our paper contributes to a fast developing branch of effective mathematics which combines methods of computable algebra [14, 1, 15] with tools of computable analysis [3, 56, 47] to advance both subjects. The main tools of such studies are the notions of computability of algebraic and topological structures; Turing [54, 55], Fröhlich and Shepherdson [17], Maltsev [34], Rabin [48] and others suggested various notions of computability for infinite mathematical structures and spaces.

The standard approach to computability in effective algebra is as follows. A computable presentation of a (discrete, countably infinite) algebraic structure is its isomorphic copy upon the domain \mathbb{N} in which the operations and relations are Turing computable [17, 34, 48]. In topology and Banach space theory the situation is more complex since structures are almost never countable. However, at least in the separable case one can use a dense countable set to define computability, as follows. Following the early ideas of Turing [54, 55], say that a Polish space is computably metrized (computable Polish) if there is a countable dense subset $(x_i)_{i \in \mathbb{N}}$ and a complete metric d compatible with the topology such that $d(x_i, x_j)$ can be uniformly computed with precision 2^{-n} . For Banach spaces [47] one usually fixes the metric associated with some complete norm and also additionally requires the operations to be computable (to be clarified). In the context of this paper, we follow [38, 37] and allow the metric to be compatible but not fixed (that is, different presentations can have different metrics).

Initially, computably presented structures and spaces were mainly used to formally illustrate that certain proofs and procedures in the classical literature can or cannot be performed algorithmically; see [1, 14, 47]. For example, it is well-known that the algebraic closure \overline{F} of a computable field F is computable via a computable embedding $\phi: F \rightarrow \overline{F}$, but $\phi(F)$ does not have to be a computable (decidable) subset of \overline{F} [17].

Beginning with [21], these methods have found applications in the study of classification problems not necessarily restricted to computable mathematics. We discuss some of these below; see [10, 8, 2, 9] for further discussion. The intuition is that for

many common non-trivial classes of objects, the problem of characterising computably presentable members of the class tends to be as hard as describing arbitrary (not necessarily computable) members of the class by invariants of some kind. This intuition is formalized using definability techniques, hierarchies, and computability-theoretic relativisation. Results of this sort are somewhat akin to those in descriptive set theory [19], but these two approaches sometimes provide slightly different complexity estimates for the same class; e.g., compare [16] with the corresponding results in [19, 25]. In some cases the computability-theoretic versions of results are more “constructive” and fine-grained. In particular, computable results typically can be relativized to an arbitrary oracle, and thus they imply the respective “boldface” topological estimates.

For example, Downey and Montalbán [8] proved that the isomorphism problem for computable torsion-free abelian groups is Σ_1^1 -complete. This means that any problem which involves an exhaustive search through the uncountably many members of the Baire space $\mathbb{N}^{\mathbb{N}}$ can be computably transformed into the problem of deciding whether two computable torsion-free abelian groups are isomorphic. The result of Downey and Montalbán can be uniformly relativised to any oracle X , which means that the problem of deciding whether two X -computable torsion-free abelian groups are isomorphic is a $\Sigma_1^1(X)$ -complete problem. In particular, it follows that the isomorphism problem for (the set of reals coding) such groups is analytic complete under continuous reducibility, i.e., it is Σ_1^1 -complete in the boldface hierarchy. This provides strong evidence that countable torsion-free abelian groups are unclassifiable up to isomorphism; see [21, 8] for a detailed discussion¹.

Our goals include the study from the perspective outlined above of the algorithmic content of Pontryagin - van Kampen duality, and its application to classification problems.

1.3. Pontryagin - van Kampen duality. Given a topological abelian group G , the character group \widehat{G} of G is the collection of all continuous homomorphisms from G to the unit circle group \mathbb{R}/\mathbb{Z} under the compact-open topology (the topology of uniform convergence on compact sets), with pointwise addition. Note that \widehat{G} is abelian as well. Unless otherwise stated, *we will assume that all our groups are Polish and abelian*. Pontryagin - van Kampen duality states that, if G is locally compact, then \widehat{G} is also locally compact; furthermore $\widehat{\widehat{G}}$ is topologically isomorphic to G via the map sending $g \in G$ to the evaluation map $\phi \mapsto \phi(g)$. It follows that, similarly to Stone duality in the case of Boolean algebras, the character group \widehat{G} contains all the information about G . For a locally compact abelian group G , the character group \widehat{G} is usually called the Pontryagin - van Kampen dual of G , or simply the dual of G if there is no danger of confusion. In Section 4 we provide the details about Pontryagin - van Kampen duality necessary for our proofs. We refer the reader to the books [46, 40] for more on this subject. For now, we note that G is discrete countable iff \widehat{G} is compact Polish. In that case, G is torsion-free iff \widehat{G} is connected, and G is torsion iff \widehat{G} is totally disconnected.

Note that the definition of \widehat{G} seems to be non-algorithmic. Nonetheless, there is one (and, perhaps, only one) instance of Pontryagin - van Kampen duality that is somewhat evidently computable, namely the case of profinite and discrete torsion abelian groups. Smith [52] was perhaps the first to note this. However, for almost 40 years after the publication of [52] essentially no progress had been done towards understanding the algorithmic content of duality beyond this special case.

Recently, the second author has initiated a systematic investigation of the computability-theoretic aspects of Pontryagin - van Kampen duality, with applications to classification problems in topological group theory [37]. The paper [37] focusses on the classes of

¹We also note that, while the present article was under review, Paolini and Shelah [43] announced that the isomorphism problem for torsion-free abelian groups is Borel complete.

compact and discrete abelian groups. Among other results, it formally clarifies the conjecture of Smith.

The main tool in [37] is *the computable version of Pontryagin - van Kampen duality* when passing from discrete to compact groups. More specifically, for a computable discrete G , its dual can be computably metrized. As a consequence of the aforementioned result of Downey and Montalbán [8], the isomorphism problem for compact connected abelian groups (represented as completions) is Σ_1^1 -complete. However, neither the result nor the techniques developed in [37] help to “construct” the dual \widehat{G} of a given computably metrized compact G . Our first goal is to completely settle the compact connected case.

1.4. Connected compact groups: the first main result. Recall that the definition of the character group seems rather non-constructive in the connected case, in the following sense. Since the dual of a compact group G is discrete, it can be viewed as a collection of isolated paths through the Baire space. One can use Σ_1^1 bounding (see e.g. [50]) to see that \widehat{G} has a Δ_1^1 presentation. So the complexity of \widehat{G} could potentially belong to an arbitrary high level of the hyperarithmetical hierarchy. Can we obtain a better complexity estimate and at least establish an upper bound (such as, e.g., $\Delta_{\aleph_2}^0$) for the hyperarithmetical level? This question was raised in the conclusion of [37]; see also Problem 21(3) of [11].

The estimate that we give below was unexpected since our initial conjecture was that the complexity of \widehat{G} has to be non-arithmetical. Recall that a compact computably metrized Polish space is *effectively compact* if the set of all 2^{-n} -covers by basic open balls of the space is computably enumerable uniformly in n . The theorem is the desired computability-theoretic version of Pontryagin – van Kampen duality in the compact connected/discrete torsion-free case.

Theorem 1.1. *For a torsion-free abelian group G , the following are equivalent:*

- (1) G is computably presentable as a discrete group;
- (2) \widehat{G} admits an effectively compact presentation.

It is well-known that, in general, $0'$ can list all covers of a given compact computably metrized Polish space (see, e.g., [24, 29] for the technical details). It follows that, without the assumption of effective compactness, the dual of a computably metrized connected G is Δ_2^0 ; this is a significant improvement over the crude Δ_1^1 bound. Soon we will see that the bound Δ_2^0 is sharp and that the assumption of effective compactness cannot be removed in Theorem 1.1; this is Theorem 1.5.

The proof of Theorem 1.1 is rather indirect. In (2), we naturally assume that the group operations are computable; however, this assumption is actually not necessary for the proof of (2) \rightarrow (1) to work. It relies on a new constructive definition of Čech cohomology and a result from algebraic topology asserting that the first Čech cohomology group of the underlying space of a connected abelian Polish group is isomorphic to its dual. The proof also makes use of a theorem of Khisamiev [31] stating that every torsion-free abelian group that admits a computably numerable presentation also admits a computable presentation.

We also note that (1) \rightarrow (2) strengthens the main result of [37], and that (2) \rightarrow (1) is uniform in the case when the group is non-zero; this will be important in applications. We also conjecture that (2) \rightarrow (1) is non-uniform in general.

1.5. Beyond compact groups: the t.d.l.c. case. Can we extend Theorem 1.1 to cover locally compact groups which are neither compact nor discrete? To answer this question, we will need new ideas and techniques. The class of totally disconnected locally compact (t.d.l.c.) groups is perhaps the narrowest class extensively studied in the literature

(recent papers include [58, 57, 20, 4, 28]) which contains both the countable discrete and the profinite Polish groups.

In Definition 3.4(2) we will introduce a notion of computability for abelian t.d.l.c. groups. The problem is that the dual of a t.d.l.c. group is not totally disconnected in general. The duals of t.d.l.c. abelian groups are exactly the extensions (in the sense of super-groups here and throughout) of compact abelian groups by discrete torsion abelian groups; see, e.g., [5]. Such groups are sometimes called locally elliptic. The most commonly accepted notion of computability for general Polish groups is computable metrizability, i.e., there exists a computable complete metric with respect to which the operations are computable. Thus, the second main result of this article stated below is perhaps the best result one could hope for in the t.d.l.c. case.

Theorem 1.2. *Suppose G is a computable abelian t.d.l.c. group. Then its dual \widehat{G} is computably metrized Polish.*

We believe that the proof of Theorem 1.2 is of independent interest. To circumvent a difficulty in constructing a computable metric in Theorem 1.2, we extend a well-known result of Dobritsa [7] (see Theorem 4.2) from computable abelian group theory to arbitrary computable abelian t.d.l.c. groups; this is Proposition 5.1. Of course, the result of Dobritsa has no direct analog in the theory of computable topological groups. Thus, it is not surprising that the technical proof of Proposition 5.1 relies on some novel strategies specific to the subject; see §5.1.1 for a detailed discussion.

How about a converse of Theorem 1.2? As we have already mentioned above, the obvious obstruction is that \widehat{G} does not have to be t.d.l.c. We restrict ourselves to G such that both G and \widehat{G} are t.d.l.c.; see Subsection 1.7 for a discussion of the general case. The abelian groups such that both the group and its dual are t.d.l.c. are exactly the extensions of profinite groups by torsion groups. Equivalently, these are exactly the locally compact abelian protorsion groups. A more careful analysis of the proof of Theorem 1.2 in this special case shows:

Theorem 1.3. *Suppose G is an abelian t.d.l.c. group such that \widehat{G} is also t.d.l.c.. Then G has a computable presentation if, and only if, \widehat{G} has a computable presentation.*

We note that, in contrast with Theorem 1.2, the proof of Theorem 1.3 is (computably) uniform.

1.6. Consequences. We now discuss several applications of our computable duality results and of the techniques used to prove them. First, in §1.6.1 we apply Theorem 1.1 to measure the complexity of the isomorphism problem for special classes of compact abelian groups. Then in §1.6.2 we discuss an application of the methods developed in the proof of Theorem 1.2. More specifically, it follows that the assumption of effective compactness in Theorem 1.1 cannot be removed. With a bit of extra work this result can be applied to simultaneously answer several open questions about computable metric spaces.

1.6.1. Applications to classification problems. Our first corollary is concerned with the complexity of the effective classification problem for various subclasses of compact Polish abelian groups. We follow [21, 11] and measure this using the special index sets which are called the characterisation problem and the isomorphism problem (to be defined in Subsection 8.2). Informally, we use the universal Turing machine to list all (partially) computably metrized group presentations and ask which indices (pairs of indices) correspond to members of a certain class of groups (respectively, to isomorphic members of the class). We attack Problem 21(3) of [11]:

Measure the complexity of the effective classification problem

for natural subclasses of compact Polish groups.

There are many potential applications of Theorem 1.1 to various subclasses of connected compact abelian groups. We state only three such applications below:

Corollary 1.4. *For each of the following classes, both the characterization problem and the isomorphism problem are arithmetical:*

- (1) *compact abelian Lie groups;*
- (2) *direct products of solenoid groups;*
- (3) *connected compact abelian groups of finite covering dimension.*

As usual, the corollary can be relativized to an arbitrary oracle. The corollary can be informally interpreted as follows. Given a presentation of a group, we can use only local properties of the presentation (such as finite covers and their first-order or computable properties) to recognise whether a given group is, say, homeomorphic to a product of solenoid groups. Also, given two such groups, we can arithmetically decide whether they are (algebraically) homeomorphic. Note that the usual definition involves at least one set-quantifier.

While (1) and (3) are relatively straightforward consequences of Theorem 1.1, (2) is non-trivial since it relies on the technical main result of [10]. In our proof we establish only somewhat crude arithmetical upper bounds. Perhaps with some extra work at least some of our estimates can be turned into optimal completeness results; we leave this open.

1.6.2. *Applications to the foundations of computable topology.* We begin this paragraph with the natural question: Is the assumption of effective compactness necessary to prove (2) \rightarrow (1) of Theorem 1.1? In view of (2) \rightarrow (1) of Theorem 1.1, this is equivalent to answering the following more general question in the abelian case:

- (Q1) Is every computably metrized compact connected group *homeomorphic* to an effectively compact one?

We will answer this question below in the negative. Interestingly, our next theorem which answers this question also solves several further problems which we state and discuss next.

One of the first tasks in any emerging theory is to establish the equivalence (or non-equivalence) of some of the most basic definitions and assumptions which lie at the foundations of the theory. Point-set topology is notorious for its zoo of various notions of regularity of spaces, the most fundamental of which are known to be non-equivalent via relatively straightforward but clever counterexamples. In stark contrast, *computable* topology seems to be essentially completely missing the proofs that many of its computability-theoretic notions are (non-)equivalent. This is partially explained by the fact that proving (non-)equivalence of such notions presents a significant challenge. For instance, the problem of comparing effectively compact and computably metrized presentation has recently attracted a considerable attention. It follows from [24, 29] that every computably metrized Stone space is homeomorphic to an effectively compact one; see [24] for an explanation. In contrast, Lemma 3.21 of [29] gives an example of an effectively metrized compact Polish space which is not homeomorphic to an effectively compact Polish space. Ng has also recently announced that every effectively compact metric space is homeomorphic to a computably metrized space which is not effectively compact. All these results rely on advanced modern techniques. In a recent personal communication with the second author, Ng has raised the following question:

- (Q2) Is every computably metrized compact *connected* Polish space homeomorphic to an effectively compact one?

This is similar to our question above stated for connected groups, but it is concerned with arbitrary connected Polish spaces.

Somewhat unexpectedly, it takes much effort to prove that there exists a Δ_2^0 -metrized Polish space not homeomorphic to a computably metrized one (answering a question posed by Selivanov [51]); see [29, 24] for three substantially different proofs all of which are non-trivial. All known examples share the same feature, namely they use connected components of the constructed spaces to “code” an undecidable set. Kihara in his recent CiE2020 talk asked:

- (Q3) Is every Δ_2^0 -metrized *connected* compact Polish space homeomorphic to a computably metrized one?

This question was also independently raised in [24] (Question 2). Also, [24] (Question 3) asks the analogous question for Δ_2^0 -metrized connected Polish groups.

We will see in Corollary 1.6 that, with the help of Theorem 1.1, our result stated below can be used to answer all these questions in the negative using a unified approach.

Theorem 1.5. *There exists a computably metrized connected group G such that \widehat{G} has no computable presentation as a discrete group.*

The proof of Theorem 1.5 is quite different from the proof of a similar profinite counterexample in [37]; it relies on a new diagonalization strategy. Even though the constructed group is connected compact, the proof is much more technically related to Theorem 1.2 than to Theorem 1.1. With some extra work, we will derive:

Corollary 1.6.

- (1) There exists a computably metrized connected compact Polish group not homeomorphic to any effectively compact Polish space. (This simultaneously answers (Q1) and (Q2).)
- (2) There exists a Δ_2^0 -metrized connected compact Polish group not homeomorphic to any computably metrized Polish space. (This simultaneously answers (Q3) and Question 3 of [24].)

Note that in the corollary, although both counterexamples happen to be computably metrized compact groups, we diagonalize against all *spaces*. Jason Rute gave an elegant argument showing that a computably metrized compact Polish group G is effectively compact iff the left Haar probability measure is computable iff the right Haar probability measure is computable. See his post [12, Section 17]. It thus follows from (1) of Corollary 1.6 that there is a computably metrized connected compact group not homeomorphic to any computably metrized compact group with computable Haar probability measure. Willem Fouche asked whether every computable compact Polish group has computable Haar measure. A fixed presentation of a group can have this property (see [37] and [12, Theorem 15.1]). Our corollary gives the strongest possible negative answer to this question, in the sense that no computable presentation of the constructed group can have computable Haar measure.

1.7. The locally elliptic case. Recall that the duals of t.d.l.c. groups are called locally elliptic groups; they are exactly the extensions of compact abelian groups by discrete torsion abelian groups. Although the terminology is perhaps not self-explanatory, this property can be viewed as a generalization of local finiteness of a discrete group. Platonov [44] initiated the systematic study of not necessarily abelian locally elliptic groups in the 1960s. We thank Yves Cornuier for this bit of history and cite [6] for some recent results on locally elliptic groups.

We leave open whether our methods can be combined and extended to prove computability (or non-computability) of the duality between arbitrary t.d.l.c. and locally elliptic abelian groups, we conjecture that it should be possible. Perhaps, one needs a suitable general notion of effective local compactness, such as a strong version of effective σ -compactness, to make the converse of Theorem 1.2 work. However, at the

current state of the theory we do not seem to possess enough techniques to cover this more general case, and thus leave it to be investigated in the future.

2. PRELIMINARIES

2.1. Computable topological spaces. A *computable topological space* is a pair (X, ν) , where $\nu : \mathbb{N} \rightarrow \tau$ is a numbering of a countable basis of the topological space X , so that

$$\nu(i) \cap \nu(j) = \bigcup_{(i,j,k) \in R} \nu(k),$$

where $R \subseteq \mathbb{N}^3$ is c.e. More generally, an open set U of X is c.e. or effectively Σ_1^0 (relative to ν) if there is a c.e. set I such that $U = \bigcup_{i \in I} \nu(i)$. In a computable topological space, the intersection of two basic open sets is a c.e. open set in a uniform way.

Definition 2.1. Given a computable topological space (X, ν) , we call

$$N^x = \{i : x \in \nu(i)\}$$

the name of x . A point x is called *computable* if it has a computably enumerable name.

Definition 2.2. A function $f : X \rightarrow Y$ between two computable topological spaces is *effectively continuous* if there is a c.e. family F of pairs of (indices of) basic open sets such that:

- (C1): for every $(U, V) \in F$, $f(U) \subseteq V$;
- (C2): for every $x \in X$ and basic open $E \ni f(x)$ in Y there exists a basic open $D \ni x$ in X such that $(D, E) \in F$.

Definition 2.3. A function $f : X \rightarrow Y$ between two computable topological spaces is *effectively open* if there is a c.e. family F of pairs of basic open sets such that

- (O1): for every $(U, V) \in F$, $f(U) \supseteq V$;
- (O2): for every $x \in X$ and any basic open $E \ni x$ there exists a basic open $D \ni f(x)$ such that $(E, D) \in F$.

2.2. Effectively compact metric spaces. Recall that a ball in a computably metrized space is basic if it has a positive rational radius and is centered in a special point.

Definition 2.4. A computably metrized Polish space is *effectively compact* if there is a uniformly computable procedure which, given $\varepsilon \in \mathbb{Q}^+$, enumerates all covers of the space by basic open ε -balls.

Remark 2.5. We note that an apparently weaker definition that also occurs in the literature is via an effective version of total boundedness: one asks that given $\varepsilon \in \mathbb{Q}^+$ one can compute *some* cover of the space by basic open ε -balls. However, this definition is in fact equivalent to the one we use. To obtain our version from the weaker formulation, the idea is as follows. Take a finite collection $(B_\varepsilon(x_i))$ of basic open ε -balls. If it is a cover, the values of the continuous function $z \mapsto \sup_i (\varepsilon - d(z, x_i))$ are all positive, and hence all greater than 2δ for a sufficiently small but unknown δ . So we can wait for a rational $\delta > 0$ so that for the cover $(B_\delta(y_k))$ by finitely many δ -balls we are effectively given by hypothesis, for each k there is i such that $d(y_k, x_i) < \varepsilon - \delta$. This implies that $B_\delta(y_k) \subseteq B_\varepsilon(x_i)$, and hence verifies that $(B_\varepsilon(x_i))$ is a cover. So we add it to the list.

2.3. Computable Polish spaces and groups. Examples of computable topological spaces can be obtained using the following concept.

Definition 2.6. A *computable (Polish) metric space* is a triple $(M, d, (x_i)_{i \in \mathbb{N}})$, where (M, d) is a Polish metric space and $(x_i)_{i \in \mathbb{N}}$ is a dense sequence in M such that, for any $i, j \in \mathbb{N}$, $d(x_i, x_j)$ is a computable real uniformly in i, j .

The points x_i in the dense sequence $(x_i)_{i \in \mathbb{N}}$ are called *special points*. A topological space X is *computably metrizable* if there exists a computable metric on the set of natural numbers such that the completion of this metric space is homeomorphic to X . It is clear that such a space is computable as a topological space: let $\langle v(k) \rangle_{k \in \mathbb{N}}$ be an effective listing of the open balls around special points with a radius of the form 2^{-r} .

Definition 2.7. [38] A Polish group G is *computably metrizable* if there is a dense set $(x_i)_{i \in \mathbb{N}}$ in G and a metric d compatible with the topology of G such that:

- $(G, d, (x_i)_{i \in \mathbb{N}})$ is a computable (Polish) metric space;
- the group operations of G are effectively continuous

A pair $((x_i)_{i \in \mathbb{N}}, d)$ as in the definition above is called a *computable Polish presentation*, or a *computably metrized presentation*, of G .

3. COMPUTABLE PRESENTATIONS OF T.D.L.C. ABELIAN GROUPS

In Section 1.5 we discussed totally disconnected locally compact (t.d.l.c.) abelian groups. In this section we give a formal definition of computability for such groups. Our approach in this abelian case is via a definition of computability for the larger class of procounable groups, which we recall next.

Suppose we are given a sequence of groups $(A_i)_{i \in \mathbb{N}}$ such that each A_i is countable discrete. Suppose we are also given epimorphisms $\phi_i : A_i \rightarrow A_{i-1}$ for each $i > 0$. The inverse limit $\varprojlim (A_i, \phi_i)$ is concretely defined as the closed subgroup of the topological group $\prod_{i \in \mathbb{N}} A_i$ consisting of those g such that $\phi_i(g(i)) = g(i-1)$ for each $i > 0$.

Definition 3.1. A topological group G is called *procounable* if $G \cong \varprojlim (A_i, \phi_i)$ for some sequence $(A_i, \phi_i)_{i \in \mathbb{N}}$ as above.

It is well-known [?, Thm. 1.5.1] that a Polish group G is isomorphic to a closed subgroup of S_∞ if, and only if, it has a neighbourhood basis of the neutral element consisting of open subgroups. Furthermore, G is procounable if, and only if, it has such a neighbourhood basis where the open subgroups are normal. In particular, each abelian group isomorphic to a closed subgroup of S_∞ is procounable.

Among the procounable (not necessarily abelian) groups, being t.d.l.c. can be characterized easily.

Fact 3.2. Suppose $G \cong \varprojlim (A_i, \phi_i)$ for some sequence $(A_i, \phi_i)_{i \in \mathbb{N}}$ as above. Then
 G is locally compact $\Leftrightarrow \ker \phi_i$ is finite for all sufficiently large i .

Proof. For each i let $N_i \leq G$ be the kernel of the natural epimorphism $G \rightarrow A_i$, an open normal subgroup of G . Identifying A_i with G/N_i via this epimorphism, we have $\phi_i(gN_i) = gN_{i-1}$ for each $g \in G$, and in particular $\ker \phi_i = N_{i-1}/N_i$ for $i > 0$.

For the implication “ \Rightarrow ”, note that G as a t.d.l.c. group has a compact open subgroup K by a theorem of van Dantzig. Since the N_i form a basis of neighbourhoods of 1 in G , we have $N_{i-1} \leq K$ for sufficiently large i . So N_{i-1} is compact, whence N_i has finite index in it.

For the implication “ \Leftarrow ”, we may assume that *each* ϕ_i has a finite kernel. For $i > 0$ write $B_i = N_0/N_i$, a finite subgroup of A_i . Let $\phi'_i = \phi_i|_{B_i}$. Note that $\phi'_i : B_i \rightarrow B_{i-1}$ is onto for each i , and the group N_0 is topologically isomorphic to $\varprojlim_{i>0} (B_i, \phi'_i)$. So N_0 is profinite and hence compact. Since N_0 is open in G , this shows that G is locally compact. \square

Shortly, in Def. 3.4 we will consider computable procounable presentations, and introduce an effective version of the finiteness condition in Fact 3.2 in order to obtain a notion of a computable locally compact procounable group. Each t.d.l.c. group G is isomorphic to a closed subgroup of S_∞ , and if G is abelian then it is also procounable

as noted above. Thus, in taking our approach to computability via the procountable groups, we include the case of *abelian* t.d.l.c. groups, which suffices for this paper.

3.1. Computable t.d.l.c. presentations. In computability theory, a *strong index* k for a finite set $F \subseteq \mathbb{N}$ is a direct encoding of the set by a single natural number; e.g. one can take $k = \sum_{n \in F} 2^n$. A *computable index* for a set S is the code for a Turing program computing S . By a strong index for a finite group with domain a subset of \mathbb{N} , we will mean a pair of strong indices as above, one for the domain and one for the group operation.

Towards Def. 3.4, we first recall a definition going back to LaRoche [33] and Smith [52].

Definition 3.3. A *computable profinite presentation* of a profinite group P is a sequence of finite groups A_i and epimorphisms $\phi_i : A_i \rightarrow A_{i-1}$ (for $i > 0$), all given by uniformly strong indices, such that $P \cong \varprojlim (A_i, \phi_i)$.

We now widen the definition of La Roche and Smith in that we allow the A_i to be computable groups, and require that the A_i and the ϕ_i have computable indices uniformly in i .

Definition 3.4. (1) A computable presentation of a procountable group G is a sequence $(A_i, \phi_i)_{i \in \mathbb{N}}$, of discrete groups A_i and epimorphisms $\phi_i : A_i \rightarrow A_{i-1}$ (for $i > 0$) such that $G \cong \varprojlim (A_i, \phi_i)$, each group A_i is uniformly computable as a discrete group, and the sequence of maps $(\phi_i)_{i \in \mathbb{N}^+}$ is uniformly computable. (For notational ease, we also let ϕ_0 be the identity on A_0 .)

(2) Suppose that $\ker \phi_i$ is finite for each i (so that G is locally compact by Fact 3.2). We say that $(A_i, \phi_i)_{i \in \mathbb{N}}$ is a *computable t.d.l.c. presentation* of G if in addition, from i one can compute a strong index for $\ker \phi_i$ as a subset of A_i .

Remark 3.5. Note that each computable profinite presentation in the sense of La Roche and Smith is a computable t.d.l.c. presentation. Recall that a computable discrete group G is a group upon the domain \mathbb{N} in which the operations are represented by computable functions (Maltsev [34], Rabin [48]). Letting all the maps ϕ_i be the identity, our definition subsumes the definition of a computable group.

Remark 3.6. Given a computable procountable presentation of an abelian group G , we can always refine the sequence $(A_i)_{i \in \mathbb{N}}$ so that kernels of the projections $\phi_i : A_i \rightarrow A_{i-1}$ are cyclic subgroups of A_i . Then, saying that the (strong) indices of $(\ker \phi_i)_{i \in \mathbb{N}}$ are computable is equivalent to saying that the generators of these kernels and their orders can be computed from i .

Remark 3.7. It is clear that each computably presented procountable group is computably metrizable (Def. 2.7). The converse of this fails even for profinite groups; this follows from Cor. 1.6 of [37].

3.2. Discussion. The second and third author in forthcoming work [36] define computability for a t.d.l.c. group G without the restriction to being abelian. They begin by showing that two rather different possible definitions are equivalent. The first asks that a certain computable subgroup of S_∞ satisfying an effective local compactness condition is isomorphic to G . The second says, roughly speaking, that a (countable) ordered groupoid defined canonically on the compact open cosets of G is computable. [36, Section 8] shows that in the abelian case the definition in [36] is equivalent to the one given here. This evidences that the definition of computability for t.d.l.c. groups used here is not *ad hoc*; rather, it is equivalent to a general definition, but stated here in such a way as to be appropriate in our technical proofs below.

We provide further evidence that our definition of computability for t.d.l.c. abelian groups is a natural one. We give a characterisation of this class based on computable extensions of a profinite group by a discrete group.

First we need a little background; for more detail see e.g. [18]. Given abstract groups A, L , an *extension* of A by L is an exact sequence $0 \rightarrow A \rightarrow E \rightarrow L \rightarrow 0$. (Note that in homological algebra this is usually called an extension of L by A .) The extensions of A by L are given by equivalence classes of *cocycles*. A cocycle from L to A is a function $c: L \times L \rightarrow A$ that is symmetric and satisfies the condition that $c(u, v) + c(u + v, w) = c(v, w) + c(u, v + w)$. By these conditions, on $L \times A$, the operation

$$(u, a) + (v, b) := (u + v, a + b + c(u, v))$$

defines an abelian group E . The inverse of (u, a) is $(-u, -a - c(u, -u))$. A monomorphism $A \rightarrow E$ is defined by $a \mapsto (0, a)$. An epimorphism $E \rightarrow L$ is defined by $(u, a) \mapsto u$. So we have an exact sequence as above. We write $E = \text{ext}_c(L, P)$. If A is a topological group and L is countable discrete, then E is naturally a topological group with the product topology, and A is open in E .

Given a uniformly computable sequence of groups $(A_i)_{i \in \mathbb{N}}$, by a computable element of $\prod A_i$ we mean a computable function f such that $f(i) \in A_i$ for each i . Let L be a computable discrete group and P a computable profinite group as defined in 3.3 (identified with its presentation). We say that a 2-cocycle $c: L \times L \rightarrow P$ is computable if $c(x, y)$ is a computable element of P uniformly in $x, y \in L$. In this case, we call the group $\text{ext}_c(P, L)$ a *computable extension* of P by L .

Proposition 3.8. *Let G be a t.d.l.c. abelian group. The following are equivalent.*

- (1) *G has a computable t.d.l.c. presentation.*
- (2) *G is homeomorphic to a computable extension of a profinite group P by a discrete group L .*

Proof. (1) \rightarrow (2): Let $(A_i, \phi_i)_{i \in \mathbb{N}}$ be a computable t.d.l.c. presentation of G as in Def. 3.4(2). Write $L = A_0$. Let α denote the canonical projection $G \rightarrow L$. Let P denote the kernel of α .

We claim that P has a computable profinite presentation $(B_i, \phi'_i)_{i \in \mathbb{N}}$. The groups B_i and the maps ϕ'_i are defined as in the proof of the implication " \Leftarrow " in Fact 3.2. We show how to recursively compute strong indices for the groups B_i . Fix a strong index for the finite group B_1 . If we have a strong index for B_{i-1} , since the epimorphism ϕ_i is computable uniformly in i , we can compute a strong index for a finite set $D \subseteq A_i$ such that $\phi_i(D) = B_{i-1}$. By hypothesis, from i we can compute a strong index for the set $\ker \phi_i$. So we can compute a strong index for $D \cdot \ker \phi_i$, which equals the domain of B_i . Since A_i is a uniformly computable group, we can therefore obtain a strong index for the group B_i .

Since the maps ϕ_i are uniformly computable, we automatically obtain strong indices for the restricted maps $\phi_i|_{B_i}$. This verifies the claim.

We next define a computable 2-cocycle $c: L \times L \rightarrow P$. Since the maps ϕ_i are onto and uniformly computable, given $x \in L$ we can uniformly determine a computable element $\tau(x) \in G$ such that $\alpha(\tau(x)) = x$. Now let $c(x, y) = \tau(x) + \tau(y) - \tau(x + y)$. Then c is as required.

(2) \rightarrow (1): Let (B_i, ϕ'_i) be a computable profinite presentation of P . Let $A_0 = L$. For each i let $\beta_i: P \rightarrow B_i$ be the canonical projection. Note that $c_i := \beta_i \circ c: L \times L \rightarrow B_i$ is a 2-cocycle. Let $A_i = \text{ext}_{c_i}(L, B_i)$ be the corresponding extension. Define maps $\phi_i: A_i \rightarrow A_{i-1}$ by $\phi_i(u, a) = (u, \phi'_i(a))$. Using that $\phi'_i \circ c_i = c_{i-1}$, it is easily verified that the ϕ_i are epimorphisms.

We show that $G \cong \varprojlim (A_i, \phi_i)$. By our concrete definition of inverse limits given at the beginning of this section, and our concrete definition of extensions, $H_0 = \text{ext}_c(L, P)$ is a closed subgroup of $L \times \prod_i B_i$. Furthermore, $H_1 = \varprojlim_i (\text{ext}_{c_i}(L, B_i), \phi_i)$ is the closed subgroup of $\prod_i (L \times B_i)$ consisting of those g such that the first component $g(i)_0 \in L$ does not change with i , and for the second components we have $\phi'_i(g(i)_1) = g(i-1)_1$ for each

$i > 0$. Define a topological isomorphism $\Phi: H_0 \cong H_1$ by letting $\phi(u, f)$ be the function g such that $g(i)_0 = u$ and $g(i)_1 = f(i)$ for each i .

Note that (viewing B_i as a subgroup of A_i) one has $\ker(\phi_i) = \ker(\phi'_i)$, so $\ker(\phi_i)$ is finite and given by a strong index uniformly in i . Thus (A_i, ϕ_i) is a computable t.d.l.c. presentation of G . \square

4. PONTRYAGIN - VAN KAMPEN DUALITY

All topological groups in this section will be abelian, separable, and Hausdorff. As usual, we use additive notation for abelian groups and view them as \mathbb{Z} -modules. Recall that the Pontryagin - van Kampen dual (or Pontryagin dual, or just dual for short) \widehat{G} of a locally compact abelian group G is the collection of all topological group homomorphisms from G to the unit circle group T endowed with the topology of uniform convergence (the compact-open topology). Pontryagin - van Kampen duality states that, for a locally compact G , applying the dualization process twice yields a group topologically isomorphic to G . Pontryagin [45] proved the duality for the important special cases of compact and discrete abelian groups. Thereafter, van Kampen [30] in a 3-page paper showed how the result can be extended to arbitrary locally compact abelian groups.

4.1. Compact and discrete torsion cases. It is not hard to see that G is compact iff \widehat{G} is discrete and G is profinite iff \widehat{G} is discrete torsion; see [40] for details. More generally, for a Polish t.d.l.c. abelian G , its dual is a topological extension of a separable compact group N by a countable torsion discrete group X . That is, the abelian group \widehat{G} has an open compact subgroup N such that $\widehat{G}/N \cong X$. It follows that \widehat{G} is equal to the union of its compact subgroups and it does not have to be t.d.l.c. in general. Recall that the strongest notion of computability known beyond the class of t.d.l.c. groups is computable metrizability (Def. 2.7). Recall also that the class of t.d.l.c. groups includes discrete and profinite groups.

Proposition 4.1. *Suppose G is a Polish abelian t.d.l.c. group which is either compact or discrete torsion. Then G has computable t.d.l.c. presentation if, and only if, \widehat{G} has computable t.d.l.c. presentation.*

Proof. A profinite group G has a computable profinite presentation iff its dual \widehat{G} , which is a discrete torsion group, has a computable discrete presentation; see Theorem 1.9 of [37]. This suffices by Remark 3.5. \square

For any computable discrete A , its dual \widehat{A} is computably metrizable [37]. We first explain why the dual of a discrete group can be viewed as a closed subgroup of $\mathbb{A} = \mathbb{T}^{\mathbb{N}}$, where \mathbb{T} is the unit circle group.

4.1.1. The group $\text{Hom}(G, \mathbb{T})$. Let \mathbb{T} be the group \mathbb{R}/\mathbb{Z} , which is isomorphic as topological group to the multiplicative group of complex numbers having norm 1. We say that a point $x \in \mathbb{T}$ is *rational* if the respective point of the unit interval is a rational number. Then \mathbb{T} equipped with rational points is a computable Polish group. The direct product

$$\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i,$$

of infinitely many identical copies \mathbb{T}_i of \mathbb{T} carries the natural product-metric

$$D(\chi, \rho) = \sum_{i=0}^{\infty} \frac{1}{2^{-i-1}} d_i(\chi_i, \rho_i),$$

where each of the d_i stands for the shortest arc metric on \mathbb{T}_i . Under this metric and the component-wise operation \mathbb{A} is a computably metrized Polish abelian group. The dense sets are given by sequences (a_i) , where a_i is a rational point in \mathbb{T}_i , and almost all a_i are equal to zero. The basic open sets in $\prod_{i \in \mathbb{N}} \mathbb{T}_i$ are direct products of intervals with rational end-points such that a.e. interval in the product is equal to the respective

\mathbb{T}_i . Clearly, we can effectively list all such open sets. (The exact choice of this basic system of balls is not crucial, but it will be convenient to assume that the end-points of the intervals are rational.) Every compact abelian group can be realised as a closed subgroup of \mathbb{A} , as explained below.

Suppose $G = \{g_0 = 0, g_1, g_2, \dots\}$ is a countably infinite discrete group. Let $\text{Hom}(G, \mathbb{T})$ be the subset of $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i$, (each \mathbb{T}_i is a copy of \mathbb{T}) consisting of tuples $\chi = (\chi_0, \chi_1, \dots)$, where each such tuple represents a group-homomorphism $\chi : G \rightarrow \mathbb{T}$ such that $\chi(g_i) = \chi_i \in \mathbb{T}_i$. Since G is discrete, every group homomorphism $\chi : G \rightarrow \mathbb{T}$ is necessarily continuous. Thus, $\widehat{G} \cong \text{Hom}(G, \mathbb{T})$. Since being a group-homomorphism is a universal property, $\text{Hom}(G, \mathbb{T})$ is a closed subspace of \mathbb{A} . Pontryagin duality implies that every separable compact abelian group is homeomorphic to a closed subgroup of \mathbb{A} .

In our later proofs we will need a more detailed understanding of the dual of a discrete G within $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i$. We can “build” a closed subgroup of \mathbb{A} homeomorphic to \widehat{G} ; this is explained in detail below.

4.2. Constructing \widehat{G} within \mathbb{A} . In this subsection we give a $0'$ -computable procedure which, given a computable discrete G , builds a closed subgroup of \mathbb{A} isomorphic to \widehat{G} . We outline the main idea, then we explain why this “naive” construction is not computable in general. Then we explain how to modify the construction to make it computable.

Fix some enumeration of all elements of G ; to be consistent with the notation in the previous subsection, suppose $G = \{g_0 = 0, g_1, g_2, \dots\}$. Recall, $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i$ where each \mathbb{T}_i is a copy of \mathbb{T} . Let π_i be the projection of $\text{Hom}(G, \mathbb{T})$ onto \mathbb{T}_i .

4.2.1. Performing a few steps. We describe an idea behind the construction. We do this by performing a few steps and blend these steps with informal explanation.

At stage 0, set $\pi_0(\text{Hom}(G, \mathbb{T})) = 0$; clearly, $\chi(g_0) = \chi(0) = 0$ for any character χ of G .

At stage 1, we consider the next element g_1 . Suppose, for example, $\langle g_1 \rangle \cong \mathbb{Z}$, and therefore its dual $\widehat{\langle g_1 \rangle}$ is homeomorphic to \mathbb{T} . We then declare $\pi_1(\text{Hom}(G, \mathbb{T})) = \mathbb{T}_1$ and go to the next stage.

At stage 2, consider g_2 and suppose, say, $2g_2 = g_1$. Then the order of g_2 is infinite, and thus $\widehat{\langle g_2 \rangle} \cong \mathbb{T}$. For any $x \in \mathbb{T}$, $\chi(g_2) = x$ implies $\chi(g_1) = 2x$. We then declare $\pi_2(\text{Hom}(G, \mathbb{T})) = \mathbb{T}_2$, but we also require $2\chi_3 = \chi_2$ for each character $\chi = (\chi_0, \chi_1, \dots)$ of G .

We can proceed in this manner to define the closed set representing \widehat{G} within \mathbb{A} .

4.2.2. The naive construction.

At stage 0, set $\pi_0(\text{Hom}(G, \mathbb{T})) = 0$.

At stage s , if the order of g_s is infinite then declare $\pi_s(\text{Hom}(G, \mathbb{T})) = \mathbb{T}_s$, and if it is finite and $\langle g_s \rangle \cong \mathbb{Z}_k$, then declare $\pi_s(\text{Hom}(G, \mathbb{T})) = \langle \frac{1}{k} \rangle \leq \mathbb{T}_s$. Check if there exist integers n, m_0, \dots, m_{s-1} such that

$$ng_s = \sum_{i < s} m_i g_i,$$

where $n \neq 0$, and if at least one of the $m_i \neq 0$ then $\text{GCD}(n, m_0, \dots, m_{s-1}) = 1$. If such a relation holds for g_0, \dots, g_s , then also require that

$$n\chi_s = \sum_{i < s} m_i \chi_i,$$

for any sequence (χ_0, χ_1, \dots) in the closed set that we construct.

The naive construction described above builds a closed subset of \mathbb{A} . Together with the computable operations inherited from \mathbb{A} it forms a closed subgroup of \mathbb{A} topologically isomorphic to \widehat{G} .

4.2.3. The main issue with the naive construction. It is not difficult to show that searching for a linear combination of the form $ng_s = \sum_{i < s} m_i g_i$ in a computable discrete G requires $0'$ in general. Proposition 3.3 of [37] shows that this difficulty cannot be circumvented in the following sense. The closed subgroup $\text{Hom}(G, \mathbb{T}) \leq \mathbb{A}$ does not necessarily have a dense sequence of points uniformly computable with respect to \mathbb{A} . This is the only difficulty, because if $\text{Hom}(G, \mathbb{T})$ had a dense computable subset of points, then we could use the metric of \mathbb{A} and its computable operations to computably metrize $\text{Hom}(G, \mathbb{T})$.

4.2.4. Fixing the issue. The idea is to replace G with a computable $H \cong G$ such that $\text{Hom}(H, \mathbb{T})$ does contain a computable dense subset. As was verified in [37], for the naive construction to work it is sufficient that H has a computable maximal linearly independent set over integers².

Dobritsa [7] proved the following, building on the earlier work of Nurtazin [42]:

Theorem 4.2 (Dobritsa). *Every computable discrete abelian group has a computable presentation with a computable maximal linearly independent set.*

Using Dobritsa's theorem, we can replace G with a computable H that admits a computable maximal linearly independent set; in H the conditions required to run the naive construction are computable. This gives a computably metrized presentation of $\widehat{H} \cong \widehat{G}$.

5. DETERMINING THE DUAL

5.1. Proof of Theorem 1.2.

5.1.1. Discussion and intuition. Recall that Theorem 1.2 states that the Pontryagin dual \widehat{G} of a computable abelian t.d.l.c. group G is computably metrizable. Fix a computable t.d.l.c. presentation (A_i, ϕ_i) , where each A_i is discrete and each $\text{Ker } \phi_i$ is finite and given by its strong index as in Def. 3.4(2). Since the kernel of each projection is finite, the index of \widehat{A}_{i+1} in \widehat{A}_i will also be finite. The dual of G will be the direct limit of \widehat{A}_i .

As discussed in the previous section, each \widehat{A}_i can be computably metrized. Nonetheless, to build the computably metrized copy of \widehat{A}_i , we need to pass to a new computable discrete presentation B_i of A_i that has a computable maximal linearly independent set. The isomorphism between A_i and B_i does not have to be computable in general. Consequently, it will no longer be clear how B_{i+1} projects onto B_i , and thus we will have difficulties in defining a computable metric on the direct limit of \widehat{B}_i .

It seems that the only reasonable solution to this problem is to build all these B_i simultaneously and additionally maintain the projections between them. Once the result of Dobritsa is extended to computable procountable abelian groups in this sense in Proposition 5.1, the rest of the proof becomes somewhat routine. We will use the specific properties of the computably metrized \widehat{B}_i as described in the previous section to put them together in a coherent way.

In the subsection below we prove Theorem 1.2 assuming Proposition 5.1. We will prove Proposition 5.1 in Section 9, which requires a considerable effort.

5.1.2. Formal proof of Theorem 1.2.

Proposition 5.1. *Suppose (A_i, ϕ_i) is a computable t.d.l.c. presentation of an abelian group G as in Def. 3.4(2). Then G has a computable t.d.l.c. presentation (B_i, ψ_i) such that each B_i has a computable maximal linearly independent set, uniformly in i .*

²Recall that elements g_1, \dots, g_k of an additive abelian group G are *linearly independent* (over \mathbb{Z}) if, for any choice of $n_1, \dots, n_k \in \mathbb{Z}$, $n_1 g_1 + \dots + n_k g_k = 0$ implies $n_i = 0$ for all i . Note that torsion elements are "dependent on themselves". All maximal linearly independent subsets of G have the same cardinality which is called the (Prüfer or torsion-free) rank of A .

We postpone the technical proof of the proposition to Section 9. Assuming the proposition, we prove the theorem. By the proposition, we can assume that each A_i has a uniformly computable maximal linearly independent set. As we explained in Subsection 4.2, the duals of \widehat{A}_i have uniformly computable metrizations; this is Theorem 1.4 of [37]. As explained in §4.2.4, the procedure described in §4.2.2 becomes computable if the group has a computable maximal linearly independent set.

Furthermore, it follows from the description of the construction in §4.2.2 that, within each version \mathbb{T}_j of the unit circle in $\mathbb{A} = \prod_i \mathbb{T}_i$, we can use only rational points when we define a projection of A_i to \mathbb{T}_j . The image of this projection will be equal to the completion of this set of rational points. More specifically, if we decide that $\pi_j(A_i) \cong \mathbb{Z}_m$ then we add the rational points $\frac{k}{m}$, $k \leq m$, all at once. Otherwise, if we decide that $\pi_j(A_i) = \mathbb{T}_j$, then we initiate the enumeration of all rational points in \mathbb{T}_j .

Recall that the kernel of each $\phi_i : A_i \rightarrow A_{i-1}$ in the computable procountable presentation is given by its strong index. We can uniformly rearrange the enumeration of A_i to make sure that the elements of the kernel are listed first, and then the rest of the elements of A_i listed.

It follows from Subsection 4.2 that, after this re-enumeration, the dual of A_i can be realised as a computable closed subgroup of $\prod_i \mathbb{T}_i$, where the first k projections π_i ($i \leq k$) correspond to the elements of the kernel. Note that each of these k projections is finite. In particular, the collection of sequences of the form

$$(0, \dots, 0, \chi_{k+1}, \chi_{k+2}, \dots)$$

will be a computably metrized presentation of A_{i-1} . Similarly, we take the finite kernel of the composition $\hat{\phi}_i$ of ϕ_j , $i < j$, then the dual of A_i can be represented as a computable closed subgroup of $\prod_i \mathbb{T}_i$, where the first m_i projections π_i ($i \leq m_i$) correspond to the elements of $\text{Ker } \hat{\phi}_i$, and the sequences with the first m_i coordinates equal to 0 represent a computably metrized presentation of A_0 . Furthermore, it follows from §4.2.2 that each \widehat{A}_i will be a disjoint union of finitely many cosets of \widehat{A}_0 , and *uniformly computably* so.

The computably metrized groups \widehat{A}_i can be viewed as nested under inclusion. Modify the metric on this computably metrized presentation of \widehat{A}_i to make it more suitable for metrizing the direct limit of all \widehat{A}_i . Declare the distance between any two points coming from different cosets of \widehat{A}_0 equal to 2, and use the product metric of $\mathbb{T}^{\mathbb{N}}$ (which is always ≤ 1) to compare elements in each individual coset mod \widehat{A}_0 . This is a computable metric which is furthermore effectively compatible with the original metric on \widehat{A}_i . Each \widehat{A}_i is a disjoint union of finitely many cosets of \widehat{A}_0 , in a tractable way (see §4.2.2). It follows that the operations on \widehat{G} are effectively open and effectively continuous, since for any input the group operations can be computed within \widehat{A}_i for a sufficiently large i . This finishes the proof of Theorem 1.2.

5.2. Proof of Theorem 1.3. Recall that, by assumption, G and \widehat{G} are t.d.l.c. abelian. Since $\widehat{\widehat{G}} \cong G$, it is sufficient to prove that computability of G implies computability of \widehat{G} in the sense of Definition 3.4(2). By hypothesis G admits a computable procountable presentation (A_i, ϕ_i) in which every A_i is torsion discrete. Each A_i has a maximal computable linearly independent set, namely the empty set, and therefore Proposition 5.1 is vacuously true for these A_i . It remains to note that the computable metric on each totally disconnected compact \widehat{A}_i produced in the proof of Theorem 1.2 is effectively compatible with the standard ultra-metric on the respective compact subset of $\mathbb{N}^{\mathbb{N}}$; see also §4.2.2 and §4.2.4. Furthermore, since there are only finitely many points in each \mathbb{T}_i and they are enumerated instantly (i.e., as a strong index), it follows that we can produce a computable profinite presentation of \widehat{A}_i , uniformly in i . Following the same argument

as in the proof of Theorem 1.2, computably metrize the direct limit of $\widehat{A_i}$. Since each individual compact $\widehat{A_i}$ is computable t.d.l.c., the resulting computable ultra-metric makes \widehat{G} computable t.d.l.c.

Remark 5.2. The correspondence given by Theorem 1.3 is uniform, since the only non-uniform step in Theorem 1.2 is checking if the rank of A_0 is finite or infinite; this will be explained in the proof of Proposition 5.1.

6. THE CONNECTED CASE: PROOF OF THEOREM 1.1

6.1. Effective compactness of the dual. (1) \rightarrow (2): Recall from Section 4 that \widehat{G} can be represented as a solenoid-type closed subgroup H of

$$\prod_n \mathbb{T}_n,$$

where each \mathbb{T}_n is the standard computable presentation of the unit circle group. It is sufficient to show that this presentation H is effectively compact. It consists of sequences (a_n) such that, for some of the n 's, we additionally require that $q_n a_n = \sum_{s < n} m_{n,s} a_s$, where q_n is a prime and $m_{n,s}$ are integers which depend on n . Furthermore, we can decide for which n such a restriction occurs, and in this case we also can compute the respective q_n and $m_{n,s}$. Recall that in Subsection 4.2.4 we also explained that for this to work, we picked a computable presentation of G with a computable maximal linearly independent set. Note that the map $x \rightarrow q_n x$ is surjective and computably maps basic intervals to basic intervals in a strong sense (i.e., maps names to names), and the same can be said about the standard operations of addition and multiplication by an integer scalar (in \mathbb{T}). In fact, there exist arbitrarily small intervals $I_s \ni a_s$ for which we have $q_n I_n = \sum_{s < n} m_{n,s} I_s$.

Given a finite potential cover C_0, \dots, C_k by basic open balls of H , uniformly compute i so large that the i' -th projection of each of the C_j covers all of the $T_{i'}$, $i' > i$. If (C_i) is indeed a cover, then every $\xi \in H$ is contained in some C_j together with a neighbourhood of the form

$$I_0 \times I_1 \times \dots \times I_i \times \prod_{j > i} \mathbb{T}_j,$$

where:

- i. each I_n is a basic interval with rational end-points;
- ii. each I_n is formally contained in the n -th projection of C_i :

$$d(\text{cntr}(I_n), \text{cntr}(C_i)) + r(I_n) < r(C_i),$$

where $\text{cntr}(I)$ denotes the center of an interval I ;

- iii. if q_n is defined, then $q_n I_n = \sum_{s < n} m_{n,s} I_s$.

Note that these conditions are Σ_1^0 . Also, by compactness, there must exist finitely many neighbourhoods of this form that cover the whole group. We argue that such finite families can be computably enumerated.

Suppose we are given a family of intervals (I_n^k) , $k \leq m$, such that for every fixed k the intervals I_n^k satisfy the properties (i)-(iii) above (with index k suppressed). If q_n is defined and $\{\prod_{s < n} I_s^k : k \leq m\}$, is a cover of the projection of the group on $\prod_{s < n} \mathbb{T}_n$, then $\{\prod_{s \leq n} I_s^k : k \leq m\}$, is a cover of the projection of the group on $\prod_{s \leq n} \mathbb{T}_n$; this is because every ξ (viewed as a sequence) that projects into $\prod_{s < n} I_s^k$ will also project into I_n^k . Thus, by induction on m , the problem of covering the group by neighbourhoods satisfying (i)-(iii) is reduced to the problem of covering of the product of those \mathbb{T}_n for which q_n is not defined, as follows. Let J be the set of all indices $n \leq i$ for which q_n is not defined. For a family

$$\{\prod_{n \leq i} I_n^k \times \prod_{j > i} \mathbb{T}_j : k \leq m\}$$

satisfying (iii) to cover the group, it is necessary and sufficient that $\{\prod_{n \leq i} I_n^k : k \in J\}$ is a cover of $\prod_{k \in J} T_k$. Since the spaces T_i are uniformly effectively compact, such neighbourhoods can be effectively enumerated. Thus, if C_0, \dots, C_k is indeed a cover, then we will eventually see it.

(2) \rightarrow (1): First, we will informally explain the main idea behind the proof. Then we give some background from algebraic topology. After that we introduce and verify a new method of calculating the Čech cohomology groups. Then we check that it can be used to produce a computable presentation of the discrete torsion-free dual.

6.2. Informal explanation of the proof. Suppose we are given a computably metrized presentation of the compact connected abelian \widehat{G} ; we can assume it is effectively compact. Recall that $\widehat{\widehat{G}} \cong G$ is the collection of all continuous homomorphisms from \widehat{G} to the unit circle \mathbb{T} under the topology of uniform convergence. Although \widehat{G} is topologically isomorphic to a closed subgroup of the computable group $\mathbb{T}^{\mathbb{N}}$, we do not necessarily have access to an isomorphism showing this. So our job is to list all of the countably many continuous homomorphisms $\chi_i : \widehat{G} \rightarrow \mathbb{T}$ without repetition in such a way that the operation of addition is computable:

$$\chi_i + \chi_j = \chi_{f(i,j)},$$

where f is a computable function. Naively, one could try to approximate the Π_1^0 class of all elements in $\widehat{\widehat{G}}$ by means of finite partial maps from \widehat{G} to \mathbb{T} . Since the space of all continuous maps from \widehat{G} to \mathbb{T} is not compact, the best we can hope using this brute-force approach is to reduce the problem to the problem of uniformly computing isolated members of a Π_1^0 class in the Baire space. By a uniform version of Σ_1^1 -bounding (see [50], Thm. 6.2.III), this will give a Δ_1^1 presentation of $\widehat{\widehat{G}} \cong G$.

We see that the brute-force approach gives an upper bound on the complexity of G which is too crude. The idea is to use algebraic topology (which is “algorithmic” in its nature) to significantly improve this estimate. A peculiar consequence of the structural theory of compact abelian groups is the following fact: *for a compact connected abelian group G , its homeomorphism type determines its algebraic homeomorphism type* (see Part 5 of Chapter 8 of [26]). To prove the theorem, one shows that the first Čech cohomology group (to be defined) of the underlying space of a given compact connected abelian group is isomorphic to its discrete dual.

To calculate the Čech cohomology group of \widehat{G} , we need to list its 2^{-n} -covers and compute their nerves; recall that the nerve of a cover is the simplicial complex in which the faces are the collections of balls that intersect non-trivially. The covers form an inverse system under refinement maps, and so do the associated nerves under the induced simplicial maps. For each nerve, it makes sense to calculate its cohomology groups. For a fixed i , the direct limit of the i th cohomology groups under the homomorphisms induced by the refinement maps on nerves is exactly the i th Čech cohomology group of \widehat{G} , which for $i = 1$ turns out to be isomorphic to G . The process described above is readily seen to be arithmetical, which is already a significant improvement over the crude Δ_1^1 bound that we established earlier. It takes quite a bit of work to prove that, indeed, we can *computably* recover the Čech cohomology groups from the given effectively compact presentation of \widehat{G} .

There are three main obstacles in effectively computing the Čech cohomology groups. First, we need to have access to covers of \widehat{G} . This is where effective compactness will be useful. Second, for each cover we will need to calculate its nerve. This is problematic since we need to decide whether balls intersect or not; this is naturally Σ_1^0 . In fact, it seems that in general this difficulty cannot be resolved by means of manipulating with the given presentation. We also suggest that the reader take some time to check that the

mere computable enumerability of intersection for basic open balls is insufficient for the cohomology machinery to work computably. This is not immediately obvious since seeing this involves examining multiple layers of definitions; we omit the unpleasant tedious details. To circumvent this obstacle, we will replace the standard notion of a Čech nerve with a different notion of a metric nerve which yields the same cohomology groups. Even though this will not be difficult, it will involve some subtle choice of numerical parameters to check that the new notion works the way we expect. Third, even assuming that the first two issues are resolved, the 1st Čech cohomology group is merely c.e. presented rather than computable. Recall that a group is c.e. presented if its operation is computable but equality is merely computably enumerable. Luckily, it was conveniently proven by Khisamiev in the 1980s that every c.e. presented torsion-free abelian group has a computable presentation which can be produced with sufficient degree of uniformity. This uniformity will be important in applications. We will state the result below and we will also give a sketch of its proof.

We are now ready to give the details.

6.3. Background from algebraic topology. The material contained in this subsection can be found in [41]. The main point of this section is to set the terminology and notation.

6.3.1. Simplicial complexes. We assume that all the simplicial complexes are finite. If K is a simplicial complex, write $H_i(K)$ for its i -th homology group (with coefficients in \mathbb{Z}), and $H^i(K)$ to denote its i -th cohomology group. We omit the standard definitions of these groups and refer the reader to, say, [41] where these definitions are stated and explained in great detail. We also refer the reader to Chapter 3 of [53] and Chapter IV of [13].

Recall that a simplicial map between simplicial complexes is a function from vertices to vertices that maps simplices to simplices. Suppose that K and L are simplicial complexes and $f, g : K \rightarrow L$ are simplicial maps. We say that f, g are contiguous if for every simplex σ of K , $f(\sigma) \cup g(\sigma)$ is a simplex of L . For $d \in \mathbb{N}$, two contiguous simplicial maps $K \rightarrow L$ induce the same homomorphism $H_d(K) \rightarrow H_d(L)$ and $H^d(L) \rightarrow H^d(K)$.

We say that f, g are contiguous equivalent if there exist $\ell \in \mathbb{N}$ and f_0, \dots, f_ℓ such that $f_0 = f$, $f_\ell = g$, and f_i, f_{i+1} are contiguous for $0 \leq i < \ell$. This defines an equivalence relation among simplicial maps. From the perspective of (co)homology, simplicial maps belonging to the same contiguous equivalence class are indistinguishable. It is thus convenient to define a morphism of simplicial complexes $K \rightarrow L$ to be a contiguous equivalence class of simplicial maps $K \rightarrow L$.

Then one can consider simplicial complexes as objects of a category with morphisms defined as above. This makes $K \mapsto H_d(K)$ a covariant functor from simplicial complexes to abelian groups, and $K \mapsto H^d(K)$ is a contravariant functor from simplicial complexes to abelian groups.

6.3.2. Towers of simplicial complexes. We consider finite simplicial complexes. A tower of simplicial complexes is a sequence $(K^{(n)}, p^{(n, n+1)})$ of simplicial complexes $K^{(n)}$ and morphisms $p^{(n, n+1)} : K^{(n+1)} \rightarrow K^{(n)}$. For $n < m$ let $p^{(n, m)} : K^{(m)} \rightarrow K^{(n)}$ be the canonical composition of maps. Suppose that

$$\mathbf{K} = (K^{(n)}, p^{(n, n+1)}) \text{ and } \mathbf{L} = (L^{(n)}, p^{(n, n+1)})$$

are towers of simplicial complexes. A morphism $\mathbf{K} \rightarrow \mathbf{L}$ is represented by a sequence $(n_k, f^{(k)})_{k \in \mathbb{N}}$ where (n_k) is an increasing sequence in \mathbb{N} and $f^{(k)} : K^{(n_k)} \rightarrow L^{(k)}$ is a morphism such that $p^{(k, k+1)} f^{(k+1)} = f^{(k)} p^{(n_k, n_{k+1})}$ for $k \in \mathbb{N}$. Two such sequences $(n_k, f^{(k)})_{k \in \mathbb{N}}$ and $(n'_k, f'^{(k)})_{k \in \mathbb{N}}$ represent the same morphism if for every $k \in \mathbb{N}$ there exists $m_k \geq$

$\max\{n_k, n'_k\}$ such that $f^{(k)} p^{(n_k, m_k)} = f'^{(k)} p^{(n'_k, m_k)}$ for every $k \in \mathbb{N}$. The identity morphism of \mathbf{K} is represented by the sequence $(n_k, f^{(k)})$ where $n_k = k$ and $f^{(k)}$ is the identity of $K^{(k)}$. The composition of morphisms $(n_k, f^{(k)})$ and $(k_\ell, g^{(\ell)})$ is the morphism $(n_{k_\ell}, g^{(\ell)} \circ f^{(k_\ell)})$. This defines a category with towers of simplicial complexes as objects.

One similarly define the category of towers of abelian groups. More generally, one can define the category $\mathbf{tow}(\mathcal{C})$ of towers associated with a category \mathcal{C} . The category of inductive sequences $\mathbf{ind}(\mathcal{C})$ in \mathcal{C} is defined in the similar way. The objects in $\mathbf{ind}(\mathcal{C})$ are sequences $(A_{(n)}, \eta_{(n+1, n)})$ where $A_{(n)}$ is an object in \mathcal{C} and $\eta_{(n+1, n)} : A_{(n)} \rightarrow A_{(n+1)}$ is a morphism in \mathcal{C} . One can succinctly define $\mathbf{ind}(\mathcal{C})$ as the opposite category of $\mathbf{tow}(\mathcal{C}^{\text{op}})$, where \mathcal{C}^{op} is the opposite category of \mathcal{C} .

Given a tower of simplicial complexes $\mathbf{K} = (K^{(n)})$ and $d \in \mathbb{N}$ one defines the tower $H_d(\mathbf{K}) = (H_d(K^{(n)}))_{n \in \mathbb{N}}$ of homology groups, where the group homomorphism $H_d(K^{(n+1)}) \rightarrow H_d(K^{(n)})$ is induced by the morphism $K^{(n+1)} \rightarrow K^{(n)}$, and the inductive sequence $H^d(\mathbf{K}) = (H^d(K^{(n)}))_{n \in \mathbb{N}}$ of cohomology group, where the group homomorphism $H^d(K^{(n)}) \rightarrow H^d(K^{(n+1)})$ is induced by the morphism $K^{(n+1)} \rightarrow K^{(n)}$. This defines a covariant functor $\mathbf{K} \mapsto H_d(\mathbf{K})$ from the category of towers of simplicial complexes to the categories of towers of groups, and a contravariant functor $\mathbf{K} \mapsto H^d(\mathbf{K})$ from the category of towers of simplicial complexes to the category of inductive sequences of groups.

6.3.3. Čech nerves. Suppose that X is a compact metrizable space. We define a tower of simplicial complexes $\mathbf{K}(X)$ as follows. Fix a sequence $(\mathcal{U}^{(n)})_{n \in \mathbb{N}}$ of finite open covers of X such that, for every $n \in \mathbb{N}$, $\mathcal{U}^{(n+1)}$ refines $\mathcal{U}^{(n)}$, and for every finite open cover \mathcal{W} of X there exists $n \in \mathbb{N}$ such that $\mathcal{U}^{(n)}$ refines \mathcal{W} . For every $n \in \mathbb{N}$, fix a refinement map $p^{(n, n+1)} : \mathcal{U}^{(n+1)} \rightarrow \mathcal{U}^{(n)}$. Let $K^{(n)}$ be the Čech nerve $N(\mathcal{U}^{(n)})$ of the cover $\mathcal{U}^{(n)}$. Then $p^{(n, n+1)}$ is a simplicial map $K^{(n+1)} \rightarrow K^{(n)}$. Furthermore, any two refinement maps $\mathcal{U}^{(n+1)} \rightarrow \mathcal{U}^{(n)}$ induce *contiguous* simplicial maps $K^{(n+1)} \rightarrow K^{(n)}$. This defines a tower of simplicial complexes $\mathbf{K}(X) = (K^{(n)})_{n \in \mathbb{N}}$. For $d \in \mathbb{N}$, one obtains an inductive sequence of abelian groups $H_d(\mathbf{K}(X)) = (H_d(K^{(n)}))_{n \in \mathbb{N}}$. This defines a functor $X \mapsto H_d(\mathbf{K}(X))$ from compact metrizable spaces to inductive sequences of abelian groups. A different choice of covers yields an isomorphic functor.

6.4. Metric nerves. In this subsection we define a new notion of a metric nerve and verify that metric nerves can be used in place of Čech nerves in all our applications. The new notion we define in this subsection was inspired by [49].

Fix decreasing vanishing sequences (ε_n) and (δ_n) in \mathbb{R}_+ such that $2\varepsilon_n + \delta_{n+1} \leq \delta_n$ for every $n \in \mathbb{N}$. Suppose that X is a compact metrizable space. A subset $A \subseteq X$ is ε -dense if for every $x \in X$ there exists $a \in A$ such that $d(x, a) < \varepsilon$. Fix a sequence $(A^{(n)})$ of finite subsets of X such that, for every $n \in \mathbb{N}$, $A^{(n)}$ is ε_n -dense. Define $N(A^{(n)})$ to be the simplicial complex with set of vertices $A^{(n)}$, where one lets σ be a simplex in $N(A^{(n)})$ if and only if σ has diameter less than δ_n , i.e. $d(a, b) < \delta_n$ for every $a, b \in \sigma$. For $n \in \mathbb{N}$, fix a function $p^{(n, n+1)} : A^{(n+1)} \rightarrow A^{(n)}$ such that $d(p^{(n, n+1)}(a), a) < \varepsilon_n$ for every $a \in A^{(n+1)}$. The assumption that $2\varepsilon_n + \delta_{n+1} \leq \delta_n$ guarantees that $p^{(n, n+1)}$ is a simplicial map $N(A^{(n+1)}) \rightarrow N(A^{(n)})$. A different choice of function $p^{(n, n+1)}$ as above yields a contiguous simplicial map $N(A^{(n+1)}) \rightarrow N(A^{(n)})$. Define the tower of simplicial complexes $\mathbf{L}(X) = (N(A^{(n)}))_{n \in \mathbb{N}}$.

Suppose that X, Y are compact metrizable spaces, and $\varphi : X \rightarrow Y$ is a continuous function. Suppose that $(A^{(n)})$ and $(B^{(n)})$ are sequences of finite subsets of X, Y , respectively, obtained as above. Let $(n_k)_{k \in \mathbb{N}}$ be an increasing sequence in \mathbb{N} such that, for $a, b \in X$, $d(a, b) < \delta_{n_k}$ implies that $d(\varphi(a), \varphi(b)) < \delta_{k+1}$. For $k \in \mathbb{N}$ define a simplicial map $A^{(n_k)} \rightarrow B^{(k)}$, $a \mapsto f^{(k)}(a)$ such that $d(f^{(k)}(a), \varphi(a)) < \varepsilon_k$ for every $a \in A^{(n_k)}$. Notice that this is indeed a simplicial map, and a different choice would yield a contiguous simplicial map. The sequence $(n_k, f^{(k)})$ represents a morphism $\mathbf{L}(X) \rightarrow \mathbf{L}(Y)$. The assignment

$X \mapsto \mathbf{L}(X)$ defines a functor from compact metrizable spaces to simplicial complexes. A different choice of finite sets $(A^{(n)})$ of X and of vanishing sequences (ε_n) and (δ_n) in $(0, 1)$ such that $2\varepsilon_n + \delta_{n+1} \leq \delta_n$ for $n \in \mathbb{N}$ would yield an isomorphic functor.

6.4.1. *Comparing $\mathbf{K}(X)$ and $\mathbf{L}(X)$.* We want to show that the functors $X \mapsto \mathbf{K}(X)$ and $X \mapsto \mathbf{L}(X)$ are isomorphic. Fix vanishing sequences (ε_n) , (δ_n) , (r_n) satisfying the following for every $n \in \mathbb{N}$:

- $2\varepsilon_n + \delta_{n+1} \leq \delta_n$;
- $2\varepsilon_n + r_{n+1} \leq r_n$;
- $\varepsilon_n \leq r_n$;
- $2r_n \leq \delta_n$;
- $2\varepsilon_n + \delta_{n+1} \leq r_n$.

For example, one can set $\varepsilon_n = 2^{-8n}$ and $\delta_n = 2^{-4(n-1)}$ and $r_n = 2^{-4n+1}$ (for large enough n). Fix a compact metrizable space X , and let $(A^{(n)})$ be the corresponding sequence of finite subsets such that, for every $n \in \mathbb{N}$, $A^{(n)}$ is ε_n -dense. Let $\mathbf{L}(X) = (N(A^{(n)}))$ be defined as above with respect to $(A^{(n)})$ and the sequence (δ_n) . We can assume that $\mathbf{K}(X) = (N(\mathcal{U}^{(n)}))$ is defined in reference to the sequence of covers $\mathcal{U}^{(n)}$, where $\mathcal{U}^{(n)}$ is the cover of open balls of radius r_n with center in $A^{(n)}$. Furthermore, we can assume that the refinement map $p^{(n,n+1)} : \mathcal{U}^{(n+1)} \rightarrow \mathcal{U}^{(n)}$ is the same as the bonding map $p^{(n,n+1)} : N(A^{(n+1)}) \rightarrow N(A^{(n)})$, defined by assigning to $a \in A^{(n+1)}$ and element $p^{(n,n+1)}(a) \in A^{(n)}$ such that $d(a, p^{(n,n+1)}(a)) < \varepsilon_n$.

The identity map of $A^{(n)}$ defines a simplicial map $\varphi_n : N(\mathcal{U}^{(n)}) \rightarrow N(A^{(n)})$, as $2 \cdot r_n \leq \delta_n$. Thus, (φ_n) defines a morphism $\varphi : \mathbf{K}(X) \rightarrow \mathbf{L}(X)$. We claim that this is an isomorphism. Indeed, for every $n \in \mathbb{N}$, $p^{(n,n+1)}$ is a simplicial map $N(A^{(n+1)}) \rightarrow N(\mathcal{U}^{(n)})$. Indeed, if σ is a simplex in $N(A^{(n+1)})$, then we claim that σ is contained in the ball of center $p^{(n,n+1)}(a)$ and radius r_n for every $a \in \sigma$. Indeed, if $a, a' \in \sigma$ then we have that $d(a, a') < \delta_{n+1}$, $d(p^{(n,n+1)}(a), a) < \varepsilon_n$ and $d(p^{(n,n+1)}(a'), a') < \varepsilon_n$ and hence

$$d(a', p^{(n,n+1)}(a)) < 2\varepsilon_n + \delta_{n+1} \leq r_n.$$

Thus, the sequence $(n+1, p^{(n,n+1)})$ defines a morphism $\psi : \mathbf{L}(X) \rightarrow \mathbf{K}(X)$ that is the inverse of φ . This shows that $\varphi : \mathbf{K}(X) \rightarrow \mathbf{L}(X)$ is an isomorphism, which can be easily seen to be natural.

6.4.2. *Computability of $\mathbf{L}(X)$.* Given an effectively compact X , define $A_n \subseteq X$, $p^{(n,n+1)}$, $\delta_n, \varepsilon_n \in \mathbb{Q}$ by recursion as follows. For $n = 0$, set $\varepsilon_0 = k > \text{diam}(X)$ and $\delta_0 = 8k$, and $A_0 = \{x\}$, where x is any (first found) special point of X . Suppose $A_n, \delta_n, \varepsilon_n$ have already been defined and assume $4\varepsilon_n < \delta_n$. Search for the first found finite $A_{n+1} \subseteq X$ of special points and the first found positive rationals δ_{n+1} and ε_{n+1} , and a map $p^{(n,n+1)} : A_{n+1} \rightarrow A_n$ that satisfy:

- i. $2\varepsilon_n + \delta_{n+1} < \delta_n$ and $\delta_{n+1} < \delta_n/2$;
- ii. $4\varepsilon_{n+1} < \delta_{n+1}$;
- iii. A_{n+1} is an ε_{n+1} -dense in X ;
- iv. for every pair $a, a' \in A_{n+1}$, either $d(a, a') < \delta_{n+1}$ or $d(a, a') > \delta_{n+1}$.
- v. For each, $a \in A_{n+1}$ set $p^{(n,n+1)}(a)$ equal to the first found $b \in A_n$ such that $d(b, a) < \varepsilon_n$.

Note that condition (iii) is equivalent to saying that the ε_{n+1} -balls centered in elements of A_{n+1} cover X ; this is a Σ_1^0 because the space is effectively compact. Conditions (i), (ii), and (iv) are Σ_1^0 by definition. To see why such $A_n, \delta_n, \varepsilon_n$ can be found, fix any set of special points A_{n+1} and rationals ε_{n+1} and δ'_{n+1} as follows. Since $4\varepsilon_n < \delta_n$ by our hypothesis, so we can choose any $\delta'_{n+1} < \min\{2\varepsilon_n, \delta_n/2\}$ to satisfy (i) and then fix any $\varepsilon_{n+1} < \delta_{n+1}/8$ to meet (ii.) strongly (in the sense that any $\delta_{n+1} \in (\delta'_{n+1}/2, \delta'_{n+1})$ satisfies

condition ii. for the same ε_{n+1}). Choose some

$$\delta_{n+1} \in (\delta'_{n+1}/2, \delta'_{n+1}) \setminus \{d(a, a') : a, a' \in A_{n+1}\}$$

so that this finite set of potential equalities is avoided; this gives condition iv. while maintaining condition (ii.). It follows that such $A_n, \delta_n, \varepsilon_n$ exist and, thus, we will eventually find (perhaps, some other) such A_n, δ_n , and ε_n . Similarly, the definition of $p^{(n,n+1)}$ is computable too since for every $a \in A_{n+1}$ there is at least one $b \in A_n$ which satisfies $d(b, a) < \varepsilon_n$.

Note that the conditions (i) and (ii) imply that both sequences δ_n and ε_n are vanishing, and thus the analysis contained in the previous paragraphs implies that we will obtain a uniformly computable tower $L(X)$ and the associated maps $p^{(n,n+1)}$ which, furthermore, will be uniformly given by their strong indices. Condition (iv) guarantees that the strong indices of the finite simplices in the tower are uniformly computable.

6.5. Calculating the cohomology group. Given a compact M , let \mathcal{N} be the directed set of all its finite open covers under the refinement relation. For each member C of \mathcal{N} , define its (Čech) nerve $N(C)$ and the cohomology groups $H^*(N(C))$ (with coefficients in \mathbb{Z}). Let $H^*(M)$ be the direct limit of $H^*(N(C))$ induced by the inverse system \mathcal{N} . Remarkably, for a compact connected abelian group G ,

$$H^1(G) \cong \widehat{G},$$

where as usual \widehat{G} denotes the Pontryagin dual of G (which is torsion-free discrete); see, e.g., Part 5 of Chapter 8 of [26], and see also [27] for a detailed exposition of cohomology theory for compact abelian groups. Note that we did not use the operation of G to define $H^1(G)$; therefore homeomorphic connected compact abelian groups are necessarily (topologically) isomorphic as groups.

In the previous subsection we verified that instead of Čech nerves, one can use metric nerves to define $H^1(G)$. This is because, in the notation of the previous section, $X \mapsto K(X)$ and $X \mapsto L(X)$ are isomorphic, and thus the respective towers of abelian groups are also isomorphic and give isomorphic limits. We have also verified that, in contrast with Čech nerves, metric nerves can be effectively produced under the assumption that the underlying space is effectively compact. Our next goal is to verify that this indeed implies that $H^1(G)$ is computably presented.

Produce a computable inverse system \mathcal{N} of metric nerves. For a fixed finite set of points $C \in \mathcal{N}$ and the respective metric simplex $\tilde{N}(C)$, define the simplicial chain complex as usual:

$$\dots \rightarrow \delta_3 A_2 \rightarrow \delta_2 A_1 \rightarrow \delta_1 A_0$$

where A_i are finitely generated free abelian groups and δ_i are boundary homomorphisms, and then define the associated cochain complex $A^i = \text{Hom}(A_i, \mathbb{Z})$ and define $d_i : A^i \rightarrow A^{i+1}$ to be the dual homomorphism of δ_{i+1} . Then $H^i(\tilde{N}(C)) = \text{Ker}(d_i) / \text{Im}(d_{i-1})$ is the i th cohomology group of the simplex $\tilde{N}(C)$ which is a finitely generated abelian group which can be thought of as given by finitely many generators and relations. By the analysis from the previous subsection, the direct limit of the $H^i(\tilde{N}(C))$ is isomorphic to then i -th Čech cohomology group $H^i(G)$ of G . We are interested mainly in the case when $i = 1$.

A sequence of finitely generated uniformly computable abelian groups (B_i) is *strongly completely decomposable* if each B_i uniformly splits into a direct sum of its cyclic subgroups, and furthermore the sets of generators of the cyclic summands are given by their strong indices. We will need the lemma below which is well-known; see [18] for a proof.

Lemma 6.1. Let $G \leq F$ be free abelian groups. There exist generating sets g_1, \dots, g_k and f_1, \dots, f_m ($k \leq m$) of G and F , respectively, and integers n_1, \dots, n_k such that for each $i \leq k$, we have $g_i = n_i f_i$.

Claim 6.2. The groups $H^i(\tilde{N}(C))$ are strongly completely decomposable (uniformly in C and i).

Proof. A close examination of the definitions shows that, given C (as a finite set of parameters) and i , we can compute the generators of $A^i = \text{Hom}(A_i, \mathbb{Z})$ and compute d_i .

We can computably find the set of generators (a_j) of $\text{Ker}(d_i)$ and a set of generators (b_s) of $\text{Im}(d_{i-1})$ such that for each s there is an integer m and an index i such that $ma_i = b_s$; we know that such generators exist so we just search for the first found ones. It follows that the factor $H^i(\tilde{N}(C)) = \text{Ker}(d_i)/\text{Im}(d_{i-1})$ is strongly completely decomposable with all possible uniformity. \square

Recall that a group admits a c.e. presentation if it is isomorphic to a factor of a computable (free) group by a computably enumerable subgroup. It is equivalent to generalising the definition of a computable discrete presentation by requiring the equality to be merely c.e. (while the operation is still computable).

Claim 6.3. The direct limit $\lim_{C \in \mathcal{N}} H^i(\tilde{N}(C))$ admits a c.e. presentation.

Proof. We have checked in the previous subsection that effective compactness can be used to effectively list \mathcal{N} . The refinement map between two metric covers $C \leq C'$ in \mathcal{N} induces a simplicial map between the respective nerves $\tilde{N}(C)$ and $\tilde{N}(C')$, and this induces a computable homomorphism between the respective cohomology groups $H^i(\tilde{N}(C)) \rightarrow H^i(\tilde{N}(C'))$. By Claim 6.2, these finitely generated abelian groups are effectively completely decomposable uniformly in C and i . Note that $\text{Im}\phi$ is generated in $H^i(\tilde{N}(C'))$ by the images of the generators of $H^i(\tilde{N}(C))$. Similarly to the proof of Claim 6.2, choose new generators of $H^i(\tilde{N}(C'))$ and $\text{Im}\phi$ so that the latter are integer multiples of the former. In particular, it is easy to see that $\text{Im}\phi$ is a computable subgroup of $H^i(\tilde{N}(C'))$. This means that we can augment $\text{Im}\phi$ with extra generators in a computable way to expand it to $H^i(\tilde{N}(C'))$. It follows that $\lim_{C \in \mathcal{N}} H^i(\tilde{N}(C)) = H^i(G)$ can be consistently defined as the “union” of the $H^i(\tilde{N}(C))$, $C \in \mathcal{N}$, to obtain a group in which the operations are computable and the equality is c.e. (The equality is merely c.e. since an element $a \in H^i(\tilde{N}(C))$ can be mapped to 0 in some $H^i(\tilde{N}(C''))$ which appears arbitrarily late in the directed system.) \square

Since \hat{G} is torsion-free, to finish the proof it is sufficient to apply the result below.

Proposition 6.4 (Khisamiev [31]). *Every c.e.-presented torsion-free abelian group A has a computable presentation.*

In contrast with Dobritsa’s proof, the detailed argument contained in [31] seems complete and correct, but it is not necessarily easy to follow by current standards. We thus give an extended proof idea.

Proof idea. At every stage, we have a finitely generated partial group C_s and an embedding of C_s into a c.e. presentation U of A . Suppose C_s is generated by $b_1, b_2, \dots, b_{k(s)}$. At a later stage we may discover that the image of $h = \sum_i m_i b_i$ in U is declared equal to 0. As in the proof of Claim 6.2, we can pick a new collection of generators $g_1, \dots, g_{k(s)}$ of C_s such that $ng_{k(s)} = h$ for some n . (Notice however that here we are dealing with partial groups which adds a bit of extra combinatorial noise to the construction. In particular, strictly speaking, we cannot refer to Claim 6.2 since all our objects are finite partial groups. This is nonetheless not really a problem, but we omit the details.)

It is crucial that the image of $g_{k(s)}$ in U must be 0 as well, because A is torsion-free. We thus can safely dispose of $g_{k(s)}$ by declaring it equal to a linear combination of the generators which have not yet been declared zero using sufficiently large coefficients. For a modern clarification and verification of this strategy, see the proof of Claim 4.7 in [23]. Note that for the process to eventually stabilise, we need to have at least one non-zero element in U . It is routine to show that, if the setup of the construction is

right, the map from C to U is eventually stable (i.e., the process is Δ_2^0) and its limit is a surjective isomorphism of groups. We omit technical details. \square

Remark 6.5. It follows that there is a uniform procedure which, given a c.e.-presentation of a *non-zero* torsion-free abelian group, outputs its computable presentation. See also [39] for a discussion. (Note that the requirement of being *non-zero* was omitted in the statement of Theorem 4.6 in [39] since the zero group was viewed as torsion in [39].)

7. A CONNECTED COMPACT COUNTEREXAMPLE. PROOF OF THEOREM 1.5

Recall that the Pontryagin duals of connected compact Polish abelian groups are exactly the discrete torsion-free abelian groups; see, e.g., [40]. We prove that there exists a compact computably metrized connected Polish abelian group G such that \widehat{G} has no computable presentation.

7.1. Notation and an informal description of the proof. Recall the notation of Subsection 4.2. We build $G \leq \mathbb{A} = \prod_i \mathbb{T}_i$ so that its discrete dual \widehat{G} is isomorphic to the additive subgroup of the rationals generated by $\{\frac{1}{p_i} : i \in U\}$, where (p_i) is the standard listing of all primes and U an infinite set of natural numbers. The following fact is immediate; it is a special case of the elementary characterisation of computable subgroups of $(\mathbb{Q}, +)$ which can be traced back to Maltsev [35].

Claim 7.1. $\widehat{G} = \langle \{\frac{1}{p_i} : i \in U\} \rangle \leq \mathbb{Q}$ is computably presentable if, and only if, U is c.e.

The elementary proof can be found in [39]. Thus, it is sufficient to construct an effectively metrized G of this form such that the invariant set U of \widehat{G} is not c.e. Recall the naive construction described in §4.2.2. The plan is to build $\widehat{G} \leq \mathbb{Q}$ and, based on the naive construction, produce $G \cong \widehat{\widehat{G}} \leq \mathbb{A}$. We identify G and $\widehat{\widehat{G}}$ throughout the proof.

In the naive construction, we declare $n\chi_s = \sum_{i < s} m_i \chi_i$ for all $\chi_i \in \mathbb{T}_i \cap G$ whenever we have a relation $ng_s = \sum_{i < s} m_i g_i$ in $\widehat{G} = \{g_0 = 0, g_1 = 1, g_2, g_3, \dots\}$. In §4.2.2, we also declare $\pi_i G = \mathbb{T}_i \cap G$ to be either the whole \mathbb{T}_i or a discrete subgroup of \mathbb{T}_i depending on the order of generator g_i of \widehat{G} . In our case, each non-zero g_i have infinite order since \widehat{G} is torsion-free. Thus, we need to worry only about the rules of the form $n\chi_s = \sum_{i < s} m_i \chi_i$.

The idea is to *not* make immediate commitments and, for example, ensure that $n\chi_s = \sum_{i < s} m_i \chi_i$ holds up to error 2^{-s} at stage s . This should be understood as follows.

For example, we can begin with believing that $pg_2 = g_1$, where p is a prime. This corresponds to the rule

$$(\dagger) \quad p\chi_2 = \chi_1,$$

which must be satisfied by every $\chi_1 \in \mathbb{T}_1 \cap G$ and $\chi_2 \in \mathbb{T}_2 \cap G$. At stage s , we produce a finite list of pairs of open intervals $(I_1, I_2) \subset \mathbb{T}_1 \times \mathbb{T}_2$ such that the size of each I_i is at most 2^{-s} , and such that $pI_2 \subseteq I_1$. We can additionally require that all such I_2 listed so far (of size 2^{-s}) together cover the whole \mathbb{T}_2 , and thus the corresponding intervals I_1 cover \mathbb{T}_1 . This is equivalent to approximating, up to error 2^{-s} , the effectively continuous map $x \rightarrow px$ between \mathbb{T}_2 and \mathbb{T}_1 .

Remark 7.2. Note that, unless $x = 0$, there are p pre-images of $x \in \mathbb{T}_1$ in \mathbb{T}_2 under $x \rightarrow px$. We could explicitly computably list the name of the inverse multi-function $x \rightarrow p^{-1}x$. But we only need that these pre-images can be computed, which is obvious.

If nothing has to be changed at a later stage and $p\chi_2 = \chi_1$ is the only rule, then G is the intersection of the effectively closed sets H_s , such that H_s is composed of $(\chi_i : i \in \mathbb{N})$ where $p\chi_2 = \chi_1$ up to error 2^{-s} . For every fixed s , it is easy to list a computable dense subset of H : simply list the rational points in \mathbb{T}_1 and take their pre-images in \mathbb{T}_2 under the approximated map. We initiate such a list; at every stage we have only finitely many

points which have already been listed. For each such rational $r_1 \in \mathbb{T}_1$, as s gets larger these pre-images computably converge to finitely many points in \mathbb{T}_2 ; see Remark 7.2. This way we will obtain a computable dense subset of G in the limit.

However, we must make the set of primes U not c.e., and this means that some primes will have to be extracted from U at some stage. For example, suppose that at stage s we change our mind and decide that $p \notin U$. The idea is to take q very large which is currently outside of U , declare $q \in U$, and replace the rule $x_0 = px_i$ with the rule $x_0 = qx_i$. Since q is very large, each of the finitely many intervals of \mathbb{T}_2 of size 2^{-s} currently listed in the 2^{-s} -approximation of $x \rightarrow px$ contains at least one point of the form $\frac{k}{q-p}$, where $k \leq (q-p)$. Under $x \rightarrow qx$, the point $\frac{k}{q-p}$ is mapped to

$$(\ddagger) \quad \frac{kq}{q-p} = \frac{k(q-p+p)}{q-p} = k + \frac{kp}{q-p} = \frac{kp}{q-p}.$$

In other words, $x \rightarrow px$ and $x \rightarrow qx$ agree on all points of this form. At stage s we have enumerated only finitely many intervals approximating $x \rightarrow px$ (in the sense as explained above). Using these intervals, we have initiated the approximation of finitely many pre-images

$$y_0, \dots, y_{p-1} \in \mathbb{T}_2,$$

of finitely many rational points $r \in \mathbb{T}_1$ under the map $x \rightarrow px$.

For each such $y_i \in \mathbb{T}_1$, fix the intervals $I \ni y_i$ and $J \ni r$ of size at most 2^{-s} such that (J, I) has already been listed in the 2^{-s} -approximation of $px : \mathbb{T}_2 \rightarrow \mathbb{T}_1$. Choose a k such that $z = \frac{k}{q-p} \in J$. It follows from (\ddagger) that $qz = pz \in I$. In other words, since $r \in I$, the interval J can equivalently be viewed as a 2^{-s} -approximation to the point qr .

We thus *switch* the process of enumeration of the dense set of G by replacing (\ddagger) with the new rule

$$(*) \quad q\chi_2 = \chi_1,$$

which, from this stage on, must hold for every $\chi_1 \in \mathbb{T}_1 \cap G$ and $\chi_2 \in \mathbb{T}_2 \cap G$.

It follows that we can adjust our approximation of the dense set of G to the new rule $(*)$, and the current best approximations to the old preimages of r can be recycled as approximations to some of its new preimages. Unless $r = 0$, we also have to introduce more approximations to preimages of r since q is much larger than p ; see Remark 7.2.

In the case of many generators, we work with \mathbb{T}_1 which corresponds to $1 \in \mathbb{Q}$ and \mathbb{T}_i that corresponds to $g_i \in G$ such that $g_i = \frac{1}{p}$ for some p which depends on i and is uniquely determined by i . There is no interaction between strategies working with different generators. At the end, we will define infinite sequences of shrinking intervals, and we define a dense computable sequence on G using these infinite sequences similarly to how it is done in the Baire space: starting from some large enough index, always pick the next interval in your approximation to be the first one found inside the previous interval of the cruder approximation.

7.2. Formal proof. Take a non-c.e. set $U = \text{range } \lim_s f(i, s)$, where $f : \mathbb{N} \rightarrow \mathbb{N}$ is injective, total computable, and for every i there exists at most one s such that $f(i, s) \neq f(i, s+1)$; in this case we also require

$$f(i, s) \neq f(i, s+1) \implies f(i, s+1) > s.$$

In other words, let $U = \{\lim_s f(i, s) : i \in \mathbb{N}\}$, where the value of the total computable function $f(i, s)$ of two arguments changes at most once for each fixed i , and furthermore if this happens at s then the new value is set equal to a number larger than the stage. We note that, when $f(i, s+1) \neq f(i, s)$, the value $f(i, s+1)$ can be set as large as we desire

during the construction. For instance, $f(i, s+1)$ can be picked larger than $h(i)$ for any fixed total computable function h . It is easy to build such a U and additionally satisfy the usual requirements $W_e \neq U$ for $e \in \mathbb{N}$.

Remark 7.3. We give a bit more detail. To diagonalize against the e -th c.e. set W_e , pick a follower x_e and keep it in U . In our notation, this corresponds to setting $f(e, s) = x_e$ at stage s . If x_e never enters W_e then $W_e \neq U$. If at some later stage t the follower x_e is enumerated into W_e , then we permanently extract x_e from U ; this corresponds to setting $f(e, t) = y$ for some y larger than any number mentioned so far in the construction. In this case again $W_e \neq U$. Since we always use large numbers to redefine f , x_e will be permanently kept out of the range of $\lim_s f(\cdot, s)$. In particular, there is no interaction between the strategies working with different c.e. sets. We leave further standard details to the reader.

Since U is not c.e., the group $\widehat{G} = \langle \{\frac{1}{p_i} : i \in U\} \rangle \leq \mathbb{Q}$ has no computable presentation (Claim 7.1). We claim that $G \cong \widehat{G}$ is computably metrizable. Let $\widehat{G} = \{g_0 = 0, g_1 = 1, g_2, g_3 \dots\}$. We build G as a closed subset of $\mathbb{A} = \prod_i \mathbb{T}_i$, where \mathbb{T}_i corresponds to the value of a character $\chi \in \widehat{G}$ evaluated at g_i ; see Subsection 4.2. To define a computable metric, initiate an approximation of the dense set $(\rho^j)_{j \in \mathbb{N}}$ of G , as follows.

7.2.1. Construction. At stage s , define open intervals $U_{i,s}^j \subseteq \mathbb{T}_i$ with rational end-points, which satisfy, for every $i, j \leq s$:

- (1) $U_{i,s}^j \subseteq \mathbb{T}_i$ and $0 \in U_{0,s}^j$;
- (2) $U_{i,s}^j \subseteq U_{i,s-1}^j$ if $U_{i,s-1}^j$ is defined;
- (3) $p_{f(i,s)} U_{i,s}^j \subseteq U_{1,s}^j$;
- (4) $U_{1,s}^j$ cover \mathbb{T}_1 ;
- (5) $\text{diam } U_{i,s}^j < 2^{-s}$;
- (6) If $c_{j,s}$ is the center of $U_{1,s}^j$, then for each $y \in \mathbb{T}_i$ such that $p_{f(i,s)} y = c_{j,s}$, then there is an interval $U_{i,s}^k$ with center y . If there is no such interval, then introduce a new one with this property which also satisfies (1) – (4) (if they are applicable).

(We can have $U_{i,s}^j = U_{i,s}^l$ for $j \neq l$.) This finishes the construction.

7.2.2. Verification. Since the informal explanation was rather detailed, we give a somewhat compressed verification to avoid unnecessary repetition. Use $f(i, s) \neq f(i, s+1) \implies f(i, s+1) > h(i)$, where $h(i)$ is so large that, for every j such that $U_{1,s-1}^j$ is defined, there exists an integer k with the property

$$\frac{k}{p_{f(i,s+1)} - p_{f(i,s)}} \in U_{1,s-1}^j.$$

Remark 7.4. There is no circularity here; for instance, we could use Kleene's Recursion Theorem to argue that the index of h is known. Alternatively, we could uniformly combine the construction of f with the construction of the group, so that at every stage we have total control over the value of h when it is needed, but this would be more tedious.

It follows from the equation below that holds in \mathbb{R}/\mathbb{Z} (cf. (‡)):

$$(‡') \quad \frac{k p_{f(i,s+1)}}{p_{f(i,s+1)} - p_{f(i,s)}} = \frac{k p_{f(i,s)}}{p_{f(i,s+1)} - p_{f(i,s)}}$$

that condition (3) can be maintained at every stage of the construction, for a suitable choice of intervals. For that, pick $U_{i,s}^j$ using a small enough neighbourhood of

$$\frac{k}{p_{f(i,s+1)} - p_{f(i,s)}} \in U_{1,s-1}^j.$$

The rest of the verification is routine. Using the intervals $U_{i,s}^j$, define a sequence $(\rho^j)_{j \in \mathbb{N}}$ of points uniformly computable in \mathbb{A} , and consider the closure $cl(\rho^j)_{j \in \mathbb{N}}$ of the sequence in \mathbb{A} . It forms a computably metrized group under the operations inherited from \mathbb{A} . The group is isomorphic to the group of characters of \widehat{G} by design; see Subsection 4.2 for more details. By Pontryagin - van Kampen duality, $cl(\rho^j)_{j \in \mathbb{N}} \cong \widehat{\widehat{G}} \cong G$. By the choice of U and Claim 7.1, \widehat{G} has no computable presentation.

8. CONSEQUENCES

8.1. Proof of Corollary 1.6. Recall that the corollary states:

- (1) There exists a computably metrized connected Polish group not homeomorphic to any effectively compact Polish space. (This simultaneously answers (Q1) and (Q2).)
- (2) There exists a Δ_2^0 -metrized connected Polish group not homeomorphic to any computably metrized Polish space. (This simultaneously answers (Q3) and Question 3 of [24].)

The first clause of the corollary follows from the proof of Theorem 1.1 and the previous theorem. Indeed, recall that to produce the discrete dual of an effectively compact connected abelian group we do not need to use the group operation. Thus, if the computably metrized compact connected domain of the group constructed in Theorem 1.5 was homeomorphic to an effectively compact space, then we would be able to produce a computable presentation of its dual group.

To see why the second clause of the corollary holds, relativize the proof of Theorem 1.5 to $0'$. We obtain a Δ_2^0 -metrized Polish group G_S such that $\widehat{G_S}$ has no Δ_2^0 -presentation. Assume the underlying Polish space M of G admits a computable metrization, and thus a Δ_2^0 -effectively compact presentation. Apply the construction from the proof of Theorem 1.1 to this presentation to calculate a Δ_2^0 -presentation of $\widehat{G_S}$, which is a contradiction. (Recall that we do not need to use the group operation on M .)

8.2. Background on index sets. Let

$$M_0, M_1, \dots$$

be the list of all partially computable presentations of metrized Polish spaces with two partial operations on them (one binary and one unary), and let $\overline{M_e}$ denote the completion of M_e . It is not hard to see that

$$\{e : \overline{M_e} \text{ is a compact connected Polish group}\}$$

is arithmetical (Π_3^0); see [37]. It thus makes sense to study the complexity of index sets of various classes compact connected groups which we define below.

Let K be a class of compact topological groups. The index set of K , or the characterisation problem for K , is the set

$$\{e : \overline{M_e} \in K\}.$$

The isomorphism problem for K is the set

$$\{(e, j) : \overline{M_e}, \overline{M_j} \in K \text{ and } \overline{M_e} \cong \overline{M_j}\},$$

where in our case \cong stands for topological group isomorphism of groups.

These definitions mimic the analogous definitions for computable discrete structures which can be found in, e.g., [21]. An argument can be made that the complexity

of these sets accurately reflect the complexity of the classification problem for K , especially if the estimates that we obtain can be relativized to an arbitrary oracle; see [11] for a detailed discussion.

8.3. Proof of Corollary 1.4. Recall that the corollary states that, for each of the following classes, both the characterization problem and the isomorphism problem are arithmetical:

- (1) compact abelian Lie groups;
- (2) direct products of solenoid groups;
- (3) connected compact abelian groups of finite covering dimension.

We can arithmetically (this is Σ_1^0) check whether \overline{M}_e is a *non-zero* compact connected abelian group. Then the dual of the group has to be non-zero torsion-free abelian, and it is also isomorphic to the first Čech cohomology group which admits a Σ_2^0 presentation uniformly in e . By Remark 6.5, we can uniformly produce a Δ_2^0 -presentation of the discrete torsion-free dual of \overline{M}_e . Thus, in each case it is sufficient to check that the isomorphism and the characterisation problems of the respective discrete duals are arithmetical.

8.3.1. Compact abelian Lie groups. It is well-known that, up to topological isomorphism, every compact abelian Lie group is the product of finitely many copies of the unit circle \mathbb{T} ; e.g., [46]. The duals of such groups are exactly the direct sum of the same number of copies of \mathbb{Z} , i.e., are free abelian groups of finite rank. It is easy to see that the index set and the isomorphism problem for free abelian groups of finite rank is arithmetical; the same is true for Δ_2^0 free abelian groups of finite rank. Indeed, there is a uniformly computable list of isomorphism types of such groups, and every group in the list is relatively computably categorical. It is sufficient to ask whether there is a Δ_2^0 computable isomorphism from the given Δ_2^0 presentation of the dual to one of the groups in the computable list. This is clearly an arithmetical question. This makes the isomorphism problem arithmetical too.

8.3.2. Direct products of solenoid groups. Recall that the solenoid groups are exactly the duals of additive subgroups of \mathbb{Q} . Thus, the duals of direct products of solenoid groups are exactly the direct sums of additive subgroups of \mathbb{Q} . Such groups are called completely decomposable. The main result of [10] says that both the isomorphism problem and the characterisation problem for completely decomposable groups are arithmetical. The proof in [10] can be relativised to $0'$, thus giving (2) of the corollary.

8.3.3. Compact connected abelian groups of finite dimension. Recall that, under duality, covering dimension corresponds to Prüfer rank of the discrete dual (Thm 47 of [46]). It is clear that the property of having a finite Prüfer rank is arithmetical. It remains to observe that every such group is relatively computably categorical, so we again can check if there is a Δ_2^0 -isomorphism (between two Δ_2^0 duals) instead of an arbitrary isomorphism. It follows that both the isomorphism problem and the characterisation problem for this class are arithmetical.

9. EXTENDING THE THEOREM OF DOBRITSA TO T.D.L.C. GROUPS: PROOF OF PROPOSITION 5.1

By hypothesis G has a computable t.d.l.c. presentation as in Definition 3.4(2). In other words, we may assume G equals the inverse limit of uniformly computable discrete groups (A_i) under uniformly computable surjective projections $\phi_i : A_i \rightarrow A_{i-1}$ having finite kernels; furthermore, the kernels are given by their strong indices. Proposition 5.1 says that we can pass to a computable t.d.l.c. presentation in which each discrete B_i has a uniformly computable maximal linearly independent set. The rest of the section is devoted to the proof of this proposition.

9.1. The choice of technique. Recall that Dobritsa proved that every computable abelian group has a computable presentation with computable linearly independent set. The original, 1-page proof of Dobritsa [7] is extremely compressed and omits some important details. Dobritsa used a strategy that, as far as we know, was invented by Nurtazin [42] for a different purpose; the brief sketch in the research announcement [42] is even more compressed than the aforementioned proof of Dobritsa³. Both major surveys on the subject [32] and [39] also contain merely outlines of the proof. The only complete proof of the theorem of Dobritsa in the literature can be found in [37]. It replaces the clever combinatorial strategy of Nurtazin [42] with an application of abelian group theory. The result then becomes a special case of the more general theorem from [23] which is not restricted to groups but works for other classes too (such as differential closed fields, for instance).

Unfortunately, we cannot use the abstract techniques of [37, 23] to extend the theorem of Dobritsa to the t.d.l.c. case; this is because we have to simultaneously approximate the projections between the B_i , so the group-theoretic structure on B_i cannot be separated from the process of building a maximal linearly independent set. It has to be done all at once. We will have to use a version of Nurtazin's strategy, which tends to make the combinatorics in the proof more complex.

Not only is the theorem of Dobritsa a special case of Proposition 5.1, but the proof of Proposition 5.1 that we give below is the only complete proof of Dobritsa's result in the literature which relies on a variation of Nurtazin's strategy rather than on some other strategy. The only similar full proof of this kind can be found in [22] where Nurtazin's strategy was modified to handle the somewhat tamer class of computable ordered abelian groups.

We begin our discussion with the elementary case of finite rank, and then move on to the discussion and the proof of the general case.

9.2. Proposition 5.1 in the case of finite rank. For an abelian group A , elements are linearly independent in A iff they are independent in $A/T(A)$, where $T(A)$ is the subgroup of torsion elements of A . If $\phi : A \rightarrow B$ is a surjective homomorphism with finite kernel, then the kernel in particular consists of torsion elements. These simple facts will be used to verify the following.

Claim 9.1. Suppose $\phi : A \rightarrow B$ is a surjective homomorphism of abelian groups with finite kernel. Suppose $C \subseteq A$. If C is a maximal linearly independent in A then $\phi(C)$ is maximal linearly independent in B .

Conversely, suppose $D = \{d_0, d_1, \dots\}$ is maximal linearly independent in B . Then any set C' of the form $\{c'_0, c'_1, \dots\}$ where $\phi(c'_i) = d_i$ is a maximal linearly independent subset of A .

Proof of Claim 9.1. First, suppose $C = \{c_0, c_1, \dots\}$ is a maximal linearly independent subset of A . Assume $\sum_i n_i \phi(c_i) = 0$, i.e., $\sum_i n_i c_i \in \text{Ker } \phi$. Since $\text{Ker } \phi$ is finite, for $m > 0$ large enough we have

$$\sum_i m n_i c_i = 0,$$

and therefore $m n_i = 0 \iff n_i = 0$ for all i . Fix $z \in B$ and let $\phi(y) = z$. By maximality of C in A , there exist $c_i \in C$ and coefficients m_i and $m \neq 0$ such that $\sum_i m_i c_i + m y = 0$. It follows that $\sum_i m_i \phi(c_i) + m z = 0$. Thus, $\phi(C)$ is maximal linearly independent in B .

Now suppose that $D = \{d_0, d_1, \dots\}$ is maximal linearly independent in B , and that C' is a subset of A of the form $\{c'_0, c'_1, \dots\}$ where $\phi(c'_i) = d_i$. If $\sum_i m_i c'_i = 0$ then $\phi(\sum_i m_i c'_i) =$

³As far as we know, the volume containing [42] is stored at the basement of the Sobolev Institute of Mathematics library. Only a physical copy is available and only upon a special request.

$\sum_i m_i d_i = 0$, which implies $m_i = 0$ for all i . Fix $x \in A$; by maximality of D in B there exist m_1, \dots, m_k and $m \neq 0$ such that

$$m\phi(x) + \sum_i m_i d_i = 0.$$

Thus,

$$0 = m\phi(x) + \sum_i m_i \phi(c'_i) = \phi(mx + \sum_i m_i c'_i),$$

and therefore $mx + \sum_i m_i c'_i$ is a torsion element since $\text{Ker } \phi$ is finite. Pick $d > 0$ large enough so that

$$0 = d(mx + \sum_i m_i c'_i) = dmx + \sum_i dm_i c'_i.$$

It remains to note that $dm \neq 0$. □

Remark 9.2. The “large enough d ” at the end of the proof above can be computed uniformly from the strong index of $\text{Ker } \phi$.

We return to the proof of the proposition. By Claim 9.1, if the rank of A_0 is finite, then there is nothing to prove since the maps ϕ_i are computable. Throughout the rest of the proof, assume that the rank of A_0 (thus, of each A_i) is infinite.

9.3. Proof idea of Proposition 5.1 in the case of infinite rank. In brief, we will combine the proof of Dobritsa’s Theorem 4.2 with a dynamic version of the elementary proof of Claim 9.1 to simultaneously permute all A_i . Unfortunately, there are several technical issues that cannot be summarised in just one sentence. We give more intuition below.

9.3.1. The case of only one A_0 . The factorial trick. We first informally outline the main idea of Dobritsa’s original proof in the case when we have only one computable discrete group. Given a computable discrete A , Dobritsa transforms it into a computable discrete B having a computable maximal linearly independent set C , as follows.

Build a Δ_2^0 isomorphism $\theta : B \rightarrow A$. Initially, let θ copy A into B without any change. In B , declare that a computable set $C = \{c_0, c_1, \dots\}$ is a linearly independent set. For simplicity, pick just two $a_0, a_1 \in A$ which currently look linearly independent (recall that linear independence is a Π_1^0 -property) and interpret c_0 as the pre-image of a_0 and c_1 as the pre-image of a_0 . If these a_0 and a_1 are indeed linearly independent, then $\theta(c_i)$, $i = 0, 1$, do not have to be changed. However, at a later stage we may discover that, in A , a_0 and a_1 are linearly dependent:

$$n_0 a_0 + n_1 a_1 = 0,$$

and therefore we must pick a new image for c_1 in A .

For that, choose the first found $d \in A$ which currently looks independent of a_0 (recall independence is Π_1^0) and, following the idea of Nurtazin [42], define

$$\theta(c_1) = a_1 + t!d,$$

where t is larger than any number mentioned so far in the construction. We have to also correct θ on other elements, but we omit details. It is important to note that, if d is indeed independent of a_0 , then so is $a_1 + t!d$; furthermore $a_1 + t!d$ and d will have equal linear spans over a_0 . Otherwise, the strategy will be repeated with a fresh d' , and then perhaps d'' (etc.) until a $d^{(k)}$ truly independent over a_0 is found. In particular, it follows that this process of correcting mistakes will eventually stabilise.

We call Nurtazin’s strategy involving $t!$ “the factorial trick”; see the surveys [32, 39] and also [22, 23] for more about this strategy, its applications, and variations. Of course, the strategy can be extended to cover arbitrary collections of c_i , not only two of them. We address several questions that can potentially puzzle the reader if they are not familiar with this or similar method.

Why do we need the factorial? It is necessary to preserve the relations. We may have already defined θ on x such that

$$m\theta(x) = m_0 a_0 + m_1 a_1,$$

where m does not divide the $\text{GCD}(m_0, m_1)$. The new image of x will be

$$m_0 a_0 + m_1 (a_1 + t!d) = (m_0 a_0 + m_1 a_1) + m_1 t!d,$$

where the former summand $(m_0 a_0 + m_1 a_1)$ is divisible by m as witnessed by the previous image of x , and $m_1 t!d$ is divisible by m because $m < t$ (recall t is large). In particular, the relation $mx = m_0 c_0 + m_1 c_1$ will be preserved under θ . Of course, we do not have to use $t!$ here, but since we do not necessarily worry about the efficiency of our algorithm we can just as well stick with the factorial.

How do we make θ an onto homomorphism? Surjectivity can be achieved by specifically putting the s -th element into the range of θ at stage s . Alternatively, we can search for a_1 with the least possible index (in A) such that a_1 is linearly independent of a_0 . This will guarantee that $(c_i)_{i \in \mathbb{N}}$ is mapped to a basis of A . Since every element of A can be expressed as a linear combination of basic elements over the maximal torsion $T(A)$, all we need to do is to make sure that each such combination has a pre-image. The “factorial trick” will preserve any relation mentioned so far, and this ensures that θ respects the group structure.

Why is θ injective? This is perhaps the most subtle question which was furthermore almost completely overlooked in the original compressed proof in [7]. We must argue, by induction, that no new relations are introduced in the process of correcting θ . This will be done in Claim 9.5; note that the proof of the claim is not entirely trivial.

9.3.2. The case of two groups A_0 and A_1 . Now assume we have only two groups in the sequence, $A_0 \leftarrow_{\phi_1} A_1$, and recall that the kernel of ϕ_1 is finite and given by its strong index. In particular, we can assume that $\text{Ker } \phi_1$ is already known at stage 0. In this case, the idea is to simultaneously build computable presentations B_0 and B_1 of A_0 and A_1 , respectively, approximate Δ_2^0 isomorphisms $\theta_0 : B_0 \rightarrow A_0$ and $\theta_1 : B_1 \rightarrow A_1$, and at every stage maintain $\phi_1 \theta_1 = \theta_0$. We also build computable bases C_0 and C_1 in B_0 and B_1 , respectively.

By Claim 9.1, Nurtazin’s strategy can be used in A_1 , while the change of θ_0 is done by correcting it via ϕ_1 . The brute-force proof of Claim 9.1 can help the reader to see why this is possible. In that proof, we used a “large enough” coefficient $d > 0$ to transfer linear combinations between the two groups; see Remark 9.2. In the present proof, it is sufficient to use d larger than the order of the kernel of ϕ_1 when we compare our current best approximations to linear independence in A_0 and A_1 . For instance, we will never have to correct θ_0 independently of θ_1 (or vice versa). Then we have to argue, by induction, that the maps θ_1 and θ_0 do not have to be corrected on torsion elements. Thus, since $\text{Ker } \phi_1$ is finite, we will conclude that the induced map $\psi_1 : B_1 \rightarrow B_0$ defined by the rule $\psi_1 = \theta_0 \phi_1 \theta_1$ is computable and we can compute the strong index of its finite kernel; this is because θ_0 is corrected if, and only if, θ_1 is corrected.

9.3.3. The general case. In this case we have to deal with an infinite sequence $A_0 \leftarrow_{\phi_1} A_1 \leftarrow_{\phi_2} A_2 \leftarrow_{\phi_3} \dots$, but at every stage we work only with finitely many of these A_i . We will correct θ_i for the largest i attended so far in the construction, and we will use ϕ_j , $j \leq i$, to correct θ_j for $j < i$. The only difference is that we will have to correct $\theta_i : B_i \rightarrow A_i$ so that the indices of the new images of linearly independent elements in A_0 (not A_i) are smallest possible. This is because we need to argue that in some of these A_i the image of $C_i \subseteq B_i$ is indeed *maximal* linearly independent. In contrast with the case of only two groups described above, we cannot just work with the largest i , since this largest i will keep increasing at later stages. Nonetheless, Claim 9.1 essentially says that this minimality of indices can be checked in A_0 , since independence in A_i is effectively

coherent with independence in A_0 . The rest is similar to the case of only two groups described above. For instance, we need strong indices of the finite kernels of ϕ_i to make this work.

9.4. Formal proof of Proposition 5.1.

9.4.1. *Notation, conventions, and terminology.* At the end of stage $t - 1$ we initiate only the enumeration of B_i for $i < t$. We label elements of discrete computable groups with natural numbers; a natural number corresponding to an element will be called the index of the element. Without loss of generality, we can assume that the kernels of ϕ_i , $i \leq t$, are already computed. Furthermore, we can assume that the order of $\text{Ker } \phi_i$ is at most i ; to do that we effectively replace the system (A_i, ϕ_i) with a new system where some of the projections are the identity.

In the construction, we write $\xi[t]$ to denote the value of any parameter ξ at the end of stage t . We list B_i so that each $x \in B_i[t]$ is of the form $mx = \sum_{j < t} n_j c_{i,j} + d$, where $m, n_j \leq t$ and d is torsion such that the subgroup generated by d lies in $B_i[t]$. We call such a d *torsion as seen at stage t*. At the end of stage $t - 1$, we have $\theta_i[t - 1]$ defined on all elements of $B_i[t]$, including $c_{i,j}$, where $i \leq t$ and $j < t$.

9.4.2. *The requirements.* We build a uniformly computable sequence (B_i) of discrete groups such that the uniformly computable set $C_i = (c_{i,j})_{j \in \mathbb{N}}$ (of indices in B_i) is a maximal linearly independent set in B_i . We also define $\psi_i : B_i \rightarrow B_{i-1}$ so that $\psi_i(c_{i,j}) = c_{i-1,j}$. At every stage t , we also define partial maps $\theta_i[t] : B_i[t] \rightarrow A_i[t]$ and $\psi_i[t] : B_i[t] \rightarrow B_{i-1}[t]$ which satisfy:

$$\psi_i[t] = \theta_{i-1}^{-1}[t] \circ \phi_i[t] \circ \theta_i[t],$$

where defined. It is sufficient to meet, for every i , the requirements:

- $L_i : (\forall b \in B_i)(\exists t_0)(\forall t > t_0)\theta_i(b)[t] = \theta_i(b)[t_0]$;
- $I_i : \theta_i = \lim_t \theta_i[t] : B_i \rightarrow A_i$ is an isomorphism;
- $R_i : C_i$ is maximal linearly independent in B_i ;
- $P_i : \psi_i : B_i \rightarrow B_{i-1}$ is computable, uniformly in i ;
- $K_i : \text{Ker } \psi_i$ has computable strong index, uniformly in i .

The construction is perhaps best viewed as a *movable markers* argument, where each movable marker corresponds to $\theta_i(b) \in A_{i-1}$ for some $b \in B_i$. We will however not make movable markers explicit since the main complexity of the proof is not related to recursion-theoretic combinatorics.

9.4.3. *The notion of s-independence.* Elements g_1, \dots, g_k of an abelian group G are *s-independent* if, for any choice of $n_1, \dots, n_k \in \mathbb{Z}$ such that $|n_i| \leq s$

$$n_1 g_1 + \dots + n_k g_k = 0$$

implies $n_i = 0$ for all i . Clearly, elements are linearly independent if, and only if, they are *s-independent* for all s .

Remark 9.3. Note that g_1, \dots, g_k are *s-independent* if, and only if, the map $n_1 g_1 + \dots + n_k g_k \rightarrow (n_1, \dots, n_k) \in \bigoplus_{1 \leq i \leq k} \mathbb{Z}, |n_i| \leq s$, is injective.

9.4.4. *The construction.* At stage 0, begin with $C_0 = \emptyset$ in B_0 , and let B_0 copy A_0 via $\theta_0[0] = Id$, i.e., without any nontrivial permutation.

Stage t . We subdivide the stage into several phases:

- (a) Choose $k < t$ largest such that $\theta_i(c_{i,0})[t-1], \dots, \theta_i(c_{i,k})[t-1]$ are $2(t+1)!$ -independent in A_i , $i \leq t$. For every $i \leq t$, choose $d_{i,k+1}, \dots, d_{i,t} \in A_i$ and $a_{t,0}, \dots, a_{t,k} \in A_t$ such that:
 - (a.1) $\theta_i(c_{i,0})[t-1], \dots, \theta_i(c_{i,k})[t-1], \theta_i(c_{i,k+1})[t-1] + t!d_{i,k+1}, \dots, \theta_i(c_{i,t-1})[t-1] + t!d_{i,t-1}$ together with $d_{i,t}$ form a $2(t+1)!$ -independent set;
 - (a.2) $\phi_t(a_{t,r}) = \theta_{t-1}(c_{t-1,r})$, $r \leq k$;
 - (a.3) $d_{i-1,j} = \phi_i(d_{i,j})$, for each $1 \leq i \leq t$ and $k < u \leq t$;
 - (a.4) $d_{0,k+1}, \dots, d_{0,t} \in A_0$ have the smallest possible indices (lexicographically).
- (b) For each $j \leq t$ introduce $c_{t,j}$, and for every $i \leq t$ introduce $c_{i,t}$, and declare:
 - (b.1) $\theta_i(c_{i,t})[t] = d_{i,t}$;
 - (b.2) $\theta_t(c_{t,j})[t] = a_{t,j}$, $j \leq k$ (see (a) for the definition of $a_{t,j}$).
- (c) Redefine θ_i on each $c_{i,r}$, where $i \leq t$, $k < r < t$, by setting

$$\theta_i(c_{i,r})[t] = \theta_i(c_{i,r})[t-1] + t!d_{i,r}.$$

Declare $\theta_i(c_{i,u})[t] = \theta_i(c_{i,u})[t-1]$ for every $u \leq k$.

- (d) For each $i \leq t$ and $x \in B_i[t-1]$ such that $mx = \sum_{j \leq t} n_j c_{i,j} + d$, where d is torsion such that the subgroup generated by d lies in $B[t-1]$, set

$$\theta_i(x)[t] = \theta_i(x)[t-1] + \sum_{t \geq r > k} \frac{n_r t!}{m} d_{i,r}[t].$$

- (e) For every $x \in A_t$ such that $mx = \sum_{j \leq k} n_j a_{t,j} + \sum_{k < j \leq t} n_j d_{t,j} + d$, where $|n_j| \leq t$ and d is torsion as seen in $A_t[t]$, if x does not already have a θ_t -preimage, introduce a new element b in B_t and define $\theta_t(b) = x$. In particular, every torsion d as seen in $A_t[t]$ will get a pre-image d' in B_t under θ_t . In B_t , declare

$$mb = \sum_{j \leq t} n_j c_{t,j} + d',$$

where $d' = \theta_t(d)[t]$.

- (f) For $i < t$, if $\theta_{i+1}[t]$ has already been extended, then extend the domain and range of $\theta_i[t]$ as follows. For every element $a \in \text{range}(\phi_{i+1}[t] \circ \theta_{i+1}[t]) \setminus \text{range}(\theta_i[t])$, introduce a new element $b \in B_i$ and declare $\theta_i(b)[t] = a$. Define the operation on $B_i[t]$ to be the one induced from A_i via $\theta_i[t]$.

Finally, set $\psi_i[t] = \theta_{i-1}^{-1}[t] \circ \phi_i[t] \circ \theta_i[t]$, $i \leq t$ and go to the next stage.

9.4.5. *The verification.* Note that, in (f), each new element b in $B_i[t]$ satisfies $mb = \sum_{j \leq t} n_j c_{i,j} + d'$ for some $n_j \leq t$ and d' torsion as seen in $B_i[t]$. This is because $d_{i-1,j} = \phi_i(d_{i,j})$, for each $1 \leq i \leq t$ by (a.3), and in (b) and (c) the $\theta_i[t]$ -images of $c_{i,j}$ were specifically defined to agree with (a.3) and, thus, with ϕ_i . In other words, the instructions in (f) can be equivalently re-phrased in terms similar to the instructions in (e), but we chose a more compact presentation. This in particular justifies the convention stated in §9.4.1 about the form of each $x \in B_i[t]$ at every stage t .

We need to check that every search initiated at stage t eventually terminates.

Claim 9.4. Every stage of the construction eventually terminates.

Proof. In (a), we search for elements in A_t which are $2(t+1)!$ -independent, and whose ψ_t -images are $2(t+1)!$ -independent in A_t . If $\theta_i(c_{i,0}), \dots, \theta_i(c_{i,k})$ are indeed independent, then such elements must exist because the rank of A_t is infinite. Such elements (independent or not) will eventually be found.

In (d), for each x such that $mx = \sum_{j \leq t} n_j c_{i,j} + d$, where d is torsion such that the subgroup generated by d lies in $B[t-1]$, we set

$$\theta_i(x)[t] = \theta_i(x)[t-1] + \sum_{t \geq r > k} \frac{n_r t!}{m} d_{i,r}[t];$$

such an element exists because $m < t$ by our convention (see §9.4.1). \square

Claim 9.5. At the end of every stage t , each $\theta_i[t]$ is injective.

Proof. By induction on t . At stage 0, $\theta_0[0]$ is injective since it is the identity map. Suppose $\theta_i[t-1]$ is injective. We suppress i throughout (when possible). Every element x of $B[t-1]$ satisfies a unique reduced relation $mx = \sum_j m_j c_j + d$, where d is torsion (as already seen in $B[t]$) and $m, m_i \in \mathbb{Z}$ are reduced.

We first show that redefining θ in (c) and (d) preserves injectivity of θ . Suppose $x, z \in B[t-1]$ and therefore $\theta[t-1]$ is defined on $x, z \in \text{dom } \theta[t-1]$, where

$$\begin{aligned} nz &= \sum_j n_j c_j + l, \\ mx &= \sum_j m_j c_j + d. \end{aligned}$$

If $\theta[t-1]$ and $\theta[t]$ are equal on the domain of $\theta[t-1]$ then there is nothing to prove. Suppose θ needs to be redefined. If $\theta(z)[t] = \theta(x)[t]$, then in (d) we define $\theta(x)[t] = \sum_{r > k} \frac{m_r t!}{m} d_r + \theta(x)[t-1]$, and we declare $\theta(z)[t]$ equal to $w = \sum_{r > k} \frac{n_r t!}{n} d_r + \theta(x)[t-1]$. These values satisfy the equations:

$$\begin{aligned} m\theta(x)[t] &= \sum_{j \leq k} m_j \theta(c_j)[t-1] + \sum_{r > k} m_r (\theta(c_r)[t-1] + t! d_r) + \theta(d)[t], \\ n\theta(z)[t] &= \sum_{j \leq k} n_j \theta(c_j)[t-1] + \sum_{r > k} n_r (\theta(c_r)[t-1] + t! d_r) + \theta(l)[t]. \end{aligned}$$

(Note that, in fact, $\theta(d)[t] = \theta(d)[t-1]$ and $\theta(l)[t] = \theta(l)[t-1]$, but this is not important for this particular argument.) The orders of d and l are at most t , and $m, n \leq t$. Multiply the first equation by $nt!$ and the second by $mt!$, and then subtract the first one from the second one. By (a.1) at stage t , the values $\theta(c_j)[t-1]$ and $\theta(c_r)[t-1] + t! d_r$ form a $2(t+1)!$ -independent set. In particular, it must be that, for every $r > k$, $t!nm_r = t!mn_r$, and since both $m, n \neq 0$, we arrive at

$$\frac{m_r t!}{m} = \frac{n_r t!}{n}, \text{ for each } r > k.$$

Now recall that $\theta(x)[t] = \sum_{r > k} \frac{m_r t!}{m} d_r + \theta(x)[t-1]$ and $\theta(z)[t] = \sum_{r > k} \frac{n_r t!}{n} d_r + \theta(x)[t-1]$.

Since $\sum_{r > k} \frac{n_r t!}{n} d_r = \sum_{r > k} \frac{m_r t!}{m} d_r$ by the above remarks, and $\theta(z)[t] = \theta(x)[t]$ by our assumption, we must have that

$$\theta(z)[t-1] = \theta(x)[t-1].$$

Since $\theta[t-1]$ is injective by the inductive hypothesis, $z = x$.

It remains to argue that injectivity is maintained when we extend the domain of θ_i in (e) and (f). In other words, now suppose $\theta_i(z)[t] = \theta_i(x)[t]$, where at least one of x and z does not belong to $\text{dom } \theta_i[t-1]$ and thus was introduced in (e) or (f). Recall that in (a.1) we chose $d_{i,t}$ to be $2(t+1)!$ -independent over $\theta_i(c_{i,j})[t-1]$ and $\theta_i(c_{i,r})[t-1] + t! d_{i,r}$. In (e), where $i = t$, different choices of coefficients $n_j \leq t$ and torsion elements d in A_t will result in different linear combinations in A_t ; cf. Remark 9.3. In other words, our actions in (e) preserve injectivity because if they did not, then this would violate $2(t+1)!$ -independence in (a.1). For $i < t$, recall the instructions in (f). Equality of elements is decidable in A_i , and we adjoin new elements to B_i only if their respective images in A_i are not equal. In other words, (f) preserves injectivity by construction. \square

Claim 9.6. For every $x \in B_i$, $\lim_t \theta_i(x)[t]$ exists. (The requirement L_i is met.)

Proof. We use Claim 9.1 throughout. We suppress i . If $\theta(x)[t] \neq \theta(x)[t-1]$, then this must be because the action in (c) or (d) of stage t : $\theta(x)[t] = \sum_r \frac{t!n_r}{m} d_r + \theta(x)[t-1]$. This involves only finitely many d_j that correspond to c_j in $mx = \sum_j m_j c_j + d$, where d is torsion as seen in $B[t-1]$. The change occurs if, and only if, the $\theta[t-1]$ images of c_j which occur with $m_j \neq 0$ are discovered to be linearly dependent. In this case, we pick d_j and then set $\theta(c_j)[t] = \theta(c_j)[t-1] + t!d_j$ in (c). This implies that these d_j are linearly independent if, and only if, the respective elements $\theta(c_j)[t]$ are linearly independent. We choose the d_j to be $2(t+1)!$ -independent together with some of the c_j which stay $2(t+1)!$ -independent at stage t . We also choose these elements so that the respective elements in A_0 have the least possible indices; see (a.4). After several iterations of this process we will finally hit the truly independent elements in A_0 (thus, in A_i by Claim 9.1) at some late enough stage s_0 . It follows that $\lim_t \theta(x)[t]$ exists and is equal to $\theta(x)[s_0]$. \square

From now on, θ_i stands for $\lim_t \theta_i[t]$.

Claim 9.7. Let $a_{i,j} = \lim_t \theta_i(c_{i,j})[t]$, and let $D_i = \{a_{i,j} : j \in \mathbb{N}\}$. Then D_i is maximal linearly independent in A_i .

Proof. For each i and every stage t , $\{c_{i,j} : j \leq t\}$ are held linearly independent in $B_i[t]$, and $a_{i,j}[t] = \theta_i(c_{i,j})[t]$ are $2(t+1)!$ -independent in A_i . By the previous claim, each D_i is linearly independent. When correcting θ_i in (a), we choose $2(t+1)!$ -independent elements $d_{i,j}$ in A_i such that $d_{0,k+1}, \dots, d_{0,t} \in A_0$ have the smallest possible indices (lexicographically) in A_0 ; see (a.1). This means that, in particular, the $a_{i,j}$ which were truly independent remain independent when we redefine them in (c). (This follows from an elementary analysis of their respective linear spans; we leave the elementary details to the reader⁴.) The minimality of indices also implies that, in the limit, every element of A_0 will be in the linear span of D_0 . By Claim 9.1, each D_j is maximal linearly independent in A_j . \square

Claim 9.8. For each i , θ_i is an isomorphism of B_i onto A_i . (The requirement I_i is met.)

Proof. As before, we suppress i . For any pair of elements $x, y \in B$, $x \neq y$, there is a stage s_0 large enough such that $\lim_t \theta(x)[t] = \theta(x)[s_0]$ and $\lim_t \theta(y)[t] = \theta(y)[s_0]$. Since $\theta[s_0]$ is injective on its domain, we have that $\theta(x) \neq \theta(y)$. Since at every stage $\theta[t]$ is used to copy a finite part of A into B , it evidently respects the operations; cf. (d). If $x - y = z$ and this is preserved by $\theta[t]$ for every s , then it will also be preserved by $\theta[s_0]$ for s_0 so large that θ is stable on x, y, z . By Claim 9.7, $D_i = \{a_{i,j} : j \in \mathbb{N}\}$ is maximal linearly independent in A_i . In particular, every element of A_i lies in the linear span of D_i . Thus, in (e) and (f), we made sure that any element will eventually be put into the range of θ . It follows that θ is surjective. \square

Claim 9.9. For each i , $\{c_{i,j}\}$ is maximal linearly independent in B_i . (The requirement R_i is met.)

Proof. This follows from Claim 9.7 and Claim 9.8. \square

Recall that $\psi_i = \theta_{i-1}^{-1} \circ \phi_i \circ \theta_i$.

Claim 9.10. For every i , $\psi_i : B_i \rightarrow B_{i-1}$ is a surjective homomorphism uniformly computable in i . The strong index of its finite kernel can be computed uniformly in i . (The requirements P_i and K_i are met.)

⁴For example, suppose we have a_0, a_1, a_2 and we discover that $a_1 \in \text{Span}(a_0)$ but a_2 still looks independent over a_0 . Since the index of d_1 has to be the least possible, we can have that $d_1 \in \text{Span}(a_2)$, and indeed $d_1 = a_2$. But then we will choose d_2 to be outside of $\text{Span}\{a_0, d_1\}$. The general case of many a_i is done by a straightforward induction.

Proof. It follows from Claim 9.8 and surjectivity of ϕ_i that $\psi_i = \theta_{i-1}^{-1} \circ \phi_i \circ \theta_i$ is a surjective homeomorphism. Recall that we say that z is torsion as seen in $B_i[t]$ if it has order m and $x, 2x, \dots, (m-1)x$ lie in $B_i[t]$. Note that $\text{Ker } \phi_i$ consists of elements torsion within $B_i[i]$, because we copy the kernel of ϕ_i into B_i via $\theta_i[i]$ and identify it with the kernel of ψ_i . Furthermore, by induction, if z is torsion within $B_i[t]$, then we have that $\theta_i(z)[t] = \theta_i(z)$; this is because in (d) we have $\theta(z)[t] \neq \theta(z)[t-1]$ only if z has a non-trivial coefficient in its $c_{i,j}$ -expansion.

To see why ψ_i it is computable, use induction. Let s be the first stage at which $\hat{x} = \psi_i(x)[s] = \theta_{i-1}^{-1} \circ \phi_i \circ \theta_i(x)[s]$ is defined.

We prove by induction on $t \geq s$ that

$$\phi_i \theta_i(x)[t] = \theta_{i-1}(\hat{x})[t];$$

this property certainly holds for the stage at which B_i is first attended (see (e), (f)).

According to the instructions in (e) we have

$$\theta_i(x)[t] = \sum_j \frac{t!}{m} d_{i,j} + \theta_i(x)[t-1],$$

where the sum $\sum_j \frac{t!}{m} d_{i,j}$ can possibly be empty, i.e, there are no such $d_{i,j}$. Similarly for $i-1$ and \hat{x} :

$$\theta_{i-1}(\hat{x})[t] = \sum_j \frac{t!}{m} d_{i-1,j} + \theta_{i-1}(\hat{x})[t-1],$$

where $d_{i-1,j} = \phi_i(d_{i,j})$ according to (a.4). Recall that, by our assumption, $\phi_i \theta_i(x)[t-1] = \theta_{i-1}(\hat{x})[t-1]$.

Combine the above equations:

$$\begin{aligned} \theta_{i-1}(\hat{x})[t] &= \phi_i \left(\sum_j \frac{t!}{m} d_{i-1,j} \right) + \phi_i(\theta_i(x)[t-1]) = \\ &= \phi_i \left(\sum_j \frac{t!}{m} d_{i-1,j} + \theta_i(x)[t-1] \right) = \phi_i \theta_i(x)[t] \end{aligned}$$

to see that

$$\theta_{i-1}(\hat{x})[t] = \phi_i \theta_i(x)[t].$$

Take s_0 so large that $\theta_{i-1}(\hat{x})[s_0] = \theta_{i-1}(\hat{x}) = \phi_i \theta_i(x)[s_0] = \phi_i \theta_i(x)$. By Claim 9.5, we have $\theta_{i-1}^{-1} \phi_i \theta_i(x)[s_0] = \theta_{i-1}^{-1} \phi_i \theta_i(x) = \hat{x} = \psi_i(x)[t]$. In other words, once ψ_i is defined it never changes, even though θ_i and θ_{i-1} will perhaps change. \square

It is easy to see that the group G is isomorphic to the inverse limit of $(B_i, \psi_i)_{i \in \mathbb{N}}$. This completes the proof of Proposition 5.1.

REFERENCES

- [1] C. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.
- [2] N. Bazhenov, M. Harrison-Trainor, I. Kalimullin, Alexander G. Melnikov, and Keng Meng Ng. Automatic and polynomial-time algebraic structures. *J. Symb. Log.*, 84(4):1630–1669, 2019.
- [3] V. Brattka, P. Hertling, and K. Weihrauch. A tutorial on computable analysis. In *New computational paradigms*, pages 425–491. Springer, New York, 2008.
- [4] I. Castellano and G. Corob Cook. Finiteness properties of totally disconnected locally compact groups. *J. Algebra*, 543:54–97, 2020.
- [5] Y. Cornulier. On the Chabauty space of locally compact abelian groups. *Algebr. Geom. Topol.*, 11(4):2007–2035, 2011.
- [6] Y. Cornulier. Commability and focal locally compact groups. *Indiana Univ. Math. J.*, 64(1):115–150, 2015.
- [7] V. Dobritsa. Some constructivizations of abelian groups. *Sibirsk. Mat. Zh.*, 24(2):18–25, 1983.
- [8] R. Downey and A. Montalbán. The isomorphism problem for torsion-free abelian groups is analytic complete. *J. Algebra*, 320(6):2291–2300, 2008.

- [9] R. Downey, A. Melnikov, and K.M. Ng. Enumerating abelian p -groups. *Journal of Algebra*, 560:745–790, 2020.
- [10] R. Downey and A. Melnikov. Computable completely decomposable groups. *Trans. Amer. Math. Soc.*, 366(8):4243–4266, 2014.
- [11] R. Downey and A. Melnikov. Computable analysis and classification problems. In Marcella Anselmo, Gianluca Della Vedova, Florin Manea, and Arno Pauly, editors, *Beyond the Horizon of Computability - 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 - July 3, 2020, Proceedings*, volume 12098 of *Lecture Notes in Computer Science*, pages 100–111. Springer, 2020.
- [12] A. Nies (editor). Logic Blog 2016. Available at <http://arxiv.org/abs/1703.01573>, 2016.
- [13] S. Eilenberg and N. Steenrod. *Foundations of algebraic topology*. Princeton University Press, Princeton, New Jersey, 1952.
- [14] Y. Ershov and S. Goncharov. *Constructive models*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 2000.
- [15] Y. Ershov. Problems of solubility and constructive models [in russian]. Nauka, Moscow (1980).
- [16] E. Fokina, J. Knight, A. Melnikov, S. Quinn, and C. Safranski. Classes of Ulm type and coding rank-homogeneous trees in other structures. *J. Symbolic Logic*, 76(3):846–869, 2011.
- [17] A. Fröhlich and J. Shepherdson. Effective procedures in field theory. *Philos. Trans. Roy. Soc. London. Ser. A.*, 248:407–432, 1956.
- [18] L. Fuchs. *Infinite abelian groups. Vol. I*. Pure and Applied Mathematics, Vol. 36. Academic Press, New York, 1970.
- [19] S. Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.
- [20] H. Glöckner and C. Raja. Expansive automorphisms of totally disconnected, locally compact groups. *J. Group Theory*, 20(3):589–619, 2017.
- [21] S. Goncharov and J. Knight. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.
- [22] S. Goncharov, S. Lempp, and R. Solomon. The computable dimension of ordered abelian groups. *Adv. Math.*, 175(1):102–143, 2003.
- [23] M. Harrison-Trainor, A. Melnikov, and A. Montalbán. Independence in computable algebra. *J. Algebra*, 443:441–468, 2015.
- [24] M. Harrison-Trainor, A. Melnikov, and K.M. Ng. Computability of Polish spaces up to homeomorphism. *The Journal of Symbolic Logic*, pages 1–25, 2020.
- [25] G. Hjorth. The isomorphism relation on countable torsion free abelian groups. *Fund. Math.*, 175(3):241–257, 2002.
- [26] K. Hofmann and S. Morris. *The Structure of Compact Groups*. De Gruyter Studies in Mathematics, 25, Berlin, New York, 1998.
- [27] K. Hofmann and P. Mostert. *Cohomology theories for compact Abelian groups*. Springer-Verlag, Berlin, 1973.
- [28] K. Hofmann and G. Willis. Continuity characterizing totally disconnected locally compact groups. *J. Lie Theory*, 25(1):1–7, 2015.
- [29] M. Hoyrup, T. Kihara, and V. Selivanov. Degree spectra of homeomorphism types of Polish spaces. *Preprint*.
- [30] E. Van Kampen. Locally compact abelian groups. *Proceedings of the National Academy of Sciences of the United States of America*, 20(7):434–436, 1934.
- [31] N. Khisamiev. Hierarchies of torsion-free abelian groups. *Algebra i Logika*, 25(2):205–226, 244, 1986.
- [32] N. Khisamiev. Constructive abelian groups. In *Handbook of recursive mathematics, Vol. 2*, volume 139 of *Stud. Logic Found. Math.*, pages 1177–1231. North-Holland, Amsterdam, 1998.
- [33] P. La Roche. Effective Galois theory. *J. Symbolic Logic*, 46(2):385–392, 1981.
- [34] A. Mal'cev. Constructive algebras. I. *Uspehi Mat. Nauk*, 16(3 (99)):3–60, 1961.
- [35] A. Mal'cev. On recursive Abelian groups. *Dokl. Akad. Nauk SSSR*, 146:1009–1012, 1962.
- [36] A. Melnikov and A. Nies. Computably totally disconnected locally compact groups. Available at arxiv.org/pdf/2204.09878.pdf, 2022.
- [37] A. Melnikov. Computable topological groups and Pontryagin duality. *Trans. Amer. Math. Soc.*, 370(12):8709–8737, 2018.
- [38] A. Melnikov and A. Montalbán. Computable Polish group actions. *J. Symb. Log.*, 83(2):443–460, 2018.
- [39] A. Melnikov. Computable abelian groups. *The Bulletin of Symbolic Logic*, 20(3):315–356, 2014.
- [40] S. Morris. *Pontryagin duality and the structure of locally compact abelian groups*. Cambridge University Press, Cambridge-New York-Melbourne, 1977. London Mathematical Society Lecture Note Series, No. 29.
- [41] J. Munkres. *Elements of algebraic topology*. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.
- [42] A. Nurtazin. *Computable classes and algebraic criteria of autostability*. Summary of Scientific Schools, Math. Inst. SB USSRAS, Novosibirsk, 1974.
- [43] G. Paolini, S. Shelah. Torsion-Free Abelian Groups are Borel Complete. *Submitted*.

- [44] V. Platonov. Locally projectively nilpotent subgroups and nilelements in topological groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 30:1257–1274, 1966.
- [45] L. Pontryagin. The theory of topological commutative groups. *Ann. of Math. (2)*, 35(2):361–388, 1934.
- [46] L. Pontryagin. *Topological groups*. Translated from the second Russian edition by Arlen Brown. Gordon and Breach Science Publishers, Inc., New York-London-Paris, 1966.
- [47] M. Pour-El and J. Richards. *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989.
- [48] M. Rabin. Computable algebra, general theory and theory of computable fields. *Trans. Amer. Math. Soc.*, 95:341–360, 1960.
- [49] F. Richman. A constructive modification of Vietoris homology. *Fundamenta Mathematicae*, 91:231–240, 1976.
- [50] G. Sacks. *Higher recursion theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.
- [51] V. Selivanov. On degree spectra of topological spaces. *Lobachevskii Journal of Mathematics*, 41:252–259, 2020.
- [52] R. Smith. Effective aspects of profinite groups. *J. Symbolic Logic*, 46(4):851–863, 1981.
- [53] E. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.
- [54] A. Turing. On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.
- [55] A. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. A Correction. *Proceedings of the London Mathematical Society*, 43:544–546, 1937.
- [56] K. Weihrauch. *Computable analysis*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2000. An introduction.
- [57] P. Wesolek. Elementary totally disconnected locally compact groups. *Proc. Lond. Math. Soc. (3)*, 110(6):1387–1434, 2015.
- [58] G. Willis. The scale and tidy subgroups for endomorphisms of totally disconnected locally compact groups. *Math. Ann.*, 361(1-2):403–442, 2015.

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