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A LIE CONFORMAL SUPERALGEBRA AND DUALITY OF REPRESENTATIONS FOR $E(4, 4)$

NICOLETTA CANTARINI, FABRIZIO CASELLI, AND VICTOR KAC

ABSTRACT. We construct a duality functor in the category of continuous representations of the Lie superalgebra $E(4, 4)$, the only exceptional simple linearly compact Lie superalgebra, for which it wasn't known. This is achieved by constructing a Lie conformal superalgebra of type $(4, 4)$, for which $E(4, 4)$ is the annihilation algebra. Along the way we obtain an explicit realization of $E(4, 4)$ by vector fields on a $(4|4)$ -dimensional supermanifold.

1. INTRODUCTION

In our paper [5] we constructed a duality functor on the category of continuous modules over some linearly compact Lie superalgebras L . The main assumption on L for this construction is the existence of a Lie conformal superalgebra R of type (r, s) , whose annihilation algebra is L .

The notion of a Lie conformal superalgebra, as treated in [12], corresponds to our notion of a Lie conformal superalgebra of type $(1, 0)$. Recall that the latter is an $\mathbb{F}[\partial]$ -module R with a λ -bracket $R \otimes R \rightarrow \mathbb{F}[\lambda] \otimes R$,

$$[a_\lambda b] = \sum_{j \geq 0} (a_{(j)} b) \lambda^j / j!,$$

satisfying axioms, similar to the Lie superalgebra axioms. Here λ and ∂ are even indeterminates. For the more general Lie conformal superalgebras R of type (r, s) one takes, instead, r even and s odd indeterminates (see Section 2 for the precise definition of R and a conformal R -module).

Most of the work on representation theory of a Lie conformal superalgebra R of type $(1, 0)$ was based on the simple observation that conformal R -modules are closely related to continuous modules over the associated to R annihilation algebra $\mathcal{A}(R)$. This name comes from the fact that $\mathcal{A}(R)$ consists of the operators that annihilate the vacuum vector of the universal enveloping vertex algebra of R [12].

Recall that, if R is a finitely generated $\mathbb{F}[\partial]$ -module, then $\mathcal{A}(R)$ is a linearly compact Lie superalgebra

$$(1) \quad \mathcal{A}(R) = R[[y]] / (\partial + \partial_y) R[[y]],$$

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where y is an even indeterminate, with the (well defined) continuous bracket

$$(2) \quad [ay^m, by^n] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)}b) y^{m+n-j}.$$

Since ∂ commutes with ∂_y , it defines a continuous derivation of $\mathcal{A}(R)$, and we may consider the so-called extended annihilation algebra $\mathcal{A}^e(R) = \mathbb{F}[\partial] \ltimes \mathcal{A}(R)$. It is straightforward to see that a conformal R -module M is the same as a continuous $\mathcal{A}^e(R)$ -module [8].

In order to go from conformal R -modules to continuous $\mathcal{A}(R)$ -modules, one needs to assume that ∂ is an inner derivation of $\mathcal{A}(R)$:

$$(3) \quad \partial = \text{ad } a \text{ for some } a \in \mathcal{A}(R).$$

A conformal R -module ($= \mathcal{A}^e(R)$ -module) M is called coherent if $(\partial - a)M = 0$. Thus a continuous module over the linearly compact Lie superalgebra $\mathcal{A}(R)$ is the same as a coherent conformal R -module.

A duality functor on the category of conformal R -modules for R of type $(1, 0)$ was constructed in [3]. Therefore, under assumption (3), this functor can be transferred to the category of $\mathcal{A}(R)$ -modules.

In [5] we extended this construction of a duality functor to the category \mathcal{P} of continuous modules with discrete topology over a linearly compact Lie superalgebra L under the assumptions that there exists a Lie conformal superalgebra RL of type (r, s) , for which

$$(4) \quad \mathcal{A}(RL) \cong L,$$

satisfying an assumption analogous to (3).

The category \mathcal{P} of L -modules is similar to the BGG category \mathcal{O} , and, as in category \mathcal{O} , the most important objects in \mathcal{P} are parabolic Verma modules $M(F)$. Recall that, given an open subalgebra $L_0 \subset L$, and a finite-dimensional L_0 -module F , one defines

$$M(F) = \text{Ind}_{L_0}^L F.$$

(In [5] these modules are called generalized Verma modules.) The main result of our paper [5] is the computation of the dual to $M(F)$ L -module $M(F)^\vee$, if F is finite-dimensional and L has a D -conformal structure (see Definition 2.17), and in particular it satisfies (3) and (4). It turned out that, if L has a D -conformal structure, $M(F)^\vee$ is not $M(F^*)$, but $M(F^\vee)$, where F^\vee is the dual L_0 -module F^* shifted by the following character χ of L_0 :

$$(5) \quad \chi(a) = \text{str}(\text{ad } a|_{L/L_0}), \quad a \in L_0.$$

We proved in [5] that many simple linearly compact Lie superalgebras L (classified in [11]) with L_0 the open subalgebra of minimal codimension, have a D -conformal structure, including four (out of five) exceptional L . However, for the remaining exceptional simple linearly compact Lie superalgebra $L = E(4, 4)$, one could construct in a rather natural way a Lie conformal superalgebra of type $(4, 0)$ satisfying (3) and (4), although it does not provide a D -conformal structure for L .

The main result of the present paper is a construction of a Lie conformal superalgebra $RE(4, 4)$ of type $(4, 4)$, whose annihilation algebra is isomorphic to $E(4, 4)$, and which

provides a D -conformal structure for $E(4, 4)$ (Corollary 3.7 and Corollary 4.7). As a result, we obtain a duality functor on the category \mathcal{P} for $L = E(4, 4)$ and L_0 open subalgebra of minimal codimension, for which $M(F)^\vee$ is isomorphic to $M(F^*)$ since the shift χ defined in (5) is zero. Along the way we provide an explicit embedding of $E(4, 4)$ in the Lie superalgebra $W(4, 4)$ of all continuous derivations of the superalgebra of formal power series in four even and four odd indeterminates.

The construction of a duality functor is important for the classification of reducible (i.e., degenerate) parabolic Verma modules. In [4] and [6] we used this functor in the classification of degenerate parabolic Verma modules over $E(5, 10)$. We hope that the duality of modules over $E(4, 4)$, constructed in the present paper, will lead to the classification of such modules over $E(4, 4)$, the only exceptional linearly compact Lie superalgebra, for which this classification is not known.

The base field \mathbb{F} is assumed to be a field of characteristic 0, and tensor products are assumed to be over \mathbb{F} .

2. LIE CONFORMAL SUPERALGEBRAS OF TYPE (r, s) AND THE DUALITY FUNCTOR FOR THEIR MODULES

Here we recall some fundamental constructions of the theory of Lie conformal superalgebras of type (r, s) and their conformal modules.

Let r and s be two nonnegative integers. We will use several sets of $r + s$ variables such as $\lambda_1, \dots, \lambda_{r+s}, \partial_1, \dots, \partial_{r+s}, y_1, \dots, y_{r+s}$. We will always assume that variables with indices $1, \dots, r$ are even and variables with indices $r + 1, \dots, r + s$ are odd, and accordingly we let $p_i = \bar{0}$ if $i = 1, \dots, r$ and $p_i = \bar{1}$ if $i = r + 1, \dots, r + s$, so that $\lambda_i \lambda_j = (-1)^{p_i p_j} \lambda_j \lambda_i$ and similarly for the ∂_i and y_i . We will also use bold letters such as λ or ∂ or y to denote the set of corresponding variables. We let $\mathbb{F}[\lambda] = \mathbb{F}[\lambda_1, \dots, \lambda_r] \otimes \wedge(\lambda_{r+1}, \dots, \lambda_{r+s})$ and we similarly define $\mathbb{F}[\partial]$ or $\mathbb{F}[y]$. The completion $\mathbb{F}[[y]]$ of $\mathbb{F}[y]$ is the algebra of formal power series in y .

If R is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space we give to $\mathbb{F}[\lambda] \otimes R$ the structure of a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{F}[\lambda]$ -bimodule by letting $\lambda_i(P(\lambda) \otimes a) = \lambda_i P(\lambda) \otimes a$ and $(P(\lambda) \otimes a) \lambda_i = (-1)^{p_i p(a)} P(\lambda) \lambda_i \otimes a$, where $p(a) \in \mathbb{Z}/2\mathbb{Z}$ denotes the parity of a . We will usually drop the tensor product symbol and simply write $P(\lambda)a$ instead of $P(\lambda) \otimes a$.

Definition 2.1. A Lie conformal superalgebra of type (r, s) is a $\mathbb{Z}/2\mathbb{Z}$ graded $\mathbb{F}[\partial]$ -bimodule R such that $a\partial_i = (-1)^{p_i p(a)} \partial_i a$ for all $a \in R$ and $i \in \{1, \dots, r + s\}$, endowed with a λ -bracket, i.e. a $\mathbb{Z}/2\mathbb{Z}$ -graded linear map $R \otimes R \rightarrow \mathbb{F}[\lambda] \otimes R$, denoted by $a \otimes b \mapsto [a_\lambda b]$, that satisfies the following properties:

- (6) $[(\partial_i a)_\lambda b] = -\lambda_i [a_\lambda b], [a_\lambda (b \partial_i)] = [a_\lambda b](\partial_i + \lambda_i);$
- (7) $[b_\lambda a] = -(-1)^{p(a)p(b)} [a_{-\lambda-\partial} b];$
- (8) $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)} [b_\mu [a_\lambda c]].$

We refer to Property (6) as the conformal sesquilinearity, to Property (7) as the conformal skew-symmetry and to Property (8) as the conformal Jacobi identity.

We introduce the following notation. If $K = (k_1, \dots, k_t)$ is any sequence with entries in $\{1, \dots, r + s\}$ we let

$$\lambda_K = \lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_t},$$

and we similarly define \mathbf{y}_K , \mathbf{x}_K and so on. If $K = \emptyset$, we let $\lambda_K = 1$. We also let $p_K = p_{k_1} + \dots + p_{k_t}$ and so $p(\lambda_K) = p_K$.

Starting from a Lie conformal superalgebra R of type (r, s) one can construct a new Lie conformal superalgebra \tilde{R} of the same type, called the *affinization* of R and defined as follows. Let $\tilde{R} = R \otimes \mathbb{F}[[\mathbf{y}]]$. We consider \tilde{R} as a $\mathbb{F}[[\mathbf{y}]]$ -bimodule and also as a $\mathbb{F}[\partial_{\mathbf{y}}] = \mathbb{F}[\partial_{y_1}, \dots, \partial_{y_{r+s}}]$ -bimodule letting

$$\partial_{y_i}(a\mathbf{y}_K) = (-1)^{p_i p(a)} a(\partial_{y_i} \mathbf{y}_K) = (-1)^{p_i(p(a)+p_K)} a\mathbf{y}_K \partial_{y_i}$$

with λ -bracket given by

$$(9) \quad [(\mathbf{y}_K a)_{\lambda}(b\mathbf{y}_N)] = (\mathbf{y}_K [a_{\lambda+\partial_{\mathbf{y}}} b])\mathbf{y}_N.$$

The following proposition holds.

Proposition 2.2. [5, Proposition 2.3] *The $\mathbb{F}[\tilde{\partial}]$ -module \tilde{R} with $\tilde{\partial} = \partial + \partial_{\mathbf{y}}$ and λ -bracket given by (9) is a Lie conformal superalgebra of the same type as R .*

Definition 2.3. Given a Lie conformal superalgebra R of type (r, s) , the *annihilation algebra* associated to R is the vector super space

$$\mathcal{A}(R) = \tilde{R}/\tilde{\partial}\tilde{R},$$

with bracket given by

$$[\mathbf{y}_K a, b\mathbf{y}_N] = [(\mathbf{y}_K a)_{\lambda}(b\mathbf{y}_N)]|_{\lambda=0}.$$

Proposition 2.4. [5, Proposition 2.5] *$\mathcal{A}(R)$ is a Lie superalgebra.*

Definition 2.5. A Lie conformal superalgebra R of type (r, s) is called \mathbb{Z} -graded if $\mathbb{F}[\lambda] \otimes R = \bigoplus_{d \in \mathbb{Z}} (\mathbb{F}[\lambda] \otimes R)_d$, where $(\mathbb{F}[\lambda] \otimes R)_d$ denotes the homogeneous component of degree d , and for every homogeneous elements $a, b \in \mathbb{F}[\lambda] \otimes R$ one has:

- i) $\deg(\lambda_i a) = \deg(a) - 2$;
- ii) $\deg(\partial_i a) = \deg(a) - 2$;
- iii) $\deg[a_{\lambda} b] = \deg(a) + \deg(b)$.

Notice that if R is a \mathbb{Z} -graded Lie conformal superalgebra of type (r, s) then its annihilation algebra $\mathcal{A}(R)$ inherits a \mathbb{Z} -gradation by setting

$$(10) \quad \deg(a\mathbf{y}_K) = \deg(a) + 2\ell(K),$$

where $\ell(K)$ is the number of entries in K .

Definition 2.6. A *conformal module* M over a Lie conformal superalgebra R of type (r, s) is a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{F}[\partial]$ -module with a $\mathbb{Z}/2\mathbb{Z}$ -graded linear map

$$R \otimes M \rightarrow \mathbb{F}[\lambda] \otimes M, \quad a \otimes v \mapsto a_{\lambda} v$$

such that

- (M1) $(\partial_i a)_{\lambda} v = [\partial_i, a_{\lambda}] v = -\lambda_i a_{\lambda} v$;
- (M2) $[a_{\lambda}, b_{\mu}] v = a_{\lambda}(b_{\mu} v) - (-1)^{p(a)p(b)} b_{\mu}(a_{\lambda} v) = [a_{\lambda} b]_{\lambda+\mu} v$.

Definition 2.7. The conformal dual M^\vee of a conformal R -module M is defined as $M^\vee = \{f_\lambda : M \rightarrow \mathbb{F}[\lambda] \mid f_\lambda(\partial_i m) = (-1)^{p_i p(f)} \lambda_i f_\lambda(m), \text{ for all } m \in M \text{ and } i = 1, \dots, r+s\}$, with the structure of $\mathbb{F}[\partial]$ -module given by $(\partial_i f)_\lambda(m) = -\lambda_i f_\lambda(m)$, and with the following λ -action of R :

$$(a_\lambda f)_\mu m = -(-1)^{p(a)p(f)} f_{\mu-\lambda}(a_\lambda m), \quad a \in R, \quad m \in M.$$

Here by $p(f)$ we denote the parity of the map f_λ .

Proposition 2.8. [5, Proposition 3.7] *If M is a conformal R -module, then M^\vee is a conformal R -module.*

Proposition 2.9. [5, Proposition 3.8] *Let $T : M \rightarrow N$ be a morphism of conformal R -modules i.e. a linear map such that:*

- (1) $T(\partial_i m) = (-1)^{p_i p(T)} \partial_i T(m)$,
- (2) $T(a_\lambda m) = (-1)^{p(a)p(T)} a_\lambda T(m)$,

then the map $T^\vee : N^\vee \rightarrow M^\vee$ given by: $(T^\vee(f))_\lambda m = (-1)^{p(T)p(f)} f_\lambda T(m)$ is a morphism of conformal R -modules.

Theorem 2.10. *The assignment $M \mapsto M^\vee$, $T \mapsto T^\vee$ for every conformal R -module M and every morphism $T : M \rightarrow N$ of conformal R -modules provides a contravariant functor of the category of conformal R -modules.*

Proof. Due to Propositions 2.8 and 2.9, one only needs to verify that

- i) $(id_M)^\vee = id_{M^\vee}$;
- ii) given two morphisms $T : M \rightarrow N$ and $S : N \rightarrow P$ of conformal R -modules, one has: $(ST)^\vee = (-1)^{p(S)p(T)} T^\vee S^\vee$.

This easy check is left to the reader. □

As explained in the Introduction, conformal R -modules are used to study representations of the linearly compact Lie superalgebra $\mathcal{A}(R)$. Let $D = \langle \partial_{y_1}, \dots, \partial_{y_{r+s}} \rangle$, and consider the semi-direct sum of Lie superalgebras $\mathcal{A}^e(R) = D \ltimes \mathcal{A}(R)$. This is a natural generalization of the so-called extended annihilation algebra introduced in [8]. A key observation made in [8] is that conformal R -modules are exactly the same as continuous (called conformal in [3]) modules over the extended annihilation algebra. The following proposition extends this observation to our context.

Proposition 2.11. [5, Proposition 3.4] *A conformal R -module is precisely a continuous module over the Lie superalgebra $\mathcal{A}^e(R)$, i.e. a module M such that for every $v \in M$ and every $a \in R$, $(\mathbf{y}_K a).v \neq 0$ only for a finite number of K . The equivalence between the two structures is provided by the following relations:*

- $a_\lambda v = \sum_K (-1)^{p_K} \frac{\lambda_{\bar{K}}}{f(\bar{K})} (\mathbf{y}_K a).v$;
- $\partial_i v = -\partial_{y_i}.v$.

Here the summation is taken over K , viewed up to permutation of its entries, \bar{K} is obtained from K by reversing the order of entries, and $f(K) = \prod_i m_i(K)!$, where $m_i(K)$ is the multiplicity of i in K .

Definition 2.12. A Lie conformal superalgebra R of type (r, s) is called *regular* if the subspace D of $\mathcal{A}^e(R)$ consists of inner derivations of $\mathcal{A}(R)$. In this case for each $x \in D$ there exists $a_x \in \mathcal{A}(R)$ such that $x - a_x$ is a central element of $\mathcal{A}^e(R)$. A conformal R -module M is called *coherent* if all elements $x - a_x$ act trivially on M .

It follows from Proposition 2.11 that a coherent R -module is precisely a continuous module over $\mathcal{A}(R)$ (see also [5, §3]).

Proposition 2.13. [5, Proposition 3.7] *Let R be regular and let M be a coherent conformal R -module. Then M^\vee is also coherent.*

Theorem 2.14. *Let L be a linearly compact Lie superalgebra and let R be a regular Lie conformal superalgebra of type (r, s) , such that L is isomorphic to $\mathcal{A}(R)$. Then the duality functor exists on the category of continuous L -modules.*

Proof. Note that $\mathcal{A}(R)$ is isomorphic to the factor algebra of $\mathcal{A}^e(R)$ by the ideal generated by all elements $x - a_x$, $x \in D$. Hence the theorem follows from Propositions 2.11 and 2.13. \square

Remark 2.15. Let L be a linearly compact Lie superalgebra, for which $\text{Der } L \subseteq \text{ad } L + \mathbb{F}E$ where E is a diagonalizable operator. Most of simple L , including the five exceptional ones and $W(r, s)$, have this property [7]. Let R be a Lie conformal superalgebra of type (r, s) such that $\mathcal{A}(R) \cong L$. Then R is regular.

Example 2.16. Let \mathfrak{g} be a Lie superalgebra and let $\text{Cur } \mathfrak{g} = \mathbb{F}[\partial] \otimes \mathfrak{g}$ be the current Lie conformal superalgebra of type $(1, 0)$, for which $[a_\lambda b] = [a, b]$ for $a, b \in 1 \otimes \mathfrak{g}$. Then

$$\mathcal{A}^e(\text{Cur } \mathfrak{g}) = \mathbb{F}\partial_y \ltimes (\mathfrak{g} \otimes \mathbb{F}[[y]]).$$

Hence ∂_y is not an inner derivation of $\mathcal{A}(\text{Cur } \mathfrak{g})$, and $\text{Cur } \mathfrak{g}$ is not regular.

In what follows we assume the following technical conditions on a Lie conformal superalgebra R of type (r, s) , which turn out to be satisfied in many interesting cases (see [5]).

Definition 2.17. Let R be a Lie conformal superalgebra. We say that a linearly compact Lie superalgebra L has a D -conformal structure over R if the following conditions are satisfied:

- (1) $L \cong \mathcal{A}(R)$;
- (2) R is \mathbb{Z} -graded;
- (3) the induced \mathbb{Z} -gradation on $\mathcal{A}(R)$ has depth at most 3;
- (4) the homogeneous components $\mathcal{A}(R)_{-1}$ and $\mathcal{A}(R)_{-3}$ are purely odd (in particular they can vanish);
- (5) the map $\text{ad} : \mathcal{A}(R)_{-2} \rightarrow \text{Der}(\mathcal{A}(R))$ is injective and its image is D .

Note that such R is regular. By condition (5) of Definition 2.17, if $x \in \mathcal{A}(R)_{-2}$ then $\text{ad}(x)$ can be identified with an element in D and hence in $\mathcal{A}^e(R)$.

Definition 2.18. If \mathfrak{g} is a Lie superalgebra, $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation of \mathfrak{g} and $x \mapsto \chi_x \in \mathbb{F}$ is a character of \mathfrak{g} , we let $\varphi^\chi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be given by

$$\varphi^\chi(x)(v) = \varphi(x)(v) + \chi_x v.$$

It is clear that φ^χ is still a representation and we call it the χ -shift of φ .

Definition 2.19. Let \mathfrak{g} be a \mathbb{Z} -graded Lie superalgebra of finite depth. For $x \in \mathfrak{g}_0$ let

$$\chi_x = \text{str}(\text{ad}(x)|_{\mathfrak{g}_{<0}}),$$

where str denotes supertrace.

If V is any \mathfrak{g} -module we call the χ -shifted dual of V the χ -shift of the contragredient representation V^* and we denote by V^\vee the χ -shifted dual of V , where χ is the character introduced in Definition 2.19.

Let $L = \prod_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a linearly compact Lie superalgebra with a D -conformal structure over R . Let F be a finite-dimensional \mathfrak{g}_0 -module which we extend to $L_{\geq 0} = \prod_{j \geq 0} \mathfrak{g}_j$ by letting \mathfrak{g}_j , $j > 0$, act trivially. We let

$$M(F) = \text{Ind}_{L_{\geq 0}}^L F$$

be the parabolic Verma module, attached to F . It is natural to wonder whether the dual of a parabolic Verma module is still a parabolic Verma module. In [5] we gave an answer to this question in the following theorem.

Theorem 2.20. *Let L be a Lie superalgebra with a D -conformal structure over R and let F be a finite-dimensional \mathfrak{g}_0 -module. Then $M(F)^\vee$ is isomorphic to $M(F^\vee)$ as an L -module.*

Theorem 2.20 leads to the following natural problem for a \mathbb{Z} -graded linearly compact Lie superalgebra L : does L have a D -conformal structure over a Lie conformal superalgebra R of type (r, s) ? In [5] we provided a positive answer to this question for the Lie superalgebras $W(r, s)$, $E(3, 6)$, $E(5, 10)$ and $E(3, 8)$ with the principal \mathbb{Z} -grading, while the case of $E(1, 6)$ was essentially established in [2]. The only missing exceptional Lie superalgebra was therefore $E(4, 4)$ and the main result of this paper is to provide a positive answer also in this case.

3. THE LIE CONFORMAL SUPERALGEBRAS $RW(4, 4)$ AND $RE(4, 4)$

The prototypical example of a \mathbb{Z} -graded Lie conformal superalgebra of type (r, s) is $RW(r, s)$ which is defined as follows.

Definition 3.1. We denote by $RW(r, s)$ the free $\mathbb{F}[\partial]$ -module with even generators a_1, \dots, a_r and odd generators a_{r+1}, \dots, a_{r+s} , all of degree -2 , and the λ -bracket given by

$$[a_i \lambda a_j] = (\partial_i + \lambda_i) a_j + a_i \lambda_j, \quad i, j = 1, \dots, r + s$$

and extended on the whole $RW(r, s)$ by the sesquilinearity (6) of Definition 2.1.

In [5] we proved the following results:

Proposition 3.2. [5, Propositions 4.2, 4.3] *The Lie superalgebra $W(r, s)$ consisting of all continuous derivations of $\mathbb{F}[[\mathbf{x}]]$ has a D -conformal structure over $RW(r, s)$. The map*

$$\varphi : W(r, s) \rightarrow \mathcal{A}(RW(r, s)),$$

given by $\mathbf{x}_K \partial_{x_i} \mapsto -\mathbf{y}_K a_i$ is a \mathbb{Z} -graded Lie superalgebra isomorphism, where the \mathbb{Z} -gradation on the Lie superalgebra $W(r, s)$ of all continuous derivations of $\mathbb{F}[[\mathbf{x}]]$ is given by $\deg x_i = -\deg \partial_{x_i} = 2$.

Now we introduce a notable Lie conformal subalgebra $RE(4, 4)$ of $RW(4, 4)$. To this aim we introduce the following notation: if $u = (u_1, \dots, u_8)$ and $v = (v_1, \dots, v_8)$ we set

$$(11) \quad B_{uv} = \sum_{i=1}^4 u_i v_{i+4}, \quad C_{uv} = \sum_{i=1}^4 (u_i v_{i+4} + u_{i+4} v_i).$$

For example:

$$B_{\partial\partial} = \sum_{i=1}^4 \partial_i \partial_{i+4} \in \mathbb{F}[\partial], \quad C_{\partial a} = \sum_{i=1}^4 (\partial_i a_{i+4} + \partial_{i+4} a_i) \in RW(4, 4).$$

This is the main definition:

Definition 3.3. For $i = 1, \dots, 8$ we let

$$(12) \quad \alpha_i = \partial_i B_{\partial\partial} C_{\partial a} - 2\nu_i C_{\partial a} + \partial_i B_{\nu a} \in RW(4, 4),$$

where

$$\nu_1 = \partial_6 \partial_7 \partial_8, \quad \nu_2 = -\partial_5 \partial_7 \partial_8, \quad \nu_3 = \partial_5 \partial_6 \partial_8, \quad \nu_4 = -\partial_5 \partial_6 \partial_7, \quad \nu_5 = \nu_6 = \nu_7 = \nu_8 = 0.$$

Denote by $RE(4, 4)$ the $\mathbb{F}[\partial]$ -submodule of $RW(4, 4)$ generated by the elements $\alpha_1, \dots, \alpha_8$.

Note that all elements $\alpha_1, \dots, \alpha_8$ have degree -10 , $\alpha_1, \dots, \alpha_4$ are even and $\alpha_5, \dots, \alpha_8$ are odd. As an example, for the reader's convenience, we explicitly write the elements α_1 and α_5 :

$$\begin{aligned} \alpha_1 = & 2\partial_5 \partial_6 \partial_7 \partial_8 a_1 - \partial_1 \partial_6 \partial_7 \partial_8 a_5 - \partial_1 \partial_5 \partial_7 \partial_8 a_6 + \partial_1 \partial_5 \partial_6 \partial_8 a_7 - \partial_1 \partial_5 \partial_6 \partial_7 a_8 - 2\partial_2 \partial_6 \partial_7 \partial_8 a_6 \\ & - 2\partial_3 \partial_6 \partial_7 \partial_8 a_7 - 2\partial_4 \partial_6 \partial_7 \partial_8 a_8 + \partial_1^2 \partial_5 \partial_8 a_4 + \partial_1^2 \partial_5 \partial_6 a_2 + \partial_1^2 \partial_5 \partial_7 a_3 + \partial_1 \partial_2 \partial_6 \partial_7 a_3 \\ & - \partial_1 \partial_2 \partial_5 \partial_6 a_1 + \partial_1 \partial_2 \partial_6 \partial_8 a_4 + \partial_1 \partial_3 \partial_7 \partial_8 a_4 - \partial_1 \partial_3 \partial_5 \partial_7 a_1 - \partial_1 \partial_3 \partial_6 \partial_7 a_2 - \partial_1 \partial_4 \partial_7 \partial_8 a_3 \\ & - \partial_1 \partial_4 \partial_6 \partial_8 a_2 - \partial_1 \partial_4 \partial_5 \partial_8 a_1 + \partial_1^3 \partial_5 a_5 + \partial_1^2 \partial_2 \partial_5 a_6 + \partial_1^2 \partial_3 \partial_5 a_7 + \partial_1^2 \partial_4 \partial_5 a_8 \\ & + \partial_1^2 \partial_2 \partial_6 a_5 + \partial_1 \partial_2^2 \partial_6 a_6 + \partial_1 \partial_2 \partial_3 \partial_6 a_7 + \partial_1 \partial_2 \partial_4 \partial_6 a_8 + \partial_1^2 \partial_3 \partial_7 a_5 + \partial_1 \partial_2 \partial_3 \partial_7 a_6 \\ & + \partial_1 \partial_3^2 \partial_7 a_7 + \partial_1 \partial_3 \partial_4 \partial_7 a_8 + \partial_1^2 \partial_4 \partial_8 a_5 + \partial_1 \partial_2 \partial_4 \partial_8 a_6 + \partial_1 \partial_3 \partial_4 \partial_8 a_7 + \partial_1 \partial_4^2 \partial_8 a_8; \\ \alpha_5 = & \partial_5 \partial_6 \partial_7 \partial_8 a_5 + \partial_2 \partial_5 \partial_6 \partial_7 a_3 + \partial_1 \partial_2 \partial_5 \partial_6 a_5 + \partial_2 \partial_5 \partial_6 \partial_8 a_4 + \partial_3 \partial_5 \partial_7 \partial_8 a_4 + \partial_1 \partial_3 \partial_5 \partial_7 a_5 \\ & - \partial_3 \partial_5 \partial_6 \partial_7 a_2 - \partial_4 \partial_5 \partial_7 \partial_8 a_3 - \partial_4 \partial_5 \partial_6 \partial_8 a_2 + \partial_1 \partial_4 \partial_5 \partial_8 a_5 + \partial_2^2 \partial_5 \partial_6 a_6 + \partial_2 \partial_3 \partial_5 \partial_6 a_7 \\ & + \partial_2 \partial_4 \partial_5 \partial_6 a_8 + \partial_2 \partial_3 \partial_5 \partial_7 a_6 + \partial_3^2 \partial_5 \partial_7 a_7 + \partial_3 \partial_4 \partial_5 \partial_7 a_8 + \partial_2 \partial_4 \partial_5 \partial_8 a_6 + \partial_3 \partial_4 \partial_5 \partial_8 a_7 \\ & + \partial_4^2 \partial_5 \partial_8 a_8. \end{aligned}$$

Proposition 3.4. For all $i, j = 5, 6, 7, 8$, with $i \neq j$, we have the following relations in $RE(4, 4)$:

- (1) $\partial_i \alpha_i = 0$;
- (2) $\partial_i \alpha_j = -\partial_j \alpha_i$;
- (3) $\partial_i \alpha_{j-4} = \partial_{j-4} \alpha_i$;

- (4) $\partial_i \alpha_{i-4} = \partial_{i-4} \alpha_i - 2 \sum_{k=1}^4 \partial_k \alpha_{k+4}$;
 (5) $2\partial_i \alpha_j = \partial_p \alpha_q - \partial_q \alpha_p$, where $p, q \in \{1, 2, 3, 4\}$ are such that $(i-4, j-4, p, q)$ is an even permutation of $(1, 2, 3, 4)$.

Proof. Recalling that for $i \geq 5$ we have $\alpha_i = \partial_i(B_{\partial\partial}C_{\partial a} + B_{\nu a})$, Equations (1) and (2) trivially hold since $\partial_i^2 = 0$ and $\partial_i \partial_j = -\partial_j \partial_i$. Observing that $\partial_i \nu_{j-4} = 0$, we obtain (3):

$$\partial_i \alpha_{j-4} = \partial_i \partial_{j-4}(B_{\partial\partial}C_{\partial a} + B_{\nu a}) - 2\partial_i \nu_{j-4}C_{\partial a} = \partial_{j-4} \alpha_i.$$

Furthermore, we observe that

$$\partial_5 \nu_1 = \partial_6 \nu_2 = \partial_7 \nu_3 = \partial_8 \nu_4 = \partial_5 \partial_6 \partial_7 \partial_8$$

and we compute

$$\partial_i \alpha_{i-4} = \partial_i \partial_{i-4}(B_{\partial\partial}C_{\partial a} + B_{\nu a}) - 2\partial_i \nu_{i-4}C_{\partial a} = \partial_{i-4} \alpha_i - 2\partial_5 \partial_6 \partial_7 \partial_8 C_{\partial a}.$$

On the other hand

$$\sum_{k=1}^4 \partial_k \alpha_{k+4} = \sum_k \partial_k \partial_{k+4}(B_{\partial\partial}C_{\partial a} + B_{\nu a}) = B_{\partial\partial}(B_{\partial\partial}C_{\partial a} + B_{\nu a}) = B_{\partial\partial}B_{\nu a} = \partial_5 \partial_6 \partial_7 \partial_8 C_{\partial a},$$

hence Equation (4) follows. Finally we compute

$$\partial_i \alpha_j = \partial_i \partial_j(B_{\partial\partial}C_{\partial a} + B_{\nu a}) = \partial_i \partial_j B_{\partial\partial}C_{\partial a} = \partial_i \partial_j (\partial_p \partial_{p+4} + \partial_q \partial_{q+4})C_{\partial a}$$

and, observing that $\nu_q = -\partial_i \partial_j \partial_{p+4}$ and $\nu_p = \partial_i \partial_j \partial_{q+4}$,

$$\begin{aligned} \partial_p \alpha_q - \partial_q \alpha_p &= \partial_p \partial_q B_{\partial\partial}C_{\partial a} - 2\partial_p \nu_q C_{\partial a} + \partial_p \partial_q B_{\nu a} \\ &\quad - \partial_q \partial_p B_{\partial\partial}C_{\partial a} + 2\partial_q \nu_p C_{\partial a} - \partial_q \partial_p B_{\nu a} \\ &= -2(\partial_p \nu_q - \partial_q \nu_p)C_{\partial a} \\ &= 2\partial_i \partial_j (\partial_p \partial_{p+4} + \partial_q \partial_{q+4})C_{\partial a}, \end{aligned}$$

and Equation (5) follows. □

Corollary 3.5. $RE(4, 4)$ is a free $\mathbb{F}[\partial_1, \partial_2, \partial_3, \partial_4]$ -module with basis $\alpha_1, \dots, \alpha_8$.

Proof. Relations (1), (3), (4), (5) in Proposition 3.4 (actually (2) is a consequence of (5)) allow us to write every element $\beta \in RE(4, 4)$ in the form

$$\beta = \sum_{i=1}^8 f_i(\partial_1, \partial_2, \partial_3, \partial_4) \alpha_i$$

where $f_i(\partial_1, \partial_2, \partial_3, \partial_4) \in \mathbb{F}[\partial_1, \dots, \partial_4]$ for all $i = 1, \dots, 8$. We have to show that if $\beta = 0$ then $f_i = 0$ for all i . If we expand all α_i 's in terms of the elements a_i 's and we keep all terms divisible by $\partial_5 \partial_6 \partial_7 \partial_8$ we obtain

$$\partial_5 \partial_6 \partial_7 \partial_8 (2f_1 a_1 + 2f_2 a_2 + 2f_3 a_3 + 2f_4 a_4 + f_5 a_5 + f_6 a_6 + f_7 a_7 + f_8 a_8) = 0.$$

Therefore $f_1 = f_2 = \dots = f_8 = 0$ since a_1, \dots, a_8 are free generators of $RW(4, 4)$. □

Before stating the main result we need some further notations. We let

$$\mu_1 = \lambda_6 \lambda_7 \lambda_8, \quad \mu_2 = -\lambda_5 \lambda_7 \lambda_8, \quad \mu_3 = \lambda_5 \lambda_6 \lambda_8, \quad \mu_4 = -\lambda_5 \lambda_6 \lambda_7, \quad \mu_5 = \mu_6 = \mu_7 = \mu_8 = 0$$

and, consistently with (11), we will consider the following elements:

$$C_{\lambda\partial} = \sum_{i=1}^4 (\lambda_i \partial_{i+4} + \lambda_{i+4} \partial_i), \quad C_{\lambda\alpha} = \sum_{i=1}^4 (\lambda_i \alpha_{i+4} + \lambda_{i+4} \alpha_i),$$

$$B_{\lambda\lambda} = \sum_{i=1}^4 \lambda_i \lambda_{i+4}, \quad B_{\mu\partial} = \sum_{i=1}^4 \mu_i \partial_{i+4}, \quad B_{\mu\alpha} = \sum_{i=1}^4 \mu_i \alpha_{i+4}.$$

Theorem 3.6. *For all $i, j = 1, \dots, 8$ we have*

$$[\alpha_i \lambda \alpha_j] = (\lambda_i \lambda_j B_{\lambda\lambda} - 2\lambda_i \mu_j - 2\lambda_j \mu_i) C_{\lambda\alpha} + 3\lambda_i \lambda_j B_{\mu\alpha} + (\lambda_i C_{\mu\partial} + \lambda_i B_{\lambda\lambda} C_{\lambda\partial} - 2\mu_i C_{\lambda\partial}) \alpha_j.$$

In particular the $\mathbb{F}[\partial]$ -submodule $RE(4, 4)$ of $RW(4, 4)$ generated by $\alpha_1, \dots, \alpha_8$ is a Lie conformal subalgebra of $RW(4, 4)$.

Proof. The proof of this theorem consists of a finite number of computations that we verified with a computer¹. Recalling the definition of α_i in (12), the following formulas can be useful for a direct check:

$$(13) \quad [B_{\nu a} \lambda B_{\nu a}] = -B_{\mu\partial} B_{\nu a},$$

$$(14) \quad [C_{\partial a} \lambda C_{\partial a}] = -(2B_{\lambda\lambda} + C_{\lambda\partial}) C_{\partial a},$$

$$(15) \quad [B_{\nu a} \lambda C_{\partial a}] = \frac{1}{2} C_{\lambda\mu} C_{\lambda a} - B_{\mu\partial} C_{\partial a} - C_{\lambda\partial} B_{\mu a} - B_{\mu\partial} C_{\lambda a} + C_{\lambda\mu} C_{\partial a},$$

$$(16) \quad [C_{\partial a} \lambda B_{\nu a}] = -C_{\lambda\partial} B_{\rho a} - C_{\lambda\partial} B_{\nu a} - 2B_{\lambda\lambda} B_{\nu a} + C_{\lambda\nu} C_{\lambda a} + C_{\lambda\partial} B_{\mu a} - B_{\mu\partial} C_{\lambda a}$$

$$- \frac{1}{2} C_{\lambda\mu} C_{\partial a} + \frac{1}{2} C_{\lambda\mu} C_{\lambda a} + 2B_{\mu\partial} C_{\partial a} + C_{\lambda\partial} B_{\sigma a} + 2B_{\partial\partial} B_{\mu a},$$

where

$$\rho_1 = \lambda_6 \partial_7 \partial_8 + \partial_6 \lambda_7 \partial_8 + \partial_6 \partial_7 \lambda_8, \quad \rho_2 = -(\lambda_5 \partial_7 \partial_8 + \partial_5 \lambda_7 \partial_8 + \partial_5 \partial_7 \lambda_8),$$

$$\rho_3 = \lambda_5 \partial_6 \partial_8 + \partial_5 \lambda_6 \partial_8 + \partial_5 \partial_6 \lambda_8, \quad \rho_4 = -(\lambda_5 \partial_6 \partial_7 + \partial_5 \lambda_6 \partial_7 + \partial_5 \partial_6 \lambda_7),$$

$$\rho_5 = \rho_6 = \rho_7 = \rho_8 = 0,$$

and

$$\sigma_1 = \lambda_6 \lambda_7 \partial_8 + \partial_6 \lambda_7 \lambda_8 + \lambda_6 \partial_7 \lambda_8, \quad \sigma_2 = -(\lambda_5 \lambda_7 \partial_8 + \partial_5 \lambda_7 \lambda_8 + \lambda_5 \partial_7 \lambda_8),$$

$$\sigma_3 = \lambda_5 \lambda_6 \partial_8 + \partial_5 \lambda_6 \lambda_8 + \lambda_5 \partial_6 \lambda_8, \quad \sigma_4 = -(\lambda_5 \lambda_6 \partial_7 + \partial_5 \lambda_6 \lambda_7 + \lambda_5 \partial_6 \lambda_7),$$

$$\sigma_5 = \sigma_6 = \sigma_7 = \sigma_8 = 0.$$

¹The source code has been made from scratch using SAGE and is available upon request.

As an example we compute $[\alpha_5 \lambda \alpha_5]$, recalling that $\alpha_5 = \partial_5 B_{\partial\partial} C_{\partial a} + \partial_5 B_{\nu a}$. By Equation (13) we have

$$(17) \quad [\partial_5 B_{\nu a \lambda} \partial_5 B_{\nu a}] = \lambda_5 \partial_5 B_{\mu\partial} B_{\nu a} = 0$$

since every term in $B_{\mu\partial}$ either contains λ_5 or ∂_5 . By Equation (14), the properties of λ -products, and recalling that $B_{\lambda\lambda}$, $C_{\lambda\partial}$ and $B_{\partial\partial}$ are odd elements, we have

$$(18) \quad \begin{aligned} [\partial_5 B_{\partial\partial} C_{\partial a \lambda} \partial_5 B_{\partial\partial} C_{\partial a}] &= \lambda_5 \partial_5 B_{\lambda\lambda} (B_{\lambda\lambda} + C_{\lambda\partial} + B_{\partial\partial}) [C_{\partial a \lambda} C_{\partial a}] \\ &= -\lambda_5 \partial_5 B_{\lambda\lambda} B_{\partial\partial} C_{\lambda\partial} C_{\partial a} \end{aligned}$$

since $B_{\lambda\lambda}^2 = 0 = C_{\lambda\partial}^2$. Similarly we can compute, by Equation (15),

$$(19) \quad \begin{aligned} [\partial_5 B_{\nu a \lambda} \partial_5 B_{\partial\partial} C_{\partial a}] &= -\lambda_5 \partial_5 (B_{\lambda\lambda} + C_{\lambda\partial} + B_{\partial\partial}) [B_{\nu a \lambda} C_{\partial a}] \\ &= \lambda_5 \partial_5 B_{\partial\partial} C_{\lambda\partial} B_{\mu a}, \end{aligned}$$

since $\lambda_5 C_{\lambda\mu} = 0$, $\lambda_5 \partial_5 B_{\mu\partial} = 0$, $\lambda_5 B_{\lambda\lambda} B_{\mu a} = 0$, and, by Equation (16),

$$(20) \quad \begin{aligned} [\partial_5 B_{\partial\partial} C_{\partial a \lambda} \partial_5 B_{\nu a}] &= -\lambda_5 \partial_5 B_{\lambda\lambda} [C_{\partial a \lambda} B_{\nu a}] \\ &= \lambda_5 \partial_5 B_{\lambda\lambda} C_{\lambda\partial} B_{\rho a} + \lambda_5 \partial_5 B_{\lambda\lambda} C_{\lambda\partial} B_{\nu a}, \end{aligned}$$

since $\lambda_5 \partial_5 C_{\lambda\nu} = 0$, $\lambda_5 B_{\lambda\lambda} B_{\mu\partial} = 0$ and $\lambda_5 \partial_5 B_{\lambda\lambda} C_{\lambda\partial} B_{\sigma a} = 0$. We have, by Equations (17), (18), (19), (20),

$$[\alpha_5 \lambda \alpha_5] = -\lambda_5 \partial_5 B_{\lambda\lambda} B_{\partial\partial} C_{\lambda\partial} C_{\partial a} + \lambda_5 \partial_5 B_{\partial\partial} C_{\lambda\partial} B_{\mu a} + \lambda_5 \partial_5 B_{\lambda\lambda} C_{\lambda\partial} B_{\rho a} + \lambda_5 \partial_5 B_{\lambda\lambda} C_{\lambda\partial} B_{\nu a}.$$

Now we can observe that all terms in $B_{\partial\partial} C_{\lambda\partial} B_{\mu a} + B_{\lambda\lambda} C_{\lambda\partial} B_{\rho a}$ which are not divisible by λ_5 or ∂_5 cancel out and so we have

$$[\alpha_5 \lambda \alpha_5] = -\lambda_5 \partial_5 B_{\lambda\lambda} B_{\partial\partial} C_{\lambda\partial} C_{\partial a} + \lambda_5 \partial_5 B_{\lambda\lambda} C_{\lambda\partial} B_{\nu a} = \lambda_5 B_{\lambda\lambda} C_{\lambda\partial} \alpha_5,$$

and the proof is complete, since one can easily verify that $\lambda_5 C_{\mu\partial} \alpha_5 = 0$. \square

Corollary 3.7. *The Lie superalgebra $\mathcal{A}(RE(4, 4))$ has a D -conformal structure over $RE(4, 4)$.*

Proof. We check all the conditions of Definition 2.17.

- (1) This is obvious.
- (2) $RE(4, 4)$ is \mathbb{Z} -graded: it is generated by the homogeneous elements $\alpha_1, \dots, \alpha_8$ of degree -10 ;
- (3) the induced \mathbb{Z} -gradation on $\mathcal{A}(RE(4, 4))$ has depth 2: although elements α_i have degree -10 all elements of the form $y_j y_h y_k \alpha_i$ vanish in the annihilation superalgebra; indeed every summand in the element α_i involves four derivatives with respect to the y -variables.
- (4) The homogeneous components of $\mathcal{A}(R)$ of degree -1 and -3 vanish, and in particular they are purely odd;
- (5) for all $i = 1, \dots, 8$ we have that $y_5 y_6 y_7 y_8 \alpha_i$ is a nonzero scalar multiple of a_i in $\mathcal{A}(RE(4, 4))_{-2}$ (and therefore it acts as a scalar multiple of ∂_{y_i} on $\mathcal{A}(RE(4, 4))$) and for every set of indices i_1, i_2, i_3, i_4 we have that $y_{i_1} y_{i_2} y_{i_3} y_{i_4} \alpha_i$ is a scalar multiple of a_j for some j . Therefore the map $\text{ad} : \mathcal{A}(R)_{-2} \rightarrow \text{Der}(\mathcal{A}(R))$ is injective and its image is $\langle \partial_{y_1}, \dots, \partial_{y_8} \rangle$.

\square

Theorem 3.8. *The annihilation superalgebra $\mathcal{A}(RE(4, 4))$ is spanned by the elements $f(y_1, y_2, y_3, y_4)y_5y_6y_7y_8\alpha_i$ with $f(y_1, y_2, y_3, y_4) \in \mathbb{F}[y_1, y_2, y_3, y_4]$ and $i = 1, \dots, 8$.*

Proof. The result follows by repeated applications of the relations in Proposition 3.4. We show an explicit example to illustrate this fact:

$$\begin{aligned} y_1y_2y_5y_6y_7\alpha_6 &= y_1y_2y_5y_6y_7\partial_{y_8}y_8\alpha_6 = y_1y_2y_5y_6y_7y_8\partial_8\alpha_6 \\ &= \frac{1}{2}y_1y_2y_5y_6y_7y_8(\partial_1\alpha_3 - \partial_3\alpha_1) \\ &= -\frac{1}{2}\partial_{y_1}(y_1y_2y_5y_6y_7y_8)\alpha_3 = -\frac{1}{2}y_2y_5y_6y_7y_8\alpha_3. \end{aligned}$$

□

4. EMBEDDING $E(4, 4)$ INTO $W(4, 4)$

In this section we want to describe an explicit embedding of the exceptional Lie superalgebra $E(4, 4)$ into $W(4, 4)$. As a consequence of this fact we show that $E(4, 4) \cong \mathcal{A}(RE(4, 4))$ and in particular that $E(4, 4)$ has a D -conformal structure.

Recall that the Lie superalgebra $E(4, 4)$ is defined as follows ([9, §5.3]): its even part \mathfrak{g}_0 is the Lie algebra W_4 of vector fields in four even indeterminates with coefficients in the field of formal power series, and its odd part $E(4, 4)_{\bar{1}}$ is the space $\Omega^1(4)$ of all differential one-forms in four even indeterminates with coefficients in the field of formal power series with the following action of $E(4, 4)_{\bar{0}}$: for $X \in E(4, 4)_{\bar{0}}$, $\omega \in E(4, 4)_{\bar{1}}$,

$$[X, \omega] = L_X(\omega) - \frac{1}{2} \operatorname{div}(X)\omega,$$

where $L_X(\omega)$ denotes the Lie derivative of the one-form ω along the vector field X (see also [7, Definition 2.5]). Besides, for $\omega_1, \omega_2 \in E(4, 4)_{\bar{1}}$:

$$[\omega_1, \omega_2] = d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2,$$

where the three-forms are identified with vector fields via contraction with the standard volume form. In other words, $E(4, 4)_{\bar{1}}$ can be interpreted as the space $\Omega_1(4)^{-\frac{1}{2}}$ of formal sections of the bundle $T^* \otimes K^{-\frac{1}{2}}$ over the four disk.

Remark 4.1. In [10, §2.1.3], in the classification of finite-dimensional Lie superalgebras, it appears the Lie superalgebra $p(4)$ (denoted in [10] by $p(3)$). This is a simple finite-dimensional Lie superalgebra with the following consistent irreducible \mathbb{Z} -grading:

$$p(4) = p(4)_{-1} \oplus p(4)_0 \oplus p(4)_1$$

where $p(4)_0 \cong \mathfrak{sl}_4$ and $p(4)_{-1} \cong \Lambda^2((\mathbb{F}^4)^*)$, $p(4)_1 \cong S^2\mathbb{F}^4$, as \mathfrak{sl}_4 -modules. The Lie superalgebra $p(4)$ has a unique, up to isomorphisms, non-trivial central extension that we will denote by $\hat{p}(4)$ ([11, Example 3.6], [13]): observe that $\Lambda^2((\mathbb{F}^4)^*)$ is isomorphic to the standard \mathfrak{so}_6 -module, with scalar product (\cdot, \cdot) ; define $\varphi : \Lambda^2(p(4)) \rightarrow \mathbb{F}$ by setting, for $x \in p(4)_i, y \in p(4)_j$,

$$\varphi(x, y) := \begin{cases} (x, y) & \text{if } i = j = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{p}(4)$ is the central extension of $p(4)$ defined by this cocycle. Therefore one has the following \mathbb{Z} -graded Lie superalgebra

$$\hat{p}(4) = \hat{p}(4)_{-2} \oplus \hat{p}(4)_{-1} \oplus \hat{p}(4)_0 \oplus \hat{p}(4)_1$$

with $\hat{p}(4)_{-2} \cong \mathbb{F}$ central.

The Lie superalgebra $L = E(4, 4)$ has, up to conjugation, only one irreducible \mathbb{Z} -grading $L = \prod_{j \geq -1} \mathfrak{g}_j$, called the principal grading, defined by setting $\deg(x_i) = 1$ and $\deg d = -2$ ([7, Corollary 9.8]). It is a grading of depth 1, where \mathfrak{g}_0 is isomorphic to $\hat{p}(4)$ and $\mathfrak{g}_{-1} \cong \mathbb{F}^{4|4}$. In this isomorphism we have that $\hat{p}(4)_1$ is spanned by the one-forms $x_i dx_j + x_j dx_i$ (with $i, j = 1, 2, 3, 4$), $\hat{p}(4)_0$ is spanned by the vector fields $x_i \partial_{x_j}$ and $x_i \partial_{x_i} - x_j \partial_{x_j}$ (with $i, j = 1, 2, 3, 4, i \neq j$), $\hat{p}(4)_{-1}$ is spanned by the one-forms $x_i dx_j - x_j dx_i$ (with $i, j = 1, 2, 3, 4$) and $\hat{p}(4)_{-2}$ is spanned by $x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} + x_4 \partial_{x_4}$.

Let us fix the Borel subalgebra $\langle x_i \partial_j, h_{ij} = x_i \partial_i - x_j \partial_j \mid i < j \rangle$ of $(\mathfrak{g}_0)_0 = \mathfrak{sl}_4$ and consider the usual set of simple roots of the corresponding root system, given by $\{\alpha_{12}, \alpha_{23}, \alpha_{34}\}$. We let Λ be the weight lattice of \mathfrak{sl}_4 and we express all weights of \mathfrak{sl}_4 using their coordinates with respect to the fundamental weights $\omega_{12}, \omega_{23}, \omega_{34}$, i.e., for $\lambda \in \Lambda$ we write $\lambda = (\lambda_{12}, \lambda_{23}, \lambda_{34})$ for some $\lambda_{i+1} \in \mathbb{Z}$ to mean $\lambda = \lambda_{12}\omega_{12} + \lambda_{23}\omega_{23} + \lambda_{34}\omega_{34}$. If $\lambda = (a, b, c) \in \Lambda$ is a dominant weight we shall denote by $F(\lambda) = F(a, b, c)$ the irreducible \mathfrak{sl}_4 -module of highest weight λ .

The component \mathfrak{g}_1 of the principal \mathbb{Z} -grading of L is an irreducible \mathfrak{g}_0 -module and, as an \mathfrak{sl}_4 -module, it decomposes as follows: $\mathfrak{g}_1 = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ where $V_1 \cong F(2, 0, 1)$, $V_2 \cong F(1, 0, 0)$, $V_3 \cong F(3, 0, 0)$ and $V_4 \cong F(1, 1, 0)$.

Remark 4.2. We point out that one can construct a graded embedding of the Lie superalgebra $\mathfrak{g}_0 = \hat{p}(4)$ into the exceptional Lie superalgebra $E(5, 10)$. (For the description of $E(5, 10)$ we refer to [11] and [6]). Indeed the following map

$$\begin{aligned} \sum_{i=1}^4 x_i \partial_{x_i} &\mapsto \frac{1}{2} \partial_{x_5}, \\ x_i dx_j - x_j dx_i &\mapsto dx_i \wedge dx_j, \\ x_i \partial_{x_j} &\mapsto x_i \partial_{x_j}, && \text{for } i \neq j, \\ x_i \partial_{x_i} - x_{i+1} \partial_{x_{i+1}} &\mapsto x_i \partial_{x_i} - x_{i+1} \partial_{x_{i+1}}, && \text{for } i = 1, \dots, 3, \\ x_i dx_j + x_j dx_i &\mapsto x_i dx_j \wedge dx_5 + x_j dx_i \wedge dx_5, \end{aligned}$$

defines an embedding of $\mathfrak{g}_0 = \hat{p}(4)$ into $E(5, 10)$. However, this embedding does not extend to the whole $E(4, 4)$ [9, Section 5.5].

Definition 4.3. A \mathbb{Z} -graded Lie superalgebra $L = \prod \mathfrak{g}_i$ is called transitive if it satisfies the following property: if $x \in \mathfrak{g}_i$, $i \geq 0$, is such that $[x, \mathfrak{g}_{-1}] = 0$, then $x = 0$.

We now recall the following simple Guillemin-Sternberg embedding theorem which generalizes a standard result for Lie algebras ([1], [10, §5.4]).

Theorem 4.4. *If $L = \prod_{i \geq -1} \mathfrak{g}_i$ is a transitive \mathbb{Z} -graded Lie superalgebra with $\dim(\mathfrak{g}_{-1})_{\bar{0}} = r$, $\dim(\mathfrak{g}_{-1})_{\bar{1}} = s$, then there is an embedding*

$$\mathfrak{g} \rightarrow W(r, s)$$

mapping the \mathbb{Z} -grading of L to the principal grading of $W(r, s)$.

In order to describe the embedding of $E(4, 4)$ into $W(4, 4)$ explicitly, we will denote by $X = \{x_1, \dots, x_4\}$ the (even) variables of $E(4, 4)$ and by $Y = \{y_1, \dots, y_4\}$ the corresponding (even) variables of $W(4, 4)$; accordingly, we will denote by $\{y_5, \dots, y_8\}$ the odd variables of $W(4, 4)$.

Theorem 4.5. *The following map Φ defines an embedding of $E(4, 4)$ into $W(4, 4)$, preserving the principal grading: for $r, i = 1, \dots, 4$,*

$$\begin{aligned} f(X)\partial_{x_r} &\mapsto f(Y)\partial_{y_r} + \sum_{i=1}^4 \partial_{y_i} f(Y)\beta_1(r, i) + \sum_{i=1}^4 \partial_{y_r} \partial_{y_i} f(Y)\beta_2(i) + \sum_{i,l=1}^4 \partial_{y_r} \partial_{y_i} \partial_{y_l} f(Y)\beta_3(i, l) \\ f(X)dx_i &\mapsto f(Y)\partial_{y_{i+4}} + \sum_{j=1}^4 \partial_{y_j} f(Y)\gamma_1(i, j) + \sum_{j,l=1}^4 \partial_{y_j} \partial_{y_l} f(Y)\gamma_2(i, j, l), \end{aligned}$$

where

$$\beta_1(r, i) = \begin{cases} -y_{r+4}\partial_{y_{i+4}} & \text{if } r \neq i, \\ -y_{r+4}\partial_{y_{r+4}} + \frac{1}{2} \sum_{l=5}^8 y_l \partial_{y_l} & \text{if } r = i, \end{cases}$$

and

$$\beta_2(i) = -\frac{1}{2}(y_{j+4}y_{h+4}\partial_{y_k} + y_{h+4}y_{k+4}\partial_{y_j} + y_{k+4}y_{j+4}\partial_{y_h}),$$

$$\beta_3(i, l) = \frac{1}{2}y_{j+4}y_{h+4}y_{k+4}\partial_{y_{l+4}},$$

$$\gamma_1(i, j) = y_{h+4}\partial_{y_k} - y_{k+4}\partial_{y_h},$$

$$\gamma_2(i, j, l) = -y_{h+4}y_{k+4}\partial_{y_{l+4}},$$

for $i, j, h, k \in \{1, 2, 3, 4\}$ such that (i, j, h, k) is an even permutation of $(1, 2, 3, 4)$, and $l \in \{1, 2, 3, 4\}$.

Proof. We want to show that, for every $a \in \mathfrak{g}_i, b \in \mathfrak{g}_j$, for every $i, j \geq -1$, we have:

$$(21) \quad \Phi([a, b]) = [\Phi(a), \Phi(b)].$$

By the transitivity of the principal grading of the Lie superalgebra $W(4, 4)$, and using induction on $i + j$, it is enough to prove (21) for $i = -1$.

For $r, s = 1, \dots, 4$ we have:

$$[\partial_{x_r}, f(X)\partial_{x_s}] = (\partial_{x_r} f)\partial_{x_s}, \quad [\partial_{x_r}, f(X)dx_s] = (\partial_{x_r} f)dx_s$$

$$\Phi(\partial_{x_r}(f)\partial_{x_s}) = (\partial_{y_r} f)\partial_{y_s} + \sum_{i=1}^4 (\partial_{y_i} \partial_{y_r} f)\beta_1(s, i) + \sum_{i=1}^4 (\partial_{y_s} \partial_{y_i} \partial_{y_r} f)\beta_2(i) + \sum_{i,l=1}^4 (\partial_{y_i} \partial_{y_s} \partial_{y_l} \partial_{y_r} f)\beta_3(i, l).$$

It follows that

$$[\Phi(\partial_{x_r}), \Phi(f(X)\partial_{x_s})] = \Phi((\partial_{x_r}f)\partial_{x_s}),$$

since, for $r, s, i, l = 1, \dots, 4$,

$$[\partial_{y_r}, \beta_1(s, i)] = 0, \quad [\partial_{y_r}, \beta_2(i)] = 0, \quad [\partial_{y_r}, \beta_3(i, l)] = 0.$$

Similarly, for $r, s = 1, \dots, 4$,

$$[\Phi(\partial_{x_r}), \Phi(f(X)dx_s)] = \Phi((\partial_{x_r}f)dx_s),$$

since, for $r, s, j, l = 1, \dots, 4$, we have:

$$[\partial_{y_r}, \gamma_1(s, j)] = 0, \quad [\partial_{y_r}, \gamma_2(s, j, l)] = 0.$$

Besides, for $r \neq s$,

$$[dx_s, f(X)\partial_{x_r}] = \frac{1}{2}(\partial_{x_r}f)dx_s,$$

and

$$[\partial_{y_{s+4}}, \Phi(f(X)\partial_{x_r})] = \frac{1}{2}(\partial_{y_r}f\partial_{y_{s+4}} + \sum_{j=1}^4 \partial_{y_j}\partial_{y_r}f\gamma_1(s, j) + \sum_{l,j=1}^4 \partial_{y_j}\partial_{y_l}\partial_{y_r}f\gamma_2(s, j, l)),$$

since

$$\begin{aligned} [\partial_{y_{s+4}}, \beta_1(r, i)] &= \frac{1}{2}\delta_{i,r}\partial_{y_{s+4}}, \\ [\partial_{y_{s+4}}, \beta_2(i)] &= \gamma_1(s, i), \\ [\partial_{y_{s+4}}, \beta_3(i, l)] &= \gamma_2(s, i, l). \end{aligned}$$

Similarly, for $r = 1, \dots, 4$,

$$\Phi([dx_r, f(X)\partial_{x_r}]) = [\Phi(dx_r), \Phi(f(X)\partial_{x_r})].$$

Finally,

$$\Phi([dx_r, f(X)dx_r]) = 0 = [\partial_{y_{r+4}}, \Phi(f(X)dx_r)],$$

since $\partial_{y_{r+4}}\gamma_1(r, j) = 0 = \partial_{y_{r+4}}\gamma_2(r, j, l)$, and, for $i \neq j$,

$$[dx_j, f(X)dx_i] = -(\partial_{x_h}f)\partial_{x_k} + (\partial_{x_k}f)\partial_{x_h}$$

$$\begin{aligned} [\partial_{y_{j+4}}, \Phi(f(X)dx_i)] &= -(\partial_{y_h}f)\partial_{y_k} + (\partial_{y_k}f)\partial_{y_h} - \sum_{l=1}^4 (\partial_{y_h}\partial_{y_l}f)y_{k+4}\partial_{y_{l+4}} + \sum_{l=1}^4 (\partial_{y_k}\partial_{y_l}f)y_{h+4}\partial_{y_{l+4}} \\ &= -(\partial_{y_h}f)\partial_{y_k} + (\partial_{y_k}f)\partial_{y_h} + \sum_{l=1}^4 (\partial_{y_h}\partial_{y_l}f)\beta_1(k, l) - \sum_{l=1}^4 (\partial_{y_k}\partial_{y_l}f)\beta_1(h, l) \\ &= \Phi(-(\partial_{x_h}f)\partial_{x_k} + (\partial_{x_k}f)\partial_{x_h}). \end{aligned}$$

□

Example 4.6. For the convenience of the reader we explicitly write the image of the vector field $f(x_1, x_2, x_3, x_4)\partial_{x_1}$ and of the form $f(x_1, x_2, x_3, x_4)dx_1$ under the map Φ :

$$\begin{aligned}
f(x_1, x_2, x_3, x_4)\partial_{x_1} \mapsto & f(Y)\partial_{y_1} + \frac{1}{2}\partial_{y_1}f(Y)(-y_5\partial_{y_5} + y_6\partial_{y_6} + y_7\partial_{y_7} + y_8\partial_{y_8}) \\
& - \partial_{y_2}f(Y)y_5\partial_{y_6} - \partial_{y_3}f(Y)y_5\partial_{y_7} - \partial_{y_4}f(Y)y_5\partial_{y_8} \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_1}f(Y)(-y_6y_7\partial_{y_4} - y_7y_8\partial_{y_2} + y_6y_8\partial_{y_3}) \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_2}f(Y)(-y_5y_8\partial_{y_3} + y_7y_8\partial_{y_1} + y_5y_7\partial_{y_4}) \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_3}f(Y)(-y_5y_6\partial_{y_4} - y_6y_8\partial_{y_1} + y_5y_8\partial_{y_2}) \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_4}f(Y)(y_5y_6\partial_{y_3} - y_5y_7\partial_{y_2} + y_6y_7\partial_{y_1}) \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_1}\partial_{y_1}f(Y)y_6y_7y_8\partial_{y_5} + \frac{1}{2}\partial_{y_1}\partial_{y_1}\partial_{y_2}f(Y)y_6y_7y_8\partial_{y_6} \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_1}\partial_{y_3}f(Y)y_6y_7y_8\partial_{y_7} + \frac{1}{2}\partial_{y_1}\partial_{y_1}\partial_{y_4}f(Y)y_6y_7y_8\partial_{y_8} \\
& - \frac{1}{2}\partial_{y_1}\partial_{y_2}\partial_{y_1}f(Y)y_5y_7y_8\partial_{y_5} - \frac{1}{2}\partial_{y_1}\partial_{y_2}\partial_{y_2}f(Y)y_5y_7y_8\partial_{y_6} \\
& - \frac{1}{2}\partial_{y_1}\partial_{y_2}\partial_{y_3}f(Y)y_5y_7y_8\partial_{y_7} - \frac{1}{2}\partial_{y_1}\partial_{y_2}\partial_{y_4}f(Y)y_5y_7y_8\partial_{y_8} \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_3}\partial_{y_1}f(Y)y_5y_6y_8\partial_{y_5} + \frac{1}{2}\partial_{y_1}\partial_{y_3}\partial_{y_2}f(Y)y_5y_6y_8\partial_{y_6} \\
& + \frac{1}{2}\partial_{y_1}\partial_{y_3}\partial_{y_3}f(Y)y_5y_6y_8\partial_{y_7} + \frac{1}{2}\partial_{y_1}\partial_{y_3}\partial_{y_4}f(Y)y_5y_6y_8\partial_{y_8} \\
& - \frac{1}{2}\partial_{y_1}\partial_{y_4}\partial_{y_1}f(Y)y_5y_6y_7\partial_{y_5} - \frac{1}{2}\partial_{y_1}\partial_{y_4}\partial_{y_2}f(Y)y_5y_6y_7\partial_{y_6} \\
& - \frac{1}{2}\partial_{y_1}\partial_{y_4}\partial_{y_3}f(Y)y_5y_6y_7\partial_{y_7} - \frac{1}{2}\partial_{y_1}\partial_{y_4}\partial_{y_4}f(Y)y_5y_6y_7\partial_{y_8}.
\end{aligned}$$

and

$$\begin{aligned}
f(x_1, x_2, x_3, x_4)dx_1 \mapsto & f(Y)\partial_{y_5} + \partial_{y_2}f(Y)(y_7\partial_{y_4} - y_8\partial_{y_3}) \\
& + \partial_{y_3}f(Y)(y_8\partial_{y_2} - y_6\partial_{y_4}) + \partial_{y_4}f(Y)(y_6\partial_{y_3} - y_7\partial_{y_2}) \\
& + \partial_{y_2}\partial_{y_1}f(Y)(-y_7y_8\partial_{y_5}) + \partial_{y_2}\partial_{y_2}f(Y)(-y_7y_8\partial_{y_6}) \\
& + \partial_{y_2}\partial_{y_3}f(Y)(-y_7y_8\partial_{y_7}) + \partial_{y_2}\partial_{y_4}f(Y)(-y_7y_8\partial_{y_8}) \\
& + \partial_{y_3}\partial_{y_1}f(Y)(y_6y_8\partial_{y_5}) + \partial_{y_3}\partial_{y_2}f(Y)(y_6y_8\partial_{y_6}) \\
& + \partial_{y_3}\partial_{y_3}f(Y)(y_6y_8\partial_{y_7}) + \partial_{y_3}\partial_{y_4}f(Y)(y_6y_8\partial_{y_8}) \\
& + \partial_{y_4}\partial_{y_1}f(Y)(-y_6y_7\partial_{y_5}) + \partial_{y_4}\partial_{y_2}f(Y)(-y_6y_7\partial_{y_6}) \\
& + \partial_{y_4}\partial_{y_3}f(Y)(-y_6y_7\partial_{y_7}) + \partial_{y_4}\partial_{y_4}f(Y)(-y_6y_7\partial_{y_8}).
\end{aligned}$$

Corollary 4.7. *The map Ψ :*

$$f(x_1, x_2, x_3, x_4)\partial_{x_i} \mapsto -\frac{1}{2}f(y_1, y_2, y_3, y_4)y_5y_6y_7y_8\alpha_i \mod \tilde{\partial}\tilde{R}E(4, 4),$$

$$f(x_1, x_2, x_3, x_4)dx_i \mapsto -f(y_1, y_2, y_3, y_4)y_5y_6y_7y_8\alpha_{i+4} \mod \tilde{\partial}\tilde{R}E(4, 4),$$

defines an isomorphism between $E(4, 4)$ and $\mathcal{A}(RE(4, 4))$. In particular Verma modules over the Lie superalgebra $E(4, 4)$ satisfy the duality property. The shift χ in Definition 2.19 is 0.

Proof. One can check that the map $\Psi = \varphi \circ \Phi$ is the composition of the maps φ and Φ defined in Proposition 3.2 and Theorem 4.5, respectively, where $\mathcal{A}(RE(4, 4))$ is canonically identified with a Lie subalgebra of $\mathcal{A}(RW(4, 4))$. The image of Ψ is indeed $\mathcal{A}(RE(4, 4))$ by Theorem 3.8. The shift is 0, since $E(4, 4)_0$ coincides with its derived subalgebra. \square

Remark 4.8. As mentioned in the introduction, one can construct a Lie conformal superalgebra R of type $(4, 0)$ whose annihilation superalgebra is $E(4, 4)$. This construction, however, does not provide a D -conformal structure to $E(4, 4)$: indeed in this case the space of derivatives D has dimension 4, while $E(4, 4)_{-2}$ has dimension 8, so condition (5) in Definition 2.17 is not satisfied.

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