



ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

ARCHIVIO ISTITUZIONALE DELLA RICERCA

Alma Mater Studiorum Università di Bologna Archivio istituzionale della ricerca

Assessment of modal density and free path distribution in central-planned halls

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:

Fratoni, G., Garai, M., D'Orazio, D. (2023). Assessment of modal density and free path distribution in central-planned halls. THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA, 154(6), 3604-3614 [10.1121/10.0022569].

Availability:

This version is available at: <https://hdl.handle.net/11585/951854> since: 2024-01-03

Published:

DOI: <http://doi.org/10.1121/10.0022569>

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>).
When citing, please refer to the published version.

(Article begins on next page)

Assessment of modal density and free path distribution in central-planned halls

Giulia Fratoni, Massimo Garai, and Dario D'Orazio

Department of Industrial Engineering, University of Bologna, Viale Risorgimento 2,

Bologna, 40136, Italy^a

1 Central-planned halls are highly widespread in the historical architectures of the
2 Western world, such as rotundae, Christian baptisteries, and Roman tombs. In such
3 halls, whispering galleries, flutter echoes, and sound focusing are the acoustic phe-
4 nomena mainly investigated by scholars. Instead, modal behaviour and free path
5 distribution are generally less treated in literature. The present study explores the
6 modal density at low frequencies and the relationship with the most recurrent free
7 path lengths in three historical nearly circular spaces, here assessed as case studies.
8 Acoustic measurements allowed the collection of objective experimental data, i.e.,
9 room impulse responses and the resulting room acoustics criteria. Wave-based nu-
10 merical models allowed investigating the eigenfrequencies distribution, whilst the free
11 paths trend has been experienced through ray-based models. The main outcomes of
12 both analyses show the prominence of the circular modes, rather than the diametral
13 and the elevation ones. Moreover, the mean free path calculated using ray-tracing
14 proves to be higher than the theoretical value commonly assumed for any kind of
15 shape. The consequent longer reverberations compared to halls with other shapes
16 and the same volume justify the significant support historically provided to sound
17 signals by circular halls.

^adario.dorazio@unibo.it

18 I. INTRODUCTION

19 Among the first scientific studies on circular spaces dating back to the 1920s, one notable
20 insight is provided by W. C. Sabine – the founder of architectural acoustics as a science –
21 who analysed the domes of St. Paul Cathedral in London and the Hall of Statues of the U.S.
22 Capitol in Washington D.C. (Sabine, 1922). The former hall is known for the *whispering*
23 *galleries* effect: a phenomenon due to rigid boundaries (hard walls), a low sound power level
24 of the sound source, and grazing incidence between the sound source and the walls (Bate,
25 1938). The same role of hard walls was confirmed when the dome of the Hall of Statues
26 of the U.S. Capitol was restored after a fire and some painted coffers were replaced by
27 plaster coffers with cavities. After the restoration, the focusing effect noted by Sabine was
28 unintentionally reduced due to the diffuse reflections (Cremer and Müller, 1978). Another
29 phenomenon that is generally investigated in circular environments is related to *focusing*
30 *effects*. The first insights on such a topic dated back to Kircher’s *Phonurgia nova* (Kircher,
31 1673), who studied proto-wave guides. The sound rays analogy was well known by scholars
32 since the 17th Century (D. D’Orazio, 2019). Focusing effects were corrected by Meyer and
33 Kuttruff by placing suspended ceilings inside the Festival Hall of the Farbwerke Hoechst, and
34 by Reichardt et al. using reflecting ceilings in the Haus del Lehrers (Meyer, 1964; Reichardt,
35 1968). A further acoustic effect typically studied in circular spaces is the *flutter echo*. In
36 the presence of domes, flutter echoes are regularly repeated over time (Alberdi *et al.*, 2019;
37 Magrini and Ricciardi, 2006). This often happens in central-planned curved architectures,
38 which have time-aligned geometric reflections due to the smoothness of reflective surfaces

39 and the lack of scattering elements. With this regard, a large number of studies were focused
40 on Orthodox churches and mosques, due to their typical geometries ([Kosała and Małecki,](#)
41 [2018](#); [Shepherd *et al.*, 2005](#); [Sü Gül *et al.*, 2016](#)). Generally, since the 1930s scholars have
42 been investigating flutter echoes in the case of concave surfaces, such as domes, where
43 the reflection may be significantly increased and heard separately from the direct sound
44 ([Petzold, 1930](#)). Further research allowed to define the first analytic treatments on flutter
45 echo, proposing the taxonomy in four categories, depending on the vault curvature and the
46 specific superposition of acoustic rays ([Haas, 1951](#); [Maa, 1941](#); [Muncey *et al.*, 1953](#)). Later,
47 scholars studied the cancellation of echo phenomena in Cabanchel Boxing Pavillion (Madrid)
48 by using ray tracing techniques ([Moreno *et al.*, 1981](#)). Makrimenko stated that the critical
49 delay difference depends on the characteristics of the signal such as frequency content and
50 temporal behaviour ([Makrinenko, 1986](#)). On the other hand, an echo evaluation method
51 based on the measured impulse response accounts for the ratio between the shift of centre
52 time due to successive reflections and the delay of these reflections ([Dietsch and Kraak,](#)
53 [1986](#)). Other scholars based their approach on modulation-transfer function, and they fixed
54 the acoustic conditions required for a certain intelligibility by analysing the modulation of
55 Gregorian chant ([D’Orazio *et al.*, 2020](#); [Vitale *et al.*, 2005](#)).

56 An interesting aspect of spaces with central symmetry with curved sidewalls is the rever-
57 beration, even though it is less treated in literature. Tzekakis measured the sound behaviour
58 in the Rotunda of Thessaloniki both in occupied and unoccupied conditions finding opti-
59 mal listening conditions in the occupied state ([Tzekakis, 1975](#)). Furthermore, the acoustic
60 absorption of one surface with respect to the others may affect the sound field ([Sumarac-](#)

61 Pavlovic *et al.*, 2008). Such is the case of mosques (Prodi *et al.*, 2001; Utami, 2004), where
 62 almost all of the floor is covered by carpets, or the Byzantine churches (Fausti *et al.*, 2003),
 63 due to faithful on the floor and low absorbent walls made of stones all around.

64 Even though there are several works focused on central-planned curved halls, the role
 65 of room modes and free path distribution require further insights. The present paper aims
 66 to compensate for this lack, exploiting the opportunity given by wave-based and ray-based
 67 numerical models. The former allowed for the analysis of the eigenfrequencies modal density,
 68 while the latter permitted the actual distribution of the free path lengths to be calculated.
 69 In Section II the workflow is reported, including theoretical hints, a description of the case
 70 studies, the acoustic measurements, and the setup of numerical models. Then, Section III
 71 provides outcomes along with the consequent remarks and discussions.

72 II. METHOD

73 A. Theoretical background

74 In cylindrical enclosures, the resonant frequencies corresponding to the natural modes
 75 have the following general form (Kuttruff, 2016):

$$f(m, n, k) = \frac{c}{2} \sqrt{\left(\frac{\beta_{mn}}{a}\right)^2 + \left(\frac{k}{l_z}\right)^2} \quad (1)$$

76 where β_{mn} represents the n -th zero of the Bessel function derivative of the first kind of order
 77 m (divided by π); a is the radius of the cylinder; l_z is its height. It should be noted that
 78 m, n, k are integer numbers ($m = 0, 1, 2, \dots$; $n = 1, 2, 3, \dots$; $k = 0, 1, 2, \dots$) corresponding,
 79 respectively, to diametral, circular, and elevation modes. Figure 1 offers a 2D visualization of

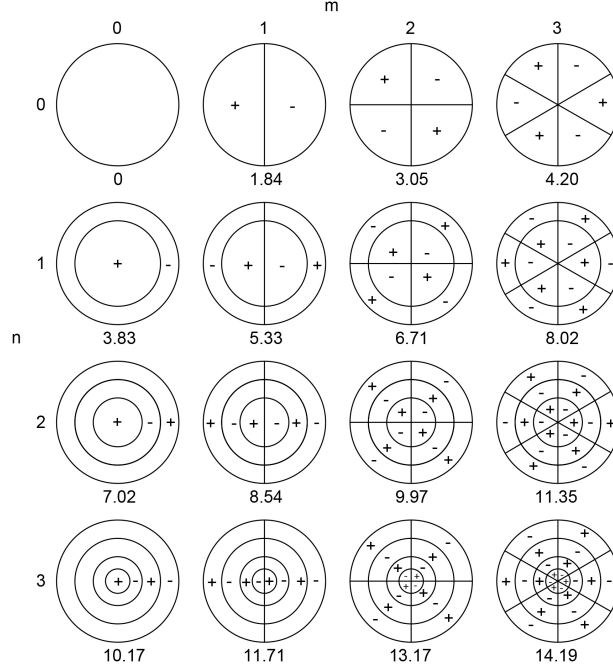


FIG. 1. Nodal lines for transverse pressure distribution in a circular space up to $m = 3, n = 3$ higher order mode. In detail, m is the order of the Bessel function, i.e., the number of pressure nodal diameters, while n is equal to the number of pressure nodal circles (Figure adapted from (Eriksson, 1980)).

80 m and n in circular spaces up to $m = 3, n = 3$ higher order mode (Eriksson, 1980). According
82 to the conventional notation, $n = 0$ is the first root of $J'_m(k_r a) = 0$ and n is its $(n + 1)^{\text{st}}$ root
83 (Lommel, 1868). As the first zero ($n = 0$) of J'_1 is at 1.84 and the second zero ($n = 1$) of J'_0
84 is at 3.83, the first diametral mode (or azimuthal mode) is at $k_{mn} a = 1.84$, whilst the first
85 circular mode is at $k_{mn} a = 3.83$. When the frequency is low enough ($f < 1.84c/\pi 2a$) or the
86 wavelength is long enough ($\lambda > \pi 2a/1.84$) the wave propagation is that of an unattenuated
87 plane wave ($p(z, t) = [C_1 e^{-jkz} + C_2 e^{+jkz}] e^{+j\omega t}$). In fact, the cut-off frequency for a circular
88 space is:

$$f_{co} = \frac{1.84}{\pi} \frac{c}{2a} = 0.5857 \frac{c}{2a}. \quad (2)$$

89 When the diameter is small, and the signal is centered at low frequencies it is sufficient to
90 perform a 1D analysis to tackle plane wave radiations because Eq. 2 is generally satisfied.
91 Conversely, when it comes to large cylindrical halls, f_{co} significantly drops so that no fre-
92 quency of interest in room acoustics can be handled with plane waves' propagation laws,
93 and a 3D analysis is required.

94 The amount of eigenfrequencies in enclosed spaces has been expressed by several scholars
95 that (Blevins, 2006; Bolt *et al.*, 1950; Maa, 1939; Walker, 1996):

$$N_f = \frac{4}{3}\pi V \left(\frac{f}{c}\right)^3 + \frac{\pi}{4} S \left(\frac{f}{c}\right)^2 + \frac{L}{8} \frac{f}{c} \quad (3)$$

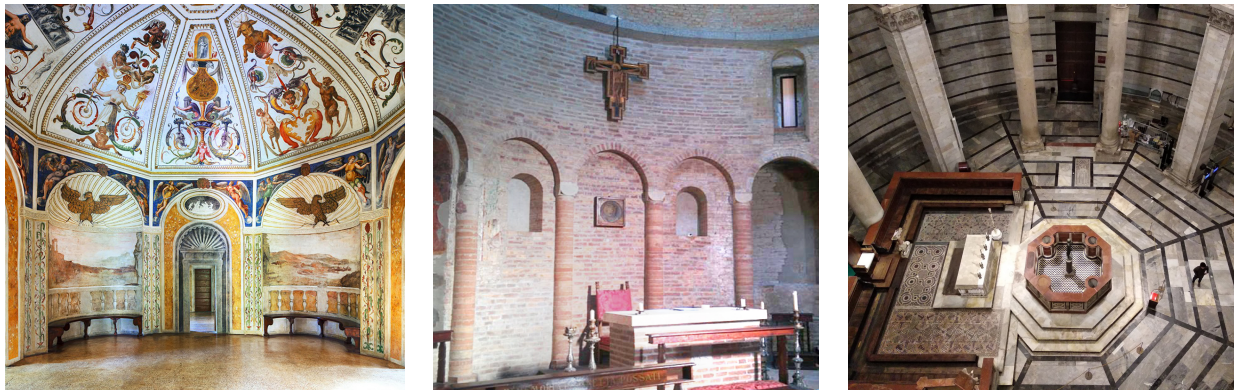
96 where N_f is total number of eigenfrequencies up to the limit frequency f , V is the volume
97 of the hall, S is the total area of all the surfaces, and L is the sum of all edge lengths of
98 the room. The distribution of the resonant frequencies is generally deemed as a continuous
99 function with a reliable approximation because the series of discrete values fluctuate above
100 and below this function (Kinzer and Wilson, 1947). Consequently, the average density of
101 eigenfrequencies at the frequency f is generally equal to

$$\frac{dN}{df} \approx \frac{4\pi V f^2}{c^3} + \frac{\pi S f}{2c^2}. \quad (4)$$

102 Even though Eqs. 3, 4 have been conceived for rectangular rooms, they are generally valid
103 also for rooms with arbitrary shapes as long as only the first term in the right hand-side
104 of the two equations are considered (Balian and Bloch, 1970; Richardson, 1912). However,
105 they are generally assumed as valid when $f \rightarrow \infty$ (Kuttruff, 2016).

106 **B. Case studies**

107 Since early Christianity, round buildings have been frequently used for Baptismal and
108 funerary rituals or by monks for solitary prayer and singing. As a result, Western culture is
109 full of these circular structures, whose shape allows for suitable voice support and intimacy
110 at the same time. Figure 2 shows the three halls taken as case studies: the Odeo Cornaro
111 (OC), the Rotunda Aldini (RA), and the Pisa's Baptistery (PB). The main geometrical
112 and acoustic features of the halls are provided in Table I. Historical references have guided
113 the whole study not only to explore the intended use of the halls but also to infer useful
114 information on the inner materials.



(a) OC

(b) RA

(c) PB

FIG. 2. (Color online) Interior view of the three well-preserved historical case studies. Courtesy of Reinhard Görner (Fig. 2(a)).

115 *a. Odeo Cornaro (OC).* The first case study is a well-preserved Renaissance music
116 space in Padua (Italy). The Odeo Cornaro (OC) is an outstanding Venetian architecture of
117 the 16th Century designed by the architect Falconetto for the nobleman Alvise Cornaro (Zara,

TABLE I. Details of the nearly cylindrical halls under study. The total volume (V), the radius of the equivalent circular plan (a), the mean height (H), the mean reverberation time value at 500 - 1000 Hz ($T_{30,M}$), the mean absorption coefficient at 500 - 1000 Hz (α_M), the mean sound strength at 500 - 1000 Hz (G_M), the Schroeder frequency (f_c), and the number of sound sources (N_S) and receivers (N_R) locations during the measurements are provided for each hall.

Hall ID	V	a	H	$T_{30,M}$	α_M	G_M	f_c	N_S	N_R
	(m ³)	(m)	(m)	(s)		(dB)	(Hz)		
OC	220	3.3	5.5	2.81	0.05	23	226	2	9
RA	715	4.7	12.2	2.78	0.08	11	125	4	4
PB	23,000	15.6	40.0	13.01	0.03	16	40	3	16

118 [2021](#)). According to the writers of that time (1537-1542), the space was intended to be the
119 music hall of Cornaro’s Renaissance mansion. The frequent occurrence of convivial moments
120 with instruments and a choir within the room is explicitly mentioned in the historical report
121 ([Moretti, 2010](#)). Since the volume of the hall is moderate ($V = 220 \text{ m}^3$), the OC hall was
122 probably reserved for small groups of erudite people only. Historical evidence also states
123 that the hall seemed to significantly support the human voice as the Vitruvius’ category of
124 *loci resonantes*. A previous study by the authors concerned the acoustic coupling between
125 the main hall and the surrounding adjacent rooms, along with an insight into the acoustic
126 role of the historical connection doors ([Fratoni et al., 2022a](#)).

127 *b. Rotunda Aldini (RA).* The second case study is a 12th Century rotunda located in
128 Bologna (Italy). The Rotunda Aldini (RA) was originally built as a central-plan worship
129 space included in a convent complex and it was used as an oratory. Between 1796 and 1802,
130 after its deconsecration, RA hall was incorporated within the lawyer Antonio Aldini’s 19th
131 Century villa. The rotunda was preserved and exploited as a music room, located in a larger
132 project that was intended to make the Villa Aldini a place dedicated to arts and culture.
133 In a previous work, the authors acquired a 3D virtual model through a laser scanner and
134 then investigated the acoustic role of the niches by means of finite-difference time-domain
135 methods (Fratoni *et al.*, 2021).

136 *c. Pisa’s Baptistery (PB).* The third case study is St John’s Baptistery in Pisa (Italy).
137 The Baptistery (PB) is an imposing architecture with a cylindrical shape and a conical dome.
138 The ground floor is split into two distinct areas by a circular columns’ array: the core of
139 the Baptistery, i.e., the baptismal font, the altar, the pulpit, and the external ambulatory.
140 The upper floor hosts the *matroneum*, a gallery intended to accommodate women, as it is
141 common in ancient worship spaces. The authors previously deepened the archaeoacoustic
142 study of the architecture, with a special focus on its liturgical use (D’Orazio *et al.*, 2020).

143 **C. Acoustic measurements**

144 Between 2017 and 2022, the authors carried out several acoustic surveys to obtain the
145 experimental room acoustics criteria in each case study. The most significant acoustic indica-
146 tors have been collected in compliance with ISO 3382 (ISO, a). During the measurements,
147 each room was furnished and unoccupied, except for the two operators necessary for the

148 acoustic survey. Impulse responses (IRs) have been acquired using a subwoofer, a high-
 149 SPL dodecahedron as an omnidirectional sound source (D’Orazio *et al.*, 2016), a half-inch
 150 free-field microphone as a monoaural receiver, a MOTU soundcard, a laptop, and the com-
 151 mercial software Dirac 6.0. Both the sound sources were previously calibrated in a certified
 152 reverberation room according to ISO 3741 (ISO, b).

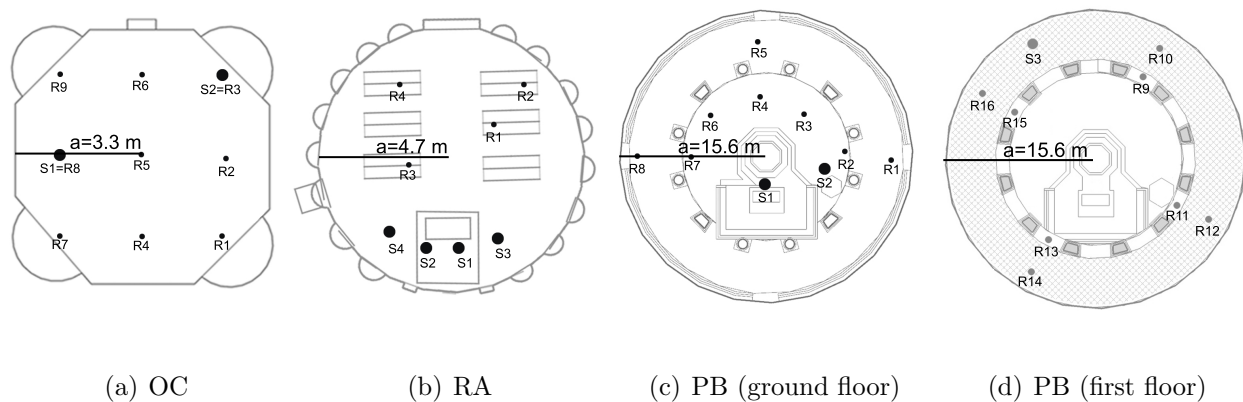


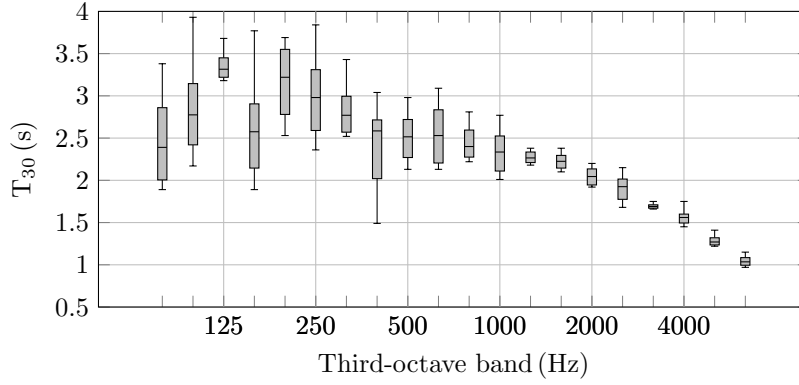
FIG. 3. Floor plans indicating the position of sources (S) and receivers (R) in the measurements.

The radius of each nearly cylindrical hall is provided (a).

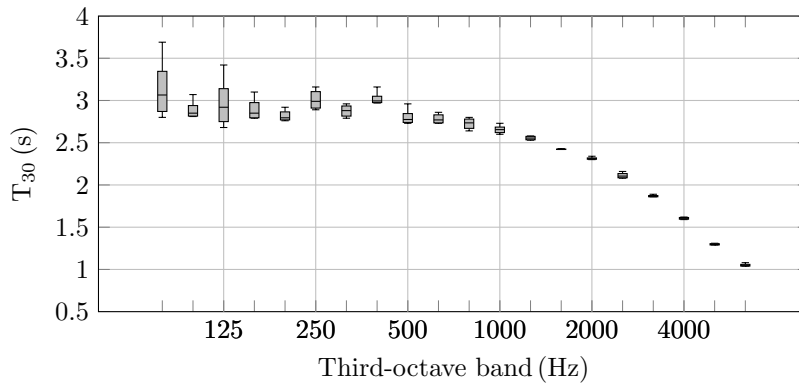
153 In OC hall, two points were selected for the location of the sound source and a regular
 154 grid of nine points was used for the receivers’ location (see Fig. 3(a)). The only pieces
 155 of furniture were four small benches inside the niches. In RA hall, the sound source was
 156 located at four positions behind the altar, corresponding to the places where the singers
 157 were supposed to perform in such an oratory/music space. Four locations were selected for
 158 the receiver points among the wooden pews present in the rotunda during the measurements
 159 (see Fig. 3(b)). In PB hall, the choice of the sound sources and the receivers’ location was
 160 determined by the spatial distribution of the volumes within the Baptistery. As it is shown
 161 in Fig. 3(c), the first two sound source positions on the ground floor - on the altar and on

162 the pulpit - are in line with the liturgical use (Martellotta *et al.*, 2009; Soeta *et al.*, 2012).
163 A third sound source was placed in the *matroneum* to understand the effect of this area on
164 the whole sound field behaviour (see Fig. 3(d)). Three monoaural receivers were placed in
165 the ambulatory, five around the baptismal font and eight in the *matroneum*.

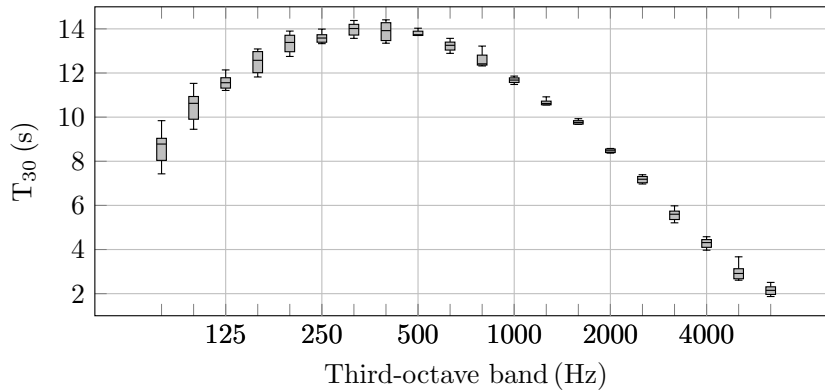
166 Table I provides the measured reverberation time and sound strength values, along with
167 the derived mean value of the absorption coefficient and the Schroeder frequency for each
168 hall. In this case, data have been averaged between the octave bands centred at 500 Hz and
169 1000 Hz. Instead, Figure 4 provides the measured T_{30} values in third-octave bands to show
170 its trend at low and mid frequencies (from 80 Hz to 6300 Hz). The box-and-whisker diagram
171 describes the spread of experimental data through a five-number summary: the minimum,
172 the lower quartile, the median, the upper quartile, and the maximum. Where the T_{30} values
173 are comparable, i.e., for OC and RA halls, the same y-axis range has been kept for easier
174 comparison between the halls. Plots show that the spread of T_{30} values is considerably
175 smaller moving forward to higher frequencies. From 1250 Hz onwards the spread turns out
176 to be lower than 10%, 6%, and 5%, respectively, in OC, RA, and PB hall. Moreover, various
177 considerations can be pointed out while comparing the T_{30} spreads of the three case studies.
178 For instance, even though at mid frequencies the experimental reverberation time values of
179 the first two case studies are almost the same (see Table I), OC hall shows a higher mean
180 spread of experimental data (16%) compared to RA hall (4%). This occurs because of the
181 moderate presence of irregular reflections in OC hall (smooth marble, lack of furniture, few
182 niches) compared to RA hall (brick walls, wooden benches, several niches), as can be seen in
183 Fig. 2. In fact, the scattering properties of the surfaces and the edge diffraction contribute



(a) OC



(b) RA



(c) PB

FIG. 4. Measured values of reverberation time (T_{30}) provided in third octave bands in each hall.

The minimum, the lower quartile, the median, the upper quartile, and the maximum values are obtained by considering the experimental results of all the source-receiver pairs employed during the measurements.

184 to increase the sound diffusion and to decrease the spread of experimental data. Moreover,
185 not only OC is the smallest hall among the case studies –and therefore the most affected by
186 modal behaviour–, but also the four rounded corners of the hall plausibly cause focussing
187 effects at the receivers and make the overall shape more similar to a square (see Fig. 3). For
188 this reason, from a global point of view, OC hall shows the highest spread (up to 35% at low
189 frequencies) among the case studies, RA hall shows a moderate spread (up to 17%), and PB
190 hall shows the lowest spread (7% as maximum). This is in line with the expectations, since
191 PB hall has a considerably greater volume compared to OC and RA halls and hosts several
192 columns, altars, and decorations increasing the sound field diffusion (Weber and Katz, 2022).

193 In the present work, the on-site measurements have been employed not only for derivations
194 of ISO 3382-1 room criteria, but also for calibrating the 3D virtual models of the halls.

195 D. Numerical models

196 *a. Wave-based models.* Recently, wave-based methods have been increasingly used for
197 3D room acoustics modelling (Fratoni *et al.*, 2022b; Pind *et al.*, 2019; Wang *et al.*, 2019).
198 For the present work, a finite-element (FE) approach has been chosen (Okuzono *et al.*, 2021;
199 Prinn, 2023). As mentioned in Section II A, determining the natural resonant frequencies in
200 real-world geometries is extremely challenging due to the underlying analytical difficulties.
201 COMSOL Multiphysics allowed exploring the modal field at low frequencies in each hall
202 (Maluski and Bougdah, 1997; Tomiku *et al.*, 2008). In particular, the modal density trend
203 has been investigated. The 3D models of OC, RA, and PB halls were built from the scratch
204 in the *Geometry* section of the COMSOL main *Component*, and the resulting models are

205 shown in Fig. 5. A single air domain was defined for each geometry by employing the linear

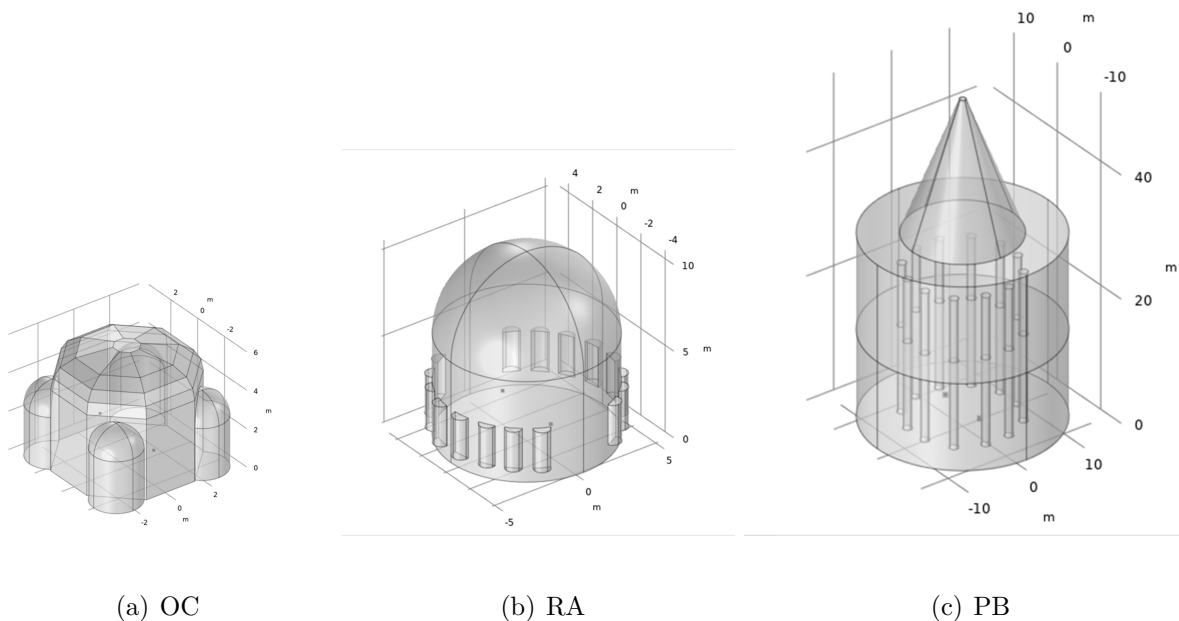


FIG. 5. View of the 3D finite-element models of the halls under study (COMSOL multiphysics).

206

207

208 elastic model. The atmosphere attenuation can be neglected even in PB hall as long as the
209 analysis is limited to low frequencies. The *Sound Hard Boundary Wall* conditions set on all
210 the surfaces involved are in line with the material features inside the halls: stone, masonry,
211 and marble represent hard and rigid boundary conditions. The mesh of the geometry has
212 been set according to the rule of thumb of 6 elements for the minimum wavelength of interest
213 (Kirkup, 2019). The *Eigenfrequency* study yielded a list of the natural resonance frequencies
214 of each geometry, allowing the calculation of the modal densities (discrete values) for each
215 frequency. The simulation has been run from 80 Hz to around twice the value of each
216 Schroeder frequency to focus on low-frequency behaviour, where natural modes are more
217 detectable and less overlapping.

218 *b. Geometrical Acoustics (GA).* Geometrical acoustics (GA) techniques (Odeon Room
 219 Acoustics) allowed investigating the free path distribution in the circular places under study
 220 (Hidaka and Nishihara, 2006; Naylor, 1993). The 3D virtual models used in previous works
 221 by the authors have been exploited for this purpose (D’Orazio *et al.*, 2020; Fratoni *et al.*,
 222 2022a, 2021). During their creation process with Sketchup software, the state-of-art guide-
 223 lines of 3D modelling have been followed, both in terms of simplification of the actual ge-
 224 ometries and the reduction of the details modelled, as it can be seen in Figure 6 (Vorländer,
 225 2020). While a FE calibration at low frequencies would yield various uncertainties due to

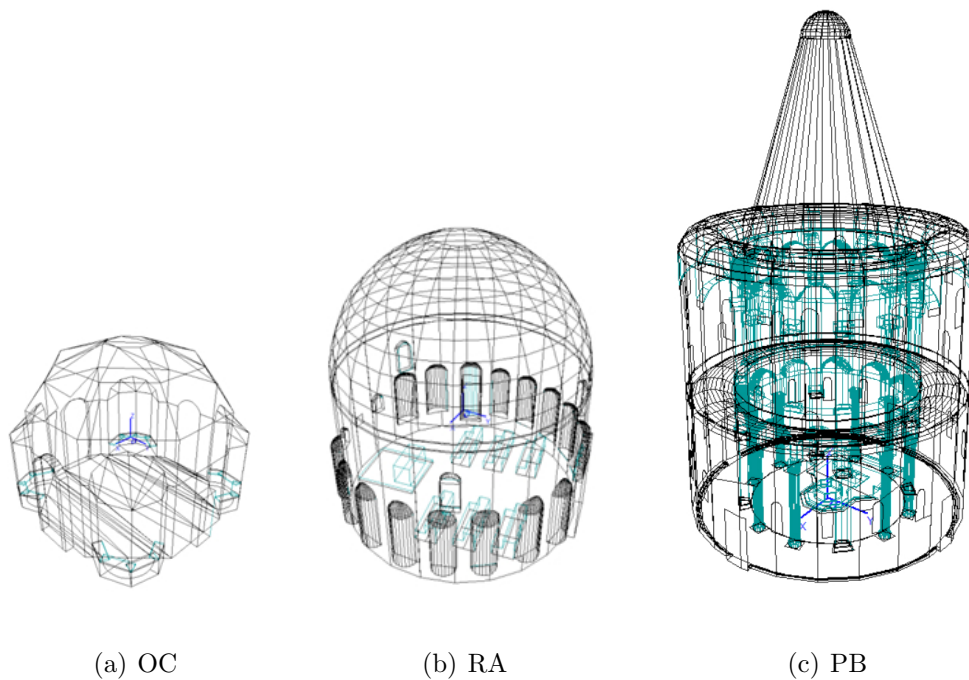


FIG. 6. View of the GA models under study (Odeon Room Acoustics).

226

227

228 the significant differences between the actual directivity of the dodecahedron and the ideal
 229 omnidirectional sound source employed in COMSOL, a complete GA calibration has been
 230 achieved according to the state-of-the-art (Pilch, 2020; Postma and Katz, 2016). Table II

TABLE II. Summary of GA calibration: measured and simulated $T_{30,M}$, EDT_M , and $C_{80,M}$ are provided, along with the corresponding differences. Room criteria have been averaged over the 500 Hz and 1000 Hz octave bands.

	$T_{30,M}$ (s)			EDT_M (s)			$C_{80,M}$ (dB)		
	Meas.	Sim.	Diff. (%)	Meas.	Sim.	Diff. (%)	Meas.	Sim.	Diff. (dB)
OC	2.74	2.62	4.4%	2.50	2.53	1.2%	-1.7	-1.9	0.2
RA	2.78	2.70	2.7%	2.78	2.71	2.5%	-2.6	-2.2	0.4
PB	12.87	13.25	2.9%	12.61	12.94	2.6%	-12.3	-12.0	0.3

231 provides a summary of GA calibration by comparing the measured and simulated $T_{30,M}$,
 232 EDT_M , and $C_{80,M}$ values at mid frequencies (500 - 1000 Hz). Those data are provided con-
 233 sidering the sound source in S1 (see Fig. 3) and the mean room criteria over all the receiver
 234 points.

235 The ray-tracing tool enabled to determine both the theoretical mean free path according
 236 to the classical kinetic theory (\bar{l}) and the mean free path evaluated by the simulations
 237 (\bar{l}_{GA}) (Prodi and Martellotta, 2014). In the former case the theoretical $\bar{l} = 4V/S$ has been
 238 calculated considering the total volume V and the total active surface area S . In the latter
 239 case the mean free path (\bar{l}_{GA}) has been obtained by the emission of 200,000 rays for each
 240 sound source location, employing the scattering coefficients assigned to the surfaces of the
 241 calibrated models.

242 III. RESULTS AND DISCUSSIONS

243 A. Eigenfrequencies distribution

244 The present section reports the simulated modal density distribution obtained through
245 FE simulations in OC, RA, and PB halls. Figure 7 provides the modal densities (discrete
246 values) evaluated by COMSOL with the procedure mentioned in Section IID. The frequency
247 range involves at least $2f_c$ for each case study to include the modal field behaviours (see
248 Table I), and the frequency axis has a linear scale according to the literature (Le Bot, 2015).
249 The Schroeder frequency is also shown with black dashed lines in each case study ($f_{c,OC}$,
250 $f_{c,RA}$, $f_{c,PB}$).

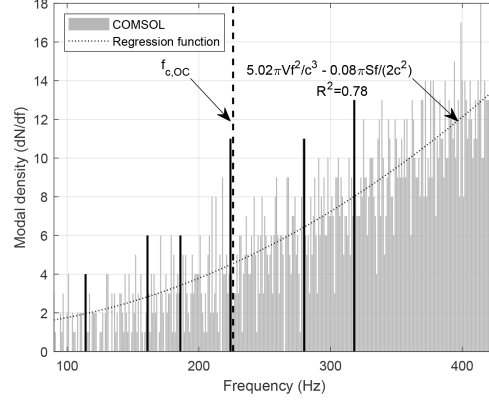
251 The first analysis concerns the general trend of the graphs. The regression functions of the
252 modal density bars have been derived with a second-degree polynomial function ($dN/df =$
253 $Af^2 + Bf$) through the Curve Fitting Tool by MATLAB to compare the modal density trend
254 obtained through COMSOL and the consolidated assumption (Eq. 4). The consequent
255 regression functions with the corresponding goodness of the fit have been obtained:

256 - $dN/df = 5.02\pi V f^2/c^3 - 0.08\pi S f/(2c^2)$ with $R^2 = 0.78$ in OC hall,

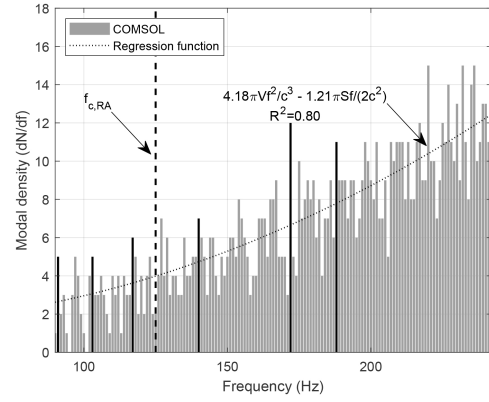
257 - $dN/df = 4.18\pi V f^2/c^3 - 1.21\pi S f/(2c^2)$ with $R^2 = 0.80$ in RA hall,

258 - $dN/df = 3.13\pi V f^2/c^3 + 3.29\pi S f/(2c^2)$ with $R^2 = 0.97$ in PB hall.

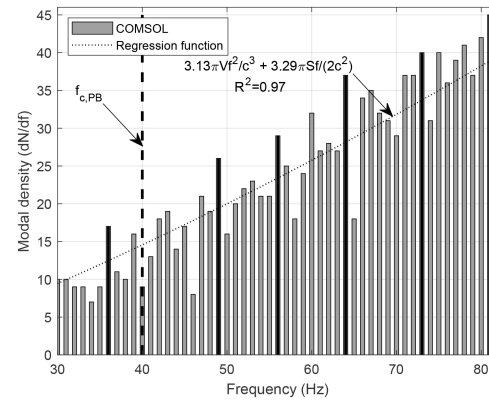
259 Those functions are plotted as dotted curves in the three panes. It is important to
260 highlight that any attempt with linear regression returned lower R^2 values in each model.
261 The difference between the f^2 multiplication factor (A coefficient) and the term $4\pi V/c^3$ in



(a) OC



(b) RA



(c) PB

FIG. 7. Simulated eigenfrequencies distribution obtained through COMSOL (frequency bins width 1 Hz). The Schroeder frequency f_c and the regression curve are also shown. The main triplets (m, n, k) corresponding to the peaks of the modal density highlighted in black are provided in Table

Eq. 4 is equal to 26% in OC hall, 5% in RA hall, and 22% in PB hall. The difference between the f multiplication factor (B coefficient) and the term $\pi S/(2c^2)$ in Eq. 4 is equal to 92% in OC hall, 21% in RA hall, and 229% in PB hall. The most significant percentage differences could be due to erroneous estimations of the halls' volume, especially in the case of PB hall, which is the largest and the most complex hall under study. The need to simplify the geometries in COMSOL (see Fig. 5) implies unavoidable discrepancies between the actual volumes and the volumes employed during the eigenfrequencies computation. On the other hand, the COMSOL model of the simplest shape, i.e. the RA hall, provides the lowest differences between theoretical and simulated eigenfrequencies distributions. However, apart from the uncertainty related to the volume estimation, the outcomes suggest that the trend of the actual modal density might diverge from the A and B coefficients of Eq. 4 at low frequencies and in central-plan halls.

The second analysis concerns the peaks in eigenfrequencies distributions highlighted in Figure 7. With regard to the theory reported in Section II A, the goal here is to derive the triplets (m, n, k) identifying the natural modes causing modal density peaks at specific frequencies. The procedure of the present study involved the following steps.

- 1 Since the sound sources (singers, musicians) and the receivers (audience) are placed along a horizontal plane, the attention is focused on the distribution of the modes throughout the circular plans of the halls rather than their elevation. Therefore, in the retrieving calculation of (m, n, k) , k values were assumed equal to zero. This assumption allows specifically investigating the prominence of the diametral (m) or the circular (n) modes causing higher modal densities.

TABLE III. List of the first four triplets (m, n, k) identifying the natural modes corresponding to the modal density peaks highlighted in black in Figure 7. The tolerance range between β_{mn} values obtained through Eq. 6 and the tabulated β_{mn} values is below 0.04.

OC	RA	PB
$f = 114$ Hz	$f = 91$ Hz	$f = 36$ Hz
(0, 2, 0) (2, 2, 0) (5, 1, 0) (6, 1, 0)	(0, 2, 0) (1, 3, 0) (3, 2, 0) (6, 1, 0)	(0, 3, 0) (2, 3, 0) (5, 2, 0) (3, 3, 0)
$f = 161$ Hz	$f = 103$ Hz	$f = 49$ Hz
(0, 3, 0) (4, 2, 0) (2, 3, 0) (5, 2, 0)	(1, 3, 0) (2, 3, 0) (4, 2, 0) (7, 1, 0)	(0, 4, 0) (1, 5, 0) (2, 4, 0) (3, 4, 0)
$f = 186$ Hz	$f = 117$ Hz	$f = 56$ Hz
(1, 4, 0) (3, 3, 0) (6, 2, 0) (9, 1, 0)	(0, 3, 0) (2, 3, 0) (4, 2, 0) (5, 2, 0)	(0, 5, 0) (2, 5, 0) (4, 4, 0) (6, 3, 0)
$f = 224$ Hz	$f = 140$ Hz	$f = 64$ Hz
(0, 4, 0) (2, 4, 0) (4, 3, 0) (7, 2, 0)	(1, 4, 0) (3, 3, 0) (4, 3, 0) (6, 2, 0)	(1, 6, 0) (3, 5, 0) (5, 4, 0) (6, 4, 0)
$f = 280$ Hz	$f = 172$ Hz	$f = 73$ Hz
(0, 5, 0) (2, 5, 0) (5, 4, 0) (7, 3, 0)	(1, 5, 0) (3, 4, 0) (6, 3, 0) (8, 2, 0)	(1, 7, 0) (3, 6, 0) (5, 5, 0) (8, 4, 0)
$f = 318$ Hz	$f = 180$ Hz	$f = 81$ Hz
(0, 6, 0) (4, 5, 0) (2, 6, 0) (6, 4, 0)	(0, 5, 0) (2, 5, 0) (4, 4, 0) (7, 3, 0)	(0, 7, 0) (2, 7, 0) (5, 6, 0) (7, 5, 0)

284 2 The inverse function of Eq. 1 has been used to obtain the triplets $(m, n, 0)$ correspond-
 285 ing to the n -th roots of the Bessel function derivatives (J'_m) of order m . Accounting
 286 for the assumption aforementioned ($k = 0$), Eq. 1 becomes:

$$f(m, n, 0) = \frac{c\beta_{mn}}{2a} \quad (5)$$

287 where $\beta_{mn}\pi$ are the roots of the Bessel function derivative of the first kind of order
 288 m (listed through MATLAB in this work). For each frequency interested by modal
 289 density peaks (f_{peak}), the corresponding $(m, n, 0)$ has been found as follows:

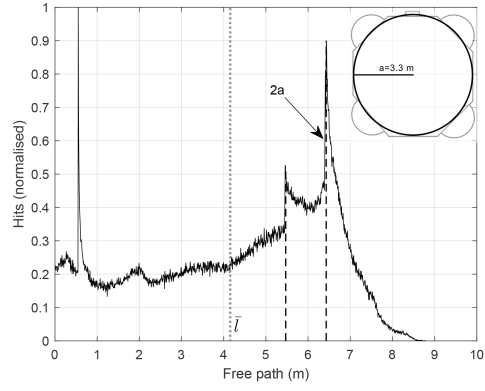
$$\beta_{mn} = \frac{2a}{c} f_{peak} \Rightarrow (m, n, 0). \quad (6)$$

290 The tolerance range between the values obtained through Eq. 6 and the tabulated
 291 values has been kept below 0.04.

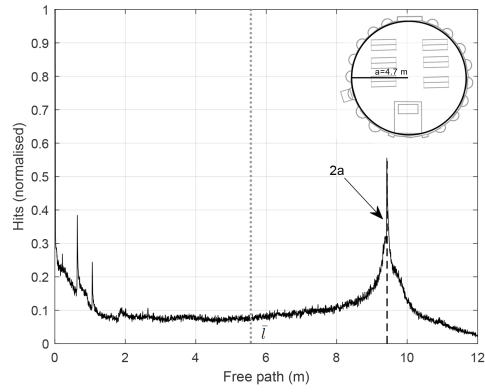
292 Table III provides the first four triplets $(m, n, 0)$ corresponding to each f_{peak} assessed. It is
 293 possible to notice that in the first pair of triplets, n index assumes higher values than m for
 294 most of the cases (92% of the pairs). This may suggest a stronger prominence of circular
 295 modes compared to the diametral ones.

296 B. Free path distribution

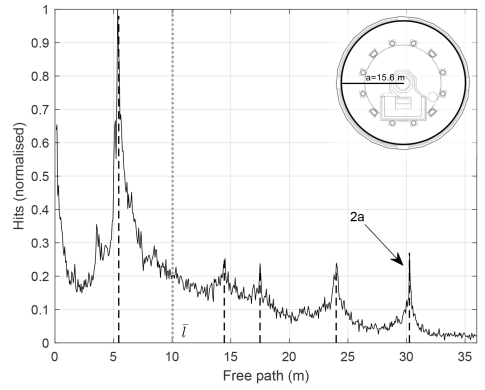
297 The present section reports the free path distribution resulting from Odeon in each case
 298 study, using the scattering coefficients assigned to the surfaces in the materials list. Figure 8
 299 provides the results in terms of the normalised frequency of surface hits versus the length of
 300 free paths in meters, derived from the study of 200,000 rays. The y-axis has been normalised



(a) OC



(b) RA



(c) PB

FIG. 8. Normalised frequency of surface hits versus the distance of free paths in meters in OC, RA, and PB hall obtained with 200,000 rays (Odeon). The prominent paths are highlighted in each geometry with dashed lines while $\bar{l} = 4V/S$ is plotted as a gray dotted line.

301 by dividing each number of hits by the maximum value, obtaining sets of values ranging from
302 0 and 1. The normalised distributions show that the highest probabilities correspond to the
303 shortest path lengths in the proximity of zero, in line with previous studies (Beranek and
304 Nishihara, 2014; Hidaka and Nishihara, 2006; Kuttruff, 2016; Šumarac-Pavlović and Mijić,
305 2007). In OC hall, the highly recurrent 0.8 m path is due to the curved surfaces discretisation,
306 as 0.8 m corresponds to the size of the segments composing each niche. A different issue is
307 the almost zero length paths in RA hall (0.01 m) and PB hall (0.10 m) because there are no
308 details of that dimension in the 3D models. A possible explanation is the accumulation of
309 short path reflections in the proximity of all the corners of the 3D models. Moreover, it is
310 plausible that discrete approaches assume a lower limit for the distance resolution to avoid
311 the distribution from diverging in the vicinity of zero (Krämer, 1997). Apart from such
312 recurrent short path lengths, the main prominent paths are highlighted in each geometry
313 with dashed lines while $\bar{l} = 4V/S$ is plotted as a gray dotted line. It is possible to notice
314 the prominence of the diameter ($2a$) of the halls in each case (6.6 m in OC hall, 9.4 m in
315 RA hall, and 30.2 m in PB hall) confirming the strong influence of the circular plan on the
316 preferred paths. However, further considerations are required for the single cases.

317 1 In OC hall, with the exception of the first peak at 0.8 m, the second peak at 5.5 m
318 is clearly visible in the distribution. This corresponds to the dimension of the height
319 of the room. However, in that case, the normalised number of hits is almost halved
320 compared to the diameter of the hall (6.6 m), confirming again the stronger influence
321 on the sound propagation of the circular shape compared to the elevation size.

322 2 In RA hall, the diameter size (9.4 m) represents the only neat peak in the whole
323 distribution of the hits occurrences. This can be due to the spherical shape of the
324 dome which significantly reduces the probability of further prominent paths.

325 3 In PB hall, the highest peak corresponds to the ambulatory width (5.4 m), highlighting
326 a prominence of the annular space between the sidewalls and the columns for the
327 preferred paths. Right after 5.4 m, the diameter (30.2 m) is the distance with the
328 highest number of hits, followed by 24 m, which is the height of the conical dome,
329 and 14 m, which is the height of each floor. Thus, PB exhibits multimodal effects not
330 shown by the other two halls.

331 Furthermore, insights into the mean free path in circular halls are here reported. In room
332 acoustics, calculating the predicted reverberation time requires the formula for the mean free
333 path, i.e., the average distance between two successive impacts of sound “rays” on the walls
334 (Kingman, 1965; Kosten, 1960). Generally, the formula used in the kinetic theory of gases is
335 employed: $\bar{l} = 4V/S$, where V is the volume of the room and S is the total internal surface
336 area of the room (Jaeger, 1911). On diffuse field assumption, this formula is supposed to
337 be independent of the shape of the hall under study. However, in the late 1920s and the
338 first years of the 1930s, approximated formulas were developed for cubic, cylindrical, and
339 spherical shapes (Schuster and Waetzmann, 1929), as follows:

340 - $\bar{l}_{\text{cub}} = 2\sqrt{3}V/S$ for the cubic shape;

341 - $\bar{l}_{\text{cyl}} = 3\sqrt{2}V/S$ for the cylindrical shape (corresponding to 4.3 m for OC hall, 5.9 m
342 for RA hall, 10.6 m for PB hall);

TABLE IV. Comparison between the mean free paths calculated according to the classical kinetic theory ($\bar{l} = 4V/S$) (Bate and Pillow, 1947; Jaeger, 1911), depending on the cylindrical shape ($\bar{l}_{\text{cyl}} = 3\sqrt{2}V/S$) (Schuster and Waetzmann, 1929), and obtained with geometrical acoustics simulations (Naylor, 1993). The mean free path and the relative variance are provided for three simulated scenarios: “Calibrated models” ($\bar{l}_{\text{GA}}, \gamma_{\text{GA}}^2$) where each layer has his own set of adequate scattering coefficients, “Scattering 1” where all the surfaces have $s = 1$ ($\bar{l}_{\text{GA},1}, \gamma_{\text{GA},1}^2$), “Scattering 0” where all the surfaces have $s = 0$ ($\bar{l}_{\text{GA},0}, \gamma_{\text{GA},0}^2$) (Kuttruff, 2016).

			Calibrated models		Scattering 1		Scattering 0	
	$\frac{4V}{S}$	$\frac{3\sqrt{2}V}{S}$	\bar{l}_{GA}	γ_{GA}^2	$\bar{l}_{\text{GA},1}$	$\gamma_{\text{GA},1}^2$	$\bar{l}_{\text{GA},0}$	$\gamma_{\text{GA},0}^2$
OC	4.1 m	4.3 m	4.3 m	0.26	3.8 m	0.39	4.2 m	0.27
RA	5.6 m	5.9 m	5.8 m	0.39	5.3 m	0.49	5.8 m	0.39
PB	10.0 m	10.6 m	11.9 m	0.61	10.8 m	0.69	11.6 m	0.61

343 - $\bar{l}_{\text{sph}} = 6V/S$ for the spherical shape.

344 Later, experimental results proved again that a good approximation for usual rooms is
345 $\bar{l} = 4V/S$ regardless of the shape of the rooms (Knudsen, 1932). The same outcomes were
346 achieved by the direct averaging of mean free paths in rectangular, spherical, and cylindrical
347 enclosures (Bate and Pillow, 1947). It is interesting to notice that up that time the scattering
348 influence on the mean free path is not explicitly mentioned in the literature. Then, Joyce
349 demonstrated that $\bar{l} = 4V/S$ is valid in case of sound field diffusion and that the circular
350 shape assists the randomizing effect of any amount of scattering (Joyce, 1975, 1978). Beranek

351 and Nishihara’s outcomes show that in almost all the concert halls $\bar{l} = 4V/S$, except for
 352 the hall with an “unusual shape” (Beranek and Nishihara, 2014). According to Kuttruff’s
 353 findings, the shape of the room and the preferred sound paths affect the way the actual free
 354 path lengths are distributed around their mean (Kuttruff, 2016). With this regard, a useful
 355 indicator is the relative variance of free path lengths, expressed as:

$$\gamma^2 = \frac{\sigma^2}{\bar{l}^2} \quad (7)$$

356 where σ^2 and \bar{l} are, respectively, the variance and the mean value of the free path lengths.
 357 Generally, γ^2 can be calculated directly only for a limited number of geometries, e.g., for a
 358 sphere ($\gamma^2 = 1/8 = 0.125$). Even though most of the shapes return $\gamma^2 \approx 0.4$, specific shapes
 359 require acoustic simulation to determine γ^2 values (Kuttruff, 2016).

360 In the present study, Odeon’s ray-tracing algorithm has been employed also with such
 361 purpose. The outcomes obtained with the method described in Section II D are provided in
 362 Table IV in terms of experienced \bar{l}_{GA} and the relative variance γ_{GA}^2 in each hall, **considering**
 363 **the scattering coefficients used to calibrate the model** (Kuttruff, 2016). Such values are
 364 compared with the mean free paths calculated according to the classical kinetic theory
 365 ($\bar{l} = 4V/S$) (Bate and Pillow, 1947; Jaeger, 1911) and depending on the cylindrical shape
 366 ($\bar{l}_{cyl} = 3\sqrt{2}V/S$) (Schuster and Waetzmann, 1929). From the comparison between the mean
 367 free paths provided by Table IV, it is possible to notice that the ratios \bar{l}_{GA}/\bar{l} assume higher
 368 values (≈ 1.05) than the ratios $\bar{l}_{GA}/\bar{l}_{cyl}$ (≈ 1) (Hidaka and Nishihara, 2006). Therefore, the
 369 values obtained through the ray-tracing method are more similar to the values depending
 370 on the cylindrical shape of the halls rather than the classical kinetic theory (Stephenson,
 371 2012). The lowest relative variance is $\gamma_{GA}^2 = 0.26$ in OC hall; a value of $\gamma_{GA}^2 = 0.39$ has been

372 found in RA hall; and the highest value experienced in this work is $\gamma_{\text{GA}}^2 = 0.60$ in PB hall,
 373 in accordance with numerical experiments (Kuttruff, 2016).

374 Moreover, the mean free path and the relative variance are provided for two further sim-
 375 ulated scenarios: “Scattering 1” with $s = 1$ assigned to all the surfaces ($\bar{l}_{\text{GA},1}$, $\gamma_{\text{GA},1}^2$), and
 376 “Scattering 0” with $s = 0$ assigned to all the surfaces ($\bar{l}_{\text{GA},0}$, $\gamma_{\text{GA},0}^2$). In all the case stud-
 377 ies assessed, the difference between the real-world scenarios corresponding to “Calibrated
 378 models” and the “Scattering 0” scenario is neglectable ($< 2\%$), whilst the “Scattering 1”
 379 scenario yields $\bar{l}_{\text{GA},1}$ values 8%-11% lower than \bar{l}_{GA} , $\bar{l}_{\text{GA},0}$, and $\gamma_{\text{GA},1}^2$ values 10%-30% higher
 380 than γ_{GA}^2 , $\gamma_{\text{GA},0}^2$. Finally, the reverberation time related to the plausible condition of the
 381 calibrated models, i.e. assuming \bar{l}_{GA} as realistic mean free path, is expressed as:

$$T = \frac{-6 \ln 10}{c} \frac{\bar{l}_{\text{GA}}}{\ln(1 - \alpha)} \approx 0.04 \frac{\bar{l}_{\text{GA}}}{\alpha} \quad [\text{s}], \quad (8)$$

382 implying that a longer \bar{l}_{GA} suggests longer reverberation time values in circular halls com-
 383 pared to rectangular halls with the same volume (Hidaka and Nishihara, 2006).

384 IV. CONCLUSIONS

385 The present work investigates the acoustics of circular ancient halls, based on measured
 386 data and numerical models. The experimental results permitted to obtain information about
 387 the amount of diffusing surfaces and the influence of the modal behaviour through the assess-
 388 ment of T_{30} values spread in third-octave bands at mid and low frequencies. Notwithstanding
 389 the similar mean value of reverberation time at mid frequencies in OC and RA halls, the
 390 significantly higher mean spread of T_{30} in OC hall (16%) than in RA hall (4%) suggests

391 uneven room acoustics criteria in the former hall. This can be related to several OC's fea-
392 tures, such as the moderate presence of diffusing surfaces, the small size, and the plausible
393 focussing effects caused by the four rounded corners. Then, from the eigenfrequencies dis-
394 tributions obtained with COMSOL, it has been found a strong relationship between the
395 peaks of modal density and the circular modes rather than the diametral or elevation modes
396 ($n > m$ in $f_{peak}(m, n, 0)$, assuming $k = 0$). Moreover, the analysis of the free path distri-
397 bution through geometrical acoustics confirmed the importance of the circular shape on the
398 horizontal plane, as the diameter size is generally the most recurrent free path. With this
399 regard, the PB hall represents an exception because it has a more composite geometry with
400 also a significant influence of the annular resonance of the ambulatory between the sidewalls
401 and the columns. Finally, similar to what has been found in previous works, the general
402 trend of the ratio \bar{l}_{GA}/\bar{l} is higher than 1 in each case study, suggesting longer reverberation
403 time values compared to halls with other shapes and the same volume. Therefore, the cir-
404 cular environments proved to adequately support the sound signals, as it was mentioned in
405 the historical reports taken as references.

406 **ACKNOWLEDGMENTS**

407 The authors would like to thank Anna Rovigatti, Domenico De Salvio, Elena Rossi,
408 Riccardo Russo, Michele Ducceschi, and Virginia Tardini for their kind support during the
409 acoustic measurements.

410 **AUTHOR DECLARATIONS**

411 **Conflict of Interest**

412 The authors have no conflicts to disclose.

413 **Ethics Approval**

414 No human subjects or animals were involved in the research.

415 **DATA AVAILABILITY**

416 The data that support the findings of this study are available from the corresponding
417 author.

418

419 (a). *ISO 3382-1:2009 Acoustics — Measurement of room acoustic parameters.*

420 (b). *ISO 3741:2010 Acoustics — Determination of sound power levels and sound energy*
421 *levels of noise sources using sound pressure — Precision methods for reverberation test*
422 *rooms.*

423 Alberdi, E., Martellotta, F., Galindo, M., and León, Á. L. (2019). “Dome sound effect in
424 the church of san luis de los franceses,” *Applied Acoustics* **156**, 56–65.

425 Balian, R., and Bloch, C. (1970). “Distribution of eigenfrequencies for the wave equation
426 in a finite domain: I. three-dimensional problem with smooth boundary surface,” *Annals*
427 *of Physics* **60**(2), 401–447.

428 Bate, A. (1938). “Note on the whispering gallery of St Paul’s Cathedral, London,” Pro-
429 ceedings of the Physical Society (1926-1948) **50**(2), 293.

430 Bate, A., and Pillow, M. (1947). “Mean free path of sound in an auditorium,” Proceedings
431 of the Physical Society (1926-1948) **59**(4), 535.

432 Beranek, L. L., and Nishihara, N. (2014). “Mean-free-paths in concert and chamber music
433 halls and the correct method for calibrating dodecahedral sound sources,” The Journal of
434 the Acoustical Society of America **135**(1), 223–230.

435 Blevins, R. (2006). “Modal density of rectangular volumes, areas, and lines,” The Journal
436 of the Acoustical Society of America **119**(2), 788–791.

437 Bolt, R., Doak, P., and Westervelt, P. (1950). “Pulse statistics analysis of room acoustics,”
438 The Journal of the Acoustical Society of America **22**(3), 328–340.

439 Cremer, L., and Müller, H. A. (1978). *Die wissenschaftlichen Grundlagen der Raumakustik*
440 (Hirzel Stuttgart).

441 D. D’Orazio, A. Rovigatti, M. G. (2019). “The proscenium of opera houses as a disappeared
442 intangible heritage: a virtual reconstruction of the 1840s original design of the Alighieri
443 Theatre in Ravenna,” in *Acoustics*, Multidisciplinary Digital Publishing Institute, Vol. 1,
444 pp. 694–710.

445 Dietsch, L., and Kraak, W. (1986). “Ein objektives Kriterium zur Erfassung von
446 Echostörungen bei Musik-und Sprachdarbietungen,” *Acta Acustica United with Acustica*
447 **60**(3), 205–216.

448 D’Orazio, D., De Cesaris, S., Guidorzi, P., Barbaresi, L., Garai, M., and Magalotti, R.
449 (2016). “Room acoustic measurements using a high SPL dodecahedron,” in *Audio Engi-*

450 *neering Society Convention 140*, Audio Engineering Society.

451 D’Orazio, D., Fratoni, G., Rossi, E., and Garai, M. (2020). “Understanding the acoustics of
452 St. John’s Baptistery in pisa through a virtual approach,” *Journal of Building Performance*
453 *Simulation* **13**(3), 320–333.

454 Eriksson, L. J. (1980). “Higher order mode effects in circular ducts and expansion cham-
455 bers,” *The Journal of the Acoustical Society of America* **68**(2), 545–550.

456 Fausti, P., Pompoli, R., and Prodi, N. (2003). “Comparing the acoustics of mosques and
457 byzantine churches,” in *19th International Symposium CIPA*, Citeseer.

458 Fratoni, G., D’Orazio, D., Ducceschi, M., and Garai, M. (2022a). “The coupled rooms of
459 Odeo Cornaro (1534) as support for Renaissance musicians and soloists,” in *Proceedings*
460 *of the 24th International Congress on Acoustics*.

461 Fratoni, G., Hamilton, B., and D’Orazio, D. (2022b). “Feasibility of a finite-difference time-
462 domain model in large-scale acoustic simulations,” *The Journal of the Acoustical Society*
463 *of America* **152**(1), 330–341.

464 Fratoni, G., Hamilton, B., and D’Orazio, D. (2021). “Rediscovering the acoustics of a XII-
465 Century Rotunda through FDTD simulation,” in *2021 Immersive and 3D Audio: from*
466 *Architecture to Automotive (I3DA)*, IEEE, pp. 1–8.

467 Haas, H. (1951). “Über den Einfluß eines Einfachechos auf die Hörsamkeit von Sprache,”
468 *Acta Acustica united with Acustica* **1**(2), 49–58.

469 Hidaka, T., and Nishihara, N. (2006). “Reverberation time, mean-free-path, and sound
470 absorption in concert halls—numerical examination by computer simulation,” *The Journal*
471 *of the Acoustical Society of America* **119**(5_Supplement), 3430–3430.

472 Jaeger, G. (1911). “Zur Theorie des Nachhalls.,” Sitzungsebr. Wiener Akad. **Kl. 120**(IIa),
473 613–634.

474 Joyce, W. (1975). “Sabine’s reverberation time and ergodic auditoriums,” The Journal of
475 the Acoustical Society of America **58**(3), 643–655.

476 Joyce, W. (1978). “Exact effect of surface roughness on the reverberation time of a uniformly
477 absorbing spherical enclosure,” The Journal of the Acoustical Society of America **64**(5),
478 1429–1436.

479 Kingman, J. (1965). “Mean free paths in a convex reflecting region,” Journal of Applied
480 Probability **2**(1), 162–168.

481 Kinzer, J., and Wilson, I. (1947). “Some results on cylindrical cavity resonators,” Bell
482 System Technical Journal **26**(3), 410–445.

483 Kircher, A. (1673). *Phonurgia nova sive conjugium mechanicophysicum artis et naturae*
484 *paranympa phonosophia concinnatum (etc.)* (Dreherr).

485 Kirkup, S. (2019). “The boundary element method in acoustics: A survey,” Applied Sciences
486 **9**(8), 1642.

487 Knudsen, V. O. (1932). *Architectural acoustics*. (Wiley).

488 Kosała, K., and Małecki, P. (2018). “Index assessment of the acoustics of orthodox churches
489 in poland,” Applied Acoustics **130**, 140–148.

490 Kosten, C. (1960). “The mean free path in room acoustics,” Acta Acustica united with
491 Acustica **10**(4), 245–250.

492 Krämer, P. S. (1997). “Mean free path length for radiating point sources in specular reflect-
493 ing enclosures,” Acta Acustica united with Acustica **83**(4), 629–634.

494 Kuttruff, H. (2016). *Room acoustics* (Crc Press).

495 Le Bot, A. (2015). *Foundation of statistical energy analysis in vibroacoustics* (OUP Oxford).

496 Lommel, E. (1868). *Studien über die Bessel'schen Functionen* (BG Teubner).

497 Maa, D.-Y. (1939). "Distribution of eigentones in a rectangular chamber at low frequency
498 range," *The Journal of the Acoustical Society of America* **10**(3), 235–238.

499 Maa, D.-Y. (1941). "The flutter echoes," *The Journal of the Acoustical Society of America*
500 **13**(2), 170–178.

501 Magrini, A., and Ricciardi, P. (2006). "The acoustic field under the dome in a central plan
502 church: measurement and simulation," *Proc 13th ICSV, Vienna, Austria* 2–6.

503 Makrinenko, L. (1986). "Acoustics of public buildings," Moskva: Stroyizdat. Revision of
504 the literature on building materials and structures **173**.

505 Maluski, S., and Bougdah, H. (1997). "Predicted and measured low frequency response of
506 small rooms," *Building Acoustics* **4**(2), 73–86.

507 Martellotta, F., Cirillo, E., Carbonari, A., and Ricciardi, P. (2009). "Guidelines for acous-
508 tical measurements in churches," *Applied Acoustics* **70**(2), 378–388.

509 Meyer, H. B. (1964). "Acoustics and people," *The Journal of the Acoustical Society of*
510 *America* **36**(10), 1967–1968.

511 Moreno, A., Zaragoza, J. G., and Alcantarilla, F. (1981). "Generation and suppression of
512 flutter echoes in spherical domes," *Journal of the Acoustical Society of Japan (E)* **2**(4),
513 197–202.

514 Moretti, L. (2010). "Quivi si essercitaranno le musiche: La sala della musica presso la corte
515 padovana di Alvise Cornaro," *Music in Art* 135–144.

516 Muncey, R., Nickson, A., and Dubout, P. (1953). “The acceptability of speech and music
517 with a single artificial echo,” *Acta Acustica united with Acustica* **3**(3), 168–173.

518 Naylor, G. M. (1993). “Odeon—another hybrid room acoustical model,” *Applied Acoustics*
519 **38**(2-4), 131–143.

520 Okuzono, T., Yoshida, T., and Sakagami, K. (2021). “Efficiency of room acoustic simula-
521 tions with time-domain fem including frequency-dependent absorbing boundary conditions:
522 Comparison with frequency-domain FEM,” *Applied Acoustics* **182**, 108212.

523 Petzold, E. (1930). “Sounding surfaces,” *The Journal of the Acoustical Society of America*
524 **2**(2), 305–308.

525 Pilch, A. (2020). “Optimization-based method for the calibration of geometrical acoustic
526 models,” *Applied Acoustics* **170**, 107495.

527 Pind, F., Engsig-Karup, A. P., Jeong, C.-H., Hesthaven, J. S., Mejling, M. S., and Strømman-
528 Andersen, J. (2019). “Time domain room acoustic simulations using the spectral element
529 method,” *The Journal of the Acoustical Society of America* **145**(6), 3299–3310.

530 Postma, B. N., and Katz, B. F. (2016). “Perceptive and objective evaluation of calibrated
531 room acoustic simulation auralizations,” *The Journal of the Acoustical Society of America*
532 **140**(6), 4326–4337.

533 Prinn, A. G. (2023). “A review of finite element methods for room acoustics,” in *Acoustics*,
534 MDPI, Vol. 5, pp. 367–395.

535 Prodi, N., Marsilio, M., and Pompoli, R. (2001). “On the prediction of reverberation time
536 and strength in mosques,” *Proceedings of the 17th ICA, Rome* .

537 Prodi, N., and Martellotta, F. (2014). “On the statistical properties of free path distribu-
538 tion as a means to investigate room acoustics of theatre halls,” in *Proceedings of Forum*
539 *Acusticum 2014*, pp. 8–12.

540 Reichardt, W. (1968). “Der Impuls-Schalltest und seine raumakustische Beurteilung,” Proc.
541 6th Int. Congr. Acoustics, Tokyo, 1968 .

542 Richardson, R. (1912). “Theorems of oscillation for two linear differential equations of the
543 second order with two parameters,” Transactions of the American Mathematical Society
544 **13**(1), 22–34.

545 Sabine, W. (1922). “Whispering galleries,” Collected Papers on Acoustics. Cambridge, MA:
546 Harvard University Press .

547 Schuster, K., and Waetzmann, E. (1929). “Über den Nachhall in geschlossenen Räumen,”
548 *Annalen der Physik* **393**(5), 671–695.

549 Shepherd, M., Leishman, T. W., and Utami, S. (2005). “Acoustics of a planetarium,” The
550 *Journal of the Acoustical Society of America* **118**(3), 1999–1999.

551 Soeta, Y., Ito, K., Shimokura, R., Sato, S.-i., Ohsawa, T., and Ando, Y. (2012). “Effects
552 of sound source location and direction on acoustic parameters in Japanese churches,” The
553 *Journal of the Acoustical Society of America* **131**(2), 1206–1220.

554 Stephenson, U. M. (2012). “Different assumptions-different reverberation formulae,” in
555 *INTER-NOISE and NOISE-CON Congress and Conference Proceedings*, Institute of Noise
556 Control Engineering, Vol. 2012, pp. 7646–7657.

557 Sü Gül, Z., Xiang, N., and Çalışkan, M. (2016). “Investigations on sound energy decays
558 and flows in a monumental mosque,” *The Journal of the Acoustical Society of America*

559 **140**(1), 344–355.

560 Šumarac-Pavlović, D., and Mijić, M. (2007). “An insight into the influence of geometrical
561 features of rooms on their acoustic response based on free path length distribution,” *Acta*
562 *Acustica united with Acustica* **93**(6), 1012–1026.

563 Sumarac-Pavlovic, D., Mijic, M., and Kurtovic, H. (2008). “A simple impulse sound source
564 for measurements in room acoustics,” *Applied Acoustics* **69**(4), 378–383.

565 Tomiku, R., Otsuru, T., Okamoto, N., and Kurogi, Y. (2008). “Direct and modal frequency
566 response analysis of sound fields in small rooms by finite element method,” *Journal of the*
567 *Acoustical Society of America* **123**(5), 3092.

568 Tzekakis, E. G. (1975). “Reverberation time of the rotunda of Thessaloniki,” *The Journal*
569 *of the Acoustical Society of America* **57**(5), 1207–1209.

570 Utami, S. S. (2004). “An acoustical analysis of a room with a concave dome ceiling element,”
571 *The Journal of the Acoustical Society of America* **115**(5), 2581–2581.

572 Vitale, R., Pisani, R., Onali, P., and Astolfi, A. (2005). “Why does the acoustic space of
573 churches exalt Gregorian chant?,” in *Proceedings of the 31th International Computer Music*
574 *Conference (ICMC), Barcelona, Spain, September*, pp. 4–10.

575 Vorländer, M. (2020). *Auralization* (Springer).

576 Walker, R. (1996). “Room modes and low-frequency responses in small enclosures,” in *Audio*
577 *Engineering Society Convention 100*, Audio Engineering Society.

578 Wang, H., Sihar, I., Pagán Muñoz, R., and Hornikx, M. (2019). “Room acoustics modelling
579 in the time-domain with the nodal discontinuous Galerkin method,” *The Journal of the*
580 *Acoustical Society of America* **145**(4), 2650–2663.

581 Weber, A., and Katz, B. F. (2022). “Sound Scattering by Gothic Piers and Columns of
582 the Cathédrale Notre-Dame de Paris,” in *Acoustics*, Multidisciplinary Digital Publishing
583 Institute, Vol. 4, pp. 679–703.

584 Zara, V. (2021). “Music, architecture, proportion and the renaissance way of thinking,”
585 *European Review* 29(2), 226–241.