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CUT ELIMINATION FOR EXTENDED SEQUENT CALCULI

Abstract

We present a syntactical cut-elimination proof for an extended sequent calculus covering the classical modal logics in the K, D, T, $\mathsf{K4}$, $\mathsf{D4}$ and $\mathsf{S4}$ spectrum. We design the systems uniformly since they all share the same set of rules. Different logics are obtained by "tuning" a single parameter, namely a constraint on the applicability of the cut rule and on the (left and right, respectively) rules for $\Box$ and $\Diamond$. Starting points for this research are 2-sequents and indexed-based calculi (sequents and tableaux). By extending and modifying existing proposals, we show how to achieve a syntactical proof of the cut-elimination theorem that is as close as possible to the one for first-order classical logic. In doing this, we implicitly show how small is the proof-theoretical distance between classical logic and the systems under consideration.

Keywords: proof theory, sequent calculus, cut elimination, modal logic, 2-sequents.

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1. Introduction

In the thirties, Gentzen introduced the sequent calculus (and natural deduction) to prove Hilbert's consistency assertion for pure logic and Peano Arithmetic. Gentzen's work marked the beginning of structural proof theory, by sanctioning its role to understand the structure of mathematical proofs and isolate and solve methodological problems in the foundations of mathematics. Proof theory is a wide research area that provides tools, methodologies, and solutions also to computer science and philosophical logic. It still offers interesting open problems, especially if we move away from classical and intuitionistic logic. Proof theory of *modal logic*, in particular, is subtle, since a uniform, technically elegant treatment of modalities ($\Box$, $\Diamond$) is generally difficult.

During the last decades, many modal systems have been introduced. Among these, some of the most interesting ones are the labeled systems [24, 26, 23], which extend ordinary calculi by explicitly mirroring in the deductive apparatus the accessibility relation of Kripke models. While such labeled frameworks provide a smart solution to represent structural properties, a more implicit representation of the semantical structure is sometimes preferable, especially if one wants to reduce the formal *iatus* between classical proof theory and the modal one.

In this regards a number of calculi have been introduced, e.g. [23] [4, 5, 7, 9, 15, 13, 18, 17, 16, 23, 21, 19, 22] (see section 5 for a detailed comparison between our proposal and some related ones).

These systems have been defined by taking into account some basic principles: *analyticity* (e.g., the subformula property), *modularity* (to be able to capture an entire family of logics instead of only one), and, if possible, an explicit *syntactical cut elimination procedure.*

Despite the number of calculi introduced and studied, *syntactical cut elimination* remains a "precious" property—many papers claim its validity but they do not exhibit detailed syntactical proofs (or do not prove it at all).

Cut elimination is often obtained either by semantical methods or by translation from other cut-free systems [14, 3]. We believe, on the contrary, that an explicit cut elimination procedure—in the spirit of Gentzen's original ideas—is still an important asset for modal proof theory.

For this reason, the present paper focuses on a syntactical cut elimination theorem, proposing a *modular* system based on *extended sequents* (in the following, simply *e–sequents*)—which allow for a uniform cut elimination argument for all the modal logics in the spectrum K, D, T, K4, D4, S4.

This paper is the natural companion of our [18], where we study a natural deduction calculus for the same family of modal logics and we prove a normalization theorems by a syntactical argument.

We pursue a strong form of modularity, since *all systems share the same set of rules*. Differently from other proposals, to obtain a specific system we do not add or drop characteristic axioms on a "kernel" calculus: we simply set some constraints on the left rule for $\Box$, on the right rule for $\Diamond$, and on the (eliminable) cut rules.

The main idea behind extended sequents is to equip formulas with a *position*—a sequence of uninterpreted tokens—which adds a *spatial dimension*. Positions allow us to interpret sequents *geometrically*, thus permitting a proof theoretical treatment of modalities as close as possible to that of first-order quantifiers.

Here are the main features of our system:

- there is exactly one left and one right rule for each connective;
- the right rule for $\Box$, and its dual left rule for $\Diamond$, are formulated using constraints on positions, with a strong analogy with the constraints on the eigenvariable of the right $\forall$ rule (and $\exists$ left rule, respectively) of standard first-order calculus;
- no direct formalization of the accessibility relation appears;
- only modal operators can change the spatial positions of formulas;
- all the logics share the same set of rules—different systems can be obtained by "tuning" some constraints on the applicability of the cut rule and on the (left and right, respectively) rules for $\Box$ and $\Diamond$.

The result is a parametric system, which we show proves the same theorems of the standard (Hilbert-style) systems for the same logics.

In Section 2 we present extended sequents (*e–sequents*); Section 3 is devoted to the syntactical proof of cut elimination; in Section 4 we show that e–sequent calculi are equivalent (they prove the same theorems) to the standard systems for the same modal logics. Comparison with related proposals and review of the state of the art are in Section 5.

2. Extended Sequent calculi

In this section we introduce *extended sequents* (briefly: *e–sequents*), an extensions of the 2–Sequents originally introduced in [20, 19] and then developed in [17, 16] (see section 5.1 for more on this approach).

To treat uniformly all the logics in the K, D, T, K4, D4 and S4 spectrum, we introduce *positions*—sequences of uninterpreted *tokens*. We start with basic notations and operations.

DEFINITION 2.1. Given a set X, X^* is the set of ordered finite sequences on X. With $\langle x_1, ..., x_n\rangle$ we denote a finite non-empty sequence such that $x_1, \ldots, x_n \in X$; $\langle\,\rangle$ is the empty sequence.

The (associative) concatenation of sequences $\circ : X^* \times X^* \to X^*$ is defined as

- $\langle x_1, ..., x_n\rangle \circ \langle z_1, ..., z_m\rangle = \langle x_1, ..., x_n, z_1, ..., z_m\rangle$,
- $s \circ \langle\,\rangle = \langle\,\rangle \circ s = s$.

For $s \in X^*$ and $x \in X$, we sometimes write $s \circ x$ for $s \circ \langle x\rangle$; and $x \in s$ as a shorthand for $\exists t, u \in X^*.\ s = t \circ \langle x\rangle \circ u$. On X^* we define the successor relation $s \lhd_X t \Leftrightarrow \exists x \in X.\ t = s \circ \langle x\rangle$. In the following:

- $\lhd^0_X$ denotes the reflexive closure of $\lhd_X$;
- $\sqsubset_X$ denotes the transitive closure of $\lhd_X$;
- $\sqsubseteq_X$ denotes the reflexive and transitive closure of $\lhd_X$.

Given three sequences $s, u, v \in X^*$, the *prefix replacement* $s[u \Rsh v]$ is so defined

$$s[u \Rsh v] = \begin{cases} v \circ t & \text{if } s = u \circ t \\ s & \text{otherwise.} \end{cases}$$

When u and v have the same length, the replacement is called *renaming* of u with v.

2.1. A class of normal modal systems

We introduce a class of systems for normal (i.e. extensions of system K) modal logics.

We first define the propositional modal language $\mathcal{L}$ which contains countably infinite *proposition symbols*, $p_0, p_1, \ldots$; the *propositional connectives* $\vee$, $\wedge$, $\rightarrow$, $\neg$; the *modal operators* $\Box$, $\Diamond$; the parenthesis as *auxiliary symbols*.

DEFINITION 2.2. The set $\mathfrak{mf}$ of propositional *modal formulas* of $\mathcal{L}$ is the least set that contains the propositional symbols and is closed under application of the propositional connectives $\rightarrow, \wedge, \vee$ (binary), $\neg$ (unary), and the modal operators $\Box$, $\Diamond$ (unary).

In the following $\mathcal{T}$ denotes a denumerable set of *tokens*, ranged by meta-variables x, y, z, possibly indexed. Let $\mathcal{T}^*$ be the sequences on $\mathcal{T}$, called *positions*; meta-variables α, β, γ, possibly indexed, range over $\mathcal{T}^*$.

Now, *extended–sequents* are tuples of finite sequences of *position-formulas*, i.e. formulas labeled with positions.

DEFINITION 2.3.

1. A *position-formula* (briefly: *p-formula*) is an expression of the form A^α, where A is a modal formula and $\alpha \in \mathcal{T}^*$; $\mathfrak{pf}$ is the set of position formulas.
2. An *extended sequent* (briefly: *e–sequent*) is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite sequences of p–formulas.

Remark 2.4. An e–sequent is a *linear notation* for the so-called *tree sequents*, or with more modern terminology, *nested sequents*. All this will be clarified in section 5.

Given a sequence Γ of p-formulas, with $\mathfrak{Init}[\Gamma]$ we mean the set $\{\beta : \exists A^\alpha \in \Gamma.\ \beta \sqsubseteq \alpha\}$.

We briefly recall the axiomatic ("Hilbert-style") presentation of normal modal systems. Let Z be a set of formulas. The normal modal logic $\mathfrak{M}[Z]$ is defined as the smallest set X of formulas verifying the following properties:

(i) $Z \subseteq X$

(ii) X contains all instances of the following schemas:

1. $A \rightarrow (B \rightarrow A)$

2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Axiom schema	Logic
$\mathbf{D}\ \Box A \to \Diamond A$	$\mathsf{K} = \mathfrak{M}[\varnothing]$
$\mathbf{T}\ \Box A \to A$	$\mathsf{D} = \mathfrak{M}[\mathbf{D}]$
$\mathbf{4}\ \Box A \to \Box\Box A$	$\mathsf{T} = \mathfrak{M}[\mathbf{T}]$
	$\mathsf{K4} = \mathfrak{M}[\mathbf{4}]$
	$\mathsf{S4} = \mathfrak{M}[\mathbf{T},\ \mathbf{4}]$
	$\mathsf{D4} = \mathfrak{M}[\mathbf{D},\ \mathbf{4}]$

Figure 1. Axioms for systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$ and $\mathsf{S4}$

3. $((\neg B \to \neg A) \to ((\neg B \to A) \to B))$

K. $\Box(A \to B) \to (\Box A \to \Box B)$

MP If $A, A \to B \in X$ then $B \in X$;

NEC If $A \in X$ then $\Box A \in X$.

We write $\vdash_{\mathfrak{M}[Z]} A$ for $A \in \mathfrak{M}[Z]$. If $N_1, .., N_k$ are names of schemas, the sequence $N_1 \dots N_k$ denotes the set $[N_1] \cup ... \cup [N_1]$, where $[N_i] = \{A : A$ is an instance of the schema $N_i\}$. Figure 1 lists the standard axioms for the well-known modal systems K, D, T, $\mathsf{K4}$, $\mathsf{S4}$. We use $\mathbb{M}$ as generic name for one of these systems.

2.2. The sequent calculi $\mathsf{E}_{\mathsf{K}}, \mathsf{E}_{\mathsf{D}}, \mathsf{E}_{\mathsf{T}}, \mathsf{E}_{\mathsf{K4}}, \mathsf{E}_{\mathsf{S4}}, \mathsf{E}_{\mathsf{D}4}$

We introduce a class of e–sequent calculi for the logics K, D, T, $\mathsf{K4}$, and $\mathsf{S4}$. The system is presented only once (Figure 2) for $\mathsf{S4}$: the other calculi are obtained by imposing some constraints on the modal rules and the cut (see Figure 3).

Observe that, as usual in sequent calculi presentations, sequences of formulas (Γ, Δ), or positions (α, β) *may be empty*, except when explicitly forbidden. The constraint on necessitation (rule $\vdash \Box$, and its dual $\Diamond \vdash$) is formulated as a constraint on position occurrences in the context, *analogously to the usual constraint on variable occurrences for $\forall$-introduction* ($\exists$-elimination, respectively).

Systems for other logics are obtained by restricting the application of some rules, exploying positions. In particular, rules $\Box \vdash$ and $\vdash \Diamond$ are constrained for all the systems but E_{S4}. Moreover, for E_{K4} and E_{K} also the cut rule is restricted. Figure 3 lists such constraints.

Axiom and cut

$$A^\alpha \vdash A^\alpha \quad Ax \qquad\qquad \frac{\Gamma_1 \vdash A^\alpha, \Delta_1 \quad \Gamma_2, A^\alpha, \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ Cut$$

Structural rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A^\alpha \vdash \Delta}\ W\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A^\alpha, \Delta}\ \vdash W$$

$$\frac{\Gamma, A^\alpha, A^\alpha \vdash \Delta}{\Gamma, A^\alpha \vdash \Delta}\ C\vdash \qquad \frac{\Gamma \vdash A^\alpha, A^\alpha, \Delta}{\Gamma \vdash A^\alpha, \Delta}\ \vdash C$$

$$\frac{\Gamma_1, A^\alpha, B^\beta, \Gamma_2 \vdash \Delta}{\Gamma_1, B^\beta, A^\alpha, \Gamma_2 \vdash \Delta}\ Exc\vdash \qquad \frac{\Gamma \vdash \Delta_1, A^\alpha, B^\beta, \Delta_2}{\Gamma \vdash \Delta_1, B^\beta, A^\alpha, \Delta_2}\ \vdash Exc$$

Propositional rules

$$\frac{\Gamma \vdash A^\alpha, \Delta}{\Gamma, \neg A^\alpha \vdash \Delta}\ \neg\vdash \qquad \frac{\Gamma, A^\alpha \vdash \Delta}{\Gamma \vdash \neg A^\alpha, \Delta}\ \vdash\neg$$

$$\frac{\Gamma, A^\alpha \vdash \Delta}{\Gamma, A \wedge B^\alpha \vdash \Delta}\ \wedge_1\vdash \qquad \frac{\Gamma, B^\alpha \vdash \Delta}{\Gamma, A \wedge B^\alpha \vdash \Delta}\ \wedge_2\vdash$$

$$\frac{\Gamma_1 \vdash A^\alpha, \Delta_1 \quad \Gamma_2 \vdash B^\alpha, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B^\alpha, \Delta_1, \Delta_2}\ \vdash\wedge \qquad \frac{\Gamma_1, A^\alpha \vdash \Delta_1 \quad \Gamma_2, B^\alpha \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B^\alpha \vdash \Delta_1, \Delta_2}\ \vee\vdash$$

$$\frac{\Gamma \vdash A^\alpha, \Delta}{\Gamma \vdash A \vee B^\alpha, \Delta}\ \vdash\vee_1 \qquad \frac{\Gamma \vdash B^\alpha, \Delta}{\Gamma \vdash A \vee B^\alpha, \Delta}\ \vdash\vee_2$$

$$\frac{\Gamma_1, B^\alpha \vdash \Delta_1 \quad \Gamma_2 \vdash A^\alpha, \Delta_2}{\Gamma_1, \Gamma_2, A \to B^\alpha \vdash \Delta_1, \Delta_2}\ \to\vdash \qquad \frac{\Gamma, A^\alpha \vdash B^\alpha, \Delta}{\Gamma \vdash A \to B^\alpha, \Delta}\ \vdash\to$$

Figure 2. Rules for the System E_{S4}

Modal rules

$$\frac{\Gamma, A^{\alpha\circ\beta} \vdash \Delta}{\Gamma, \Box A^{\alpha} \vdash \Delta}\ \Box\vdash \qquad \frac{\Gamma \vdash A^{\alpha\circ x}, \Delta}{\Gamma \vdash \Box A^{\alpha}, \Delta}\ \vdash\Box$$

$$\frac{\Gamma, A^{\alpha\circ x} \vdash \Delta}{\Gamma, \Diamond A^{\alpha} \vdash \Delta}\ \Diamond\vdash \qquad \frac{\Gamma \vdash A^{\alpha\circ\beta}, \Delta}{\Gamma \vdash \Diamond A^{\alpha}, \Delta}\ \vdash\Diamond$$

Constraints:
In rules $\vdash \Box$ and $\Diamond \vdash$, no position in Γ, Δ may start with $\alpha \circ x$; that is, $\alpha \circ x \notin \mathfrak{Init}[\Gamma, \Delta]$.

Figure 2 (cont.). Rules for the System E_{S4}

Note that both E_{K4} and E_{K}, in addition to the constraint on the main position β, have also constraints on the context: in the modal rules $\Box \vdash$ and $\vdash \Diamond$ *there must be another formula occurrence* $B^{\alpha\circ\beta\circ\eta}$ *in either* Γ *or* Δ (of course, α and/or η may be empty). This prevents the derivation of $\Box A \to \Diamond A^{\gamma}$ (the p-formula representing axiom **D**).

The notions of proof, provable sequent and height $h(\Pi)$ of a proof Π are standard.

Notation 2.5. In order to simplify the graphical representation of proofs, we will use a double deduction line to indicate application of a rule preceded or followed by a sequence of structural rules. So we will write

$$\frac{\Gamma \vdash \Delta}{\overline{\Sigma \vdash \Theta}}\ r$$

when the e–sequent $\Sigma \vdash \Theta$ has been obtained from the e–sequent $\Gamma \vdash \Delta$ by means of an application of rule r and of a finite number of structural rules.

Remark 2.6 (On the cut rule for E_{K}, E_{K4}).
The constraint is necessary for E_{K4} and E_{K}, since it prevents the derivation of the unsound schema $\Diamond(A \to A)^{\langle\rangle}$ (remember that K and $\mathsf{K4}$ do not validate $\Diamond$**true**). Indeed, without the constraint we could construct the proof-tree:

Calculus	**Constraints on the rules $\Box\vdash$ and $\vdash\Diamond$**
E_{S4}	no constraints
E_{T}	$\beta = \langle\ \rangle$, or β is a singleton sequence $\langle z\rangle$
E_{D}	β is a singleton sequence $\langle z\rangle$
E_{D4}	β is a non-empty sequence
E_{K4}	β is a non-empty sequence; there is at least a formula $B^{\alpha\circ\beta\circ\eta}$ in either Γ or Δ
E_{K}	β is a singleton sequence $\langle z\rangle$; there is at least a formula $B^{\alpha\circ\beta\circ\eta}$ in either Γ or Δ

	Constraints on the cut rule
E_{D}, E_{T}, E_{S4} E_{D4}	no constraints
E_{K}, E_{K4}	$\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$

Figure 3. Constraints

$$\dfrac{\dfrac{A^{\langle x\rangle} \vdash A^{\langle x\rangle}}{\vdash A \to A^{\langle x\rangle}} \qquad \dfrac{A \to A^{\langle x\rangle} \vdash A \to A^{\langle x\rangle}}{A \to A^{\langle x\rangle} \vdash \Diamond(A \to A)^{\langle\rangle}}}{\vdash \Diamond(A \to A)^{\langle\rangle}}$$

Using the terminology we will introduce shortly in Definition 2.8, we will say that, in order to be sound for E_{K} or E_{K4}, cut formulas must have a *sentinel*. It is easy to see that *modus ponens* (from $\vdash A \to B^\alpha$ and $\vdash A^\alpha$, obtain a derivation of $\vdash B^\alpha$) remains derivable also in presence of this constraint.

Characteristic axioms of normal modal systems are easily derivable, as shown in Section 4.

We introduce now some definitions. The position $\alpha \circ x$ in the rules $\vdash\Box$ and $\Diamond\vdash$ is the *eigenposition* of the rule, by analogy to first-order sequent calculus. It is well known that in first order sequent calculus eigenvariables should be considered as bound variables. In particular, any eigenvariable in a derivation may always be substituted with a *fresh* one (that is, a variable which does not occur in any other place in that derivation), without affecting the provable end sequent (up to renaming of its bound variables). Indeed, one may guarantee that each eigenvariable in a derivation is the eigenvariable of exactly one right $\forall$ or left $\exists$ rule (and, moreover, that variable occurs in the derivation only above the rule of which it is eigenvariable, and it never occurs as a bound variable.) We will show analogous properties for the eigenpositions of e–sequents, in order to define in a sound way a notion of *prefix replacement for proofs* (that we defined at the end of Section 2 for positions). We denote with $\Gamma[\alpha \Rsh \beta]$ the obvious extension of prefix replacement to a sequence Γ of p-formulas.

Fact 2.7. Let $\alpha \circ z$ be an eigenposition. It is always possible to rename the eigenposition as $\alpha \circ z_0$, where z_0 fresh token w.r.t the whole derivation in which the eigenposition occurs.
This assumption ensures that, after a renaming, we cannot have e–sequents $\Gamma \vdash \Delta$ containing both a formula $A^{\alpha \circ z_0}$ with $\alpha \circ z_0$ as eigenposition and other formulas of the shape B^β with $\beta \in \mathfrak{Init}[\alpha \circ z_0]$.

Definition 2.8. An occurrence of a formula A^α in an e–sequent $\Gamma \vdash \Delta$ is said *guarded* if there exists in $\Gamma \vdash \Delta$ an occurrence of a formula $B^{\alpha \circ \delta}$ (δ possibly empty) different from A^α. The formula $B^{\alpha \circ \delta}$ is the *sentinel* of A^α.

Proposition 2.9. Let $\Gamma \vdash \Delta$ be an e–sequent. If a formula A^α is guarded in $\Gamma \vdash \Delta$, then, for any substitution $[\delta \circ z \Rsh \delta \circ \tau]$, the formula $A^{\alpha[\delta \circ z \Rsh \delta \circ \tau]}$ is guarded in $\Gamma[\delta \circ z \Rsh \delta \circ \tau] \vdash \Delta[\delta \circ z \Rsh \delta \circ \tau]$.

Proof: Let B^β a sentinel formula of A^α (so $\beta = \alpha \circ \gamma$). We distinguish some cases:

1. $A^{\alpha[\delta \circ z \Rsh \delta \circ \tau]} = A^\alpha$ (the prefix of α is different from $\delta \circ z$) and then $B^{\alpha \circ \gamma[\delta \circ z \Rsh \delta \circ \tau]} \equiv B^{\alpha \circ \gamma}$. The are two subcases:

   (a) if $\delta \circ z \notin \mathfrak{Init}[\alpha \circ \gamma]$ then $B^{\alpha \circ \gamma[\delta \circ z \Rsh \delta \circ \tau]} \equiv B^{\alpha \circ \gamma}$.

   (b) if $\delta \circ z \in \mathfrak{Init}[\alpha \circ \gamma]$, since $\alpha[\delta \circ z \Rsh \delta \circ \tau] \equiv \alpha$, then $\alpha \circ \gamma = \alpha\alpha' z \gamma'$ where $\alpha\alpha' = \delta$. Then $\alpha \circ \gamma[\delta \circ z \Rsh \delta \circ \tau] = \alpha\alpha'\tau\gamma'$ and $B^{\alpha\alpha'\tau\gamma'}$ is a supervisor of A^α.

2. $\alpha = \delta \circ z \circ \mu$. In this case we have $A^{\alpha[\delta \circ z \Rsh \delta \circ \tau]} = A^{\delta \circ \tau \circ \mu}$ and $B^{\alpha \circ \gamma[\delta \circ z \Rsh \delta \circ \tau]} = B^{\alpha \circ \tau \circ \mu \circ \gamma}$ and then $A^{\delta \circ \tau \circ \mu}$ is still guarded. $\square$

We now extend the notion of prefix replacement to proofs. The lemmas are valid for all the systems (that is, in presence of the constraints) of Figure 3.

LEMMA 2.10. *Let* Π *be an e–sequent proof with conclusion* $\Gamma \vdash \Delta$. *Let* $\delta \circ z$ *be a position, and let* b *be a fresh token (that is, not occurring in either* Π *or* $\delta \circ z$*). Then we may define the* prefix replacement $\Pi[\delta \circ z \Rsh \delta \circ b]$, *a proof with conclusion* $\Gamma[\delta \circ z \Rsh \delta \circ b] \vdash \Delta[\delta \circ z \Rsh \delta \circ b]$.

PROOF: If Π is an axiom $A^\alpha \vdash A^\alpha$, than $\Pi[\delta \circ z \Rsh \delta \circ b]$ is $A^{\alpha[\delta \circ z \Rsh \delta \circ b]} \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ b]}$.
All inductive cases are trivial, except the modal rules.

If the last rule of Π is

$$\frac{\Gamma \vdash A^{\alpha \circ x}, \Delta}{\Gamma \vdash \Box A^\alpha, \Delta} \quad \vdash \Box$$

let Π' be the subproof rooted at this rule. We have two cases, depending on whether the position $\delta \circ z$ is the eigenposition of the rule. (i) If $\alpha \circ x = \delta \circ z$, obtain by induction the proof $\Pi'[\alpha \circ x \Rsh \alpha \circ b]$ with conclusion $\Gamma \vdash A^{\alpha \circ b}, \Delta$ (remember that $\alpha \circ x \notin \mathfrak{Init}[\Gamma, \Delta]$). Then $\Pi[\delta \circ z \Rsh \delta \circ b]$ is obtained from $\Pi'[\alpha \circ x \Rsh \alpha \circ b]$ by an application of $\vdash \Box$. (ii) If $\alpha \circ x \neq \delta \circ z$, obtain by induction the proof $\Pi'[\delta \circ z \Rsh \delta \circ b]$ with conclusion $\Gamma[\delta \circ z \Rsh \delta \circ b] \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ b] \circ x}, \Delta[\delta \circ z \Rsh \delta \circ b]$. Observe now that $\alpha[\delta \circ z \Rsh \delta \circ b] \circ x$ cannot be an initial segment of a formula in $\Gamma[\delta \circ z \Rsh \delta \circ b], \Delta[\delta \circ z \Rsh \delta \circ b]$. Indeed, if for some B^γ in Γ, Δ we had $\alpha[\delta \circ z \Rsh \delta \circ b] \circ x \sqsubseteq \gamma[\delta \circ z \Rsh \delta \circ b]$, since b is fresh, this could only result from $\alpha \circ x$ being a prefix of γ, which is impossible. Therefore, we may conclude with an application of $\vdash \Box$, since its side-condition is satisfied.

If the last rule of Π is

$$\frac{\Gamma \vdash A^{\alpha \circ \beta}, \Delta}{\Gamma \vdash \Diamond A^\alpha, \Delta} \quad \vdash \Diamond$$

let, as before, Π' be the subproof rooted at this rule and construct by induction the proof $\Pi'[\delta \circ z \Rsh \delta \circ b]$ with conclusion $\Gamma[\delta \circ z \Rsh \delta \circ b] \vdash A^{\alpha \circ \beta[\delta \circ z \Rsh \delta \circ b]}, \Delta[\delta \circ z \Rsh \delta \circ b]$. It is easy to verify that any side condition of the $\vdash \Diamond$ rule (which depends on the specific system, according to the table above), is still verified after the prefix replacement. We may then conclude with a $\vdash \Diamond$ rule.

The left modal rules are analogous. $\Box$

By repeatedly using the previous lemma, we obtain the following.

Proposition 2.11 (Eigenposition renaming). Given a proof Π of an e–sequent $\Gamma \vdash \Delta$, we may always find a proof Π' ending with $\Gamma \vdash \Delta$ where all eigenpositions are distinct from one another.

Π' differs from Π only for the names of positions. In practice we will freely use such a renaming all the times it is necessary (or, in other words, proofs are de facto equivalence classes modulo renaming of eigenpositions). In a similar way to the previous lemmas we may obtain the following, which allows the prefix replacement of arbitrary positions (once eigenpositions are considered as bound variables, and renamed so that any confusion is avoided). When we use prefix replacement for proofs we will always assume that the premises of the following lemma are satisfied, implicitly calling for eigenposition renaming if this is not the case.

Lemma 2.12 (Sequents Prefix Replacement). *Let $\mathbb{M}$ be one of the modal systems* K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$*, and let β a position taken according to the constraint for β of figure 3. Let $\delta \circ z$ be a position, and let Π be an $\mathsf{E}_{\mathbb{M}}$ proof of $\Gamma \vdash \Delta$, where all eigenpositions are distinct from one another, and are different from $\delta \circ z$. Then we may define the* prefix replacement *$\Pi[\delta \circ z \Rsh \delta \circ \beta]$, an $\mathsf{E}_{\mathbb{M}}$ proof with conclusion $\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Delta[\delta \circ z \Rsh \delta \circ \beta]$.*

Proof: The proof proceeds by induction on the length of the proof and by cases on the last rule. Propositional cases are trivial. We focus on modal rules and in particular on the non-serial cases E_{K} and E_{K4}.

System E_{K}: in this case β is a single token.

1. The last rule is $\vdash \Box$. We have two cases:
   (a) The proof has the structure

$$\dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash A^{\alpha \circ z}, \Delta\end{array}}{\Gamma \vdash \Box A^{\alpha}, \Delta}$$

   We can exclude this case by eigenposition renaming.

   (b) The proof has the structure

$$\dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash A^{\alpha \circ y}, \Delta\end{array}}{\Gamma \vdash \Box A^{\alpha}, \Delta}$$

By inductive hypothesis, we have a proof

$$\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ z[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}$$

Since $\alpha \circ y \notin \mathfrak{Init}[\Gamma, \Delta]$ and we can assume that the token $y \notin \mathfrak{Init}[\Gamma, \Delta, \beta]$, we have $\Gamma[\delta \circ z \Rsh \delta \circ \beta] = \Gamma$, $\Delta[\delta \circ z \Rsh \delta \circ \beta] = \Delta$. If not, by renaming we can replace $\alpha \circ y$ with $\alpha \circ y_0$ with y_0 fresh. Therefore we have that $\alpha \circ y \notin \mathfrak{Init}[\Gamma[\delta \circ z \Rsh \delta \circ \beta], \Delta[\delta \circ z \Rsh \delta \circ \beta]]$ ($\alpha \circ y$ can not appear in the substitution). We can conclude by applying the $\vdash \Box$ rule:

$$\dfrac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ y[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Box A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]} \vdash \Box$$

There are no additional constraints to satisfy, so this case is clearly sound for $\mathsf{E_K}$

2. The last rule is $\Diamond \vdash$: symmetric to the previous case.

3. The last rule is $\vdash \Diamond$, so we have a proof

$$\dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash A^{\alpha \circ c}, \Delta\end{array}}{\Gamma \vdash \Diamond A^{\alpha}, \Delta}$$

where c is a token and there exists at least a formula $B^{\alpha \circ c \circ \mu}$ in either Γ or Δ. Notice that, since we are in $\mathsf{E_K}$ the $\vdash \Diamond$ rule always modifies the position of the main formula $A^{\alpha \circ c}$ (i.e. in the conclusion we have the formula A^{α}).

By i.h., we apply the prefix replacement on the subproof Π_1:

$$\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}$$

We have some cases:

(a) $\delta \circ z \notin \mathfrak{Init}[\alpha \circ c]$, so $\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta] = \alpha \circ c$.
We apply the substitution:

$$\frac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

By Proposition 2.9, derivation is sound for K.

(b) $\delta \circ z \in \mathfrak{Init}[\alpha \circ c]$ and $\delta \circ z \in \mathfrak{Init}[\alpha]$, so $\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta] = \alpha[\delta \circ z \Rsh \delta \circ \beta] \circ c$.
Then

$$\frac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ \beta] \circ c}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

The proof is sound for K thanks to Proposition 2.9.

(c) $\delta \circ z \in \mathfrak{Init}[\alpha \circ c]$ and $\delta \circ z \notin \mathfrak{Init}[\alpha]$. We have $\alpha \circ c = \delta \circ z$ and $\alpha = \delta$ and $\beta = \langle c \rangle$.
We apply the inductive hypothesis, and we obtain the following proof:

$$\frac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

since $\Diamond A^{\delta} = \Diamond A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]} = \Diamond A^{\alpha}$
The last step of the derivation is sound for K, by means of Proposition 2.9.

4. The last rule is $\Box \vdash$: symmetric to the previous case.

5. The last rule is a cut. Let A^{α} be the cut-formula. In E_{K} we have the constraint $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^{\alpha}]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^{\alpha}, \Delta_2]$.

By i.h., we obtain the proofs

$$\dfrac{\Pi_1[\delta\circ z \Rsh \delta\circ\beta]}{\Gamma_1[\delta\circ z \Rsh \delta\circ\beta] \vdash A^{\alpha[\delta\circ z\Rsh\delta\circ\beta]}, \Delta_1[\delta\circ z \Rsh \delta\circ\beta]}$$

and

$$\dfrac{\Pi_2[\delta\circ z \Rsh \delta\circ\beta]}{\Gamma_2[\delta\circ z \Rsh \delta\circ\beta], A^{\alpha[\delta\circ z\Rsh\delta\circ\beta]} \vdash \Delta_2[\delta\circ z \Rsh \delta\circ\beta]}$$

And therefore we can conclude with a cut

$$\dfrac{\dfrac{\Pi_1[\delta\circ z \Rsh \delta\circ\beta]}{S_1} \qquad \dfrac{\Pi_2[\delta\circ z \Rsh \delta\circ\beta]}{S_2}}{\Gamma\vdash\Delta}\ Cut$$

where:
$S_1 = \Gamma_1[\delta\circ z \Rsh \delta\circ\beta] \vdash A^{\alpha[\delta\circ z\Rsh\delta\circ\beta]}, \Delta_1[\delta\circ z \Rsh \delta\circ\beta]$ and $S_2 = \Gamma_2[\delta\circ z \Rsh \delta\circ\beta], A^{\alpha[\delta\circ z\Rsh\delta\circ\beta]} \vdash \Delta_2[\delta\circ z \Rsh \delta\circ\beta]$.
Notice that the proof above is sound: the constraint on the cut rule ensures that there is at least a formula $B^{\alpha\circ\gamma}$ which is a sentinel for A^α. This still holds after the replacement $[\delta\circ z \Rsh \delta\circ\beta]$ by means of Proposition 2.9.

System E_{K4}: in this case β is an arbitrary non void position.

1. The last rule is $\vdash\Box$: as for E_{K}, case 1.
2. The last rule is $\Diamond\vdash$: as for E_{K}, case 2.
3. The last rule is $\vdash\Diamond$, so we have a proof

$$\dfrac{\dfrac{\Pi_1}{\Gamma\vdash A^{\alpha\circ\tau}, \Delta}}{\Gamma\vdash\Diamond A^\alpha, \Delta}$$

where τ is a non-empty sequence and there exists at least a formula $B^{\alpha\circ\tau\circ\mu}$ in either Γ or Δ. Since we are in E_{K4}, as in the previous case the $\vdash\Diamond$ rule always modifies the position of the main formula $A^{\alpha\circ\tau}$ (i.e. in the conclusion we have the formula A^α).

By i.h., we apply the prefix replacement on the subproof Π_1:

$$\Pi_1[\delta \circ z \Rsh \delta \circ \beta]$$
$$\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]$$

We have some cases:

(a) $\delta \circ z \notin \mathfrak{Init}[\alpha \circ \tau]$, so $\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta] = \alpha \circ \tau$. As for system $\mathsf{E_K}$ case 3a.

(b) $\delta \circ z \in \mathfrak{Init}[\alpha \circ \tau]$ and $\delta \circ z \in \mathfrak{Init}[\alpha]$, so $\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta] = \alpha[\delta \circ z \Rsh \delta \circ \beta] \circ \tau$. As for system $\mathsf{E_K}$ case 3b.

(c) $\delta \circ z \in \mathfrak{Init}[\alpha \circ \tau]$ and $\delta \circ z \notin \mathfrak{Init}[\alpha]$.
The position $\alpha \circ \tau$ has the shape $\alpha \circ \tau = \alpha \circ \tau_1 \circ z \circ \tau_2$ (so $\alpha \circ \tau_1 = \delta$) and $\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta] = \alpha \circ \tau_1 \circ \beta \circ \tau_2$.
We apply the inductive hypothesis, and we obtain the following proof:

$$\dfrac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha\tau_1 \circ \beta \circ \tau_2}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

We know that there exists at least a formula $B^{\alpha \circ \tau \circ \mu}$ in either Γ or Δ and thanks to Proposition 2.9 the formula $A^{\alpha\tau_1 \circ \beta \circ \tau_2}$ is still guarded in $\Gamma[\delta \circ z \Rsh \delta \circ \beta]$ or $\Delta[\delta \circ z \Rsh \delta \circ \beta]$ by some $B^{\alpha \circ \tau \circ \mu}[\delta \circ z \Rsh \delta \circ \beta]$. Then the proof is sound.

4. The last rule is $\Box \vdash$: as for system $\mathsf{E_K}$, case 4.

5. The last rule is a cut formula: as for system $\mathsf{E_K}$, case 5 $\square$

3. The cut elimination theorem

We prove the cut-elimination theorem for the e–sequent systems, by adapting the standard techniques for the classical predicate calculus [11]. In particular the reader could appreciate the strong similarity, in the proofs of the mix lemmas, between positions in e–systems and first-order terms in classical logic.

Thanks to the modularity of our proposal, we can prove the mix lemmas *only twice*, once for serial systems and once for non-serial ones.

We start with the usual notions of *subformula* and *degree*. Observe that, as the set of first-order (Gentzen) subformulas of $\forall x A(x)$ contain all the term-instances of $A(x)$, here the set of (position, modal) subformulas of $\Box A^\alpha$ contain all the extensions of the position α in A^α.

Definition 3.1 (subformula). The set $Sub(A^\alpha)$ of *subformulas* of a formula A^α is recursively defined as follows:

$Sub(p^\alpha) = \{p^\alpha\}$ if p is a proposition symbol;

$Sub(\neg A^\alpha) = \{\neg A^\alpha\} \cup Sub(A^\alpha)$;

$Sub(A\#B^\alpha) = \{A\#B^\alpha\} \cup Sub(A^\alpha) \cup Sub(B^\alpha)$, when $\# \in \{\to, \vee, \wedge\}$;

$Sub(\#A^\alpha) = \{\#A^\alpha\} \cup \{Sub(A^{\alpha\circ\beta}) : \beta \in P\}$, when $\# \in \{\Box, \Diamond\}$.

Definition 3.2 (degree). The *degree* of modal formulas, p-formulas, and e–sequent proofs are defined as follows.

1. The *degree of a modal formula* A, $\mathsf{dg}(A)$, is recursively defined as:
   (a) $\mathsf{dg}(p) = 0$ if p is a proposition symbol;
   (b) $\mathsf{dg}(\neg A) = \mathsf{dg}(\Box A) = \mathsf{dg}(\Diamond A) = \mathsf{dg}(A) + 1$;
   (c) $\mathsf{dg}(A \wedge B) = \mathsf{dg}(A \vee B) = \mathsf{dg}(A \to B) = \max\{\mathsf{dg}(A), \mathsf{dg}(B)\} + 1$.
2. The *degree of a p-formula* A^α, $\mathsf{dg}(A^\alpha)$, is just $\mathsf{dg}(A)$.
3. The *degree of a proof* Π, $\delta[\Pi]$, is the natural number defined as follows:

$$\delta[\Pi] = \begin{cases} 0 & \text{if } \Pi \text{ is cut-free;} \\ \sup\{\mathsf{dg}(A^\alpha) + 1 : A^\alpha \text{ is a cut formula in } \Pi\} & \text{otherwise.} \end{cases}$$

Let Γ be a sequence of formulas. We denote by $\Gamma - A^\alpha$ the sequence obtained by removing all occurrences of A^α in Γ. When writing $\Gamma, \Gamma' - A^\alpha$ we actually mean $\Gamma, (\Gamma' - A^\alpha)$. In the sequel, ordered pairs of natural numbers are intended to be lexicographically ordered. Hence one can make proofs by induction on pairs of numbers. The height $h(\Pi)$ of a proof Π is defined in the usual way.

We will prove two different "mix lemmas", to take into account that the cut-rule for the systems E_K and E_{K4} have special constraints, which are mirrored into the hypothesis of the lemma.

LEMMA 3.3 (Mix Lemma for $\mathsf{E_D}$, $\mathsf{E_T}$, $\mathsf{E_{S4}}$). *Let $\mathcal{S}$ be one of the systems $\mathsf{E_D}$, $\mathsf{E_T}$, $\mathsf{E_{S4}}$. Let $n \in \mathbb{N}$ and let A^α be a formula of degree n. Let now Π, Π' be proofs of the e–sequents $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$, respectively, satisfying the property $\delta[\Pi], \delta[\Pi'] \leq n$. Then one can obtain in an effective way from Π and Π' a proof $\mathsf{Mix}(\Pi, \Pi')$ of the e–sequent $\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'$ satisfying the property $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$.*

PROOF: The proof proceeds in a standard way, by induction on the pair $\langle h(\Pi), h(\Pi')\rangle$. We highlight only the main points. Let Π and Π' be

$$\frac{\left\{\begin{array}{c}\Pi_i\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i\in I}}{\Gamma \vdash \Delta}\ r \qquad \text{and} \qquad \frac{\left\{\begin{array}{c}\Pi'_j\\ \Gamma'_j \vdash \Delta'_j\end{array}\right\}_{j\in I'}}{\Gamma' \vdash \Delta'}\ r'$$

respectively, where I and I' are $\varnothing$ (in case of an axiom), $\{1\}$ or $\{1,2\}$. We proceed by cases.

1. r is Ax.

   If $\Gamma \vdash \Delta$ is $A^\alpha \vdash A^\alpha$, then one gets $\mathsf{Mix}(\Pi, \Pi')$ from Π' by means of a suitable sequence of structural rules.

   If $\Gamma \vdash \Delta$ is $B^\beta \vdash B^\beta$, for $B \neq A$ or $\beta \neq \alpha$, then one gets $\mathsf{Mix}(\Pi, \Pi')$ from Π by a suitable sequence of structural rules.

2. r' is Ax. This case is symmetric to case 1.

3. r is a structural rule. Apply the induction hypothesis to the pair $\langle \Pi_1, \Pi'\rangle$, then apply a suitable sequence of structural rules to get the conclusion.

4. r' is a structural rule. This case is symmetric to 3.

5. r is a cut or a logical rule not introducing A^α to the right.
   Apply the induction hypothesis to each pair $\langle \Pi_i, \Pi'\rangle$, so obtaining the proof $\mathsf{Mix}(\Pi_i, \Pi')$, for $i \in I$. The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\frac{\left\{\begin{array}{c}\mathsf{Mix}(\Pi_i, \Pi')\\ \Gamma_i, \Gamma' - A^\alpha \vdash \Delta_i - A^\alpha, \Delta'\end{array}\right\}_{i\in I}}{\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'}\ r$$

6. r' is a cut or a logical rule not introducing A^α to the left. This case is symmetric to 5.

7. r is a logical rule introducing A^α to the right and r' is a logical rule introducing A^α to the left.

   (a) r is a propositional rule. This subcase is treated as in the first order case (see, for instance, [11] or [25]).

   (b) A is $\Box B$. Let Π and Π' be

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash B^{\alpha\circ x}, \Delta_1\end{array}}{\Gamma \vdash A^\alpha, \Delta_1} \qquad \text{and} \qquad \dfrac{\begin{array}{c}\Pi'_1\\ \Gamma'_1, B^{\alpha\circ\beta} \vdash \Delta'\end{array}}{\Gamma'_1, A^\alpha \vdash \Delta'}$$

   respectively. Apply the induction hypothesis to the pairs of proofs $\langle \Pi_1[\alpha\circ x \Rsh \alpha\circ\beta], \Pi'\rangle$ and $\langle \Pi, \Pi'_1\rangle$, obtaining $\mathsf{Mix}(\Pi_1[\alpha\circ x \Rsh \alpha\circ\beta], \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$, respectively (both of degree less or equal n). The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\begin{array}{c}\mathsf{Mix}(\Pi_1[\alpha\circ x \Rsh \alpha\circ\beta], \Pi')\\ \Gamma, \Gamma'_1 - A^\alpha \vdash B^{\alpha\circ\beta}, \Delta_1 - A^\alpha, \Delta'\end{array} \qquad \begin{array}{c}\mathsf{Mix}(\Pi, \Pi'_1)\\ \Gamma, \Gamma'_1 - A^\alpha, B^{\alpha\circ\beta} \vdash \Delta_1 - A^\alpha, \Delta'\end{array}}{\Gamma, \Gamma'_1 - A^\alpha, \Gamma, \Gamma'_1 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta', \Delta_1 - A^\alpha, \Delta'}\,Cut}{\Gamma, \Gamma'_1 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta'}$$

   (c) A is $\Diamond B$. This subcase is symmetric to the case $\Box B$.

In all cases involving new cuts, since the additional cuts are performed on strict subformulas of A^α with degree less than n, we immediately get $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$. $\Box$

The above proof does not go through for the systems E_{K} and E_{K4}, because of the constraint on the context for the rules $\Box\vdash$ and $\vdash\Diamond$. Indeed, the case (5) of the proof would fail, as shown by the following two proof fragments. Let $\alpha = \beta \circ x$ be the position of the statement of the lemma,

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdash B^{\beta\circ x}, A^{\beta\circ x}\end{array}}{\vdash \Diamond B^\beta, A^{\beta\circ x}} \qquad \text{and} \qquad \begin{array}{c}\Pi'\\ A^{\beta\circ x} \vdash C^\beta\end{array}$$

If we apply the induction hypothesis to the pair $\langle \Pi_1, \Pi' \rangle$ we obtain

$$\dfrac{\mathsf{Mix}(\Pi, \Pi')}{\vdash B^{\beta \circ x}, C^{\beta}}$$

and now it is impossible to conclude with the $\vdash \Diamond$ rule, because via the induction hypothesis we deleted the only formula essential to validate the $\vdash \Diamond$ rule. To fix the problem, we need a stronger statement of the lemma, which mirrors the constraint of the cut rule of $\mathsf{E_K}$ and $\mathsf{E_{K4}}$.

Lemma 3.4 (Mix Lemma for $\mathsf{E_K}$, $\mathsf{E_{K4}}$). *Let $\mathcal{S}$ be one of the systems $\mathsf{E_K}$ or $\mathsf{E_{K4}}$. Let $n \in \mathbb{N}$ and let A^α be a formula of degree n. Let now Π, Π' be proofs of the e–sequents $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$, respectively, satisfying the properties:*

- $\delta[\Pi], \delta[\Pi'] \leq n$;
- $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$, *or* $\alpha \in \mathfrak{Init}[\Gamma' - A^\alpha, \Delta']$

Then one can obtain in an effective way from Π and Π' a proof $\mathsf{Mix}(\Pi, \Pi')$ of the e–sequent $\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'$ satisfying the property $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$.

Proof: The proof proceeds as for the previous lemma, with special care for cases (5) and (7).

1.–4. As in Lemma 3.3

5. r is a cut or a logical rule not introducing A^α to the right.
   Apply the induction hypothesis to each pair $\langle \Pi_i, \Pi' \rangle$, so obtaining the proof $\mathsf{Mix}(\Pi_i, \Pi')$, for $i \in I$. The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\left\{ \dfrac{\mathsf{Mix}(\Pi_i, \Pi')}{\Gamma_i, \Gamma' - A^\alpha \vdash \Delta_i - A^\alpha, \Delta'} \right\}_{i \in I}}{\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'}\ r$$

   Notice that in the case of r is a cut rule one has the further constrains from Figure 3: $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$.

6. r' is a cut or a logical rule not introducing A^α to the left.
   This case is symmetric to 3.

7. r is a logical rule introducing A^α to the right and r' is a logical rule introducing A^α to the left.

   (a) r is a propositional rule.
   This subcase is treated as in the classical logic case (see, for instance, [11] or [25]). Here we show only the case when A is of the form $B \to C$.

   Let Π and Π' be respectively

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma, B^\alpha \vdash C^\alpha, \Delta\end{array}}{\Gamma \vdash B \to C^\alpha, \Delta} \quad \text{and} \quad \dfrac{\begin{array}{cc}\Pi'_1 & \Pi'_2\\ \Gamma'_1, C^\alpha \vdash \Delta'_1 & \Gamma'_2 \vdash B^\alpha, \Delta'_2\end{array}}{\Gamma'_1, \Gamma'_2, B \to C^\alpha \vdash \Delta'_1, \Delta'_2}$$

   Apply the induction hypothesis to the pairs of proofs $\langle \Pi, \Pi'_2 \rangle$, $\langle \Pi_1, \Pi' \rangle$ and $\langle \Pi, \Pi'_1 \rangle$, obtaining the following proofs:

   - $\mathsf{Mix}(\Pi, \Pi'_2)$ of the sequent $\Gamma, \Gamma'_2 - A^\alpha \vdash B^\alpha, \Delta - A^\alpha, \Delta'_2$ with constraints $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma'_2 - A^\alpha, \Delta'_2, B^\alpha]$ (B^α acts as a sentinel formula for A^α).
   - $\mathsf{Mix}(\Pi_1, \Pi')$ of the sequent $\Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha B^\alpha \vdash C^\alpha, \Delta - A^\alpha, \Delta'_1, \Delta'_2$ with constraints $\alpha \in \mathfrak{Init}[\Gamma, B^\alpha, \Delta - A^\alpha, C^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha, \Delta'_1, \Delta'_2]$ (both B^α and C^α act as sentinel formulas for A^α).
   - $\mathsf{Mix}(\Pi, \Pi'_1)$ of the sequent $\Gamma, \Gamma'_1 - A^\alpha, C^\alpha \vdash \Delta - A^\alpha, \Delta'_1$ with constraints $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma'_1 - A^\alpha, C^\alpha, \Delta'_1]$ (C^α acts as a sentinel formula for A^α).

   The proof $\mathsf{Mix}(\Pi, \Pi')$ is then obtained as follows. Cut first $\mathsf{Mix}(\Pi, \Pi'_2)$ against $\mathsf{Mix}(\Pi_1, \Pi')$ to obtain the following proof Υ:

$$\dfrac{\begin{array}{cc}\mathsf{Mix}(\Pi, \Pi'_2) & \mathsf{Mix}(\Pi_1, \Pi')\\ \Gamma, \Gamma'_2 - A^\alpha \vdash B^\alpha, \Delta - A^\alpha, \Delta'_2 & \Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha, B^\alpha \vdash C^\alpha, \Delta - A^\alpha, \Delta'_1, \Delta'_2\end{array}}{\Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha \vdash C^\alpha, \Delta - A^\alpha, \Delta'_1, \Delta'_2}\ Cut$$

Cut now Υ against $\mathsf{Mix}(\Pi, \Pi'_1)$, obtaining the final proof

$$\dfrac{\begin{array}{c}\Upsilon\\ \Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha \vdash C^\alpha, \Delta - A^\alpha, \Delta'_1, \Delta'_2\end{array} \qquad \begin{array}{c}\mathsf{Mix}(\Pi, \Pi'_1) + \text{right wekenings of } \Delta'_2\\ \Gamma, \Gamma'_1 - A^\alpha, C^\alpha \vdash \Delta - A^\alpha, \Delta'_1, \Delta'_2\end{array}}{\Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha \vdash \Delta - A^\alpha, \Delta'_1, \Delta'_2}\; Cut$$

The cut in Υ is soundly applied, since at least C^α acts as a sentinel for the rule, so the constraints are verified. As for the last cut, let as check that in all possible subcases there exists a sentinel formula for the cut formula C^α. In building $\mathsf{Mix}(\Pi, \Pi')$ we know there is a sentinel formula for A^α, of shape $D^{\alpha\circ\mu}$, somewhere in the contexts: either $D^{\alpha\circ\mu}$ is in Γ, or is in $\Delta - A^\alpha$, or is in $\Gamma'_1 - A^\alpha$, or is in $\Gamma'_2 - A^\alpha$, or is in Δ'_1, or finally is in Δ'_2.

In these cases:

i. if $D^{\alpha\circ\mu}$ is in Γ, or is in $\Gamma'_1 - A^\alpha$, or is in $\Gamma'_2 - A^\alpha$, then $D^{\alpha\circ\mu}$ is a sentinel for C^α, because it appears on the left of $\vdash$ in the first premise of the cut;

ii. if $D^{\alpha\circ\mu}$ is in $\Delta - A^\alpha$, or is in Δ'_1, then $D^{\alpha\circ\mu}$ is a sentinel for C^α, because it appears on the right of $\vdash$ in the second premise of the cut;

iii. if $D^{\alpha\circ\mu}$ is in Δ'_2, note that we have added Δ'_2 (with suitable right weakenings) to the conclusion of $\mathsf{Mix}(\Pi, \Pi'_1)$, so that $D^{\alpha\circ\mu}$ could be a sentinel for the cut formula (observe that the conclusion of the whole proof does not change, since Δ'_2 is already present there.)

(b) A is $\Box B$.

Let Π, r, Π' and r' be respectively

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash B^{\alpha\circ x}, \Delta\end{array}}{\Gamma \vdash A^\alpha, \Delta}\vdash\Box \qquad \text{and} \qquad \dfrac{\begin{array}{c}\Pi'_1\\ \Gamma', B^{\alpha\circ\beta} \vdash \Delta'\end{array}}{\Gamma', A^\alpha \vdash \Delta'}\Box\vdash$$

Recall that for r we have the constraint for the $\vdash \Box$ rule $\alpha \circ x \notin \mathfrak{Init}[\Gamma, \Delta]$ and, since we are in K or in $\mathsf{K4}$, we have also the constraints on $r' = \Box \vdash$, namely there must be a sentinel for $B^{\alpha \circ \beta}$.

Let us suppose to be in $\mathsf{E_K}$.
In this case β is a singleton, i.e. $\beta = \langle z \rangle$ and one also requires that there exists at least a formula $D^{\alpha \circ z \circ \mu}$ in Γ' or in Δ'. Apply the induction hypothesis to the pairs of proofs $\langle \Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi' \rangle$ and $\langle \Pi, \Pi'_1 \rangle$, obtaining a proof $\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')$ of $\Gamma[\alpha \circ x \Rsh \alpha \circ z], \Gamma' - A^\alpha \vdash B^{\alpha \circ z}, \Delta[\alpha \circ x \Rsh \alpha \circ z] - A^\alpha, \Delta'$ and a proof $\mathsf{Mix}(\Pi, \Pi'_1)$ of $\Gamma, \Gamma' - A^\alpha, B^{\alpha \circ z} \vdash, \Delta - A^\alpha, \Delta'$.

Thanks to the constraints on the $\vdash \Box$ rule and Proposition 2.11 (eigenposition renaming), it holds that $\Gamma[\alpha \circ x \Rsh \alpha \circ z] = \Gamma$ and $\Delta[\alpha \circ x \Rsh \alpha \circ z] = \Delta$, so we can drop the substitution from the contexts Γ and Δ.

Notice that we soundly applied the induction hypothesis. To check this, it enough to verify the constraint $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma' - A^\alpha, \Delta']$ is satisfied both by $\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$.

For the proof $\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')$ of the sequent $\Gamma, \Gamma' - A^\alpha \vdash B^{\alpha \circ z}, \Delta - A^\alpha, \Delta'$ we know there is at least a formula $D^{\alpha \circ z \circ \mu}$ in Γ' or in Δ', and so the constraint is verified. This holds also for for the proof $\mathsf{Mix}(\Pi, \Pi'_1)$ of the sequent $\Gamma, \Gamma' - A^\alpha, B^{\alpha \circ z} \vdash \Delta - A^\alpha, \Delta'$, thanks to the presence of $B^{\alpha \circ z}$.

The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\dfrac{\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')}{\Gamma, \Gamma' - A^\alpha \vdash B^{\alpha \circ z}, \Delta - A^\alpha, \Delta'} \qquad \dfrac{\mathsf{Mix}(\Pi, \Pi'_1)}{\Gamma, \Gamma' - A^\alpha, B^{\alpha \circ z} \vdash \Delta - A^\alpha, \Delta'}}{\Gamma, \Gamma' - A^\alpha, \Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta', \Delta - A^\alpha, \Delta'}\, Cut}{\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'}$$

Notice that the application of the cut rule is sound, i.e. the following constraints is satisfied: $\alpha \circ z \in \mathfrak{Init}[\Gamma, \Gamma' - A^\alpha, (\Delta - A^\alpha, \Delta') - B^{\alpha\circ z}]$ or $\alpha \circ z \in \mathfrak{Init}[(\Gamma, \Gamma' - A^\alpha) - B^{\alpha\circ z}, \Delta - A^\alpha, \Delta']$. We have two cases: if there exists in Δ' a formula $C^{\alpha\circ z\circ\mu}$ (constraint of $\mathsf{K}$to the $\Box \vdash \Gamma'$) we are done; if there is a formula $C^{\alpha\circ z}$ that belongs to Γ', it also belongs to $\Gamma' - A^\alpha$ and we can conclude.

If we are in $\mathsf{E}_\mathsf{K}4$, then $\beta = \delta$ with $\delta \neq \langle\rangle$ and we proceed exactly as for E_K.

(c) A is $\Diamond B$. This subcase is symmetric to the previous one.

In all cases involving new cuts, since the additional cuts are performed on strict subformulas of A^α, with degree less than n we immediately get $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$. $\Box$

Theorem 3.5 (Cut elimination for E_D, E_T, $\mathsf{E}_{\mathsf{D}4}$ $\mathsf{E}_{\mathsf{S}4}$). *Let $\mathbb{M}$ be one of the modal systems E_D, E_T, $\mathsf{E}_{\mathsf{D}4}$ and $\mathsf{E}_{\mathsf{S}4}$. If Π is a $\mathsf{E}_\mathbb{M}$–proof of $\Gamma \vdash \Delta$, then there exists a cut-free $\mathsf{E}_\mathbb{M}$–proof Π^* of $\Gamma \vdash \Delta$.*

Proof: By induction on the pair $\langle \delta[\Pi], h(\Pi)\rangle$. Suppose Π is not cut-free and let r be the last rule applied in Π. We distinguish two cases:

1. r is not a cut. Let Π be

$$\frac{\left\{\begin{array}{c}\Pi_i\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i\in I}}{\Gamma \vdash \Delta}\, r,$$

   where I is one of $\{1\}$, $\{1,2\}$ Apply the induction hypothesis to each Π_i, obtaining cut-free proofs Π_i^*, for $i \in I$. A cut-free proof Π^* of $\Gamma \vdash \Delta$ is then

$$\frac{\left\{\begin{array}{c}\Pi_i^*\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i\in I}}{\Gamma \vdash \Delta}\, r$$

2. r is a cut. Let Π be

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \vdash A^\alpha, \Delta_1\end{array} \qquad \begin{array}{c}\Pi_2\\ \Gamma_2, A^\alpha \vdash \Delta_2\end{array}}{\Gamma \vdash \Delta}\ Cut$$

We apply the induction hypothesis to Π_1 and Π_2 in order to obtain cut-free proofs Π_1^* and Π_2^* of $\Gamma_1 \vdash A^\alpha, \Delta_1$ and $\Gamma_2, A^\alpha \vdash \Delta_2$ respectively.

Applying Lemma 3.3 to the pair $\langle \Pi_1^*, \Pi_2^* \rangle$, one gets a proof Π_0 of sequent $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ such that $\delta[\Pi_0] \leq \mathsf{dg}(A^\alpha) < \delta[\Pi]$.

Finally one gets a cut-free proof of $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ from Π_0 by induction hypothesis and, from it, a cut-free proof of $\Gamma \vdash \Delta$ by application of a suitable sequence of structural rules. $\square$

Theorem 3.6 (Cut elimination for E_K, E_{K4}). *Let $\mathbb{M}$ be one of the modal systems E_K and E_{K4}. If Π is a $\mathsf{E}_\mathbb{M}$–proof of $\Gamma \vdash \Delta$, then there exists a cut-free $\mathsf{E}_\mathbb{M}$–proof Π^* of $\Gamma \vdash \Delta$.*

Proof: By induction on the pair $\langle \delta[\Pi], h(\Pi) \rangle$. Suppose Π is not cut-free and let r be the last rule applied in Π. We distinguish two cases:

1. r is not a cut. Let Π be

$$\dfrac{\left\{\begin{array}{c}\Pi_i\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i\in I}}{\Gamma \vdash \Delta}\ r,$$

where I is one of $\{1\}$, $\{1,2\}$ Apply the induction hypothesis to each Π_i, obtaining cut-free proofs Π_i^*, for $i \in I$. A cut-free proof Π^* of $\Gamma \vdash \Delta$ is then

$$\dfrac{\left\{\begin{array}{c}\Pi_i^*\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i\in I}}{\Gamma \vdash \Delta}\ r$$

2. r is a cut. Let Π be

$$\dfrac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ \Gamma_1 \vdash A^\alpha, \Delta_1 & \Gamma_2, A^\alpha \vdash \Delta_2 \end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ Cut$$

where we know that $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$.

Apply the induction hypothesis to Π_1 and Π_2 to obtain cut-free proofs Π_1^* and Π_2^* of $\Gamma_1 \vdash A^\alpha, \Delta_1$ and $\Gamma_2, A^\alpha \vdash \Delta_2$ respectively. Notice that $\delta[\Pi_1^*], \delta[\Pi_2^*] \leq \delta[A^\alpha] = n$.

Applying Lemma 3.4 to the pair $\langle \Pi_1^*, \Pi_2^* \rangle$, one gets a proof Π_0 of sequent $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ such that $\delta[\Pi_0] \leq \delta[A^\alpha] < \delta[\Pi]$ and $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$, or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$. Notice that this is the same as we had for the last rule of Π.

Finally one gets a cut-free proof of $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ from Π_0 by induction hypothesis and, from it, a cut-free proof of $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ via a suitable sequence of structural rules. $\square$

Let $\mathbb{M}$ be one of the systems $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4}$. Subformula Property and Consistency follows as immediate corollaries of cut-elimination.

Corollary 3.7 (Subformula Property). Each formula occurring in a cut-free $\mathsf{E}_{\mathbb{M}}$-proof Π is a subformula of some formula occurring in the conclusion of Π.

Corollary 3.8 (Consistency). $\mathsf{E}_{\mathbb{M}}$ is consistent, namely there is no $\mathsf{E}_{\mathbb{M}}$-proof of the empty sequent $\vdash$.

4. E–sequent calculi are equivalent to standard calculi

The systems introduced in the previous sections prove the same theorems of the Hilbert-style presentation of the corresponding logics. Let $\mathbb{M}$ be one of the logics K, $\mathsf{K4}$, D, $\mathsf{D4}$ and $\mathsf{S4}$. We start with a proof that, if $\mathbb{M}$ proves A, then $\mathsf{E}_{\mathbb{M}}$ proves $\vdash A^{\langle\rangle}$. We show the derivations for the modal axioms. Observe that the proof of each axiom satisfies the constraints on $\Box\vdash$ and $\vdash\Diamond$ of the corresponding e–system.

Axiom K

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^{\langle x\rangle} \vdash B^{\langle x\rangle} \quad A^{\langle x\rangle} \vdash A^{\langle x\rangle}}{A^{\langle x\rangle}, A \to B^{\langle x\rangle} \vdash B^{\langle x\rangle}} \to\vdash}{A^{\langle x\rangle}, \Box(A \to B)^{\langle\ \rangle} \vdash B^{\langle x\rangle}} \Box\vdash}{\Box A^{\langle\ \rangle}, \Box(A \to B)^{\langle\ \rangle} \vdash B^{\langle x\rangle}} \Box\vdash}{\Box A^{\langle\ \rangle}, \Box(A \to B)^{\langle\ \rangle} \vdash \Box B^{\langle\ \rangle}} \vdash\Box}{\Box(A \to B)^{\langle\ \rangle} \vdash \Box A \to \Box B^{\langle\ \rangle}} \vdash\to}{\vdash \Box(A \to B) \to (\Box A \to \Box B)^{\langle\ \rangle}} \vdash\to
$$

Axiom D

$$
\dfrac{\dfrac{\dfrac{A^{\langle x\rangle} \vdash A^{\langle x\rangle}}{\Box A^{\langle\ \rangle} \vdash A^{\langle x\rangle}} \Box\vdash}{\Box A^{\langle\ \rangle} \vdash \Diamond A^{\langle\ \rangle}} \vdash\Diamond}{\vdash \Box A \to \Diamond A^{\langle\ \rangle}} \vdash\to
$$

Axiom T

$$
\dfrac{\dfrac{A^{\langle\ \rangle} \vdash A^{\langle\ \rangle}}{\Box A^{\langle\ \rangle} \vdash A^{\langle\ \rangle}} \Box\vdash}{\Box A \to A^{\langle\ \rangle}} \vdash\to
$$

Axiom 4

$$
\dfrac{\dfrac{\dfrac{\dfrac{A^{\langle y,x\rangle} \vdash A^{\langle y,x\rangle}}{\Box A^{\langle\ \rangle} \vdash A^{\langle y,x\rangle}} \Box\vdash}{\Box A^{\langle\ \rangle} \vdash \Box A^{\langle y\rangle}} \vdash\Box}{\Box A^{\langle\ \rangle} \vdash \Box\Box A^{\langle\ \rangle}} \vdash\Box}{\vdash \Box A \to \Box\Box A^{\langle\ \rangle}} \vdash\to
$$

Closure under **MP** is trivially obtained by means of the cut rule. We provide a similar construction in [18] where we study a natural deduction formulations of e–systems.

Finally, closure under **NEC** is obtained by showing that all positions in a provable sequent may be "lifted" by any prefix. Observe first that, for $\Gamma = A_1^{\gamma_1}, \ldots, A_n^{\gamma_n}$, we have $\Gamma[\langle\rangle \Rsh \beta] = A_1^{\beta\circ\gamma_1}, \ldots, A_n^{\beta\circ\gamma_n}$.

PROPOSITION 4.1 (lift). Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$, and let β be a position. If $\Gamma \vdash \Delta$ is provable in $\mathsf{E}_{\mathbb{M}}$, so is the e–sequent $\Gamma[\langle\rangle \Rsh \beta] \vdash \Delta[\langle\rangle \Rsh \beta]$.

PROOF: Like Lemma 2.12: Standard induction on derivations (with a suitable renaming of eigenpositions). It is easily verified that the constraints on the modal rules remain satisfied. $\square$

COROLLARY 4.2 (closure under **NEC**). Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. If $\vdash A^{\langle\ \rangle}$ is provable in $\mathsf{E}_{\mathbb{M}}$ so is the e–sequent $\vdash \Box A^{\langle\ \rangle}$.

We can finally state the first direction of the equivalence result.

THEOREM 4.3. *Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. If $\vdash_{\mathbb{M}} A$, the e–sequent $\vdash A^{\langle\ \rangle}$ is provable in $\mathsf{E}_{\mathbb{M}}$.*

As for the other direction, Fitting introduced tableaux systems for a large class of modal logics (see also Section 5) and proved their equivalence to the corresponding Hilbert style systems [8, pages 398–400]. Labels in Fitting's tableaux play the same role as our positions, and the semantics he proposes works for our systems. In particular, his proof of soundness also readily gives the proof we need. We simply state the result:

THEOREM 4.4. *Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. If the e–sequent $\vdash A^{\langle\ \rangle}$ is provable in $\mathsf{E}_{\mathbb{M}}$, then $\vdash_{\mathbb{M}} A$.*

Alternatively, a direct proof can be found in [18], although formulated in an equivalent natural deduction presentation of our e–sequents.

5. Related work

We discuss in this section some alternative proposals of modal systems, related to our extended sequents.

We start from the 2–sequents/linear nested sequents tradition. We then analyse the work of Fitting [9, 7, 8], Mints [22] and Cerrato [4, 5]. We also make a quick comparison with the so-called *Labeled Deductive Systems*, which represent an important field of studies. A more in-depth comparison, in the setting of natural deduction systems, may be found in [18].

5.1. Starting point: 2–sequents and linear nested sequents

The systems we studied in this paper are, in their current formulation, strongly similar to the ones proposed by Fitting [9, 7, 8] and by Mints [22]. However, our research started from other grounds, that of 2–Sequents [19, 27], especially as presented in [17]. In that paper the first and second authors propose a natural deduction system for the negative fragment $(\to, \wedge, \Box)$ of modal logic, towards a proof theory for the normal modal logics D,T,D4 and S4. If we rephrase in a sequent calculus setting the natural deduction rules of that paper, we obtain the following (intuitionistic) rules for $\Box$, where each formula is decorated with a natural number, representing its *level*:

$$\frac{\Gamma, A^{n+k} \vdash B}{\Gamma, \Box A^{n} \vdash B}\ \Box\vdash \qquad\qquad \frac{\Gamma \vdash A^{n+1}}{\Gamma \vdash \Box A^{n}}\ \vdash\Box$$

In the rule $\vdash \Box$ one requires that, for each formula C^k in Γ, $k \leq n$. Different modal systems are obtained by suitable restrictions of the $\Box\vdash$ rule. For example, if $k = 1$ we have D; if $k < 0$ we have D4, and so on.

The idea works fine for the negative $\bot$-free fragments of the modal logics D, T, D4 and S4, and for the corresponding MELL (Multiplicative Exponential Linear Logic) subsystems [12].

Unfortunately, at that time we could not extend this formulation of 2–Sequents to the full classical modal logics considered in this paper, since the notion of level of a formula is too simple and does not interact well with a standard cut elimination procedure.

Few years ago Lellmann et al. reinterpret 2–Sequents as *Linear Nested Sequents* (LNS) [15], a restricted form of Nested Sequents (in their turn a generalization of relational Hypersequents, see Section 5.5 for some references) where the tree-structure is restricted to a no-branching (linear) structure. Lellmann's reformulation allows to extends 2–Sequents to a large class of logics, also avoiding the complexity of nested sequent calculi. In [15]

Lellmann et al. state a cut elimination theorem through an indirect argument. They prove cut elimination for the standard formulation (no levels, no nested sequents) of the considered modal logics and then obtain a cut elimination statement for LNS by means of a translation into the standard cut free formalisms.

The extended sequents of the present paper result from the realization that to obtain a direct syntactical proof of cut elimination, we must enrich the notion of level in 2–Sequents/LNS, moving to a set of uninterpreted names. Instead of indexed formula A^n, at level n we should have (position) formulas of the shape A^α, with *positions* α of length n (namely $A^{\langle x_1,\dots,x_n\rangle}$). The constraints on levels of [17]—the key point of the system design—can be naturally translated (and extended) in constraints on positions.

Non-surprisingly we obtain a system with interesting similarities with those of Fitting [9, 7, 8], Mints [22] and Cerrato [4, 5]. We focus now on these authors, starting with Fitting's research. Even if Cerrato's system is antecedent to Mints' one, to simplify the presentation we introduce first Mints' tableaux and then we discuss Cerrato's by analogy.

5.2. Fitting's indexed tableaux

From now on, we use the standard notion of tableau for classical logic as given in Bell and Machover's textbook [2].

In Fitting's prefixed tableaux, formulas are labeled with a prefix α. Intuitively, one could think of $\alpha.A$ as saying A is true at the world named by α; and that a prefix $\alpha.\beta$ is a naming for a world that is accessible from the world that is named by α. Despite the semantical meaning and intent of prefixes, we can identify the notion of prefix with that of position—a prefixed formula $\alpha.A$ can be viewed[1] as a position formulas A^α. Fitting introduces two kinds of rules, that he calls π and ν rules that model the behaviour of modal operators and of their negations. Here we use directly the symbols $\Box$ and $\Diamond$ and we take into account only the rules for $\Box$ in the basic normal system K (obviously the rules for $\Diamond$ are symmetric). To facilitate the reading, we adopt our notation:

[1]In Fitting's [8] tokens are natural numbers, but this is simply syntactic sugar.

$$\textbf{$\Box$–rule:} \qquad \Pi = \left\{ \begin{array}{c} \vdots \\ \Box A^{\alpha} \\ \vdots \end{array} \right. \atop {\begin{array}{c} | \\ A^{\alpha\circ\beta} \end{array}} \qquad\qquad \textbf{$\neg\Box$–rule:} \qquad \Pi = \left\{ \begin{array}{c} \vdots \\ \neg\Box A^{\alpha} \\ \vdots \end{array} \right. \atop {\begin{array}{c} | \\ \neg A^{\alpha\circ x} \end{array}}$$

where Π is the branch that is extended by means of the rule and the following constraints hold: (i) in the $\Box$–rule the prefix $\alpha \circ \beta$ is not new in the branch of the tableaux; (ii) in the $\neg\Box$–rule the prefix $\alpha \circ x$ has to be fresh.

The constraints above match the constraints we introduced in e–sequents for K. In particular, for $\Box\vdash$ we require that β is a singleton, and that the main formula has at least a sentinel in the context (notice the analogy with Fitting's $\Box$–rule); for $\vdash\Box$ we force the usual constraint on the quantification, by requiring that $\alpha \circ x$ does not appear as a prefix in any formula of the contexts. Also in this latter case we can observe the analogy with the $\neg\Box$–rule.

As shown in [9], indexed tableau can be easily "overturned" to obtain prefixed sequent and translated to obtain nested sequents, and conversely. Fitting's tableaux (as well as their reformulation and translations) enjoy a form of modularity. Prefixed tableaux can be easily reconfigured to move from K to D: if one no longer requires that on the rule $\vdash\Box$ above the prefix $\alpha \circ \beta$ is not new in the branch, it is possible to derive the D axiom. This happens very similarly in our framework: constraints on $\Box\vdash$ for K (and $\mathsf{K4}$) prevents the derivation of the p-formula $(\Box A \rightarrow \Diamond A)^{\gamma}$. Instead, for the other normal logics, specific rules (modeling the characteristic axioms) must be added to those for K.

5.3. Mints' sequents

In [22] G. Mints introduces a calculus for a family of modal logic inspired by Kripke's semantic tableaux. Even if the paper uses the term "tableau", the framework is technically a sequent calculus. To facilitate the comparison, we show how to reformulate the calculus of Mints in terms of our e–sequents. To avoid misunderstanding, from now on we call *sequents* the standard (i.e. non indexed) sequents and *indexed sequents* (in Mints' terminology) expressions of the kind $\alpha(\Gamma \vdash \Delta)$, where α is a position and

$\Gamma \vdash \Delta$ is a sequent (in particular a sequent may be seen as an indexed sequent with $\langle\rangle$ as index.)

The objects of Mints' calculus finite are multisets of indexed sequents called *tableaux*. A tableau is therefore a multiset of the shape

$$\Gamma_0 \vdash \Delta_0; \alpha_1(\Gamma_1 \vdash \Delta_1); \cdots ; \alpha_n(\Gamma_n \vdash \Delta_n)$$

where each $(\Gamma_i \vdash \Delta_i)$ is a sequent. In spite of the name, a tableaux is then an e–sequent, under the following translation (up to exchange rules):

$$\Gamma_0 \vdash \Delta_0; \alpha_1(\Gamma_1 \vdash \Delta_1); \cdots ; \alpha_n(\Gamma_n \vdash \Delta_n) \rightsquigarrow \\ \Gamma_0, \Gamma_1^{\alpha_1}, \ldots \Gamma_n^{\alpha_n} \vdash \Delta_0, \Delta_1^{\alpha_1}, \ldots \Delta_n^{\alpha_n}$$

Under this interpretation, we can now compare Mints' rules with those of e–sequents.

The rules for K are the same as our rules, although modularity is obtained differently than in e–sequents. Mints defines a relation r between positions that at first glance seems similar to our notion of $\mathfrak{Init}[\cdot]$. This is not the case. For example, it is true that $\alpha \ r \ \alpha \circ z$, but $(\alpha, \alpha \circ \beta) \notin r$ if the length of β is greater than 1. In particular Mints forces that r is not transitive. The way r is defined and used to formulate the different logics do not allow to use it to handle directly the transitivity.

Transitivity (axiom $\mathsf{4}$) is obtained by "adding" to the basic rule of K the following one (expressed in our notation):

$$\frac{\Gamma, \Box A^{\alpha \circ z} \vdash \Delta}{\Gamma, \Box A^{\alpha} \vdash \Delta}$$

Also the system for KT is quite different. In fact, Mints introduces two rules for $\Box \vdash$. The basic rule for K (with the constraint that there must be a sentinel in the sequent) plus a new one:

$$\frac{\Gamma, A^{\alpha} \vdash \Delta}{\Gamma, \Box A^{\alpha} \vdash \Delta}$$

The two rules cannot be merged, since the basic rule for $\Box \vdash$ has to satisfy a suitable constraint.

Mints' formulation of the rule allows to prove a cut elimination theorem, at the price of having a proof that does not follow the standard steps of cut elimination for classical first order logic.

5.4. Cerrato's modal tree sequents

In [4, 5] Cerrato proposes *modal tree-sequents* as a formalism for a family of normal modal logics, from K to $\mathsf{S5}$. A modal tree-sequent is indeed a tree of sequents. In spite of a heavy graphical formalism, tree sequents correspond (modulo a direct simple translation) to Mints' tableaux, and, more interestingly, with the same rules. All systems share the same right $\Box$ rule, while dedicated left $\Box$ rules allow the derivation of the characteristic axioms of the different logics. Moreover, both Mints and Cerrato manage transitivity in the same way. If $\mathcal{F}$ is the set of formulas, one can define a function $\mathsf{index} : \mathcal{P}(\mathcal{F}) \to X^*$ that returns the position of the node in the tree. Starting from a "labeled version" of Cerrato's tree sequents, one can easily define a translation into Mints-style multiset of sequents. Therefore what we said for Mints also applies to Cerrato's tree sequents. In particularly, differently from Cerrato, we insist that we obtain a syntactical proof of cut elimination via the same standard argument which is used for first order logic, by leaning on a Mix Lemma (see [25]).

5.5. Other systems

In the previous subsections, we focused on systems strongly similar to our proposal. In the literature, of course, there are many other proof-theoretical approaches to modal logics. Among these, *display calculi* [28], (relational) *hypersequents* [1, 23, 6], and *labelled deductive systems* (LDS) [10, 24, 26, 23], on which we conclude our review.

At a first glance, our system (or those of Fitting-Cerrato-Mints) seems just a syntactical variant (a "rephrasing") of LDS. One can define a translation (objectively, quite cumbersome) of our extended sequents into the formalism of LDS. We present a detailed comparison (formulated in a natural deduction version of the present system) in our [18]. That one system could be translated into another one does not mean that the two are the same, or that one of them is uninteresting (think, for example, about natural deduction and the calculus of sequents).

The basic idea of the translation is to associate a new label a_i to each position and then define suitable relational formulas: each position $\langle x_1, \ldots, x_n \rangle$ is translated into a set of formulas $\{a_0 R a_1, \ldots, a_{n-1} R a_n\}$[2]. These relational formulas are treated in LDS with explicit logical rules,

[2] The simpler $\{x_1 R x_2, \ldots, x_{n-1} R x_n\}$ would not work.

whereas in our e–systems positions are treated in the same way as the terms of the first-order logic, thus with no need for additional special machinery.

For example, seriality and transitivity are handled in LDS through the following rules (for details, see e.g. [23]):

$$\frac{\Gamma, aRb \vdash c : A, \Delta}{\Gamma, \vdash c : A, \Delta}\ \mathit{seriality} \qquad\qquad \frac{\Gamma, aRb, bRc \vdash c : A, \Delta}{\Gamma, aRc \vdash c : A, \Delta}\ \mathit{trans}$$

Dispensing from ad hoc rules like these is the very purpose of e–systems, see [18] for more details.

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