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THE CARDINALITY OF KISELMAN'S SEMIGROUPS GROWS DOUBLE-EXPONENTIALLY

ALESSANDRO D'ANDREA AND SALVATORE STELLA

ABSTRACT. Let K_n denote Kiselman's semigroups as in [4]. We show that the sequence $2^{-n/2} \cdot \log |K_n|$ admits finite limits as n grows to infinity both on odd and even values.

1. INTRODUCTION

Kiselman's semigroup K_n is a generalization [4] of a monoid introduced by Kiselman [3] in a convexity theory setting. It is related to 0-Hecke algebras [5], arises as a quotient of the Richardson-Springer monoid [6], describes the evolution properties of certain graph dynamical systems [2], and exhibits interesting combinatorics.

In [4], Kudryavtseva and Mazorchuk show that K_n for $n \in \mathbb{N}$, is always finite by providing the explicit, quite broad, upper bound

$$|K_n| \leq 1 + n^{L(n)}$$

for their cardinality, where

$$L(n) = \begin{cases} 2^{k+1} - 2, & \text{if } n = 2k \\ 3 \cdot 2^k - 2, & \text{if } n = 2k + 1. \end{cases}$$

However, data from [7] suggest that the growth of $|K_n|$ is sharply double exponential in n .

In this short note, we would like to provide easy double exponential upper and lower bounds for $|K_n|$ and show that the sequence $2^{-n/2} \log |K_n|$ has a finite limit when restricted to either even or odd values of n . Throughout the paper, \log denotes the base 2 logarithm.

2. CANONICAL WORDS

Henceforth A will be a totally ordered set. The Kiselman semigroup $K(A)$ is defined by the monoid presentation

$$(1) \quad K(A) = \langle a \in A \mid a^2 = a, aba = bab = ab, \text{ for all } a < b \rangle.$$

We will be concerned with the case where A is finite. Then $K(A)$ depends, up to isomorphism, only on the cardinality of A . When $A = \{1, \dots, n\}$ with the standard total order, then $K(A)$ is usually denoted K_n ; in this case, generators of $K(A)$ are denoted by a_i to ease readability.

Let $W(A)$ be the monoid of finite words on the alphabet A . Then for every $x \in K(A)$ there exists a unique $[1, 4]$ shortest element (i.e., reduced expression) in $\pi^{-1}(x)$, where

$$\pi : W(A) \rightarrow K(A)$$

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is the canonical projection. Such reduced expressions admit a combinatorial description: $w \in W(A)$ is the shortest element in some $\pi^{-1}(x)$ if and only if whenever aua is a subword of w , where $u \in W(A)$ and $a \in A$, then u contains both some letter $b > a$ and some letter $c < a$; elements of $W(A)$ satisfying the above conditions are called *canonical words* and elements in $K(A)$ are thus in bijection with canonical words in $W(A)$.

Notice that, due to the presentation (1), the set of letters showing up in each word lying in $\pi^{-1}(x)$ only depends on $x \in K(A)$. Also denote by $less(a)$ and $more(a)$ the number of elements in A that are less or more than a . The following easy facts are proved in [4].

Proposition 2.1. *Let $w \in W(A)$ be the unique reduced expression of an element $x \in K(A)$. Then:*

- *each letter $a \in A$ occurs in w at most $2^{less(a)}$ times;*
- *each letter $a \in A$ occurs in w at most $2^{more(a)}$ times;*
- *the length of w is less or equal than $L(|A|)$.*

3. A BOUND FROM BELOW

Providing a double exponential lower bound for the cardinality $|K_n|$ of Kiselman's semi-groups is easy.

Proposition 3.1. *One has $|K_{n+2}| \geq 2|K_n|^2$ for every $n \in \mathbb{N}$. In particular,*

$$(2) \quad 2^{-(n+2)/2} \log(2|K_{n+2}|) \geq 2^{-n/2} \log(2|K_n|),$$

for every $n \in \mathbb{N}$.

Proof. For every choice of canonical words $w, w' \in W(\{a_2, \dots, a_{n+1}\})$, words of the form $wa_1a_{n+2}w'$ and $wa_{n+2}a_1w'$ are still canonical. This shows that the number of canonical words in $n+2$ letters is at least twice the square of the number of canonical words in n letters, thus yielding $|K_{n+2}| \geq 2|K_n|^2$.

We may write this as $2|K_{n+2}| \geq (2|K_n|)^2$, or equivalently $\log(2|K_{n+2}|) \geq 2\log(2|K_n|)$, which immediately proves (2). \square

Corollary 3.2. *The cardinality of K_n grows at least double exponentially.*

Proof. As $|K_0| = 1, |K_1| = 2$, we immediately see that

$$|K_n| \geq \frac{1}{2} \cdot 2^{2^{\lceil n/2 \rceil}}.$$

\square

4. A BOUND FROM ABOVE

We will use the following well known estimates.

Lemma 4.1. *One has:*

- (1) $\binom{2N}{N} \leq \frac{2^{2N}}{\sqrt{\pi N}};$
- (2) $\binom{4N}{N} \binom{3N}{N} \leq \frac{2^{6N}}{\pi N \sqrt{2}};$
- (3) $\binom{3N}{N} \leq \left(\frac{27}{4}\right)^N,$

for every positive $N \in \mathbb{N}$.

Proof. Part (1) is standard, and a well known proof¹ follows from observing that

$$\binom{2N}{N} = \frac{2^{2N}}{\pi} \int_{-\pi/2}^{\pi/2} (\cos x)^{2N} dx \leq \frac{2^{2N}}{\pi} \int_{-\pi/2}^{\pi/2} e^{-Nx^2} dx \leq \frac{2^{2N}}{\pi} \int_{-\infty}^{+\infty} e^{-Nx^2} dx = \frac{2^{2N}}{\sqrt{\pi N}}.$$

Then claim (2) is a consequence of part (1) as in

$$\binom{4N}{N} \binom{3N}{N} = \binom{4N}{2N} \binom{2N}{N} \leq \frac{2^{4N}}{\sqrt{2\pi N}} \frac{2^{2N}}{\sqrt{\pi N}} = \frac{2^{6N}}{\pi N \sqrt{2}},$$

whereas (3) is immediately proved by induction. \square

Lemma 4.2. *Let $n = 2k$ be an even number. The number of words in $W(\{a_1, \dots, a_n\})$ of length $L(n)$ in which each a_i , $1 \leq i \leq n$, occurs $\min(2^{i-1}, 2^{n-i})$ times is bounded from above by*

$$\frac{2^{6 \cdot (2^k - 1)}}{(\pi \sqrt{2})^k 2^{k(k-1)/2}}.$$

Proof. We need to prove the above upper bound for the multinomial coefficient

$$(3) \quad \binom{L(n)}{2^{k-1}, 2^{k-1}, 2^{k-2}, 2^{k-2}, \dots, 2, 2, 1, 1} = \prod_{h=1}^k \left(\binom{2 \cdot 2^h - 2}{2^{h-1}} \cdot \binom{3 \cdot 2^{h-1} - 2}{2^{h-1}} \right).$$

Now, notice that

$$\binom{2 \cdot 2^h - 2}{2^{h-1}} \binom{3 \cdot 2^{h-1} - 2}{2^{h-1}} \leq \binom{2 \cdot 2^h}{2^{h-1}} \binom{3 \cdot 2^{h-1}}{2^{h-1}} = \binom{4 \cdot 2^{h-1}}{2^{h-1}} \binom{3 \cdot 2^{h-1}}{2^{h-1}}.$$

We may then use Lemma 4.1(2) to obtain

$$\binom{4 \cdot 2^{h-1}}{2^{h-1}} \binom{3 \cdot 2^{h-1}}{2^{h-1}} \leq \frac{2^{3 \cdot 2^h}}{\pi 2^{h-1} \sqrt{2}}.$$

Substituting this into (3) yields the claim. \square

Proposition 4.3. *Let $n = 2k$ be an even number. Then $|K_n| \leq 2^{6 \cdot 2^k}$.*

Proof. The cardinality of K_n coincides with the number of canonical words on n letters. Thanks to Proposition 2.1, we know that each such canonical word arises, non uniquely, as a prefix of a word of length $L(n)$ in which a_i occurs $\min(2^{i-1}, 2^{n-i})$ times. We may thus bound $|K_n|$ with the number of all such prefixes, possibly with repetition.

As a word of length ℓ has $\ell + 1$ prefixes, when $n = 2k$ is even, we obtain

$$\begin{aligned} |K_n| &\leq (L(n) + 1) \cdot \binom{L(n)}{2^{k-1}, 2^{k-1}, 2^{k-2}, 2^{k-2}, \dots, 2, 2, 1, 1} \\ &\leq 2^{k+1} \cdot \frac{2^{6 \cdot 2^k - 6}}{(\pi \sqrt{2})^k 2^{k(k-1)/2}} = 2^{6 \cdot 2^k - (k^2 - 2k + 10)/2 - k \log \pi} \leq 2^{6 \cdot 2^k}. \end{aligned}$$

\square

Proposition 4.4. *Let $n = 2k + 1$ be an odd number. Then $|K_n| \leq 2^{c \cdot 2^{n/2}}$, where $c = \log(432)/\sqrt{2}$.*

¹Some elementary proofs for stronger, and asymptotically optimal, estimates for the middle binomial coefficients may be found in [8].

Proof. Starting with the same considerations as in the previous proof, one can use Lemma 4.2, Lemma 4.1(3), and the identity $L(2k+1) = L(2k) + 2^k = 3 \cdot 2^k - 2$ to compute

$$\begin{aligned}
|K_n| &\leq (L(n) + 1) \cdot \binom{L(n)}{2^k, 2^{k-1}, 2^{k-1}, 2^{k-2}, 2^{k-2}, \dots, 2, 2, 1, 1} \\
&= (L(n) + 1) \cdot \binom{3 \cdot 2^k - 2}{2^k} \cdot \binom{L(2k)}{2^{k-1}, 2^{k-1}, 2^{k-2}, 2^{k-2}, \dots, 2, 2, 1, 1} \\
&\leq 3 \cdot 2^k \cdot \binom{3 \cdot 2^k}{2^k} \cdot \frac{2^{6 \cdot (2^k - 1)}}{(\pi\sqrt{2})^k 2^{k(k-1)/2}} \\
&\leq 3 \cdot 2^k \cdot \left(\frac{27}{4}\right)^{2^k} \cdot \frac{2^{6 \cdot (2^k - 1)}}{(\pi\sqrt{2})^k 2^{k(k-1)/2}} \leq 2^{c \cdot 2^{n/2}},
\end{aligned}$$

where $c = (\log 432)/\sqrt{2} \sim 6.1906$. □

Theorem 4.5. *The following asymptotics hold:*

$$\log |K_{2k}| \sim c_{\text{even}} 2^k, \quad \log |K_{2k+1}| \sim c_{\text{odd}} 2^{(2k+1)/2},$$

where $c_{\text{even}} \leq 6$ and $c_{\text{odd}} \leq (\log_2 432)/\sqrt{2}$ are positive constants.

Proof. The sequence $2^{-k} \log(2|K_{2k}|)$ — respectively $2^{-(2k+1)/2} \log(2|K_{2k+1}|)$ — is increasing due to Proposition 3.1, so it converges to a finite limit as soon as it is bounded. However, the asymptotically equivalent sequence $2^{-k} \log |K_{2k}|$ (resp. $2^{-(2k+1)/2} \log |K_{2k+1}|$) is bounded by Proposition 4.3 (resp. Proposition 4.4), and the claim then follows easily. □

We expect, but cannot prove, that $c_{\text{even}} = c_{\text{odd}}$, i.e., that $\log |K_n| \sim c 2^{n/2}$ for some positive $c \in \mathbb{R}$. Notice that the embeddings $K_{2k-1} \subset K_{2k} \subset K_{2k+1}$ imply that

$$\frac{\log |K_{2k-1}|}{\log |K_{2k}|} \leq 1 \leq \frac{\log |K_{2k+1}|}{\log |K_{2k}|}$$

and in the limit

$$c_{\text{odd}}/\sqrt{2} \leq c_{\text{even}} \leq c_{\text{odd}}\sqrt{2}.$$

5. STRUCTURE OF LONGEST WORDS IN K_{2k+1}

In this section, we prove a technical statement showing that the lower bound in Corollary 3.2 is established, in the odd $n = 2k + 1$ case, by the precise number of longest words in K_{2k+1} . First of all, we recall that Proposition 2.1 is easily proved by induction from the following observation.

Lemma 5.1. *Let $w \in W(A)$ be a canonical word. Then the number of occurrences of $a \in A$ in w is less or equal to the total number of occurrences of letters less than a in w , increased by 1. A similar bound holds for letters more than a in w .*

Proof. As w is canonical, between any two occurrences of a in w there must lie both a lower and a higher letter. □

Proposition 5.2. *Let $n = 2k + 1 > 1$ be an odd number. Then all canonical words of length $L(n)$ in $W(\{a_1, \dots, a_n\})$ are of the form*

$$wa_1a_nw' \quad \text{or} \quad wa_na_1w',$$

where w, w' are canonical words of length $L(n-2)$ in $W(\{a_2, \dots, a_{n-1}\})$.

Proof. First of all, if w, w' are canonical words of length $L(n-2)$ in $W(\{a_2, \dots, a_{n-1}\})$ then wa_1a_nw' is still canonical, of length $2L(n-2) + 2 = L(n)$. As $L(1) = 1$, and since words of length 1 are trivially canonical, an easy induction shows that $W(\{a_1, \dots, a_n\})$ indeed contains words of length $L(n)$. Moreover, in any such word, each letter a_i occurs precisely $2^{\min(i-1, n-i)}$ times, as these numbers add up to $L(n)$.

Let now u be a canonical word of length $L(n)$ in $W(\{a_1, \dots, a_n\})$. Both a_1 and a_n occur exactly once, and we may assume without loss of generality that a_1 occurs on the left of a_n , so that $u = wa_1u'$, where a_n occurs in the subword u' . Notice that both w and u' need to be canonical words.

The number of occurrences of a_i for $2 \leq i \leq k+1$ in u is exactly 2^{i-1} , whereas it occurs in $w \in W(\{a_2, \dots, a_{n-1}\})$ and in $u' \in W(\{a_2, \dots, a_n\})$ at most 2^{i-2} times. This shows that, for $2 \leq i \leq k+1$, a_i occurs in both w and u' precisely 2^{i-2} times. By Lemma 5.1, a_{k+1} occurs in $w \in W(\{a_2, \dots, a_{n-1}\})$ at most the number of total occurrences in w of higher letters, increased by 1. As a consequence, each a_{n-1-i} , $0 \leq i \leq k-2$, occurs in w precisely 2^i times. We conclude that w has length $L(n-2)$.

We may repeat this argument for a_n in order to show that the subword on the right of a_n has length $L(n-2)$. As $L(n) = 2L(n-2) + 2$, this shows that a_1 and a_n sit next to each other, so that u is of the form wa_1a_nw' for some choice of canonical words w, w' of length $L(n-2)$. \square

Notice that this statement is false in the even n case, as the canonical word $a_2a_3a_1a_2a_4a_3$ of length $L(4) = 6$ provides an immediate counterexample.

Corollary 5.3. *Let $n = 2k + 1$ be an odd number. Then the number of canonical words of length $L(n)$ in $W(\{a_1, \dots, a_n\})$ equals $\frac{1}{2} \cdot 2^{2^{\lceil n/2 \rceil}}$.*

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