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Modeling economic growth with spatial migration: A stability analysis of the long-run equilibrium based on semigroup theory

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Abstract

We develop a new model of economic growth that fully accounts for the spatial dependence of the flows of workers and capital on salaries and returns. Considering a rather general setting in which we do not require the knowledge of the exact expressions of the production function and the population growth rate, we allow for a strictly positive steady state in which labor, capital, wages, and returns on capital are constant in space. Then, we establish the stability of such an equilibrium using the theory of abstract non-linear parabolic problems and analyzing the spectral properties of the derivative of the operator that describes the coupled dynamics of labor and capital. We present numerical simulations which agree with our theoretical investigation and show that the proposed model allows us to detect interesting transitional dynamics that the standard Solow model does not capture. In particular, we see how the migration of labor can slow down the process of wage convergence at some spatial locations and how the flow of workers, interacting with population dynamics, can reduce economic growth in countries where both developed and underdeveloped regions are present.

Keywords: Economic growth; spatial analysis; labor and capital migration; stability; abstract evolution problem.

1 Introduction

Traditional (neoclassical) models of economic growth explain the development of countries based on an aggregate production function in which capital and labor are the main inputs. Production factors are not homogeneously distributed nor constant across space, and an emerging discipline, the so-called economic geography, points out the importance of considering space variations too (see e.g., Fujita et al. (1999), Fujita and Thisse (2002), Isard and Liossatos (1979)).

Therefore, in recent years, models of economic growth have been proposed that account for the non-uniform spatial distribution of the main macroeconomic variables. In Ballestra (2016), Boucekkine et al. (2013a), Boucekkine et al. (2013b), Boucekkine et al. (2009), Brito (2004), Brito (2011), Camacho and Zou (2004), Camacho et al. (2008), Fabbri (2016), Xepapadeas and Yannacopoulos (2016). Xepapadeas and Yannacopoulos (2023) and Zhong and Huang (2018), capital (either total or per capita), which is assumed to be the only production factor, is allowed to diffuse across space. This approach has been extended in several directions: for example, Capasso et al. (2010) considers the dependence of production on technological progress, whereas Aniţa et al. (2013), Aniţa et al. (2015), and La Torre et al. (2015) include the effect of pollution.

Besides capital, another variable that plays an important role in aggregate production is labor, the geographically varying pattern of which is dealt with by some authors. For instance, Boucekkine et al. (2019) investigates the extent to which an inhomogeneous spatial distribution of labor and technology causes capital agglomeration, even though they specify the spatial distribution of labor exogenously, without modeling the migration of workers. One of the very earliest socio-demographic models that try to explain migration on a sound mathematical basis is that proposed by Hotelling (1921), who assumes that population growth follows a logistic law and migration takes place from regions with higher densities of individuals to regions with lower densities of individuals. Puu (1985) and Puu (1989) have successively extended this model in various directions, whereas Mossay (2003) and Mossay (2013) have proposed migration models taking also into account the higher utility of consumption that workers can find in neighboring locations.

It is worth noting that in Hotelling (1921), Mossay (2003), Mossay (2013), Puu (1985) and Puu (1989), the spatial dynamics of capital is not taken into account. Instead, Brito (2005), Neto and Claeyssen (2015) and Neto et al. (2014) have proposed models of economic growth that take into account the migration of both labor and capital, which is assumed to follow a Fourier-type diffusion across space. However, none of the aforementioned approaches to migration considers the effect of salaries on the spatial distribution of labor. In addition, in the spatial models of economic growth developed so far, the mutual interaction between the migration of labor and the level of wages is not accounted for (at least to the very best of our knowledge).

Nevertheless, as pointed out in several studies, salaries are one of the main determinants of labor migration. For example, Hicks, in his book (Hicks (1932)), which dates back to 1932 and is still considered a reference point in labor economics nowadays, states that 'differences in net economic advantages, chiefly differences in wages, are the main causes of migration'. Moreover, according to the so-called human capital theory of migration (see, e.g., Harris and Todaro (1970) and Sjaastad (1962)), an individual moves from a given place to another one if the extra earnings associated with the new location exceed the cost of moving. Finally, in Camacho (2013), a model of spatial migration is proposed in which skilled workers move in search of the best lifetime earning, and Kennan and Walker (2011) finds that migration decisions are determined to a substantial extent by income prospects.

In addition, in the spatial models of economic growth available in the literature, the (nonlinear) dependence of the flux of capital on the cost of capital is not fully taken into account, even if several scholars (see, e.g., Labban (2008), Uribe and Schmitt-Grohé (2017), Webber (1987)) agree on the fact that is one of the main determinants of capital movements.

In the present paper, we develop an economic growth model of the Solow type (with exogenous saving) in which both capital and workers can move across space in search of better earnings. More precisely, workers move from regions with lower salaries to regions with higher salaries, and capital moves from regions with lower returns on capital to regions with higher returns on capital. Moreover, we fully endogenize wages and returns on capital by applying the marginal revenue productivity theory, i.e., we link the spatial gradients of labor and capital to wages and returns on capital by assuming that productive firms are profit maximizers.

By taking into account constant returns to scale and a law of population growth with an equilibrium point, we allow for a (strictly positive) steady state in which labor, capital, wages, and returns on capital are constant in space. Moreover, considering a very general setting in which the knowledge of the exact functional forms of the production function and of the population growth rate is not required, we prove that such an equilibrium is asymptotically stable. In particular, to investigate the stability of the spatially uniform steady state, we follow the theory of abstract evolution problems that has been developed in Da Prato and Grisvard (1979), Da Prato and Lunardi (1988), Lunardi (1988), Lunardi (1995), and we regard the model as an infinite-dimensional abstract problem in a suitable Banach space. Specifically, we show that the Fréchet derivative of the operator that governs the evolution of capital and labor is a sectorial operator (so, it generates an analytic semigroup) and its spectrum is formed by eigenvalues with negative (and isolated from zero) real part. Then, even though the proposed model is nonlinear, the so-called principle of linearized stability holds, i.e., for the purpose of proving stability, we can linearize the full model around the stationary solution with no loss of rigor.

It is worth pointing out that the theory of semigroups of differential operators has been applied to a model of economic growth also by Calvia et al. (2021), who propose an ingenious semigroup approach for investigating a family of optimal control problems in infinite-dimensional spaces under positivity state constraints. However, we also highlight the following differences between the work of Calvia et al. (2021) and the present paper. First, Calvia et al. (2021) consider linear state equations (see equation (2.1) in Calvia et al. (2021)), whereas the state equations of the model we propose are non-linear. Moreover, when applying the semigroup approach to economic problems, Calvia et al. (2021) consider the so-called AK growth model, which takes only into account the space-time evolution of capital, whereas in the present paper we deal with the (coupled) dynamics of both capital and labor.

From an economic modeling standpoint, even though the uniform steady state solution coincides with that of the standard (non-spatial) Solow model, the present manuscript brings totally new contributions to the existing literature. First, we show that the economic-driven migration of labor and capital does not alter the local stability of the long-run equilibrium. Regarding this point, we note that a stability analysis of the Solow model with migration of labor and capital was still lacking, even though it is not a-priori guaranteed that the steady state remains stable when workers and capital are allowed to move across space. Second, we find out and investigate interesting transitional dynamics that the standard Solow model does not describe. For instance, we see how the migration of labor, even though it accelerates the process of salary equalization, can slow down the convergence of wages at some space locations. Furthermore, we also consider the case of a country in which both a developed and an underdeveloped region are present, showing how the migration of workers can cause a decline in overall economic growth. This phenomenon, which tends to disappear if capital is allowed to move, is due to the interaction between the migration of workers and natural population growth. It is also worth pointing out that the transitional dynamics we highlight in this paper agree with previous empirical studies (see Section 6).

The paper is organized as follows: in Section 2, we develop the spatial model (the partial differential problem that governs the dynamics of labor and capital is derived in Appendix A); in Section 3, we introduce the spatially homogeneous equilibrium; in Section 4, we formalize the abstract evolution problem and states the main stability result; in Section 5, we give the proof of the stability theorem introduced in Section 4; in Section 6, we perform a numerical investigation of the model, highlighting some interesting dynamics that can arise in finite time intervals; finally, in Section 7, we conclude.

2 The economic model

Following, e.g., Boucekkine et al. (2013a), Boucekkine et al. (2013b), Calvia et al. (2021), Mossay (2003), let us consider an economy where the population extends along the unit circle in the plane, i.e., $\boldsymbol{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Then, we set $\Omega = [0, 2\pi]$, and let $\theta \in \Omega$ denote the curvilinear abscissa along the circle (measured counter-clockwise starting from a fixed given point), with $\theta = 0$ and $\theta = 2\pi$ being identified. At any space location θ, at any time t the capital $K(\theta, t)$ and the labor $L(\theta, t)$ are used to produce an aggregate good $Y(\theta, t)$ through some production function $F(L(\theta, t), K(\theta, t))$. For clear economic reasons, we should have $K(\theta, t) \geq 0$, $L(\theta, t) \geq 0$ and $F(L(\theta, t), K(\theta, t)) \geq 0$ for every $\theta \in [0, 2\pi]$ and $t \in [0, +\infty)$.

In the present paper, we do not need to specify the exact functional form of F, but we shall only require very standard assumptions about it. In particular, we assume that productive firms operate in a competitive market and production has constant returns to scale. This implies that

$$F(L, K) = Lf(k), \tag{1}$$

where

$$k = \frac{K}{L}, \qquad f(k) = F(1, k). \tag{2}$$

We make the assumption that f is a continuous, non-negative function defined on the domain $[0, +\infty)$, and has continuous derivatives up to the fifth order in $(0, +\infty)$. Furthermore, we assume that f is strictly increasing, strictly concave, and satisfies the Inada conditions (see Inada (1963))

$$\lim_{k \to 0^+} f'(k) = +\infty, \quad \lim_{k \to +\infty} f'(k) = 0. \tag{3}$$

For instance, the often employed Cobb-Douglas production function (see Acemoglu (2008)) satisfies all the above requirements.

Then, let w and r denote the wage and the return on capital, which, according to the theory of production, represent the costs of labor and capital, i.e., the remunerations that workers and capital owners require for supplying their labor force and capital to producing firms (see, e.g., Zhang et al. (2008)). Following a neoclassical economic perspective (Acemoglu (2008), Zhang et al. (2008)), we assume that the labor and capital markets are frictionless (all the labor force and capital are employed for production) and firms are profit maximizers. Therefore, given w and r, the labor L and the capital K are those that maximize the profit function

$$\Pi(L, K) = F(L, K) - wL - rK. \tag{4}$$

By taking the derivatives of $\Pi(L, K)$ and setting them equal to zero, we obtain:

$$w = \frac{\partial F(L, K)}{\partial L}, \quad r = \frac{\partial F(L, K)}{\partial K}. \tag{5}$$

2.1 The dynamics of labor

To model the spatial dynamics of labor, we make the following two assumptions:

Assumption (L1): As a consequence of births and deaths, the population grows at a rate $\eta(L)$ that depends on the population level itself. We make the assumptions that η has continous derivatives up to the second order in $(0, +\infty)$ and admits a unique and stable (strictly) positive equilibrium, i.e., there exists a unique $\overline{L} > 0$ such that

$$\eta(\overline{L}) = 0, \quad \eta'(\overline{L}) < 0. \tag{6}$$

This assumption is common to several models of population growth, e.g., the logistic, the Gompertz, and the Richards models (see, for instance, Kot (2001) and Simpson et al. (2022)).

Assumption (L2): Workers can migrate from one space location to the other. Following, for example, Camacho (2013), Harris and Todaro (1970), Hicks (1932), Sjaastad (1962), we assume that migration is driven by wage gradients, that is, workers move from regions with lower wages to regions with higher wages.

To formalize this assumption, let us consider any $\theta \in (0, 2\pi)$ and $t \in (0, +\infty)$. Moreover, let $N(\theta, t, t + \Delta t)$ denote the net amount of labor migration in the time interval $[t, t + \Delta t)$ at location θ. More precisely, $N(\theta, t, t + \Delta t)$ is the number of workers that in the time interval $[t, t + \Delta t)$ move counter-clockwise along the circle crossing location θ. Let us consider the net flux of labor at the time t at location θ, which is defined as follows:

$$\tau_L(\theta, t) = \lim_{\Delta t \to 0^+} \frac{N(\theta, t, t + \Delta t)}{\Delta t}, \quad \theta \in (0, 2\pi), \quad t \in (0, +\infty). \tag{7}$$

We assume that the flux of labor is proportional to the wage gradient:

$$\tau_L(\theta, t) = b \frac{\partial w(\theta, t)}{\partial \theta}, \quad \theta \in (0, 2\pi), \quad t \in (0, +\infty), \qquad b > 0. \tag{8}$$

Relation (8) implies that at any location θ, in the time interval $[t, t + dt)$ the amount $b\frac{\partial w(\theta,t)}{\partial \theta} dt$ of workers migrates in search of better earnings.

According to (1), (2), and the first of equations (5), we have

$$w(\theta,t) = f(k(\theta,t)) - k(\theta,t)f'(k(\theta,t)), \quad \theta \in (0,2\pi), \quad t \in (0,+\infty), \tag{9}$$

and, using (8) and (9), we obtain

$$\tau_L(\theta,t) = -bk(\theta,t)f''(k(\theta,t))\frac{\partial k(\theta,t)}{\partial \theta}, \quad \theta \in (0,2\pi), \;\; t \in (0,+\infty). \tag{10}$$

Note that the parameter b controls the amount of the labor force that migrates due to differences in salaries. Then, this parameter measures the extent to which differences in salary impact the flow of migrating workers.

As shown in Appendix A, upon all the previous assumptions, the dynamics of labor is governed by the following partial differential equation:

$$\frac{\partial L(\theta,t)}{\partial t} = b\frac{\partial}{\partial \theta}\left(k(\theta,t)f''(k(\theta,t))\frac{\partial k(\theta,t)}{\partial \theta}\right) + \eta(L(\theta,t)), \quad \theta \in (0,2\pi), \;\; t \in (0,+\infty). \tag{11}$$

Equation (11) highlights the two (non-linear) contributions that govern the dynamics of labor, i.e., natural population growth and spatial migration.

2.2 The dynamics of capital

To derive the dynamics of capital, let us consider any $\theta \in (0,2\pi)$ and $t \in (0,+\infty)$. In the time interval $[t, t+\Delta t)$, the capital at the (generic) spatial location θ varies due to a) the output saved at location θ, which increases the stock of capital; b) the flow of capital from (to) location θ to (from) neighboring locations; c) the deterioration of capital at location θ.

Concerning the flow of capital, we assume that capital moves from regions with a lower return on capital to regions with a higher return on capital. This approach is followed, for example, in Anţita et al. (2015), Boucekkine et al. (2013a), Boucekkine et al. (2013b), Neto et al. (2014), Neto and Claeyssen (2015), Xepapadeas and Yannacopoulos (2023). However, in the present paper we explicitly link the return on capital (and hence the flux of capital) to the production factors through the production function itself. To the best of our knowledge, none of the spatial models of economic growth proposed so far uses the above approach, at least to the best of our knowledge.

Specifically, we make the following three assumptions:

Assumption (K1): The flux of capital is proportional to the gradient of the return on capital:

$$\tau_K(\theta,t) = a\frac{\partial r(\theta,t)}{\partial \theta}, \quad \theta \in (0,2\pi), \;\; t \in (0,+\infty). \tag{12}$$

Note that the parameter a controls the amount of the capital that migrates in search of better investment opportunities. Thus, this parameter measures the extent to which differences in returns on investments impact the flow of migrating capital.

According to (1), (2), and the second of equations (5), we have

$$r(\theta,t) = f'(k(\theta,t)), \qquad \theta \in (0,2\pi), \;\; t \in (0,+\infty), \tag{13}$$

so that, using (12), we obtain

$$\tau_K(\theta,t) = af''(k(\theta,t))\frac{\partial k(\theta,t)}{\partial\theta}, \qquad \theta\in(0,2\pi),\ \ t\in(0,+\infty). \tag{14}$$

Assumption (K2): As in the classical Solow model (see Solow (1956)), the aggregate production Y is partly re-invested in production (thus forming new capital) and partly consumed. Specifically, consumption is assumed to be equal to

$$C(\theta,t) = (1-s)Y(\theta,t), \quad \theta\in(0,2\pi),\ \ t\in(0,+\infty), \tag{15}$$

where s is the (exogenous) saving rate. It then follows that the amount of aggregate output that is re-invested in production is equal to $sY(\theta,t) = sF(L(\theta,t),K(\theta,t))$, $\theta\in(0,2\pi)$, $t\in(0,+\infty)$.

Assumption (K3): Capital deteriorates at the constant depreciation rate δ.

It can be easily shown (see Appendix A) that all the above assumptions lead to the following partial differential equation:

$$\frac{\partial K(\theta,t)}{\partial t} = -a\frac{\partial}{\partial\theta}\left(f''(k(\theta,t))\frac{\partial k(\theta,t)}{\partial\theta}\right) + sF(L(\theta,t),K(\theta,t)) - \delta K(\theta,t),$$
$$\theta\in(0,2\pi),\ \ t\in(0,+\infty). \tag{16}$$

As is usually done for the classical Solow-Swan model with no spatial effects, equation (16) can be conveniently rewritten based on the capital per capita k defined as in (2).

In particular, by straightforward differentiation in the first of (2), we have

$$\frac{\partial K(\theta,t)}{\partial t} = L(\theta,t)\frac{\partial k(\theta,t)}{\partial t} + k(\theta,t)\frac{\partial L(\theta,t)}{\partial t}, \qquad \theta\in(0,2\pi),\ \ t\in(0,+\infty). \tag{17}$$

Then, if we substitute (17) in (16) and we replace $\frac{\partial L(\theta,t)}{\partial t}$ with the right hand side of (11), we obtain

$$\frac{\partial k(\theta,t)}{\partial t} = -\frac{a}{L(\theta,t)}\frac{\partial}{\partial\theta}\left(f''(k(\theta,t))\frac{\partial k(\theta,t)}{\partial\theta}\right) - \frac{bk(\theta,t)}{L(\theta,t)}\frac{\partial}{\partial\theta}\left(k(\theta,t)f''(k(\theta,t))\frac{\partial k(\theta,t)}{\partial\theta}\right)$$
$$-\frac{k(\theta,t)}{L(\theta,t)}\eta(L(\theta,t)) + sf(k(\theta,t)) - \delta k(\theta,t), \qquad \theta\in(0,2\pi),\ \ t\in(0,+\infty). \tag{18}$$

2.3 The partial differential problem

The partial differential equations (11) and (18) must be equipped with suitable conditions for $\theta = 0$ and $\theta = 2\pi$. Since $\theta = 0$ and $\theta = 2\pi$ correspond to the same point of the circle along which the economy extends, labor, capital, the fluxes of labor and capital, and the spatial derivatives of the fluxes of labor and capital (which measure the amount of labor and capital migrating through any given location) at $\theta = 0$ and $\theta = 2\pi$ must coincide. Therefore, we impose:

$$L(0,t) = L(2\pi,t), \quad k(0,t) = k(2\pi,t), \quad t \in (0,+\infty),$$
$$\left.\frac{\partial k(\theta,t)}{\partial \theta}\right|_{(0,t)} = \left.\frac{\partial k(\theta,t)}{\partial \theta}\right|_{(2\pi,t)}, \left.\frac{\partial^2 k(\theta,t)}{\partial \theta^2}\right|_{(0,t)} = \left.\frac{\partial^2 k(\theta,t)}{\partial \theta^2}\right|_{(2\pi,t)}, \quad t \in (0,+\infty). \tag{19}$$

Finally, labor and capital are assumed to be known at some initial time $t = 0$, i.e., we set:

$$L(\theta,0) = L_0(\theta), \quad k(\theta,0) = k_0(\theta), \quad \theta \in [0,2\pi], \tag{20}$$

where $k_0(\theta) = \frac{K_0(\theta)}{L_0(\theta)}$, and L_0 and K_0 denote the initial distributions of labor and capital, respectively. By putting together (11), (18), (19) and (20), we obtain the following partial differential problem:

$$\begin{cases} \dfrac{\partial L(\theta,t)}{\partial t} = b\dfrac{\partial}{\partial \theta}\left(k(\theta,t)f''(k(\theta,t))\dfrac{\partial k(\theta,t)}{\partial \theta}\right) + \eta(L(\theta,t)), \quad \theta \in (0,2\pi), \ t \in (0,+\infty), \\ \dfrac{\partial k(\theta,t)}{\partial t} = -\dfrac{a}{L(\theta,t)}\dfrac{\partial}{\partial \theta}\left(f''(k(\theta,t))\dfrac{\partial k(\theta,t)}{\partial \theta}\right) - \dfrac{bk(\theta,t)}{L(\theta,t)}\dfrac{\partial}{\partial \theta}\left(k(\theta,t)f''(k(\theta,t))\dfrac{\partial k(\theta,t)}{\partial \theta}\right) \\ \qquad -\dfrac{k(\theta,t)}{L(\theta,t)}\eta(L(\theta,t)) + sf(k(\theta,t)) - \delta k(\theta,t), \quad \theta \in (0,2\pi), \ t \in (0,+\infty), \\ L(0,t) = L(2\pi,t), \quad k(0,t) = k(2\pi,t), \ t \in (0,+\infty), \\ \left.\dfrac{\partial k(\theta,t)}{\partial \theta}\right|_{(0,t)} = \left.\dfrac{\partial k(\theta,t)}{\partial \theta}\right|_{(2\pi,t)}, \left.\dfrac{\partial^2 k(\theta,t)}{\partial \theta^2}\right|_{(0,t)} = \left.\dfrac{\partial^2 k(\theta,t)}{\partial \theta^2}\right|_{(2\pi,t)}, \quad t \in (0,+\infty), \\ L(\theta,0) = L_0(\theta), \quad k(\theta,0) = k_0(\theta), \quad \theta \in [0,2\pi]. \end{cases} \tag{21}$$

3 The spatially homogeneous equilibrium

The model of economic growth developed in the previous section has a (strictly positive) spatially homogeneous equilibrium. This is a solution of (21) of the form

$$L(\theta,t) = \overline{L}, \quad k(\theta,t) = \overline{k}, \quad \theta \in [0,2\pi], \ t \in [0,+\infty), \quad \overline{L} > 0, \ \overline{k} > 0, \tag{22}$$

according to which labor and capital per capita are constant across both space and time. This is a remarkable property from the socio-economic standpoint, since it implies that all the macroeconomic quantities are uniform all across the economy, with no spatial disparities. In particular, setting $\overline{K} = \overline{k}\,\overline{L}$, it immediately follows that capital, wages and returns on capital are constant in the space and equal to

$$K(\theta,t) = \overline{K}, \quad w(\theta,t) = F_L(\overline{L},\overline{K}), \quad r(\theta,t) = F_K(\overline{L},\overline{K}), \quad \theta \in [0,2\pi], \ t \in [0,+\infty), \tag{23}$$

respectively. Moreover, $L(\theta,t)$ and $k(\theta,t)$ as in (22) satisfy problem (21) if and only if

$$L_0(\theta) = \overline{L}, \quad k_0(\theta) = \overline{k}, \ \theta \in [0,2\pi], \tag{24}$$

and

$$\eta(\overline{L}) = 0, \quad sf(\overline{k}) - \delta\overline{k} = 0. \tag{25}$$

The second of equations (25) is actually the same equation that yields the equilibrium in the classical Solow model with no spatial effects. Therefore, it is satisfied by a unique strictly positive $\overline{k}$. In particular, let us consider the function $\hat{f} = sf(k) - \delta k$. Then, $\hat{f}$ is strictly concave and such that $\hat{f}(0) \geq 0$ (as f is non-negative and strictly concave). Moreover, since f' is strictly decreasing and thanks to the Inada conditions (3), the derivative of $\hat{f}$ with respect to k changes sign only once, from positive to negative, in $(0, +\infty)$, and tends to $-\delta$ as $k \to +\infty$. It then follows that there exists a unique $\overline{k} \in (0, +\infty)$ such that $sf(\overline{k}) - \delta\overline{k} = 0$. Moreover, we have:

$$sf'(\overline{k}) - \delta < 0. \tag{26}$$

Then, from assumptions (6) it follows that a unique equilibrium of the form (22) exists.

4 Stability of the spatially homogeneous steady-state

An important point to address is the stability of the spatially homogeneous equilibrium. That is, we would like to determine whether the steady state (22) is stable to shocks on the initial distribution of labor and capital. To accomplish this task, we shall apply the theory of abstract non-linear evolution problems that has been initiated by Da Prato and Grisvard (1979), has been pursued, among others, by Da Prato and Lunardi (1988), Lunardi (1988), Lunardi (1991) and Lunardi (1992), and is thoroughly and rigorously explained in Lunardi (1995) and Lunardi (2002). For the sake of clarity, we divide the theoretical analysis in two sections. In this section, we present and discuss our main finding (Theorem 1), whereas its proof, which is based on the theory of semigroups of operators and requires tools of functional analysis, is given in Section 5.

Here, we emphasize that the approach followed is based on a suitable linearization of the parabolic problem that governs the evolution of labor and capital, but, according to the results in Lunardi (1995), the stability of the equilibrium holds for the full (non-linear) model.

To measure the deviations of labor and capital per capita from the spatially homogeneous equilibrium we set:

$$\hat{L}(\theta, t) = L(\theta, t) - \overline{L}, \quad \hat{k}(\theta, t) = k(\theta, t) - \overline{k}, \quad \theta \in [0, 2\pi], \ t \in [0, +\infty). \tag{27}$$

$$\hat{L}_0(\theta) = L_0(\theta) - \overline{L}, \quad \hat{k}_0(\theta) = k_0(\theta) - \overline{k}, \quad \theta \in [0, 2\pi], \ t \in [0, +\infty). \tag{28}$$

By using the change of variables (27), the partial differential problem (21) is transformed as follows:

$$
\begin{cases}
\dfrac{\partial \hat{L}(\theta,t)}{\partial t} = b\dfrac{\partial}{\partial \theta}\left((\overline{k}+\hat{k}(\theta,t))f''(\overline{k}+\hat{k}(\theta,t))\dfrac{\partial \hat{k}(\theta,t)}{\partial \theta}\right)+\eta(\overline{L}+\hat{L}(\theta,t)), \\
\qquad\qquad\qquad\qquad \theta\in(0,2\pi),\ \ t\in(0,+\infty), \\
\dfrac{\partial \hat{k}(\theta,t)}{\partial t} = -\dfrac{a}{\overline{L}+\hat{L}(\theta,t)}\dfrac{\partial^2}{\partial \theta^2}\left(f'(\overline{k}+\hat{k}(\theta,t))\right) \\
\qquad -\dfrac{b(\overline{k}+\hat{k}(\theta,t))}{\overline{L}+\hat{L}(\theta,t)}\dfrac{\partial}{\partial \theta}\left((\overline{k}+\hat{k}(\theta,t))f''(\overline{k}+\hat{k}(\theta,t))\dfrac{\partial \hat{k}(\theta,t)}{\partial \theta}\right) \\
\qquad -\dfrac{\overline{k}+\hat{k}(\theta,t)}{(\overline{L}+\hat{L}(\theta,t))}\eta(\overline{L}+\hat{L}(\theta,t))+sf(\overline{k}+\hat{k}(\theta,t))-\delta(\overline{k}+\hat{k}(\theta,t)), \\
\qquad\qquad\qquad\qquad \theta\in(0,2\pi),\ \ t\in(0,+\infty), \\
\hat{L}(0,t)=\hat{L}(2\pi,t), \quad \hat{k}(0,t)=\hat{k}(2\pi,t), \ \ t\in(0,+\infty), \\
\left.\dfrac{\partial \hat{k}(\theta,t)}{\partial \theta}\right|_{(0,t)} = \left.\dfrac{\partial \hat{k}(\theta,t)}{\partial \theta}\right|_{(2\pi,t)}, \quad \left.\dfrac{\partial^2 \hat{k}(\theta,t)}{\partial \theta^2}\right|_{(0,t)} = \left.\dfrac{\partial^2 \hat{k}(\theta,t)}{\partial \theta^2}\right|_{(2\pi,t)}, \quad t\in(0,+\infty), \\
\hat{L}(\theta,0)=\hat{L}_0(\theta), \quad \hat{k}(\theta,0)=\hat{k}_0(\theta), \quad \theta\in[0,2\pi].
\end{cases}
$$

4.1 Formulation of the abstract evolution problem

We regard problem (29) as an abstract evolution problem, and, to this aim, we must determine a suitable function space in which seeking its solution. Now, the choice of the L^2 space is somehow problematic, due to the difficulty of imposing the pointwise constraints $L(\theta,t)\geq 0$ and $k(\theta,t)\geq 0$. The treatment of pointwise constraints is a well-known technical issue when working with the L^2 space (see, e.g., Calvia et al. (2021)), and thus, in our analysis we consider spaces of continuous functions. Specifically, let $C^0(\Omega)$ denote the space of real-valued continuous functions on Ω, with norm:

$$
\|h\|_{C^0(\Omega)} = \max_{\theta\in\Omega}|h(\theta)|, \tag{29}
$$

and let $C^2(\Omega)$ denote the space of real-valued functions on Ω with continuous derivatives up to the second order, with norm:

$$
\|h\|_{C^2(\Omega)} = \|h^{(0)}\|_{C^0(\Omega)} + \|h^{'}\|_{C^0(\Omega)} + \|h^{''}\|_{C^0(\Omega)}. \tag{30}
$$

Moreover, to account for both labor and capital and the requirement $L(0,t)=L(2\pi,t)$, $k(0,t)=k(2\pi,t)$, let us consider the following function space:

$$
X = \left\{ \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in C^0(\Omega), \quad x_1(0)=x_1(2\pi), \quad x_2(0)=x_2(2\pi) \right\}, \tag{31}
$$

which we endow with the norm:

$$
\|\boldsymbol{x}\|_X = \left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right). \tag{32}
$$

Note that we are naming a vector and its elements with the same letter, using bold fonts for the vector and non-bold fonts for its elements.

Finally, since problem (29) involves also the second derivative of k with respect to θ, we also consider the following subspace of X:

$$D = \left\{ \boldsymbol{x} \in X : x_2 \in C^2(\Omega),\ \left.\frac{dx_2(\theta)}{d\theta}\right|_{\theta=0} = \left.\frac{dx_2(\theta)}{d\theta}\right|_{\theta=2\pi},\ \left.\frac{d^2x_2(\theta)}{d\theta^2}\right|_{\theta=0} = \left.\frac{d^2x_2(\theta)}{d\theta^2}\right|_{\theta=2\pi} \right\}, \tag{33}$$

endowed with the norm:

$$\|\boldsymbol{x}\|_D = \|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^2(\Omega)}. \tag{34}$$

The function space D is dense in X, since for any $\boldsymbol{x} \in X$ we can find an element of D that is arbitrarily close to $\boldsymbol{x}$ in the topology induced by the norm in X (we can take a truncated Fourier series of $\boldsymbol{x}$, the details are left to the reader).

We set

$$u_1(t) = \hat{L}(\cdot, t), \quad u_2(t) = \hat{k}(\cdot, t), \quad u_{0,1} = \hat{L}_0(\cdot), \quad u_{0,2} = \hat{k}_0(\cdot). \tag{35}$$

That is, u_1 and u_2 are to be considered as functions of time only, such that, for a generic time t, the images $u_1(t)$ and $u_2(t)$ are the spatial distributions of labor and capital at time t, respectively. Then, let us define:

$$\boldsymbol{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad \boldsymbol{u}_0 = \begin{pmatrix} u_{0,1} \\ u_{0,2} \end{pmatrix}. \tag{36}$$

By using (35) and (36), problem (29) can be conveniently rewritten as a (non-linear) abstract parabolic problem:

$$\frac{d\boldsymbol{u}(t)}{dt} = H(\boldsymbol{u}(t)), \quad t > 0; \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{37}$$

where

$$H(\boldsymbol{x}) = \begin{pmatrix} H_1(\boldsymbol{x}) \\ H_2(\boldsymbol{x}) \end{pmatrix}, \tag{38}$$

$$H_1(\boldsymbol{x}) = b\frac{d}{d\theta}\left((\overline{k} + x_2) f''(\overline{k} + x_2(\theta)) \frac{dx_2(\theta)}{d\theta} \right) + \eta(\overline{L} + x_1), \tag{39}$$

$$\begin{aligned} H_2(\boldsymbol{x}) = &-\frac{a}{\overline{L} + x_1} \frac{d^2}{d\theta^2} \left(f'(\overline{k} + x_2(\theta)) \right) \\ &-\frac{b(\overline{k} + x_2)}{\overline{L} + x_1} \frac{d}{d\theta} \left((\overline{k} + x_2(\theta)) f''(\overline{k} + x_2(\theta)) \frac{dx_2(\theta)}{d\theta} \right) \\ &-\frac{\overline{k} + x_2}{\overline{L} + x_1} \eta(\overline{L} + x_1(\theta)) + sf(\overline{k} + x_2) - \delta(\overline{k} + x_2). \end{aligned} \tag{40}$$

The operator H should be regarded as a non-linear mapping defined in a set $\mathcal{O} \subset D$ with values in X. Precisely, since we will study the stability of the equilibrium to small perturbations (see Theorem 1 below), we can choose $\mathcal{O}$ to be any neighborhood of $\mathbf{0}$ in D, provided that labor and capital remain positive, so we can consider

$$\mathcal{O} = \left\{ \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D : \|\boldsymbol{x}\|_D < \epsilon \right\}, \tag{41}$$

for some positive ϵ such that $\overline{L} - \epsilon > 0$ and $\overline{k} - \epsilon > 0$. Moreover, $H : \mathcal{O} \to X$ is twice Fréchet differentiable in $\mathcal{O}$, thanks to the regularity assumptions on the production function and the population growth rate.

To investigate the stability of the null solution of problem (37), we first need to consider the Fréchet derivative of the operator H at zero, which we denote by A. That is $A = H'(\mathbf{0})$, and by differentiation of (39) and (40) we obtain

$$A\boldsymbol{x} = \begin{pmatrix} a_{1,2}\frac{d^2x_2(\theta)}{d\theta^2} + a_1x_1 \\ a_{2,2}\frac{d^2x_2(\theta)}{d\theta^2} + a_2x_1 + a_3x_2 \end{pmatrix}, \quad \boldsymbol{x} \in D, \tag{42}$$

where

$$\begin{aligned} a_{1,2} = b\overline{k}f''(\overline{k}), \quad a_{2,2} = -\frac{f''(\overline{k})}{\overline{L}}(a + b(\overline{k})^2), \\ a_1 = \eta'(\overline{L}), \quad a_2 = -\frac{\overline{k}}{\overline{L}}\eta'(\overline{L}), \quad a_3 = sf'(\overline{k}) - \delta. \end{aligned} \tag{43}$$

It can be easily checked that A is a closed linear operator from D to X.

According to (Lunardi, 2002, Theorem 2.6) and (Lunardi, 1995, Theorems 8.1.1 and 9.1.2), we will find a solution $\boldsymbol{u}$ of problem (37) that belongs to suitable function spaces. Specifically, let $\mathcal{I}$ denote any (either bounded or unbounded) interval of real numbers. Moreover, let $C(\mathcal{I}; D)$ denote the space of functions from $\mathcal{I}$ to D which are continuous in $\mathcal{I}$, and let $C^1(\mathcal{I}; X)$ denote the space of functions from $\mathcal{I}$ to X which are continuously differentiable in $\mathcal{I}$. In addition, let β denote any real number in $(0, 1)$ and T denote any real number in $(0, +\infty)$. Then, let $C^{\beta}_{\beta}((0, T]; D)$ denote the space of functions from $(0, T]$ to D which are bounded and $\beta-$Hölder continuous in $(0, T]$, that is (see Cuesta and Makridakis (2008) and (Lunardi, 1995, Chapter 3, p. 137))

$$\sup_{\substack{t, s \in (0, T], \\ s < t}} s^{\beta} \frac{\|\boldsymbol{v}(t) - \boldsymbol{v}(s)\|_D}{(t - s)^{\beta}} < +\infty. \tag{44}$$

The uniform equilibrium solution, being such that $\hat{L}$ and $\hat{k}$ are identically null, corresponds to $\boldsymbol{u} = \mathbf{0}$. Therefore, to determine whether the spatially homogeneous steady state is stable, we shall analyze the solution $\boldsymbol{u}$ of problem (37) in response to an initial shock $\boldsymbol{u}_0 \neq \mathbf{0}$. in particular, if $\boldsymbol{u}$ tends to $\mathbf{0}$, then labor and capital tend to their equilibrium values.

As stated by the following theorem, if $\boldsymbol{u}_0$ is sufficiently small (in the norm $\|\cdot\|_D$), then the solution of problem (37) exponentially tends to zero as time tends to infinity.

Theorem 1 *The equilibrium solution is non-linearly locally stable. Precisely, there exists a constant* $\delta > 0$ *such that if* $\|\boldsymbol{u}_0\|_D < \delta$*, problem (37) has a solution* $\boldsymbol{u}$ *that belongs to*

$$C([0, +\infty); D) \cap C^1([0, +\infty); X) \cap C^{\beta}_{\beta}((0, T]; D) \tag{45}$$

for every $\beta \in (0, 1)$ *and* $T \in [0, +\infty)$*. Moreover,* $\boldsymbol{u}$ *is the unique solution of problem (37) belonging to*

$$\bigcup_{0<\beta<1} C^{\beta}_{\beta}((0, T]; D) \cap C([0, T]; D) \tag{46}$$

for every $T \in (0, +\infty)$*. Finally, we have*

$$\|\boldsymbol{u}(t)\|_D \leq Me^{-\omega t}\|\boldsymbol{u}_0\|_D \quad \forall t \geq 0, \tag{47}$$

for some (strictly) positive constants ω *and* M*.*

The proof of this theorem is provided in the next section. Here we simply observe that, according to (45), the solution $t \to \boldsymbol{u}(t)$ is continuous with values in D and continuously differentiable with values in X also in $t = 0$. As shown in (Lunardi, 1995, Chapter 9.1), we obtain a solution with such a regularity in $t = 0$ because the so-called compatibility condition (9.1.6) in Lunardi (1995) is satisfied, that is

$$G(\boldsymbol{u}_0) \in \overline{D}. \tag{48}$$

Indeed, the domain D of the operator A is dense in X and $\boldsymbol{u}_0 \in D$, from which (48) follows immediately.

According to Theorem 1, if the initial shocks on labor and capital are small enough, both the functions $\hat{L}(\theta, t)$ and $\hat{k}(\theta, t)$ tend to zero as t tends to infinity (uniformly in θ), i.e., the spatially homogeneous equilibrium is stable. That is, if the initial distribution of labor and capital is sufficiently close to the spatially homogeneous equilibrium, the model has a unique solution that asymptotically tends to the steady state.

We remark that in this paper we do not analyze the global stability of the spatially homogeneous equilibrium. Similarly, we do not investigate if the spatially homogeneous equilibrium is the only steady state of the model, or if, instead, other steady states exist in which labor and capital are not uniformly distributed across space.

5 A proof of stability based on semigroups of operators

5.1 Resolvent and spectrum of the operator A

To prove Theorem 1, we shall analyze the spectral properties of the operator A. To this aim, let us recall the definitions of resolvent set and spectrum of a linear operator.

First, let us perform the so-called complexification of the space X and of the operator A: for any two elements $\boldsymbol{x}$ and $\boldsymbol{y} \in X$ (each of which, we recall, is a vector of two functions), we consider the complex vector $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$, where i is the imaginary unit, and the complexified function spaces:

$$X_{\mathbb{C}} = \{\boldsymbol{z} : \boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}, \quad \boldsymbol{x} \in X, \quad \boldsymbol{y} \in X\}, \tag{49}$$

$$D_{\mathbb{C}} = \{\boldsymbol{z} : \boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}, \quad \boldsymbol{x} \in D, \quad \boldsymbol{y} \in D\}. \tag{50}$$

The norm (32) can be extended to $X_{\mathbb{C}}$ in a very natural fashion. For any $\boldsymbol{z} \in X_{\mathbb{C}}$, with $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \boldsymbol{x} + i\boldsymbol{y}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ we define the norm

$$\|\boldsymbol{z}\|_{X_{\mathbb{C}}} = \|z_1\|_{X_{\mathbb{C},1d}} + \|z_2\|_{X_{\mathbb{C},1d}} \tag{51}$$

where

$$\|z_i\|_{X_{\mathbb{C},1d}} = \max_{\theta \in \Omega} |z_i(\theta)| = \max_{\theta \in \Omega} \left| \sqrt{x_i^2(\theta) + y_i^2(\theta)} \right|, \quad i = 1, 2. \tag{52}$$

Analogously, we can extend the linear operator A from X to $X_{\mathbb{C}}$ by simply setting, for every $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$, with $\boldsymbol{x}$ and $\boldsymbol{y} \in D$,

$$A\boldsymbol{z} = A\boldsymbol{x} + iA\boldsymbol{y}. \tag{53}$$

Definition 1 *Let I denote the identity operator (which associates any element of $X_{\mathbb{C}}$ to itself). The resolvent set of the closed operator A, which we denote by $\rho(A)$, is the set of all the complex numbers λ such that the operator $\lambda I - A : D_{\mathbb{C}} \to X_{\mathbb{C}}$ is a bijection between $D_{\mathbb{C}}$ and $X_{\mathbb{C}}$. For $\lambda \in \rho(A)$, we set*

$$R(\lambda, A) = (\lambda I - A)^{-1}, \tag{54}$$

and $R(\lambda, A)$ is called resolvent operator. By contrast, the spectrum of A, which we denote by $\sigma(A)$, is the set of all the complex numbers that do not belong to the resolvent set of A.

In the following, we will use the notation $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ to denote the real and the imaginary parts of any complex number.

5.2 Preliminary results

To prove Theorem 1, we will use the following lemma, which follows directly from our assumptions on the production function and the population growth rate.

Lemma 1 *We have:*

$$a_{1,2} < 0, \quad a_{2,2} > 0, \quad a_1 < 0, \quad a_2 > 0, \quad a_3 < 0, \tag{55}$$

$$a_{1,2}a_2 - a_{2,2}a_1 > 0. \tag{56}$$

Proof of Lemma 1. Relations (55) are direct consequences of definitions (43). Moreover, always based on (43), we have

$$a_{1,2}a_2 - a_{2,2}a_1 = \frac{\eta(\overline{L})f''(\overline{k})}{\overline{L}}, \tag{57}$$

and thus, by using the fact that $f''(k) < 0$ and (6), we obtain (56).

Furthermore, to analyze the spectrum of the operator A, we will use the following lemma.

Lemma 2 *Let us consider the function spaces*

$$X_{\mathbb{C},1d} = \{z = x + iy, \quad x, y \in C(\Omega), \quad z(0) = z(2\pi)\}, \tag{58}$$

$$D_{\mathbb{C},1d} = \left\{ z = x + iy, \quad x, y \in C^2(\Omega), \quad z(0) = z(2\pi), \quad \left.\frac{dz(\theta)}{d\theta}\right|_{\theta=0} = \left.\frac{dz(\theta)}{d\theta}\right|_{\theta=2\pi}, \right.$$
$$\left. \left.\frac{d^2z(\theta)}{d\theta^2}\right|_{\theta=0} = \left.\frac{d^2z(\theta)}{d\theta^2}\right|_{\theta=2\pi} \right\}, \tag{59}$$

where, as usual, we set: $\frac{dz(\theta)}{d\theta} = \frac{dx(\theta)}{d\theta} + i\frac{dy(\theta)}{d\theta}$, $\frac{d^2z(\theta)}{d\theta^2} = \frac{d^2x(\theta)}{d\theta^2} + i\frac{d^2y(\theta)}{d\theta^2}$, $\theta \in [0, 2\pi]$.

Moreover, let q *be any complex number such that* $Re(q) > 0$. *Then, for any function* $\widetilde{f} \in X_{\mathbb{C},1d}$ *the differential problem*

$$q^2 z(\theta) - \frac{d^2 z(\theta)}{d\theta^2} = \widetilde{f}(\theta), \quad \theta \in (0, 2\pi), \quad z \in D_{\mathbb{C},1d}, \tag{60}$$

has the unique solution

$$z(\theta) = \frac{1}{2q\sinh(\pi q)} \left[\int_0^\theta \cosh\left(q(\theta - \gamma - \pi)\right) \widetilde{f}(\gamma) d\gamma + \right.$$
$$\left. + \int_\theta^{2\pi} \cosh\left(q(\theta - \gamma + \pi)\right) \widetilde{f}(\gamma) d\gamma \right], \quad \theta \in [0, 2\pi]. \tag{61}$$

Proof of Lemma 2. By straightforward differentiation it is easy to check that (61) is a solution of the differential problem (60), see also Lemma 2.1 in (Pazy, 1983, Section 8.2). To prove uniqueness, let us assume that two solutions $\widetilde{z}_1$ and $\widetilde{z}_2$ of the differential problem (60) exist, and let us define $m = \widetilde{z}_1 - \widetilde{z}_2$. Then, we must have

$$q^2 m(\theta) - \frac{d^2 m(\theta)}{d\theta^2} = 0, \quad \theta \in (0, 2\pi), \quad m \in D_{\mathbb{C},1d}. \tag{62}$$

Let m^* denote the complex conjugate of m. If in (62) we multiply by m^* , integrate over Ω and use integration by parts we obtain (taking also into account that $m \in D_{\mathbb{C},1d}$):

$$\int_0^{2\pi} q^2 |m(\theta)|^2 d\theta + \int_0^{2\pi} \left| \frac{dm(\theta)}{d\theta} \right|^2 d\theta = 0. \tag{63}$$

Using (63) and the fact that $\mathrm{Re}(q) > 0$ it can be easily checked that $m(\theta) = 0$ for every $\theta \in [0, 2\pi]$, and the lemma is proved.

Finally, we have the following proposition, which describes the basic properties of the spectrum of the operator A, which we denote $\sigma(A)$.

Proposition 1 *The set $\sigma(A)$ is not empty. Moreover, let us define*

$$\overline{\lambda} = \frac{1}{2} \max \left\{ a_1, a_3, -\frac{a_{1,2}a_2 - a_{2,2}a_1}{a_{2,2}} \right\}. \tag{64}$$

Then, $\lambda < 0$ and

$$Re(\lambda) \leq \overline{\lambda}, \quad \forall \lambda \in \sigma(A). \tag{65}$$

That is, the spectrum of the operator A contains only points whose real parts are negative and isolated from zero.

Proof of Proposition 1.

Let us focus on the resolvent set of the operator A, which we denote by $\rho(A)$. As in Definition 1, $\lambda \in \mathbb{C}$ belongs to $\rho(A)$ if and only if the equation:

$$\lambda \boldsymbol{z} - A\boldsymbol{z} = \boldsymbol{f}, \tag{66}$$

so-called resolvent equation, has, for any $\boldsymbol{f} \in X_{\mathbb{C}}$, exactly one solution $\boldsymbol{z} \in D_{\mathbb{C}}$.

Let us write:

$$\lambda = \lambda_r + i\lambda_i, \quad \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}. \tag{67}$$

The resolvent equation (66) is equivalent to the following system:

$$\begin{cases} \lambda z_1(\theta) - a_{1,2}\dfrac{d^2 z_2(\theta)}{d\theta^2} - a_1 z_1(\theta) = f_1(\theta), & \theta \in (0, 2\pi), \\ \lambda z_2(\theta) - a_{2,2}\dfrac{d^2 z_2(\theta)}{d\theta^2} - a_2 z_1(\theta) - a_3 z_2(\theta) = f_2(\theta), & \theta \in (0, 2\pi). \end{cases} \tag{68}$$

Moreover, since $\boldsymbol{z} \in D_{\mathbb{C}}$ we have

$$z_1 \in X_{\mathbb{C},1d}, \qquad z_2 \in D_{\mathbb{C},1d}, \tag{69}$$

where $X_{\mathbb{C},1d}$ and $D_{\mathbb{C},1d}$ are defined in (58) and (59), respectively.

If $\lambda = a_1$ and we take $f_1 = 0$, the first of equations (68) reduces to

$$\frac{d^2 z_2}{d\theta^2} = 0, \quad \theta \in (0, 2\pi), \tag{70}$$

which is satisfied by any constant function z_2. Then, it is immediate to check that, if $\lambda = a_1$ and $f_1 = 0$, we can find infinitely many functions $\boldsymbol{z} \in D_{\mathbb{C}}$ (such that z_2 is constant) that satisfy (68). So, the (negative) real number a_1 belongs to the spectrum of the operator A, and the fact that $\sigma(A)$ is not empty is proved.

Let us choose $\overline{\lambda}$ as in (64). Lemma 1 guarantees that $\overline{\lambda} < 0$. We are going to show that for any $\lambda \in \mathbb{C}$ such that

$$\mathrm{Re}(\lambda) > \overline{\lambda}, \tag{71}$$

system (68) has a unique solution satisfying (69) for every $\boldsymbol{f} \in X_{\mathbb{C}}$. Therefore, the spectrum of the operator A does not contain any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \overline{\lambda}$ and Proposition 1 is proved.

Let us consider $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > \overline{\lambda}$. Then, according to Lemma 1, we have $\lambda - a_1 \neq 0$ and from the first of equations (68) we obtain

$$z_1(\theta) = \frac{a_{1,2}\frac{d^2 z_2(\theta)}{d\theta^2} + f_1(\theta)}{\lambda - a_1}, \quad \theta \in (0, 2\pi). \tag{72}$$

Let the function $g : \Omega \to \mathbb{C}$ be defined as follows:

$$g(\theta) = \frac{a_2}{\lambda - a_1} f_1(\theta) + f_2(\theta), \quad \theta \in [0, 2\pi]. \tag{73}$$

Then, substitution of (72) in the second of equations (68) yields

$$(\lambda - a_3) z_2(\theta) - \left(a_{2,2} + \frac{a_{1,2} a_2}{\lambda - a_1}\right) \frac{d^2 z_2(\theta)}{d\theta^2} = g(\theta), \quad \theta \in (0, 2\pi). \tag{74}$$

By taking into account (64) and (71) and the fact that $a_1 < 0$ (see Lemma 1), we have $|\lambda - a_1| \geq \mathrm{Re}(\lambda) - a_1 \geq |\overline{\lambda}|$, and thus the denominator in (73) is bounded from below. It then follows that there exists a constant M_1 (which does not depend on λ) such that

$$\|g\|_{X_{\mathbb{C},1d}} \leq M_1 \|\boldsymbol{f}\|_{X_{\mathbb{C}}}. \tag{75}$$

We shall use (75) later on, when proving Theorem 1. We set

$$p^2(\lambda) = \frac{(\lambda - a_3)(\lambda - a_1)}{a_{2,2}\lambda + a_{1,2} a_2 - a_{2,2} a_1}, \tag{76}$$

$$h(\lambda) = \frac{\lambda - a_1}{a_{2,2}\lambda + a_{1,2} a_2 - a_{2,2} a_1}, \tag{77}$$

and rewrite equation (74) as follows:

$$p^2(\lambda) z_2 - \frac{d^2 z_2}{d\theta^2} = h(\lambda) g, \quad \theta \in (0, 2\pi). \tag{78}$$

Note that $p^2(\lambda)$ cannot be equal to zero and cannot be a negative real number too. Indeed, for any $r \geq 0$, let us consider the equation

$$\frac{(\lambda - a_3)(\lambda - a_1)}{a_{2,2}\lambda + a_{1,2} a_2 - a_{2,2} a_1} = -r, \tag{79}$$

which yields

$$(\lambda - a_3)(\lambda - a_1) = -r(a_{2,2}\lambda + a_{1,2} a_2 - a_{2,2} a_1). \tag{80}$$

If we take only the imaginary parts at both sides of (80) and equate them we obtain

$$(2\mathrm{Re}(\lambda) - a_3 - a_1)\mathrm{Im}(\lambda) = -ra_{2,2}\mathrm{Im}(\lambda), \tag{81}$$

which, because $2\mathrm{Re}(\lambda) - a_3 - a_1 > 0$ (thanks to (64) and (71)) and $a_{2,2} > 0$, implies $\mathrm{Im}(\lambda) = 0$. Then, since $\mathrm{Re}(\lambda) > a_1$, $\mathrm{Re}(\lambda) > a_3$, and $a_{2,2}\lambda + a_{1,2}a_2 - a_{2,2}a_1 > 0$, we conclude that (80) cannot hold for any (non-negative) r. It then follows that $p^2(\lambda)$ cannot be equal to zero. Moreover, according to the standard algebra of complex numbers, there is a unique value $p(\lambda)$ with $\mathrm{Re}(p(\lambda)) > 0$ that satisfies (76).

Since $\mathrm{Re}(p(\lambda)) > 0$, Lemma 2 can be used (with $q = p(\lambda)$ and $\widetilde{f} = h(\lambda)g$) to prove that the differential equation (78) with the boundary conditions given by second of (69) has the following unique solution:

$$z_2(\theta) = \frac{1}{2p(\lambda)\sinh(\pi p(\lambda))}\left[\int_0^\theta \cosh\left(p(\lambda)(\theta - \gamma - \pi)\right)h(\lambda)g(\gamma)d\gamma + \right.$$
$$\left. + \int_\theta^{2\pi} \cosh\left(p(\lambda)(\theta - \gamma + \pi)\right)h(\lambda)g(\gamma)d\gamma\right], \quad \theta \in [0, 2\pi]. \tag{82}$$

Thus, according to (72), we have a unique z_1 too. Precisely, (72) defines a unique z_1 in the domain $(0, 2\pi)$, but, since $z_2 \in D_{\mathbb{C},1d}$ and $\boldsymbol{f} \in X_{\mathbb{C}}$, we can set $z_1(\theta) = \dfrac{a_{1,2}\frac{d^2 z_2(\theta)}{d\theta^2} + f_1(\theta)}{\lambda - a_1}$ $\forall\theta \in [0, 2\pi]$, and we have $z_1 \in X_{\mathbb{C},1d}$.

Therefore, if $\mathrm{Re}(\lambda) > \overline{\lambda}$ the resolvent equation (66) has a unique solution $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in D_{\mathbb{C}}$, and also the second part of Proposition 1 is proved.

5.3 Proof of Theorem 1

We are now in the position to prove Theorem 1.

As already noticed, the operator H is twice differentiable in a neighborhood $\mathcal{O}$ of $\mathbf{0}$ in D and thus Assumptions (8.1.1) in Lunardi (1995) are satisfied. Therefore, according to the theory developed in (Lunardi, 1995, Chapter 9.1), we need to check the following three conditions:

Condition (C1): $\sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\} < 0$.

Condition (C2): There exist two constant $\hat{\lambda}$ and M_{sec} such that $\rho(A)$ contains the right half-plane $\left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \hat{\lambda}\right\}$, and, in addition, $\forall\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > \hat{\lambda}$ we have:

$$\|\lambda R(\lambda, A)\|_{L(X_{\mathbb{C}})} \leq M_{sec}, \tag{83}$$

where $\|\lambda R(\lambda, A)\|_{L(X_{\mathbb{C}})}$ is the norm of the resolvent operator $R(\lambda, A)$.

Condition (C3): The graph norm of the operator A, which, for every $\boldsymbol{x} \in D$, is defined as follows:

$$\|\boldsymbol{x}\|_A = \|\boldsymbol{x}\|_X + \|A\boldsymbol{x}\|_X, \tag{84}$$

is equivalent to the norm of D, i.e., there exist positive constants c and C such that

$$c\|\boldsymbol{x}\|_D \leq \|\boldsymbol{x}\|_A \leq C\|\boldsymbol{x}\|_D \quad \forall \boldsymbol{x} \in D. \tag{85}$$

Then, if the above three conditions are satisfied, we can apply Theorems 8.1.1 and 9.1.2 in (Lunardi, 1995, Chapters 8.1 and 9.1) and conclude that Theorem 1 in the present paper holds. In particular, since the compatibility condition (48) is satisfied, Theorem 8.1.1 guarantees that a unique solution $\boldsymbol{u}$ of problem (37) exists in a maximal time interval $[0, \tau)$, that $\boldsymbol{u}$ belongs to $C([0,\tau); D) \cap C^1([0,\tau); X) \cap C^{\beta}_{\beta}((0, \tau - \epsilon]; D)$ for every $\beta \in (0,1)$ and $\epsilon \in (0,\tau)$, and that $\boldsymbol{u}$ is the unique solution belonging to $\bigcup_{0<\beta<1} C^{\beta}_{\beta}((0, \tau - \epsilon]; D) \cap C([0,\tau); D)$ for every $\epsilon \in (0,\tau)$. Moreover, Theorem 9.1.2 allows us to replace τ with $+\infty$ (i.e., $\boldsymbol{u}$ exists for every time t), and yields the stability result (47).

Note that Condition (C1) is actually Assumption (9.1.8) in (Lunardi, 1995, Chapter 9.1), whereas Conditions (C2) and (C3) coincide with Assumptions (9.1.2) in (Lunardi, 1995, Chapter 9.1). Moreover, Condition (C2) is actually Condition (2.1.12) in (Lunardi, 1995, Chapter 2), which ensures that A is a *sectorial operator*[1].

In the following, we prove Conditions (C1), (C2), and (C3).

Proof of Condition (C1)

Condition (C1) is satisfied thanks to Proposition 1.

Proof of Condition (C2).

We prove that Condition (C2) is satisfied if we set

$$\hat{\lambda} = 1 + \max\left\{\left|\frac{a_{1,2}a_2 + a_{2,2}a_3}{a_{2,2}}\right|, -a_1, -a_3\right\}. \tag{86}$$

Indeed, according to Proposition 1, any λ such that $\mathrm{Re}(\lambda) > 0$ belongs to the resolvent set of A. Thus, recalling that a_1 and a_3 are negative numbers, it immediately follows that $\rho(A)$ contains the right half-plane $\left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \hat{\lambda}\right\}$.

Thus, it remains to prove (83). To this aim, from now on we restrict our attention to those complex λ such that

$$\mathrm{Re}(\lambda) \geq \hat{\lambda}. \tag{87}$$

Moreover, let $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ denote the unique solution $\in D_{\mathbb{C}}$ of the resolvent equation (66). According to the definition of resolvent operator (54), we can rewrite (83) as follows:

$$\sup_{\substack{\boldsymbol{f} \in X_{\mathbb{C}}, \\ \boldsymbol{f} \neq \boldsymbol{0}}} \frac{\|\lambda \boldsymbol{z}\|_{X_{\mathbb{C}}}}{\|\boldsymbol{f}\|_{X_{\mathbb{C}}}} \leq M_{sec}. \tag{88}$$

[1]For the sake of brevity, in this paper we omit the definition of sectorial operator. The interested reader is referred to Definition 2.0.1 and Proposition 2.1.11 in Lunardi (1995).

Inequality (88) can be rewritten as follows:

$$\|\boldsymbol{z}\|_{X_{\mathbb{C}}} \leq M_{sec} \frac{\|\boldsymbol{f}\|_{X_{\mathbb{C}}}}{|\lambda|}, \quad \forall \boldsymbol{f} \in X_{\mathbb{C}}. \tag{89}$$

As shown in the proof of Proposition 1, z_2 satisfies equation (78), where $p^2(\lambda)$ is given by (76). Moreover, with the choice of λ as in (87), we have $\mathrm{Re}(p^2(\lambda)) > 0$. Indeed, by directly computing (76), we obtain

$$\begin{aligned}\mathrm{Re}(p^2(\lambda)) = & \frac{(\mathrm{Re}(\lambda) - a_1)(\mathrm{Re}(\lambda) - a_3)(a_{2,2}\mathrm{Re}(\lambda) + a_{1,2}a_2 - a_{2,2}a_1)}{(a_{2,2}\mathrm{Re}(\lambda) + a_{1,2}a_2 - a_{2,2}a_1)^2 + (a_{2,2}\mathrm{Im}(\lambda))^2} \\ & + \frac{(a_{2,2}\mathrm{Re}(\lambda) - a_{1,2}a_2 - a_{2,2}a_3)(\mathrm{Im}(\lambda))^2}{(a_{2,2}\mathrm{Re}(\lambda) + a_{1,2}a_2 - a_{2,2}a_1)^2 + (a_{2,2}\mathrm{Im}(\lambda))^2},\end{aligned} \tag{90}$$

and thus from (86), (87) and the fact that $a_{1,2}a_2 - a_{2,2}a_1 > 0$ (see Lemma 1) it immediately follows that $\mathrm{Re}(p^2(\lambda)) > 0$. Note that, according to elementary algebra of complex numbers, the fact that $\mathrm{Re}(p^2(\lambda)) > 0$ implies

$$\mathrm{Re}(p(\lambda)) \geq \frac{1}{\sqrt{2}} \sqrt{|p^2(\lambda)|} = \frac{1}{\sqrt{2}} |p(\lambda)|. \tag{91}$$

As already shown, z_2 is given by (82). Using the elementary inequalities (see Pazy (1983), Section 8.2)

$$|\sinh(\pi p(\lambda))| \geq \sinh(\pi \mathrm{Re}(p(\lambda))), \tag{92}$$

$$|\cosh\left(p(\lambda)(\theta - \gamma \pm \pi)\right)| \leq \cosh\left(\mathrm{Re}(p(\lambda))(\theta - \gamma \pm \pi)\right), \tag{93}$$

we obtain

$$|z_2(\theta)| \leq \frac{|h(\lambda)|\, \|g\|_{X_{\mathbb{C},1d}}}{\mathrm{Re}(p(\lambda))|p(\lambda)|}, \quad \theta \in [0, 2\pi]. \tag{94}$$

Thus, relation (75) yields

$$\|z_2\| \leq M_1 \frac{|h(\lambda)|\, \|\boldsymbol{f}\|_{X_{\mathbb{C}}}}{\mathrm{Re}(p(\lambda))|p(\lambda)|}. \tag{95}$$

Using Lemma 1 and the definition (77) it can be easily checked that

$$|h(\lambda)| \leq M_2, \quad \frac{|\lambda - a_3|}{|\lambda - a_1|} \leq M_3, \quad |p(\lambda)|^2 \geq M_4|\lambda|, \quad |\lambda - a_1| \geq |\lambda|, \quad |\lambda - a_3| \geq |\lambda|, \tag{96}$$

for some constants M_2, M_3 and M_4 that do not depend on λ. Relations (76), (77), (91) and (95) imply that

$$\|z_2\|_{X_{\mathbb{C}}} \leq \frac{\sqrt{2}M_1M_2}{M_4}\frac{\|\boldsymbol{f}\|_{X_{\mathbb{C}}}}{|\lambda|}. \tag{97}$$

According to (72) and (74), we have

$$z_1(\theta) = \frac{a_{1,2}(\lambda - a_3)h(\lambda)}{\lambda - a_1}z_2(\theta) - \frac{a_{1,2}h(\lambda)}{\lambda - a_1}g(\theta) + \frac{1}{\lambda - a_1}f_1(\theta), \quad \theta \in [0, 2\pi]. \tag{98}$$

Then, using (96), (97) and (98) we obtain

$$\|z_1\|_{X_{\mathbb{C}}} \leq M_5\frac{\|\boldsymbol{f}\|_{X_{\mathbb{C}}}}{|\lambda|}, \tag{99}$$

where M_5 does not depend on λ. From (97) and (99) we obtain inequality (89), and Condition (C2) is proved.

Proof of Condition (C3). According to (42) and (84), for any $\boldsymbol{x} \in D$ we have:

$$\begin{aligned} \|\boldsymbol{x}\|_A &= \|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)} + \left\|a_{1,2}\frac{d^2x_2}{d\theta^2} + a_1x_1\right\|_{C^0(\Omega)} \\ &+ \left\|a_{2,2}\frac{d^2x_2}{d\theta^2} + a_2x_1 + a_3x_2\right\|_{C^0(\Omega)}, \end{aligned} \tag{100}$$

or, equivalently,

$$\begin{aligned} \|\boldsymbol{x}\|_A = \frac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right) + \frac{1}{2}\Big(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)} \\ + \left\|2a_{1,2}\frac{d^2x_2}{d\theta^2} + 2a_1x_1\right\|_{C^0(\Omega)} + \left\|2a_{2,2}\frac{d^2x_2}{d\theta^2} + 2a_2x_1 + 2a_3x_2\right\|_{C^0(\Omega)}\Big), \end{aligned} \tag{101}$$

so that the triangular inequality

$$\begin{aligned} &\left\|2a_{1,2}\frac{d^2x_2}{d\theta^2} + 2a_1x_1\right\|_{C^0(\Omega)} + \left\|2a_{2,2}\frac{d^2x_2}{d\theta^2} + 2a_2x_1 + 2a_3x_2\right\|_{C^0(\Omega)} \\ &\geq \left\|2a_{2,2}\frac{d^2x_2}{d\theta^2} + 2a_2x_1 + 2a_3x_2 - 2a_{1,2}\frac{d^2x_2}{d\theta^2} - 2a_1x_1\right\|_{C^0(\Omega)} \end{aligned} \tag{102}$$

yields

$$\begin{aligned} \|\boldsymbol{x}\|_A \geq \frac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right) + \frac{1}{2}\Big(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)} \\ + \left\|2(a_{2,2} - a_{1,2})\frac{d^2x_2}{d\theta^2} + 2(a_2 - a_1)x_1 + 2a_3x_2\right\|_{C^0(\Omega)}\Big). \end{aligned} \tag{103}$$

Let us define $\underline{a}_3 = \min\{-2a_3, 1\}$ and $\overline{a}_3 = \frac{1}{\max\{-2a_3,1\}}$. We recall that a_3 is a strictly negative real number, and so $0 < \underline{a}_3 \leq 1$ and $0 < \overline{a}_3 \leq 1$. Moreover,

$$\underline{a}_3 + 2a_3\overline{a}_3 = 0. \tag{104}$$

Then, we have

$$\begin{aligned} \|\boldsymbol{x}\|_A \geq \frac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right) + \frac{1}{2}\Big(\|x_1\|_{C^0(\Omega)} + \underline{a}_3\|x_2\|_{C^0(\Omega)} \\ +\overline{a}_3\left\|2(a_{2,2}-a_{1,2})\frac{d^2x_2}{d\theta^2} + 2(a_2-a_1)x_1 + 2a_3x_2\right\|_{C^0(\Omega)}\Big), \end{aligned} \tag{105}$$

and using the triangular inequality

$$\begin{aligned} \|\underline{a}_3x_2\|_{C^0(\Omega)} + \left\|2(a_{2,2}-a_{1,2})\overline{a}_3\frac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3x_1 + 2a_3\overline{a}_3x_2\right\|_{C^0(\Omega)} \\ \geq \left\|2(a_{2,2}-a_{1,2})\overline{a}_3\frac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3x_1 + 2a_3\overline{a}_3x_2 + \underline{a}_3x_2\right\|_{C^0(\Omega)} \end{aligned} \tag{106}$$

we obtain

$$\begin{aligned} \|\boldsymbol{x}\|_A \geq \tfrac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right) + \tfrac{1}{2}\Big(\|x_1\|_{C^0(\Omega)} \\ + \left\|2(a_{2,2}-a_{1,2})\overline{a}_3\tfrac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3x_1 + 2a_3\overline{a}_3x_2 + \underline{a}_3x_2\right\|_{C^0(\Omega)}\Big), \end{aligned} \tag{107}$$

so that (104) yields

$$\begin{aligned} \|\boldsymbol{x}\|_A \geq \frac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right) + \frac{1}{2}\Big(\|x_1\|_{C^0(\Omega)} + \\ + \left\|2(a_{2,2}-a_{1,2})\overline{a}_3\frac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3x_1\right\|_{C^0(\Omega)}\Big), \end{aligned} \tag{108}$$

Let us define $\underline{a}_{1,2} = \min\{2(a_2-a_1)\overline{a}_3, 1\}$ and $\overline{a}_{1,2} = \frac{1}{\max\{2(a_2-a_1)\overline{a}_3,1\}}$. We recall that $2(a_2-a_1)\overline{a}_3$ is a strictly positive real number (see Lemma 1), so that $0 < \underline{a}_{1,2} \leq 1$, $0 < \overline{a}_{1,2} < 1$ and

$$-\underline{a}_{1,2} + 2(a_2-a_1)\overline{a}_3\overline{a}_{1,2} = 0. \tag{109}$$

From (108) we have:

$$\begin{aligned} \|\boldsymbol{x}\|_A \geq \frac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right) + \frac{1}{2}\Big(\underline{a}_{1,2}\|x_1\|_{C^0(\Omega)} + \\ +\overline{a}_{1,2}\left\|2(a_{2,2}-a_{1,2})\overline{a}_3\frac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3x_1\right\|_{C^0(\Omega)}\Big). \end{aligned} \tag{110}$$

Using the triangular inequality,

$$\|\underline{a}_{1,2}x_1\|_{C^0(\Omega)} + \left\|2(a_{2,2}-a_{1,2})\overline{a}_3\overline{a}_{1,2}\frac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3\overline{a}_{1,2}x_1\right\|_{C^0(\Omega)}$$
$$\geq \left\|2(a_{2,2}-a_{1,2})\overline{a}_3\overline{a}_{1,2}\frac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3\overline{a}_{1,2}x_1 - \underline{a}_{1,2}x_1\right\|_{C^0(\Omega)}, \tag{111}$$

we obtain

$$\|\boldsymbol{x}\|_A \geq \tfrac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)}\right)$$
$$+\tfrac{1}{2}\left(\left\|2(a_{2,2}-a_{1,2})\overline{a}_3\overline{a}_{1,2}\tfrac{d^2x_2}{d\theta^2} + 2(a_2-a_1)\overline{a}_3\overline{a}_{1,2}x_1 - \underline{a}_{1,2}x_1\right\|_{C^0(\Omega)}\right), \tag{112}$$

so that (109) yields

$$\|\boldsymbol{x}\|_A \geq \tfrac{1}{2}\left(\|x_1\|_{C^0(\Omega)} + \|x_2\|_{C^0(\Omega)} + 2(a_{2,2}-a_{1,2})\overline{a}_3\overline{a}_{1,2}\|\tfrac{d^2x_2}{d\theta^2}\|_{C^0(\Omega)}\right). \tag{113}$$

Note that $2(a_{2,2}-a_{1,2})\overline{a}_3\overline{a}_{1,2} > 0$ (owing to Lemma 1 and to the fact that $\overline{a}_3 > 0$ and $\overline{a}_{1,2} > 0$). Since $x_2 \in C^2(\Omega)$ and $x_2(0) = x_2(2\pi)$, there exists $\theta_0 \in \Omega$ such that $\left.\frac{dx(\theta)}{d\theta}\right|_{\theta=\theta_0} = 0$. It then follows that

$$\frac{dx_2(\theta)}{d\theta} = \int_{\theta_0}^{\theta} \frac{d^2x_2(\gamma)}{d\gamma^2}d\gamma, \quad \theta \in [0, 2\pi], \tag{114}$$

from which it can be easily checked that

$$\left\|\frac{dx_2}{d\theta}\right\|_{C^0(\Omega)} \leq 2\pi \left\|\frac{d^2x_2}{d\theta^2}\right\|_{C^0(\Omega)}. \tag{115}$$

Then, from (113) and (115) it follows that there exists a positive constant c such that

$$\|\boldsymbol{x}\|_A \geq c\|\boldsymbol{x}\|_D \quad \forall \boldsymbol{x} \in D. \tag{116}$$

That is, the left part of inequality (85) is proven. The right part of inequality (85) follows immediately from (100) and from elementary norm properties.

6 A numerical analysis of the spatial model in finite time intervals

To investigate how the macroeconomic variables evolve and interact with each other on finite time intervals, we solve the partial differential problem (21) by numerical approximation.

This analysis will highlight some interesting transitional dynamics that the standard Solow model (with no spatial dependence) does not describe.

Throughout this section, we consider the following Cobb-Douglas production function:

$$F(L,K) = L^{\frac{1}{3}} K^{\frac{2}{3}}, \tag{117}$$

and the following (logistic) rate of population growth:

$$\eta(L) = 0.02L\,(1 - 0.5L)\,, \tag{118}$$

whereas the saving and capital depreciation rates are chosen as in Table 1.

Table 1

s	δ
0.3	0.2

Saving and capital depreciation rates.

By solving the first of equations (6) and equation (25), we can determine the equilibrium values $\overline{L} = 2$ for labor and $\overline{K} = 6.75$ for capital.

The space-time dynamics of labor and capital is shown in Figures 1.a and 1.b (which are obtained setting $a = 1$ and $b = 1$ and choosing L_0 and K_0 as in (120) below). In accordance to the theory developed in the previous section, $L(\theta,t)$ and $K(\theta,t)$ tend to $\overline{L}$ and $\overline{K}$, respectively, in the long-run. Consequently, wages tend to a uniform steady state as well (Figure 1.c).

Then, it is interesting to investigate the effect of migration on the process of convergence and equalization of wages, which is done in the next section.

6.1 The impact of labor migration on wages

As a measure of wage inequality, we consider the root mean-square spatial deviation of the wages:

$$\sigma_w(t) = \left(\frac{1}{2\pi}\int_0^{2\pi} (w(\theta,t) - \overline{w}(t))^2 d\theta\right)^{1/2}, \quad t \in [0,+\infty), \tag{119}$$

where $\overline{w}(t) = \frac{1}{2\pi}\int_0^{2\pi} w(\theta,t)d\theta$. The initial distributions of labor and capital are chosen as follows (see their plots in Figures 2.a and 2.b):

$$L_0(\theta) = \frac{\overline{L}}{2}, \quad K_0(\theta) = \overline{K}\left(1 + \frac{\cos(\theta)}{4}\right), \quad \theta \in [0,2\pi]. \tag{120}$$

At the time $t = 0$ wages are not uniformly distributed all over the economy (see Figure 2.c). However, as shown in Figure 3, as time increases, spatial differences in salaries decline. This is consistent with the several empirical studies that point out the role of migration in spurring the process of wage convergence (see Enflo et al. (2014) and references therein). Figure 3 also highlights the strong effect that migration has on the process of wage equalization. In particular, if workers migrate and capital does not (case $a = 0$ and $b = 1$), wage

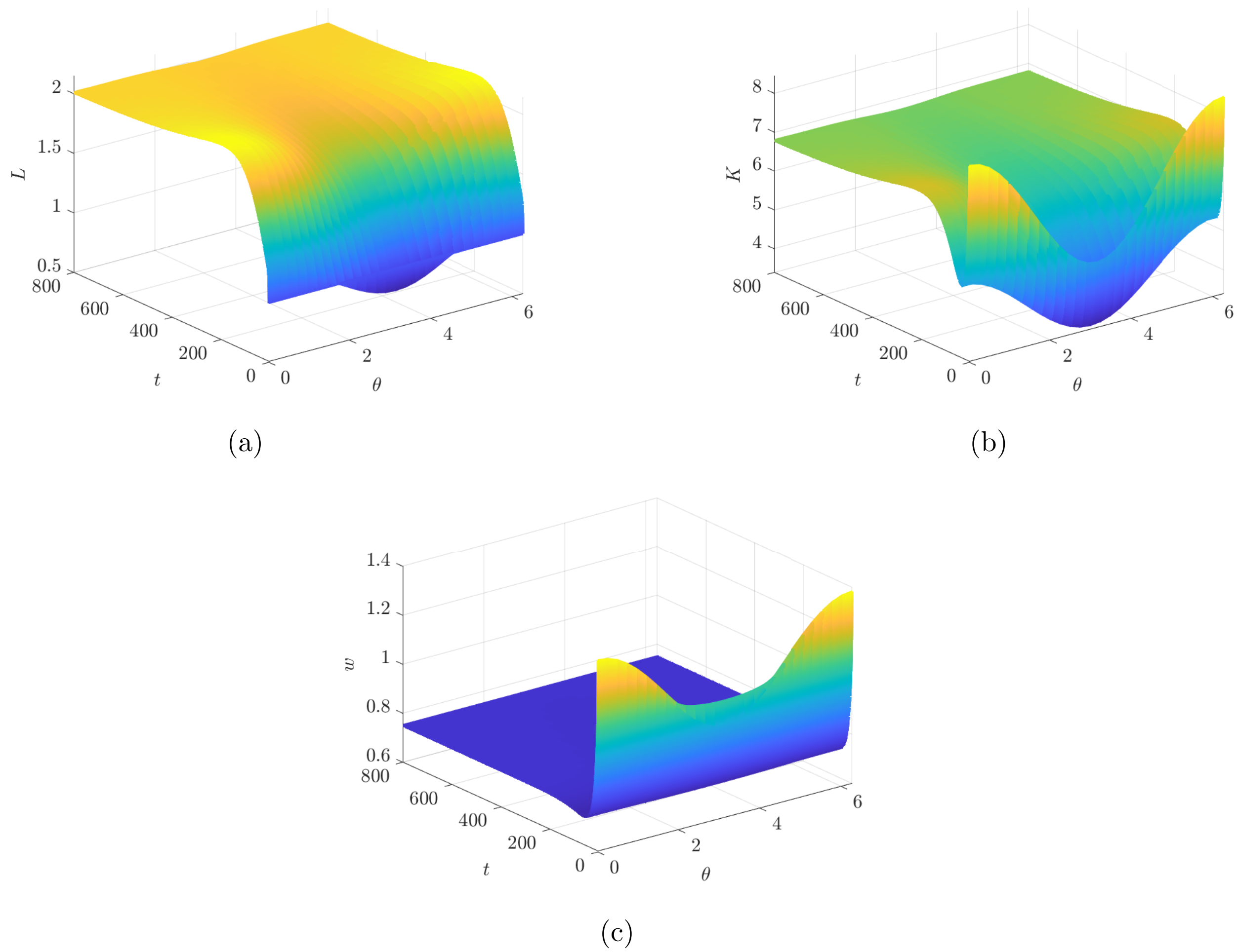
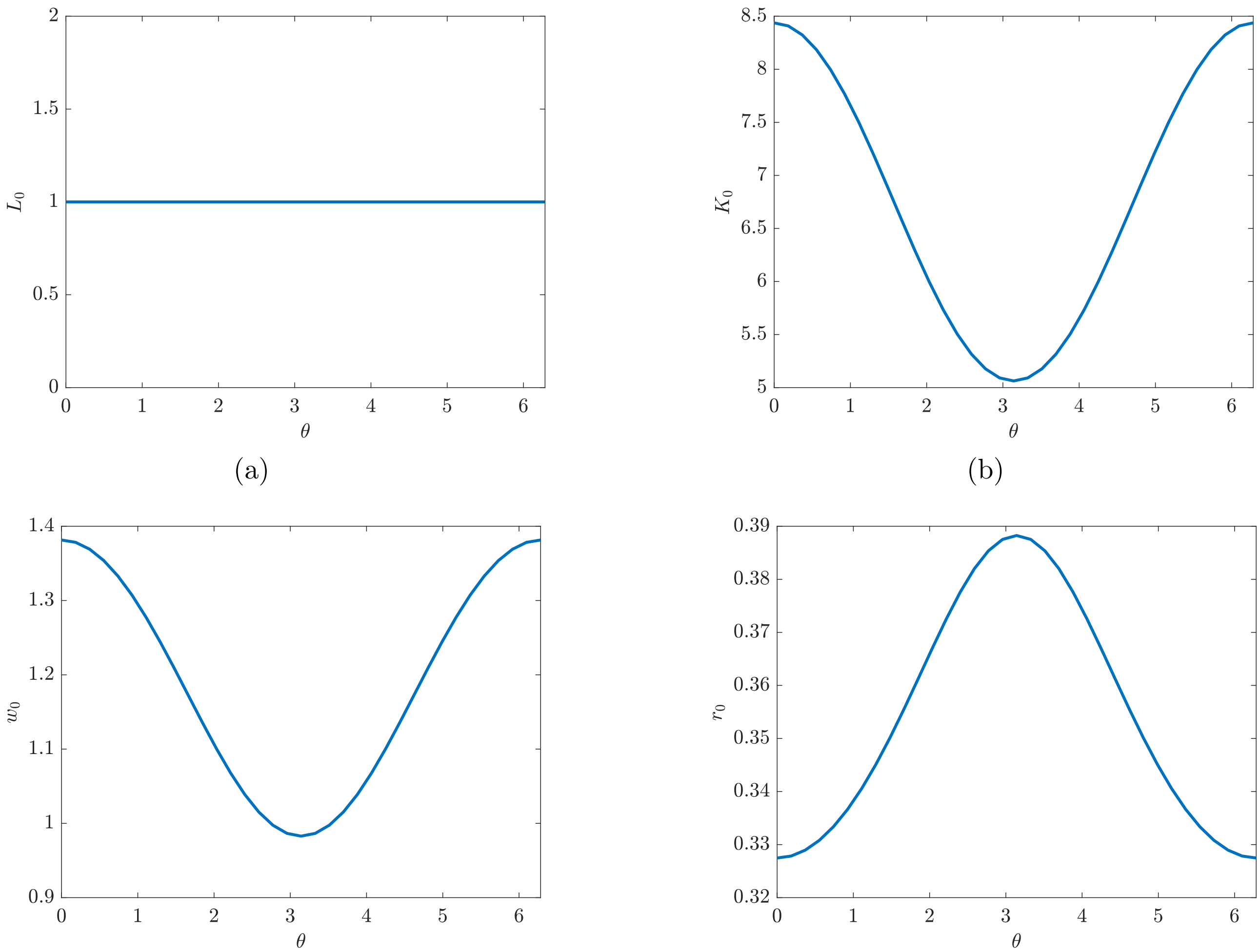


Figure 1: Dynamics of labor (a), capital (b) and wages (c). L_0 and K_0 are chosen as in (120), $a = 1$, $b = 1$ ($\overline{L} = 2$, $\overline{K} = 6.75$).

inequalities decrease at a rate that is much higher than if workers and capital do not migrate (case $a = 0$ and $b = 0$). Moreover, if capital migrates too (case $a = 50$ and $b = 1$), the process of wage homogenization is even faster.

Nevertheless, the effect of migration on the convergence of wages to the steady state is different at different spatial locations. In particular, at $\theta = 0$ (being identified with $\theta = 2\pi$) migration of labor and capital accelerates the convergence process (Figure 4.a). By contrast, at $\theta = \pi$, the migration of labor causes wages to depart from the steady state in the short run. Precisely, let us focus our attention on the case $a = 0$ and $b = 1$ (workers migrate and capital does not). Then, as we may see in Figure 4.b, salaries at the time $t = 0$ are above the steady state, but, in a time interval immediately after $t = 0$, they increase further. This fact happens because, at the time $t = 0$, the spatial distribution of wages has a minimum at $\theta = \pi$ (Figure 2.c). Therefore, as time increases, workers move away from $\theta = \pi$, and wages here increase. This effect is still more pronounced if we also consider migration of capital (case $a = 50$ and $b = 1$). In fact, at the time $t = 0$ the marginal productivity of capital has a maximum at $\theta = \pi$ (Figure 2.c), so that, if capital is allowed to migrate, immediately after $t = 0$ capital moves to $\theta = \pi$ from adjacent areas, which makes wages increase further.



(a) (b) (c) (d)

Figure 2: Initial distribution of labor (a), capital (b), wages (c) and return on investments (d) ($\overline{L} = 2$, $\overline{K} = 6.75$).

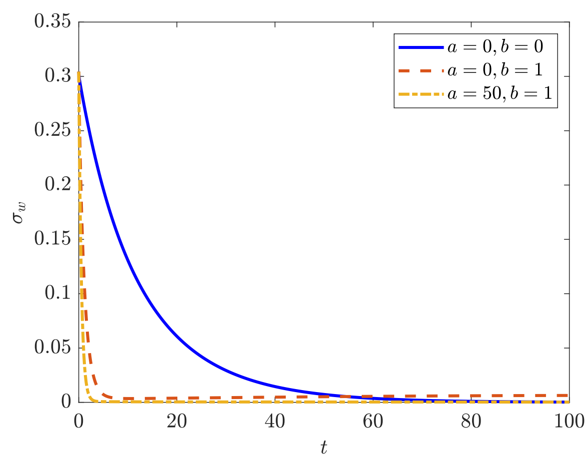


Figure 3: Root mean square spatial deviation of wages.

6.2 The impact of economic-driven migration of labor and capital on economic growth

In this section, we want to show two opposite effects that the migration of labor and capital can have on production and population growth. To this aim, we consider two different

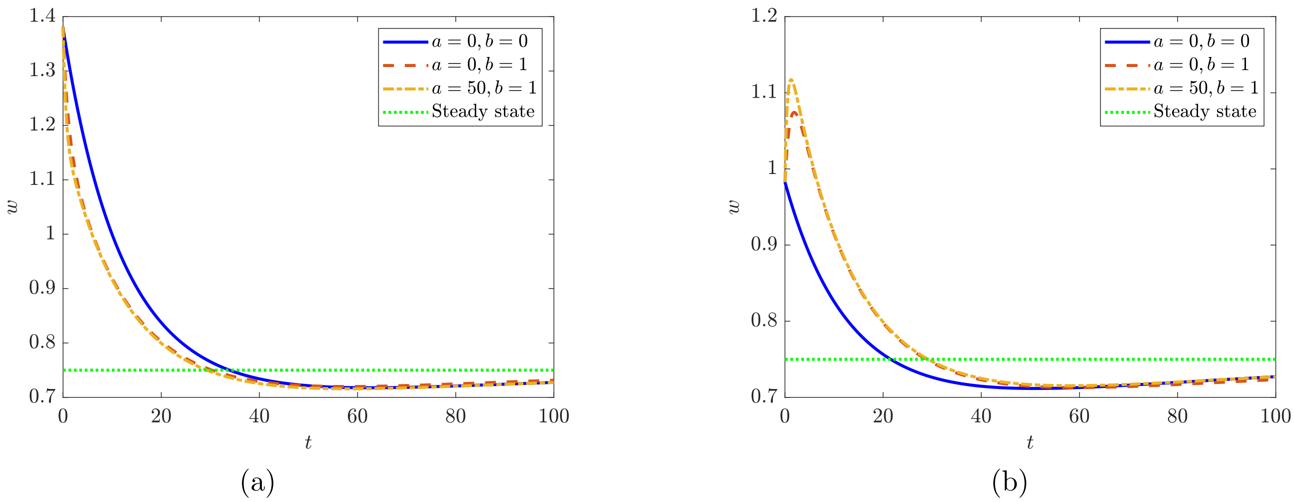


Figure 4: Convergence of wages at $\theta = 0$ (a) and $\theta = \pi$ (b).

economies. The first (Economy 1) is an economy with a *developed* region, where capital and labor are at the equilibrium level, and a *capital-rich and scarcely populated* region, where capital is at the equilibrium level and labor is significantly below the equilibrium level; the second (Economy 2) is an economy with both a developed region and an *underdeveloped* region, where capital and labor are significantly below the equilibrium level. Please notev that we use the term *developed* to refer to regions where both labor and capital have reached their steady state levels.

We will also focus on the total labor force and production (all over space), defined as follows:

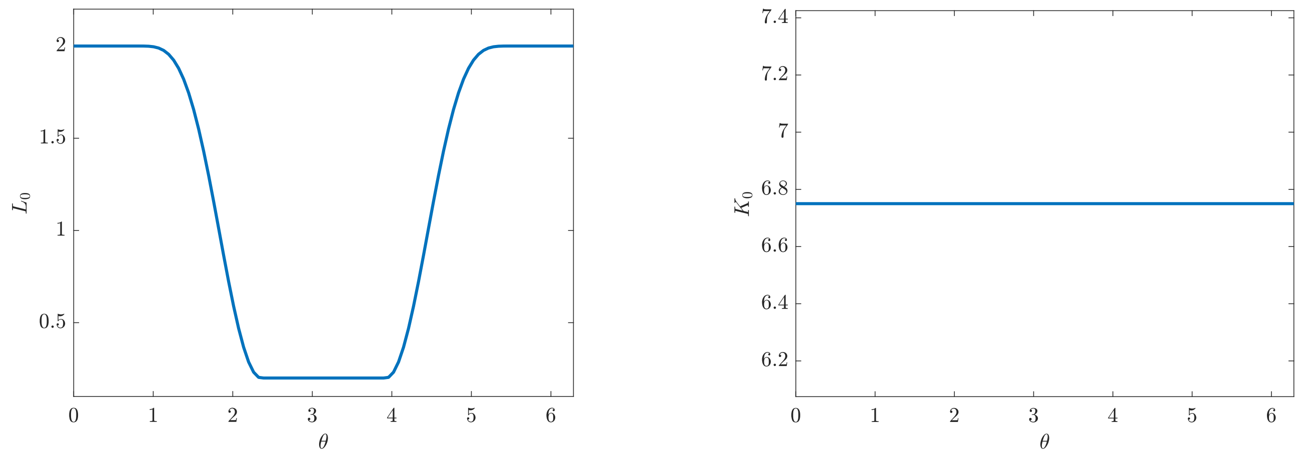
$$L_{tot}(t) = \int_0^{2\pi} L(\theta,t)d\theta, \quad Y_{tot}(t) = \int_0^{2\pi} F(L(\theta,t),K(\theta,t))d\theta, \quad t \in [0,+\infty). \tag{121}$$

6.2.1 Economy 1: the case of a developed region and a capital-rich and scarcely populated region

The initial distributions of labor and capital are as follows (see their plots in Figures 5.a and 5.b):

$$L_0(\theta) = \begin{cases} \overline{L} & \text{if } 0 \le \theta \le \frac{\pi}{4} \\ \overline{L} - \frac{9\overline{L}}{10}\sin^4(\theta - \frac{\pi}{4}) & \text{if } \frac{\pi}{4} < \theta < \frac{3\pi}{4} \\ \frac{\overline{L}}{10} & \text{if } \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4} \\ \overline{L} - \frac{9\overline{L}}{10}\sin^4(\theta - \frac{7\pi}{4}) & \text{if } \frac{5\pi}{4} < \theta < \frac{7\pi}{4} \\ \overline{L} & \text{if } \frac{7\pi}{4} \le \theta \le 2\pi \end{cases}, \qquad K_0(\theta) = \overline{K}, \quad \theta \in [0,2\pi]. \tag{122}$$

As we may see, at every spatial location capital is equal to the steady state already at the time $t = 0$. Instead, L_0 is equal to $\overline{L}$ for $\theta \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right]$ and is equal to $\frac{\overline{L}}{10}$ for $\theta \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ (with smooth variation from one spatial region to the other). Then, we have a developed



(a) (b)

Figure 5: Economy 1, initial distribution of labor (a) and capital (b) ($\overline{L} = 2$, $\overline{K} = 6.75$).

region that extends from $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$, and a capital-rich and scarcely populated region that extends from $\theta = \frac{3\pi}{4}$ to $\theta = \frac{5\pi}{4}$.

In the long run, labor, capital and production converge to the steady state, but their transitional dynamics are strongly affected by migration. In particular, let us focus our attention on the location $\theta = \pi$ (the center of the capital-rich and scarcely populated region). As shown in Figure 6.c, if $a = 0$ and $b = 0$ (labor and capital do not migrate), there is a time interval immediately after $t = 0$, which we call I_1, in which capital declines very fast (roughly, we can set $I_1 = [0, 30]$, see Figure 6.c). In fact, at the time $t = 0$, the capital per capita k is ten times higher than the equilibrium value $\overline{k}$ (owing to (122)), and thus in the time interval I_1 it declines fast to reach the steady state. Moreover, at the time $t = 0$, population grows slowly (since $L(\pi, 0)$ is small and owing to the logistic law (118)), and so the rapid decline in capital per capita results in a rapid decline in physical capital K. As a result, in the time interval I_1 production declines significantly (see Figure 6.e).

Instead, if $a = 0$ and $b = 1$ (workers migrate and capital does not), the decline in capital is significantly smaller (see Figure 6.c). In fact, in the time interval I_1, there is a massive flow of labor force from the developed region to the capital-rich and scarcely populated region, where the number of workers is smaller and wages are higher (see Figures 6.a and 6.b). Therefore, the decline in capital per capita that occurs immediately after $t = 0$ has a more limited effect on the physical capital K. As a result, production tends to increase rather than decline (see Figure 6.e). Actually, if $a = 0$ and $b = 1$, production at $\theta = \pi$ is maintained high by the labor force arriving from the developed region.

The migration of capital has the opposite effect. If, for example, $a = 50$ and $b = 1$ (case shown in Figures 6.a, 6.c and 6.e), in the time interval I_1 capital moves from the capital-rich and scarcely developed region, where the return on investments is low, to the developed region. Therefore, in the capital-rich and scarcely populated region, capital declines, and, in turn, migration of workers from the developed region declines as well.

The effect of migration of capital starts to be significant only when the parameter a is large, say, larger than 30. In other words, production is much more sensitive to the propensity of workers to migrate (parameter b) than the propensity of capital to migrate (parameter a). This is easily explained if we consider that the spatial dynamics described above stem

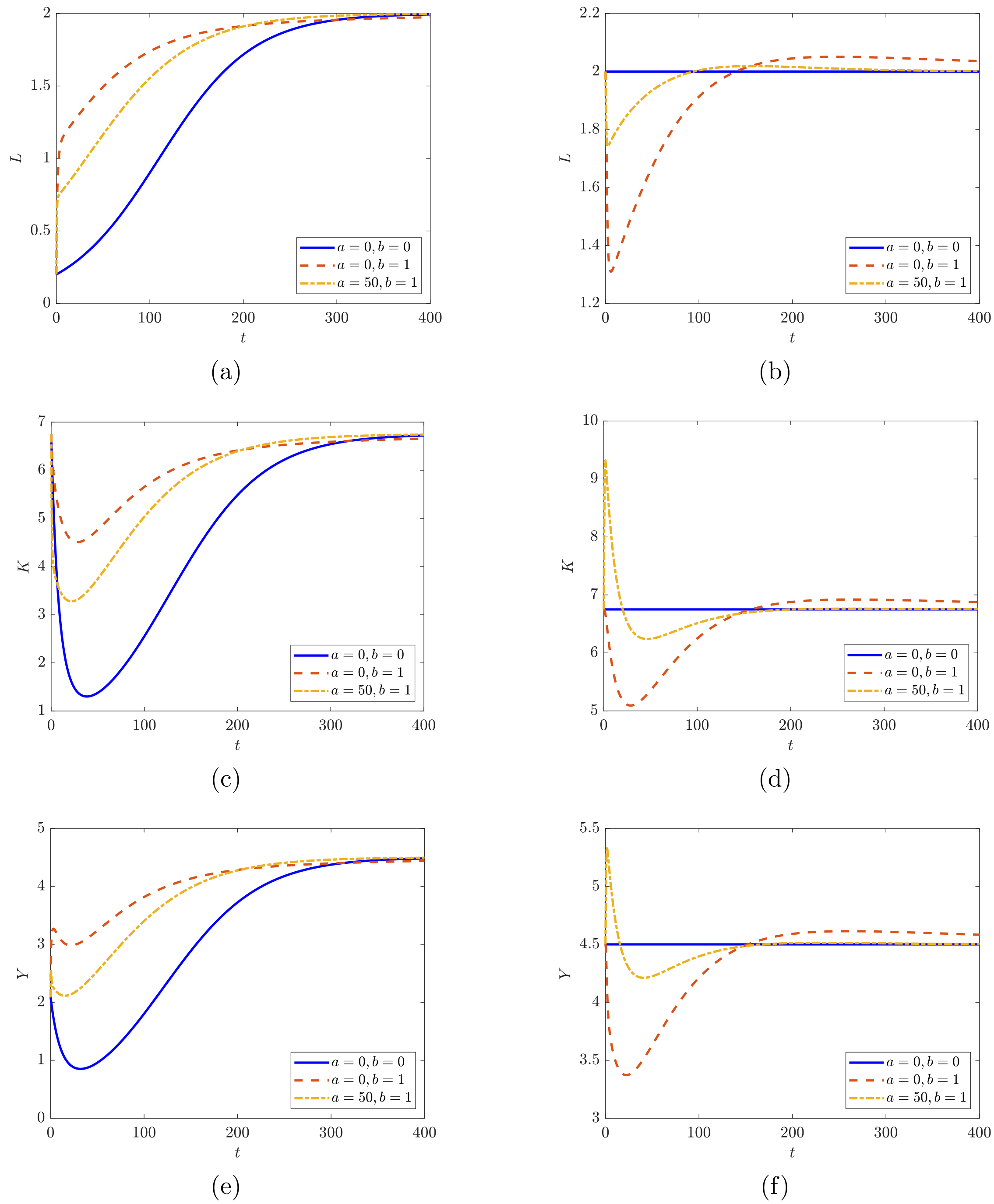
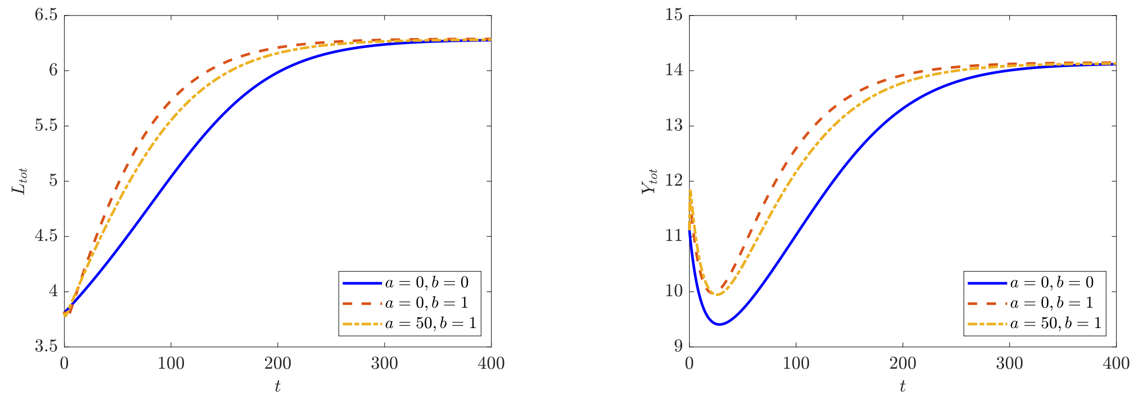


Figure 6: Economy 1, transient dynamics of labor, capital and production at $\theta = \pi$ (left column) and $\theta = 0$ (right column).

from the small number of workers at the time $t = 0$ in the developed and scarcely populated region. Therefore, the behavior of capital, labor, and production immediately after $t = 0$ is mostly affected by the migration of labor (rather than the migration of capital).

Let us briefly see what happens at $\theta = 0$ (i.e., at the center of the developed region). Here, the dynamics of labor, capital and production are somehow specular to those at $\theta = \pi$. In the case of migration of only workers ($a = 0$, $b = 1$), the labor force declines in I_1, and hence



(a) (b)

Figure 7: Economy 1, transitional dynamics of total population (a) and total production (b).

capital and production decline as well (Figures 6.b, 6.d and 6.f). Consequently, in a time interval immediately after $t = 0$ (say, for $t \in [0, 150]$), production is below the equilibrium level. Again, the migration of capital has the opposite effect. In the case $a = 50$ and $b = 1$, immediately after $t = 0$ production increases above the equilibrium value (Figure 6.f), due to the arrival of capital from the capital-rich and scarcely populated region (see Figures 6.d).

Finally, let us consider the total population and production. As we have seen, in the case of migration of only workers ($a = 0$ and $b = 1$), in the time interval I_1 we have a massive flow of workers from the developed region to the capital-rich and scarcely populated region. As a result, the rate of population growth $\eta(L)$ (which, according to the logistic relation (118), is proportional to the population level itself) increases in the capital-rich and scarcely populated region and declines in the developed region. However, the relative variation of $\eta(L)$ is greater in the developed and scarcely populated region (where $\eta(L)$ is very small) than in the developed region. Therefore, the increase in population in the capital-rich and scarcely populated region is greater than the decline in population in the developed and populated region, so that, overall, migration of workers makes the total population L_{tot} grow faster (see Figure 7.a). Accordingly, if $a = 0$ and $b = 1$, total production grows faster than in the case with no migration. Finally, again the migration of capital mitigates the effect of migration of labor (the increase in L_{tot} and Y_{tot} due to labor migration is greater for $a = 50$ and $b = 1$ than for $a = 0$ and $b = 1$, see Figures 9.a and 9.b).

6.2.2 Economy 2: the case of a developed region and an underdeveloped region

Next, we consider instead the following distribution of capital at the time $t = 0$:

$$K_0(\theta) = \begin{cases} \overline{K} & \text{if } 0 \le \theta \le \frac{\pi}{4} \\ \overline{K} - \frac{19\overline{K}}{20}\sin^4(\theta - \frac{\pi}{4}) & \text{if } \frac{\pi}{4} < \theta < \frac{3\pi}{4} \\ \frac{\overline{K}}{20} & \text{if } \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4} \\ \overline{K} - \frac{19\overline{K}}{20}\sin^4(\theta - \frac{7\pi}{4}) & \text{if } \frac{5\pi}{4} < \theta < \frac{7\pi}{4} \\ \overline{K} & \text{if } \frac{7\pi}{4} \le \theta \le 2\pi \end{cases}, \tag{123}$$

whereas we still choose the initial distribution of labor as in (122).

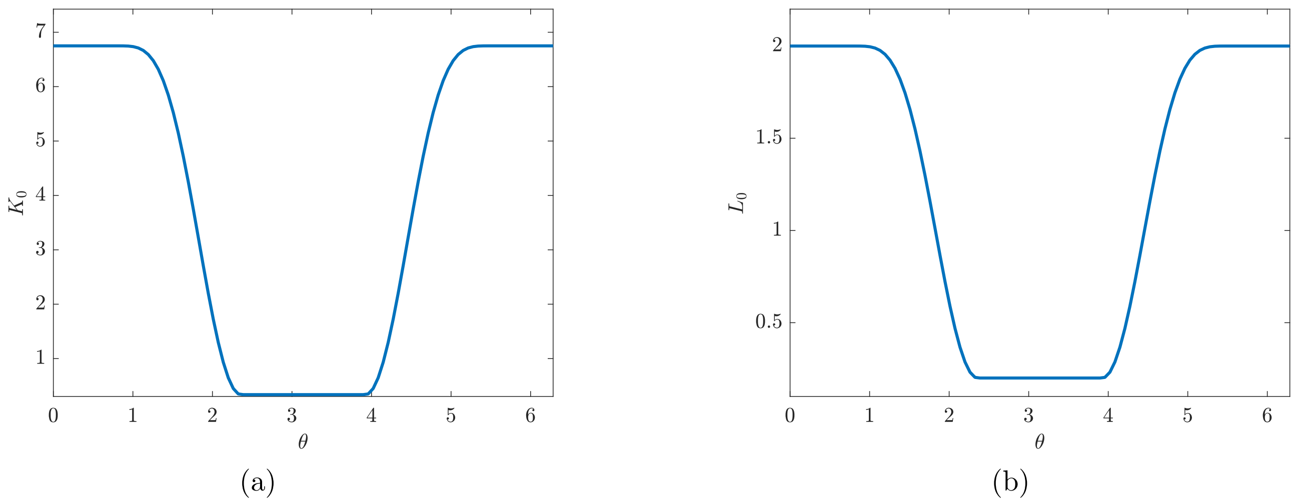


Figure 8: Economy 2, initial distribution of labor (a) and capital (b) ($\overline{L} = 2$, $\overline{K} = 6.75$).

According to (123), the initial capital is equal to $\overline{K}$ for $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, and it is equal to $\frac{\overline{K}}{20}$ for $\theta \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ (see Figure 8). Then, at the time $t = 0$ we have both an underdeveloped region where the labor force and the capital are equal to a tenth and a twentieth of their equilibrium values, respectively, and a developed region where labor and capital are equal to their equilibrium values.

Let us investigate the transitional dynamics of labor and capital at $\theta = \pi$ (the center of the underdeveloped region). Here, at the time $t = 0$ the values of labor and capital are smaller than the equilibrium values. Therefore, as time increases, L and K increase to reach the state state (see Figures 9.a and 9.c). However, growth is strongly affected by the migration of labor and capital in the short run. In fact, let us consider the case $a = 0$ and $b = 1$ (workers migrate and capital does not). Then, capital, labor, and production at $\theta = \pi$ increase very slowly, and reach the equilibrium only after a very long time (see the red line in Figures 9.a and 9.c). This fact occurs because, in a time interval immediately after $t = 0$, workers move from the underdeveloped area, where wages are lower, to the developed region, where wages are higher. As a result, at $\theta = \pi$, the labor force increases very slowly, and, in turn, production and capital increase very slowly as well. Note that our finding that in the uderdeveloped economy capital increases very slowly is consistent with the empirical investigation in Lucas (2002), highlighting a low flow of capitals from rich to poor countries.

The migration of capital reduces the impact of migration of workers, since, as shown in Figures 9.a, 9.c and 9.e, if $a = 2$ and $b = 1$, capital, labor, and production at $\theta = \pi$ grow faster than if $a = 0$ and $b = 1$. This happens because capital flows from the developed region (where returns on investments are lower) to the underdeveloped region (where returns on investments are higher). Consequently, in the underdeveloped region, wages increase, and the number of workers migrating in search of better earnings becomes smaller.

Finally, the fact that for $a = 0$ and $b = 1$, in the "short run" (say, for $t < 100$) production increases very slowly (See Figures 9.e and 9.f) contradicts the neoclassical theory predicting that poor economies would grow faster than wealthy economies. Instead, the slow growth shown in Figures 9.e and 9.f for $a = 0$ and $b = 1$ agree with the analysis in Keefer and Knack (1997), which points out on an empirical basis that in the time interval from 1960 to 1989 income has grown much slower in poor countries than in rich countries.

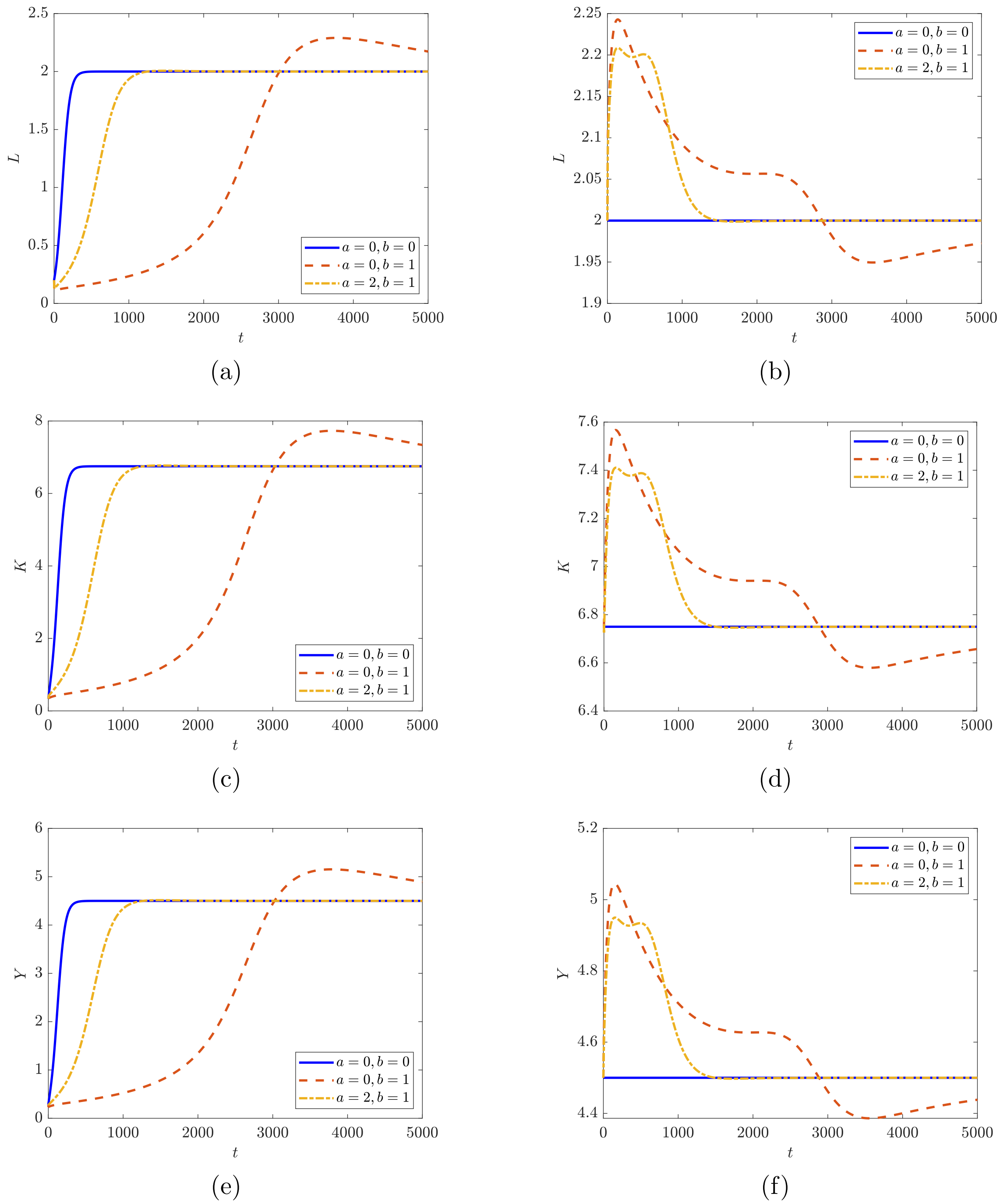


Figure 9: Transitional dynamics of labor, capital and production at $\theta = \pi$ (left column) and $\theta = 0$ (right column).

It is also interesting to see what happens at $\theta = 0$ (the center of the developed region). Here, if $a = 0$ and $b = 0$, capital, labor, and production do not vary over time, since at $t = 0$ they are already in perfect equilibrium, and labor and capital do not move across space. Instead, if $a = 0$ and $b = 1$, there is a time interval immediately after $t = 0$ (say, $t \in [0, 1300]$) in which the labor force increases above the steady state, since, as already noticed, new workers arrive from the underdeveloped region. As a consequence, production

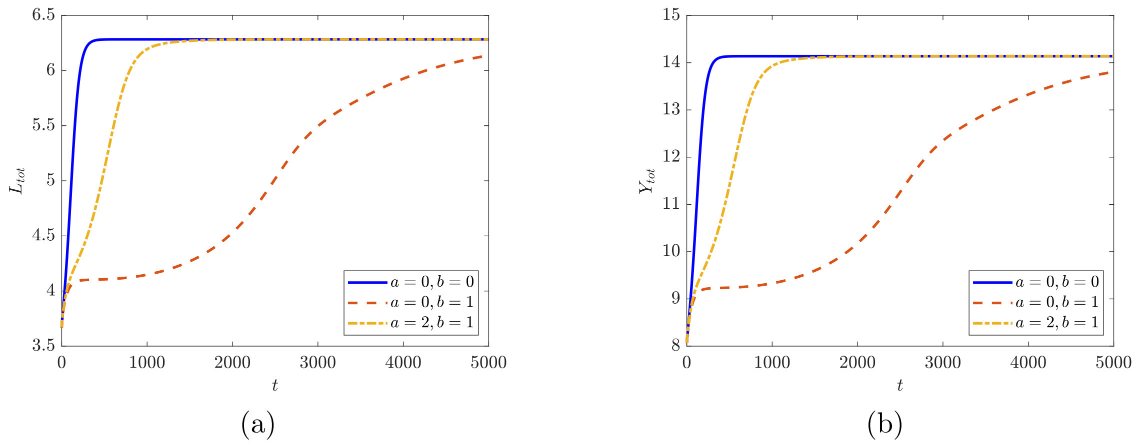


Figure 10: Economy 2, transitional dynamics of total population (a) and total production (b).

increases and, in turn, capital increases as well. Then, in the developed region, wages decline, which stops the migration of workers from the underdeveloped region. As a result, after some amount of time, labor, capital and production start to decline, to reach the steady state. Again, the migration of capital reduces the effect of the migration of labor (see the case $a = 2$ and $b = 1$ in Figures 9.b, 9.d and 9.f).

It is worth pointing out that migration has also a strong impact on the growth of the overall economy. In particular, if labor migrates and capital does not ($a = 0$ and $b = 1$), total population, and, hence, also total production, grow significantly more slowly than if labor and capital do not migrate (Figures 10.a and 10.c). Interestingly, this phenomenon can be explained based on the interaction between the migration of labor and *natural* population growth. If labor migrates and capital does not, then, as we already noticed, in some finite time interval immediately after $t = 0$ workers move from the underdeveloped region to the developed region. In the underdeveloped region, the migrant workers are substituted by the new labor force that is created because of natural population growth. However, the population level is low. Therefore, due to the (concave) parabolic shape of the logistic law (118), the decline in the number of workers (caused by migration) causes a decline in the natural rate of population growth. In the developed region, the natural rate of population growth declines as well. Indeed, here the population level is equal to the equilibrium value already at the time $t = 0$, so that, due to the arrival of new workers, the labor force rises above the steady state and the natural rate of population growth becomes negative. In summary, migration of workers reduces population growth in both the developed and the underdeveloped region, and, actually, in the developed region, the negative natural population growth wipes out the flow of workers that arrives from the developed region.

7 Conclusions and future work

Analyzing the effects of migration of labor and capital on economic development is a challenging issue that we cannot tackle based on classical models with no spatial dependence.

In this paper, we develop a new model of economic growth that accounts for the spatial migration of labor and capital. In particular, we assume that the flows of workers and capital across space depend on differences in salaries and returns on investments, and we endogenize wages and returns on investments by assuming that productive firms are profit maximizers. Furthermore, by considering constant returns to scale and a (generic) law of population growth with an equilibrium point, we allow for a spatially homogeneous time-invariant solution, which we prove to be asymptotically stable.

The contribution of this paper is three-fold. First, we show that the equilibrium of the classical Solow model remains stable even in the presence of economic-driven migration of labor and capital. Therefore, if workers and capital can freely move across space in search of better earnings, the stability of the long-run equilibrium is not altered. We point out that this fact is not a-priori guaranteed, but it requires a theoretical investigation.

Second, we develop a new model of economic growth based on which we can find out and investigate transitional dynamics that the Solow model with no spatial dependence does not capture. For instance, we can see how the migration of workers, even though it accelerates the process of salary equalization, can slow down the convergence of wages at some spatial locations. In addition, we also consider the case of a country with both a developed and an underdeveloped region, showing how the migration of workers, interacting with natural population growth, can cause a decline in the total aggregate production.

Third, from a theoretical standpoint, we use the theory of analytic semigroups generated by sectorial operators to show that the spatially homogeneous equilibrium is non-linearly and locally asymptotically stable. To the best of our knowledge, this is the first time that such a theory is applied to a spatial model of economic growth that takes into account a non-linear coupled dynamics for capital and labor.

Nevertheless, we also acknowledge that the theoretical analysis of the model performed in the current paper is not complete. Indeed, we have not answered whether the model reaches the spatially homogeneous equilibrium when the initial distribution of labor and capital is far from the spatially uniform steady state. Similarly, we have not investigated if the model admits only a spatially homogeneous equilibrium, or if, instead, other steady states may exist in which the distribution of labor and capital varies across space. The above analyses could be the object of future work. As a future investigation, one could extend the model to the case where the parameters (the capital and labor diffusion coefficient, the saving rate, and the depreciation rate) vary across space. Finally, as another future work, it would also be interesting to consider the effect of taxation or technological progress or to model the differences between skilled and unskilled workers.

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Ethical conduct: I confirm that I am the sole author of this manuscript and there are no other persons who satisfied the criteria for authorship but are not listed. I also declare that this article does not contain any studies involving animals or human participants.

Data availability: no data were used for conducting this study.

Appendix A: Derivation of equations (11) and (16)

The partial differential equation (11) can be obtained as follows. Let $\theta \in (0, 2\pi)$ and let us consider the arc of the circle that goes (counter-clockwise) from $\theta - \Delta\theta$ to $\theta + \Delta\theta$. Let t denote any time in $(0, +\infty)$. In the time interval $[t, t + \Delta t)$, the variation of labor in the above spatial interval is the sum of two contributions, one due to migration and one due to the natural population growth. In particular, the contribution due to migration is equal to the number of workers that in the time interval $[t, t + \Delta t)$ cross location $\theta - \Delta\theta$ minus the number of workers that cross location $\theta + \Delta\theta$. Therefore, we have:

$$\int_{\theta-\Delta\theta}^{\theta+\Delta\theta} \left(L(z, t + \Delta t) - L(z, t)\right) dz =$$
$$N(\theta - \Delta\theta, t, t + \Delta t) - N(\theta + \Delta\theta, t, t + \Delta t) + \int_{t}^{t+\Delta t} \int_{\theta-\Delta\theta}^{\theta+\Delta\theta} \eta(L(z, s)) dz ds. \tag{A.1}$$

If we divide both sides of (A.1) by Δt, let Δt tend to zero and use (7) we obtain:

$$\int_{\theta-\Delta\theta}^{\theta+\Delta\theta} \frac{\partial L(z, t)}{\partial t} dz = \tau_L(\theta - \Delta\theta, t) - \tau_L(\theta + \Delta\theta, t) + \int_{\theta-\Delta\theta}^{\theta+\Delta\theta} \eta(L(z, t)) dz. \tag{A.2}$$

Then, we divide both sides of (A.2) by $2\Delta\theta$ and let $\Delta\theta$ tend to zero. By doing that, and by applying the mean value theorem, we obtain:

$$\frac{\partial L(\theta, t)}{\partial t} = -\frac{\partial \tau_L(\theta, t)}{\partial \theta} + \eta(L(\theta, t)), \quad \theta \in (0, 2\pi), \quad t \in (0, +\infty). \tag{A.3}$$

Finally, by taking into account (10), the partial differential equation (11) is readily obtained. The procedure used to derive the partial differential equation (16) is analogous to that followed above, and thus we do not report it to save space.

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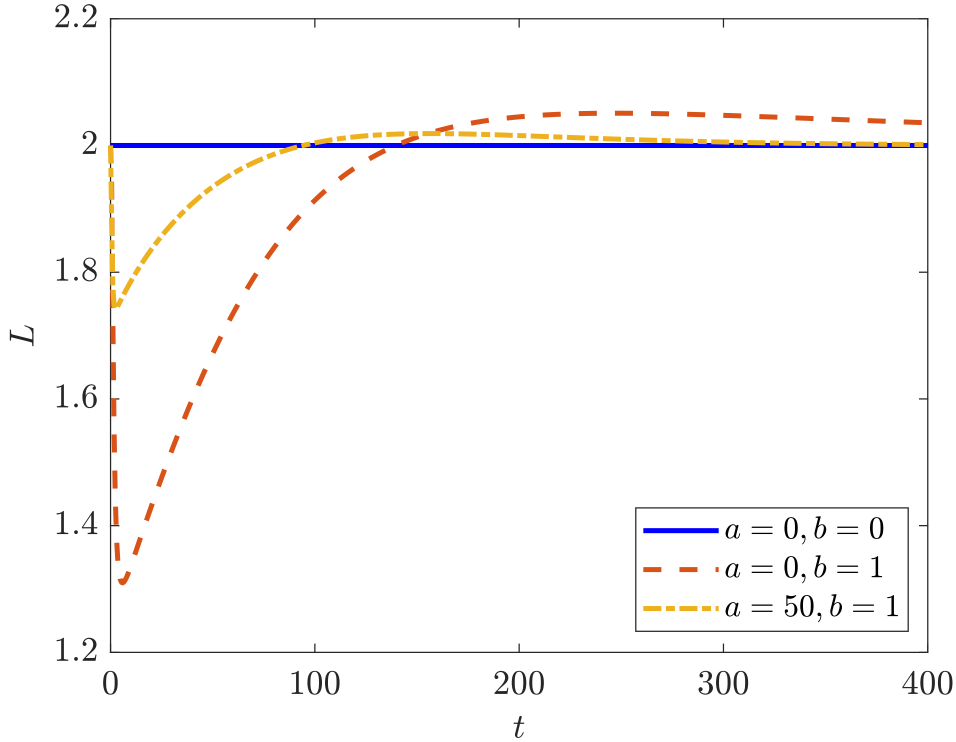
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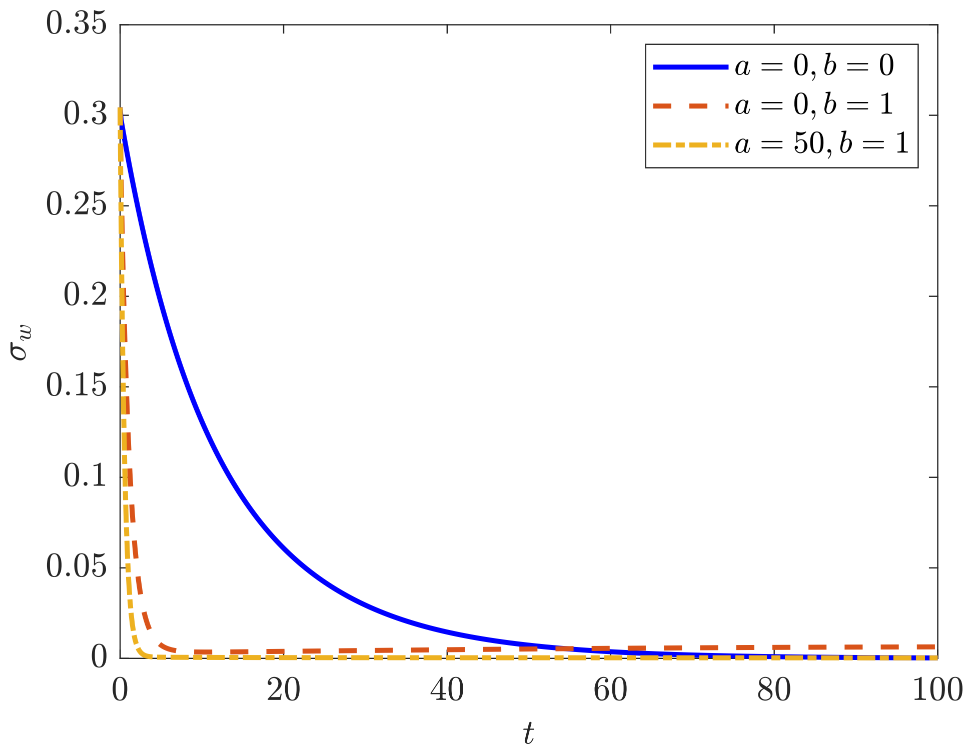

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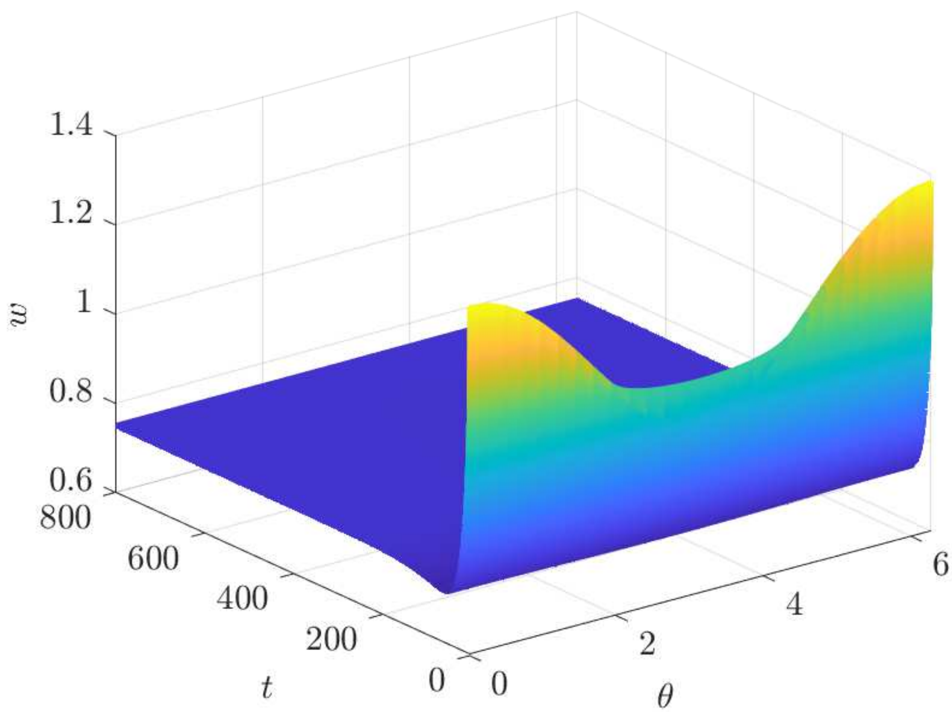
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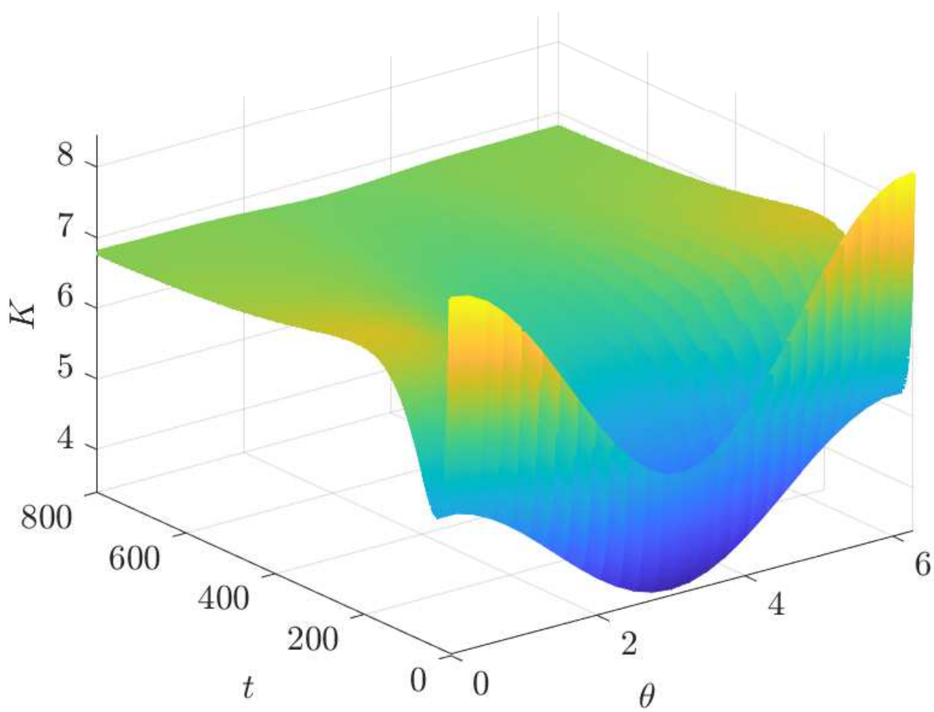
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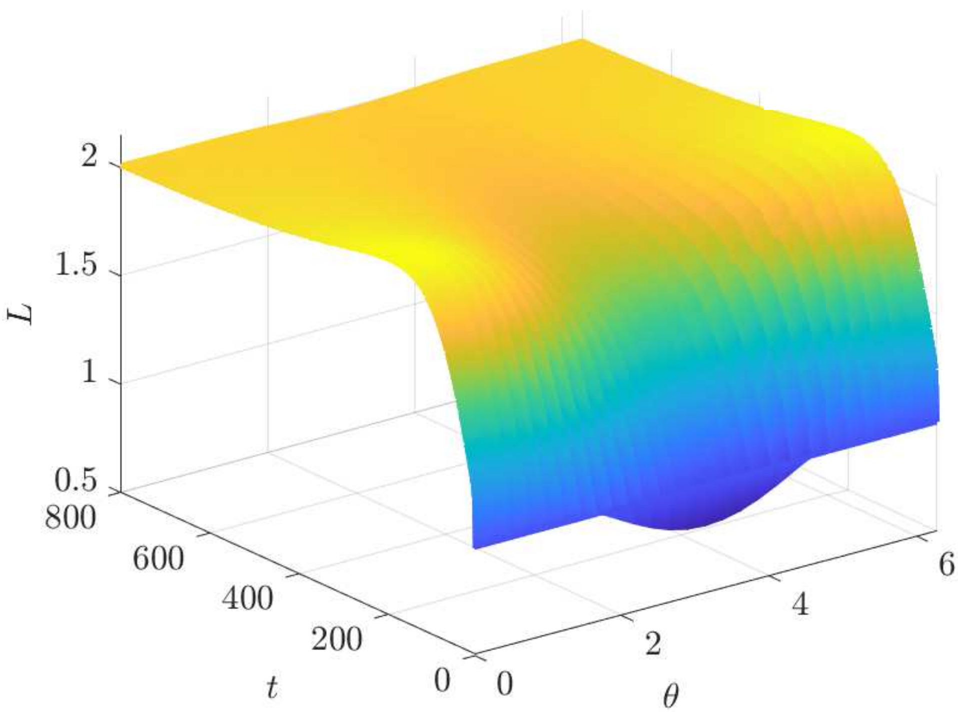

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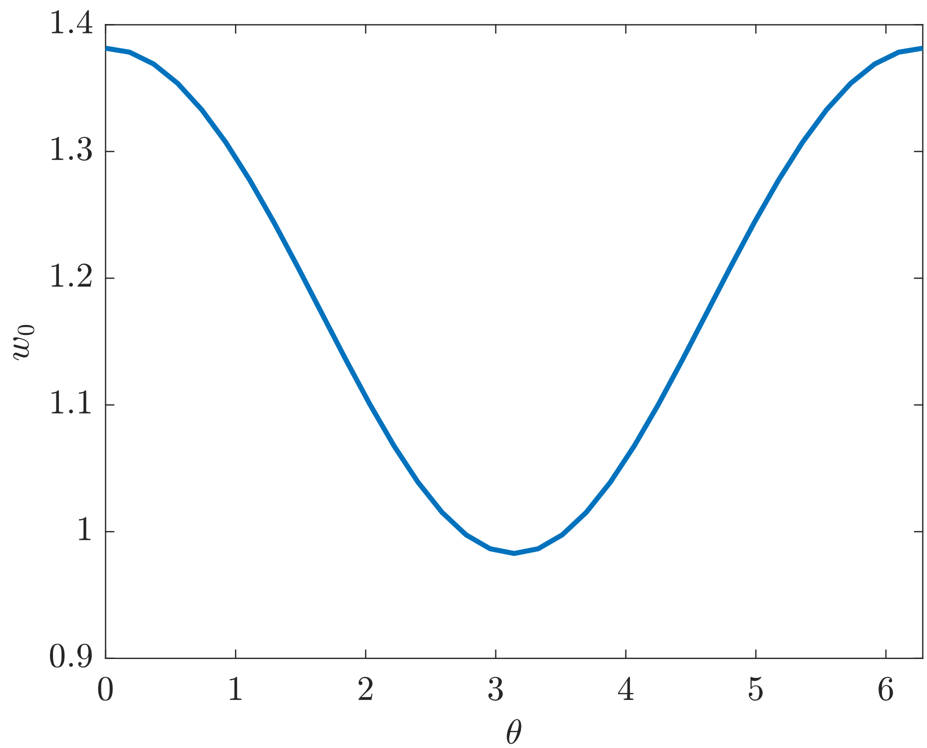
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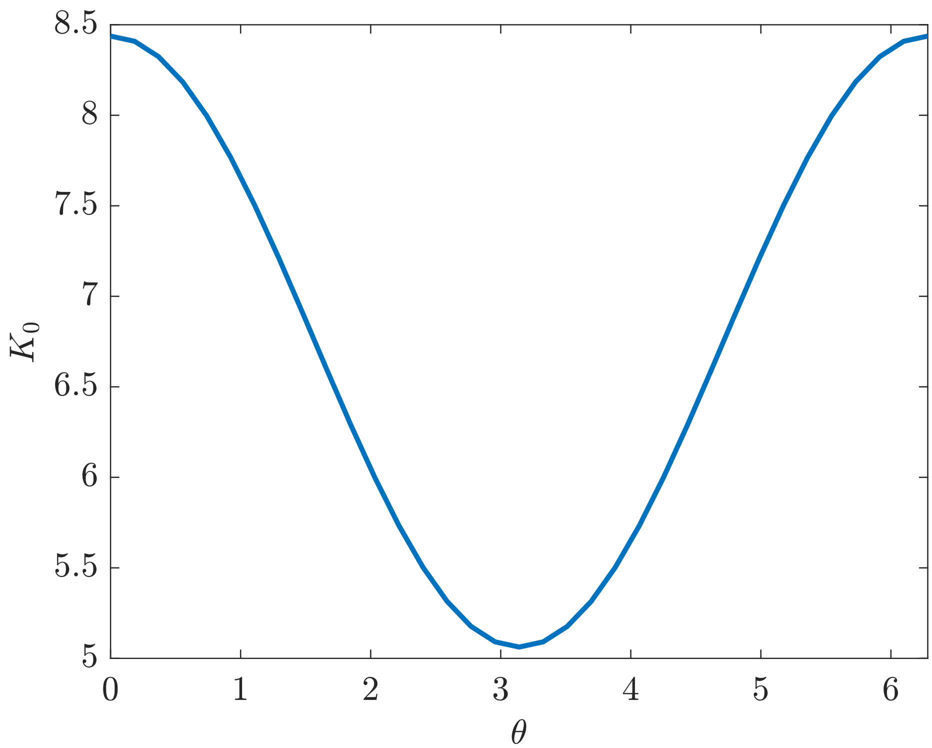
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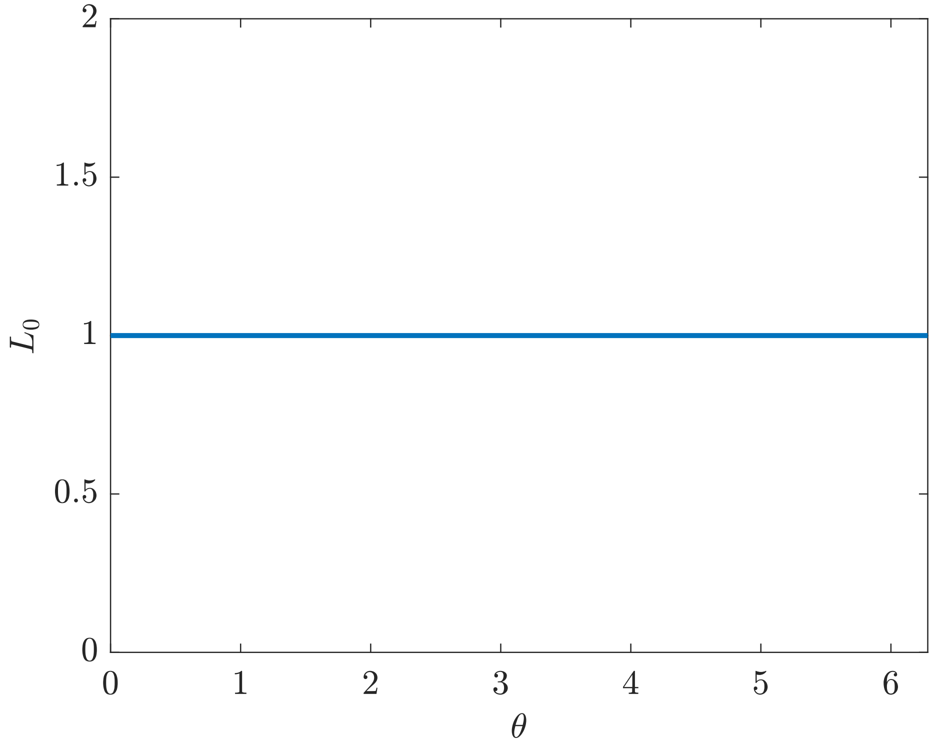
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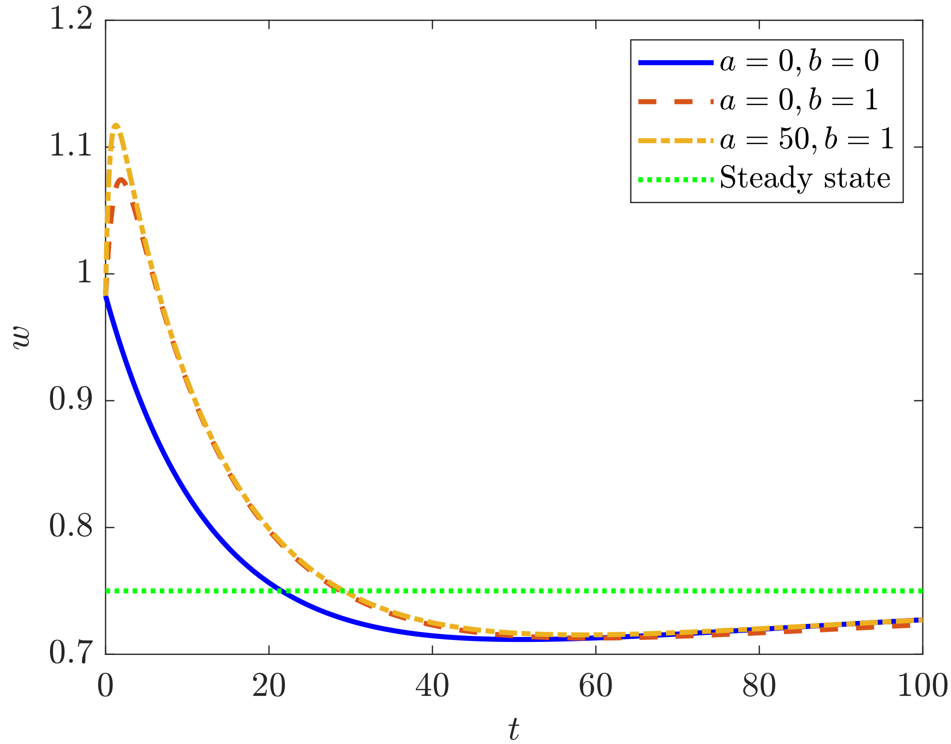
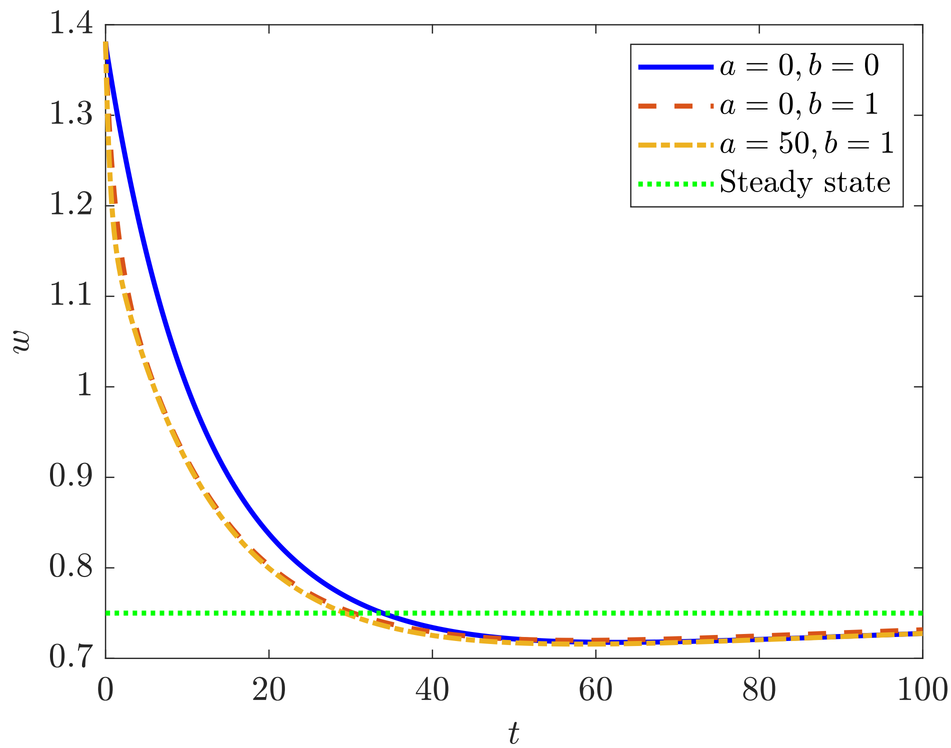
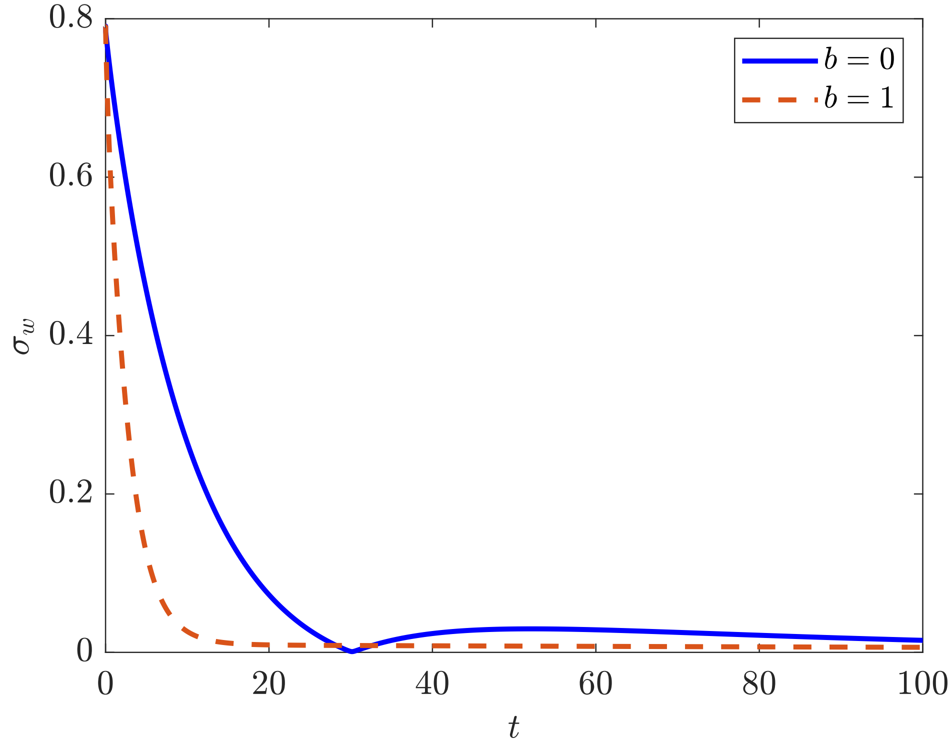
6
θ

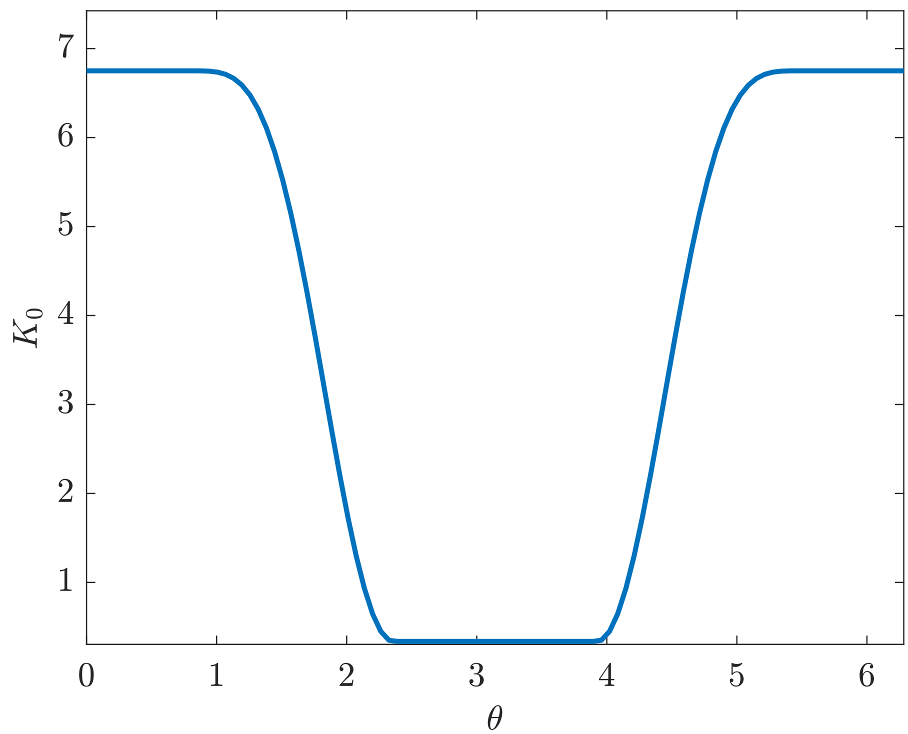


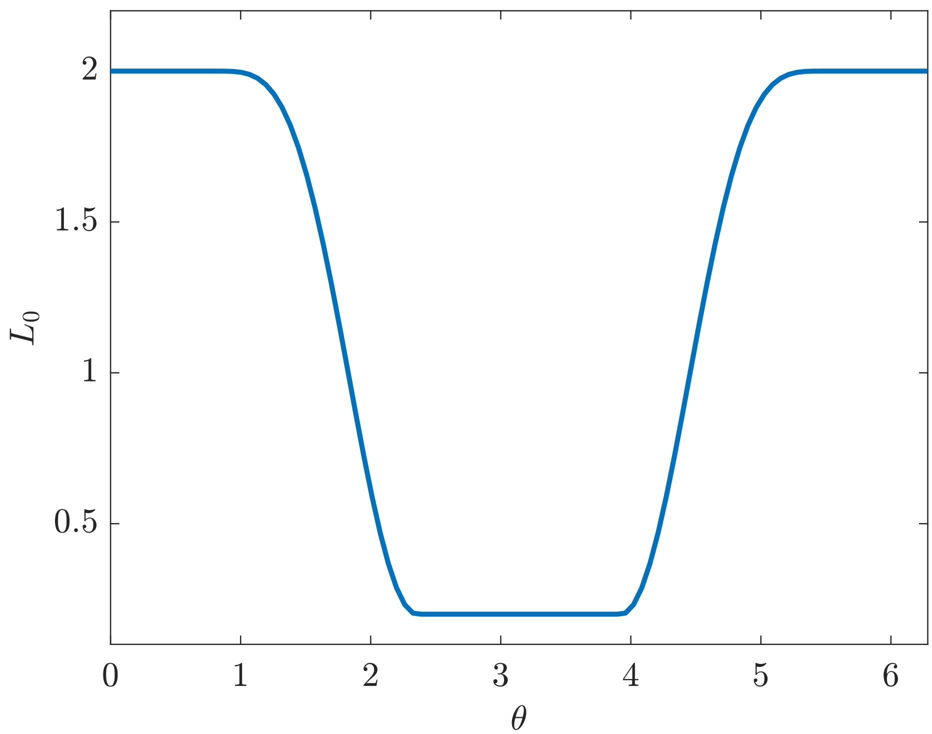
1.2
1.1
1
0.9
0.8
0.7
w
0
20
40
60
80
100
t
a = 0, b = 0
a = 0, b = 1
a = 50, b = 1
Steady state



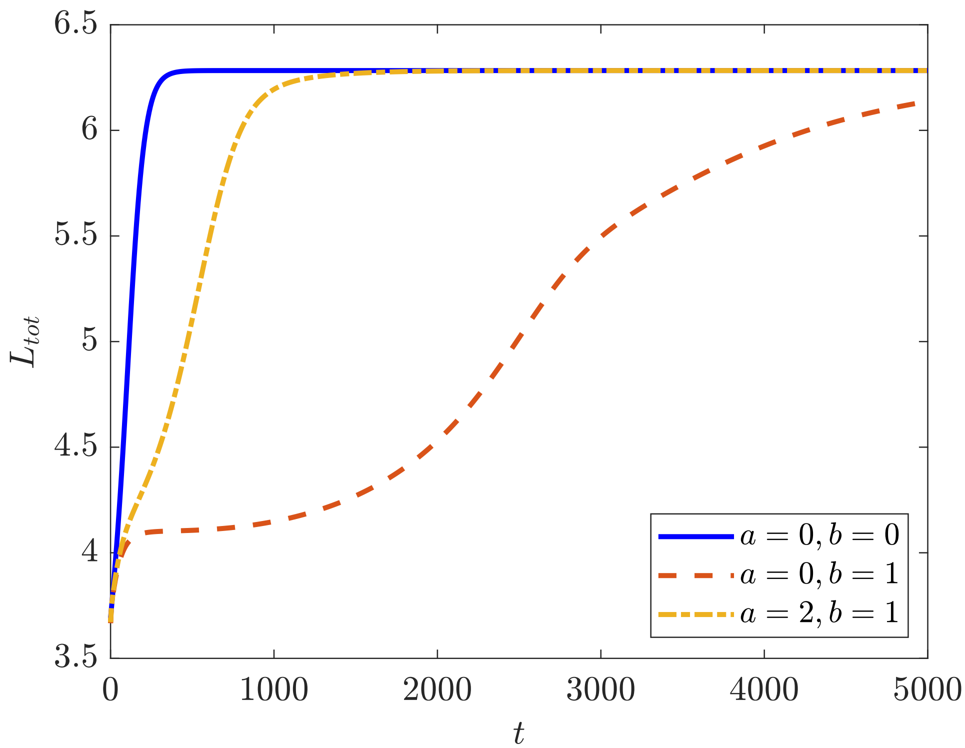
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
w
0
20
40
60
80
100
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 50, b = 1$
Steady state

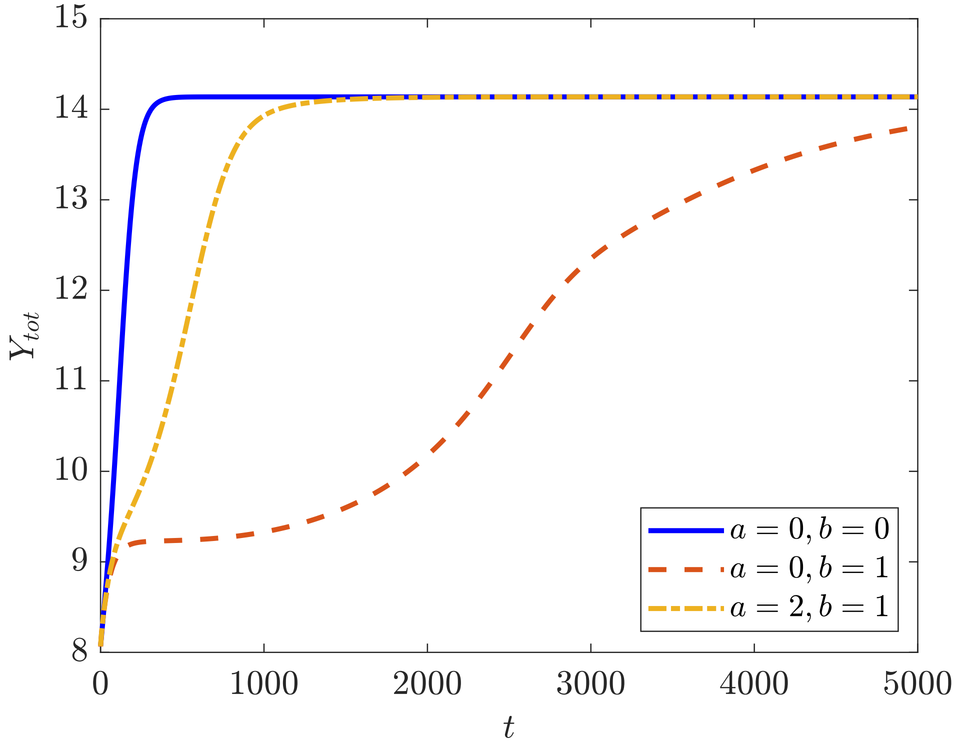

0.8
0.6
0.4
0.2
0
σ_w
0
20
40
60
80
100
t
$b = 0$
$b = 1$

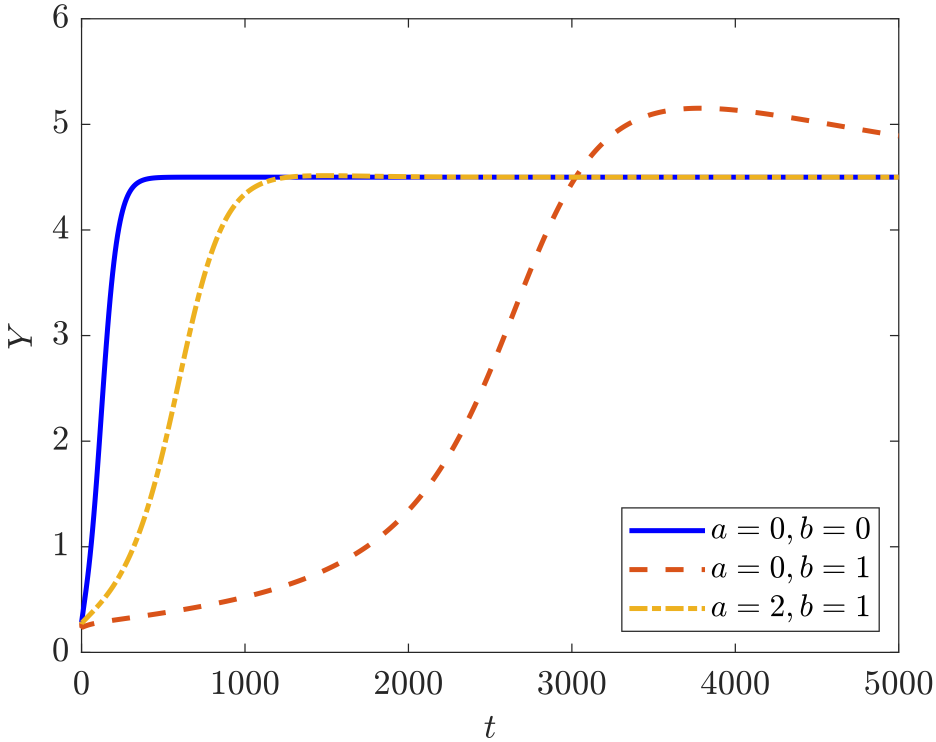

7
6
5
4
3
2
1
K_0
0
1
2
3
4
5
6
θ



2
1.5
1
0.5
L_0
0
1
2
3
4
5
6
θ


6.5
6
5.5
5
4.5
4
3.5
L_{tot}
0
1000
2000
3000
4000
5000
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 2, b = 1$


15
14
13
12
11
10
9
8
Y_{tot}
0
1000
2000
3000
4000
5000
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 2, b = 1$


6
5
4
3
2
1
0
Y
0
1000
2000
3000
4000
5000
t
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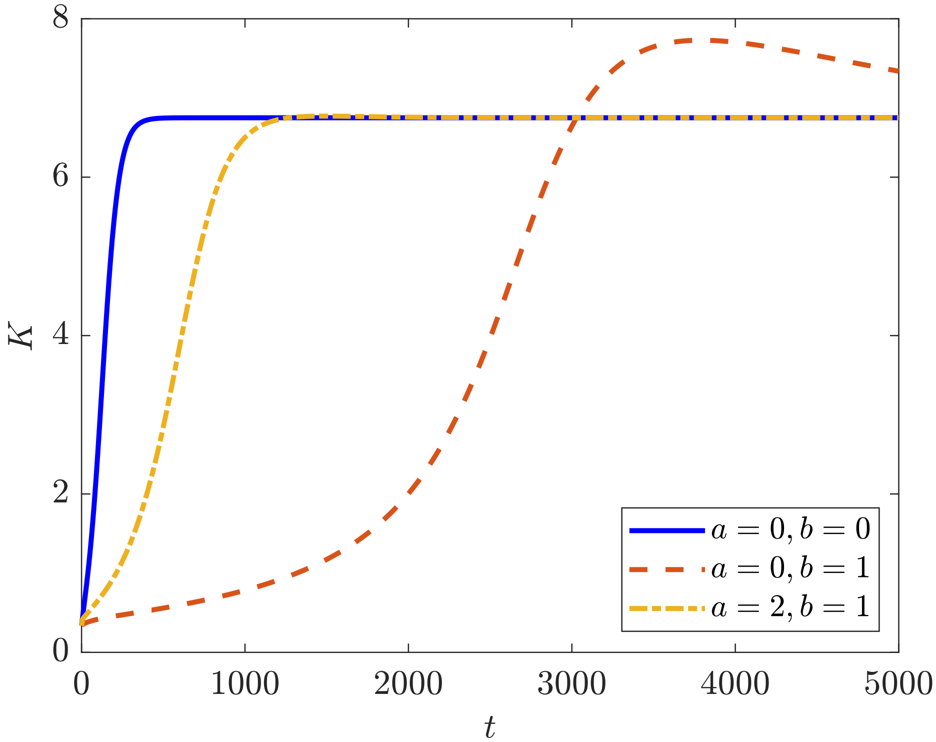
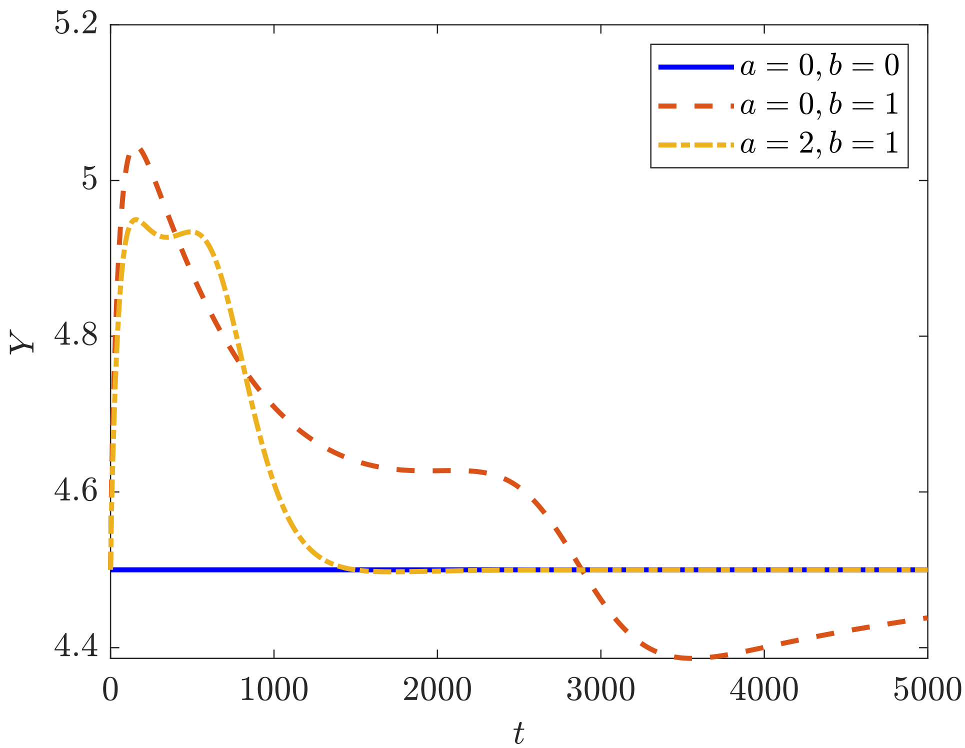
$a = 0, b = 1$
$a = 2, b = 1$



8
6
4
2
0
K
0
1000
2000
3000
4000
5000
t
a = 0, b = 0
a = 0, b = 1
a = 2, b = 1


5.2
5
4.8
4.6
4.4
Y
0
1000
2000
3000
4000
5000
t
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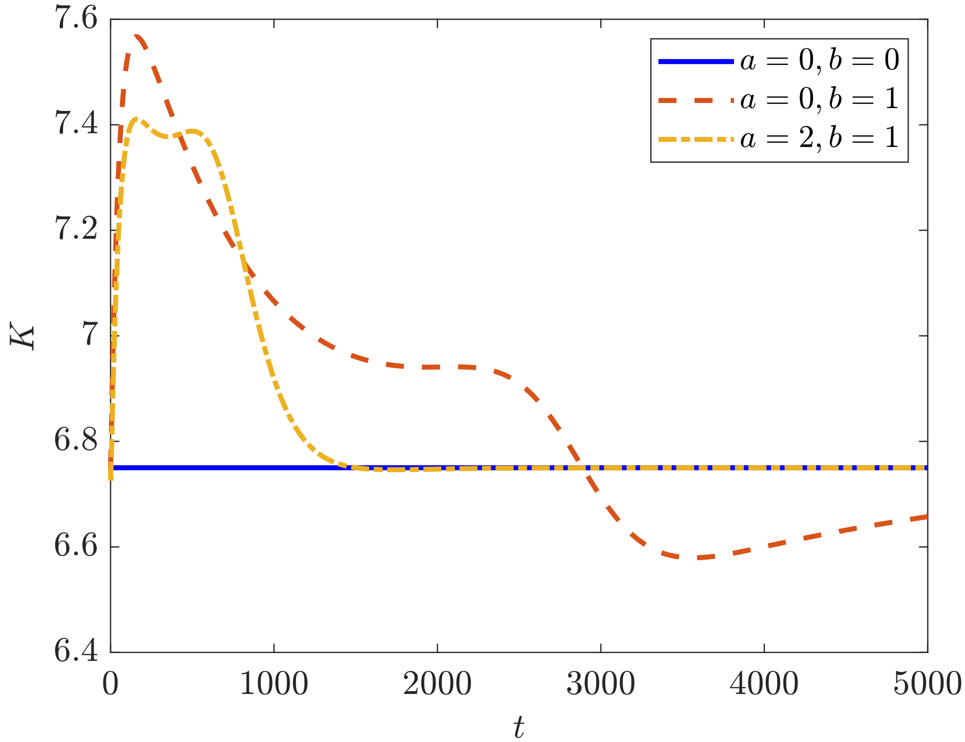
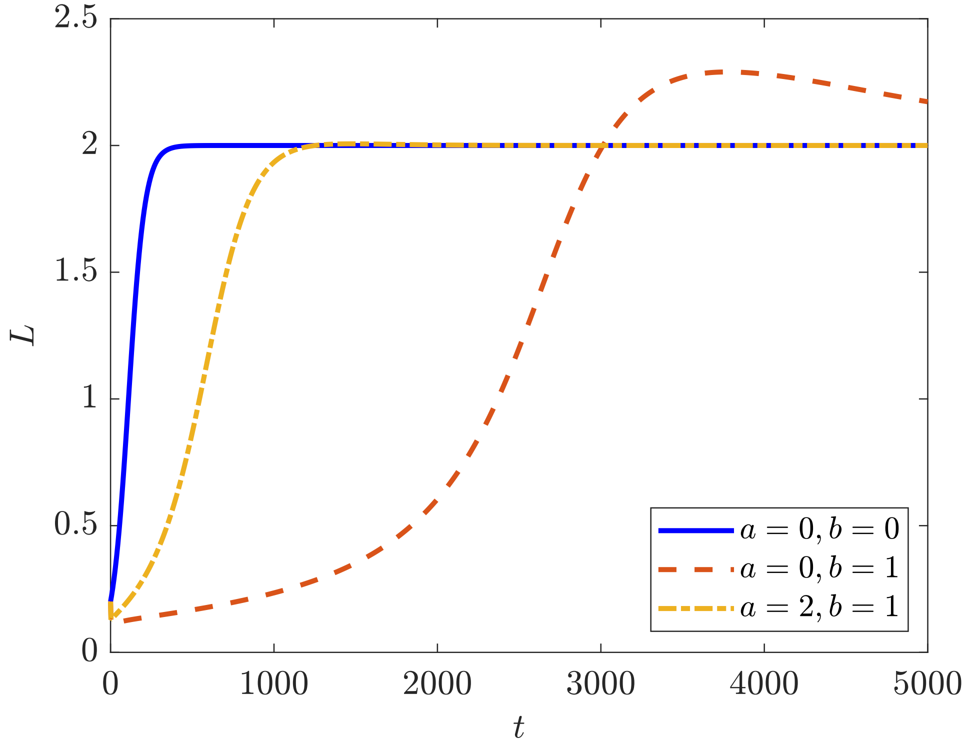
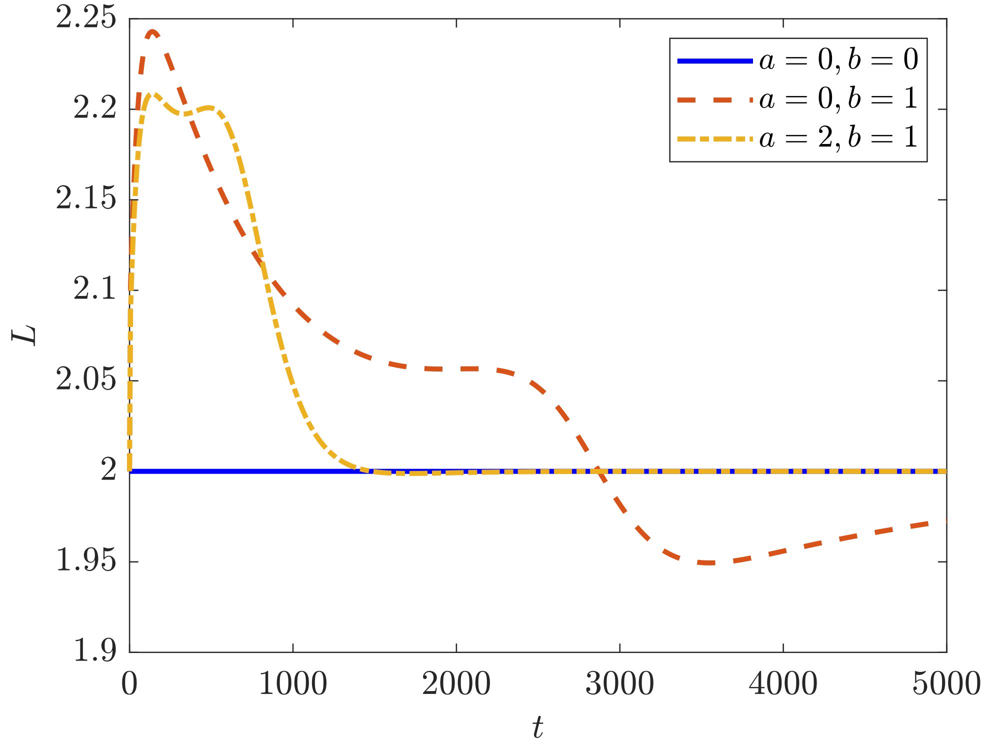
$a = 0, b = 1$
$a = 2, b = 1$


7.6
7.4
7.2
7
6.8
6.6
6.4
K
0
1000
2000
3000
4000
5000
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 2, b = 1$


2.5
2
1.5
1
0.5
0
L
0
1000
2000
3000
4000
5000
t
a = 0, b = 0
a = 0, b = 1
a = 2, b = 1


2.25
2.2
2.15
2.1
2.05
2
1.95
1.9
L
0
1000
2000
3000
4000
5000
t
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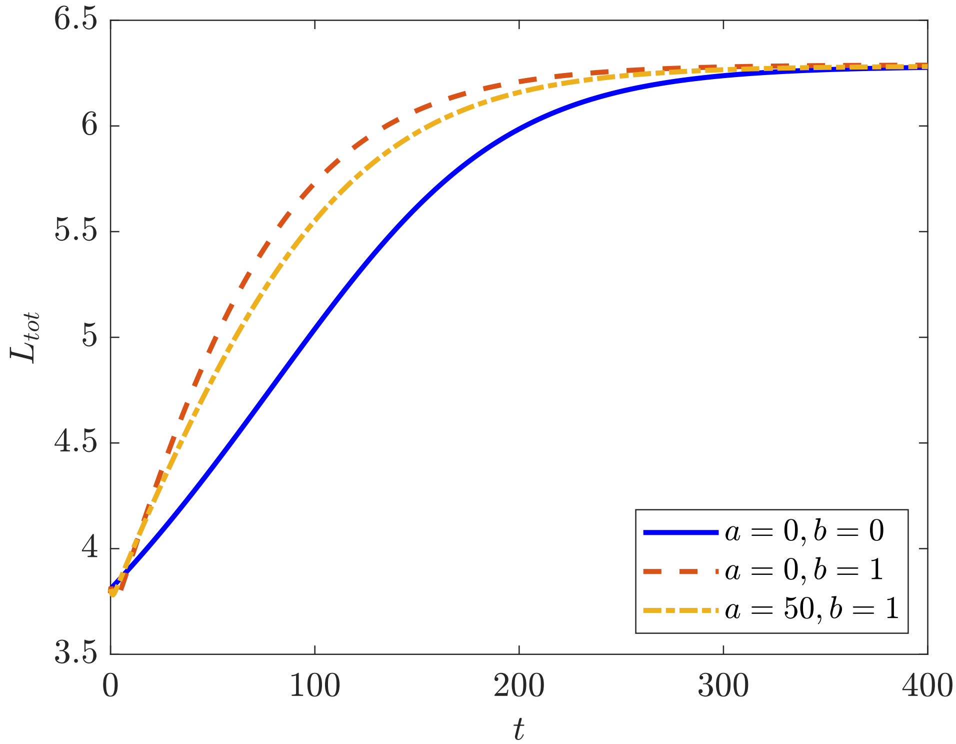
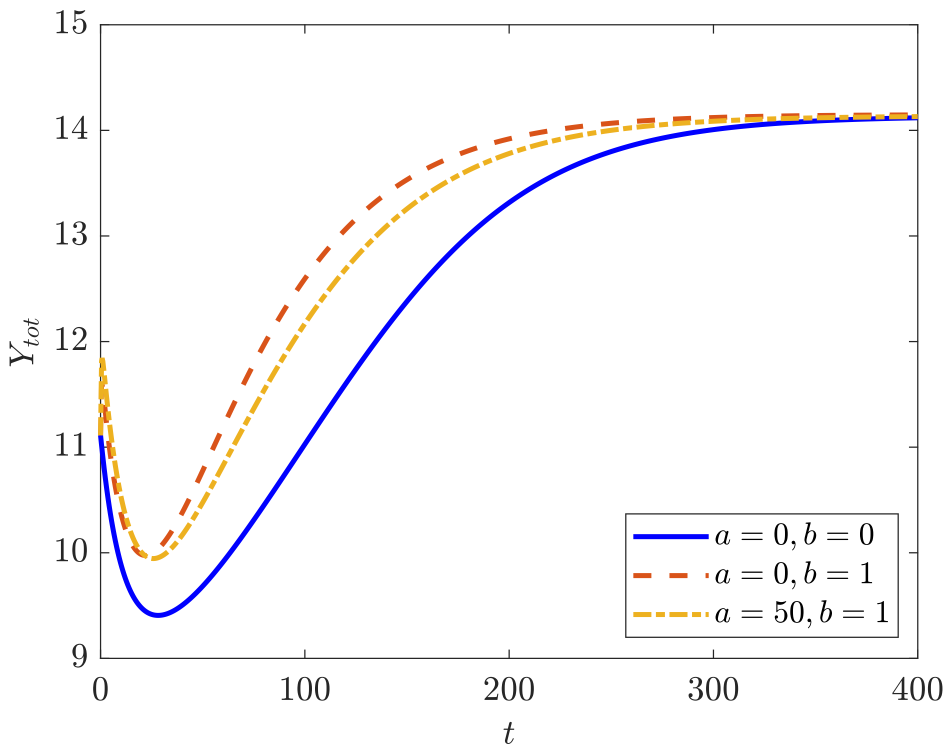
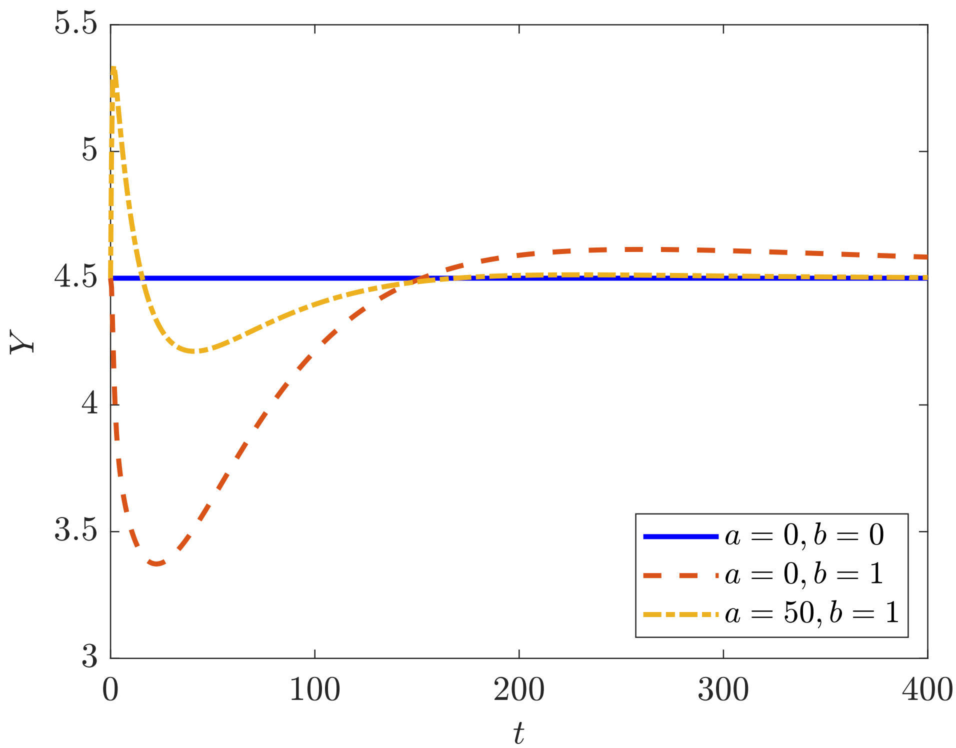
$a = 2, b = 1$



6.5
6
5.5
5
4.5
4
3.5
L_{tot}
0
100
200
300
400
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 50, b = 1$


15
14
13
12
11
10
9
0
100
200
300
400
t
Y_{tot}
$a = 0, b = 0$
$a = 0, b = 1$
$a = 50, b = 1$


5.5
5
4.5
4
3.5
3
Y
0
100
200
300
400
t
$a = 0, b = 0$
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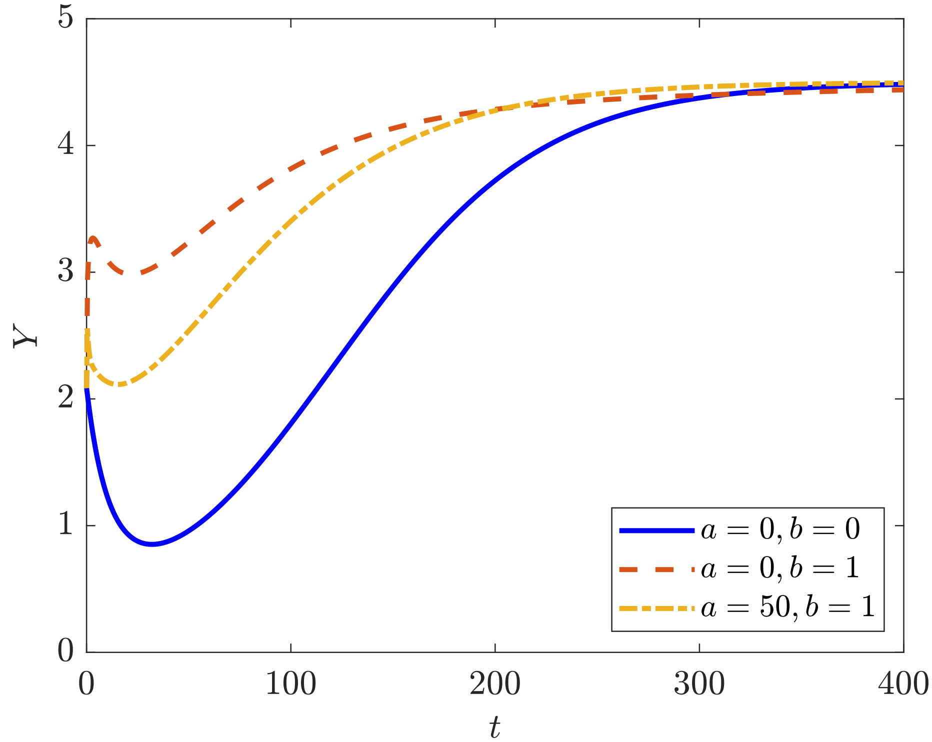
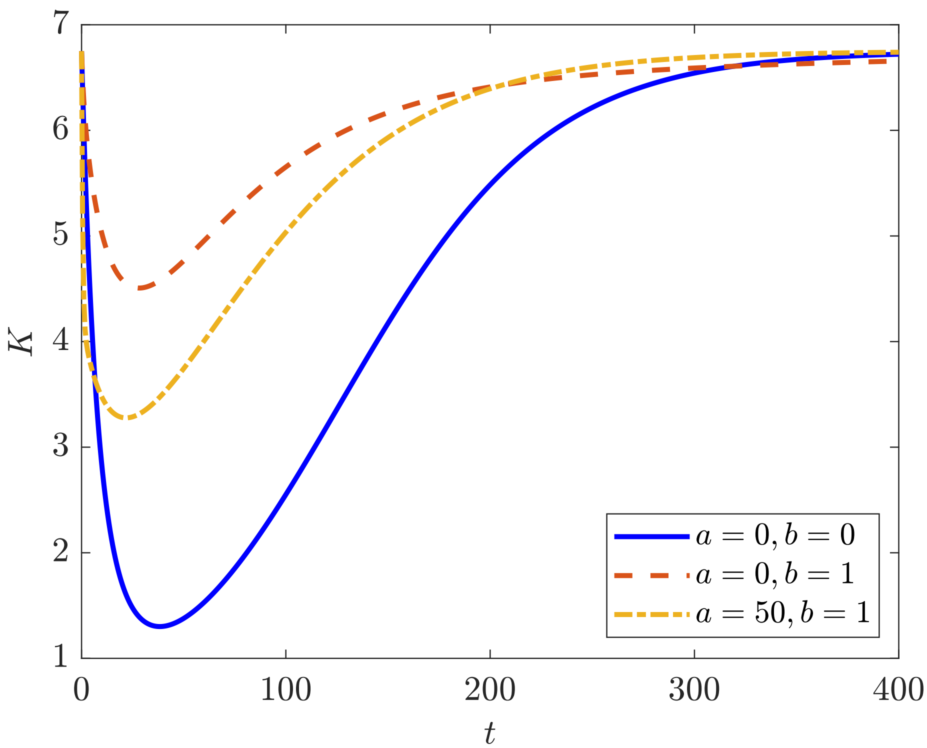
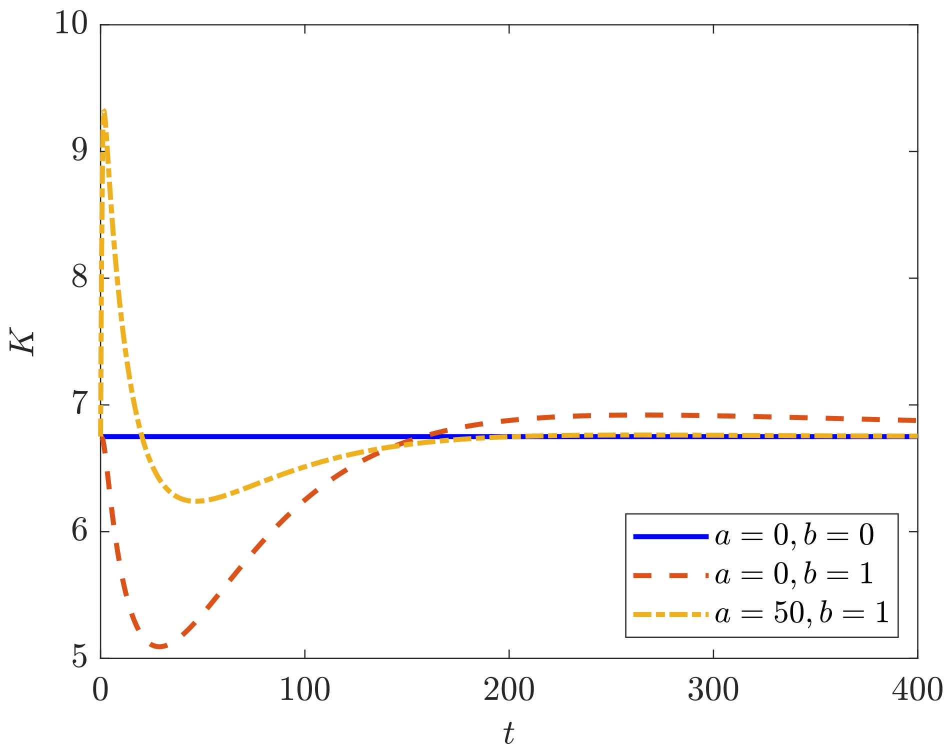
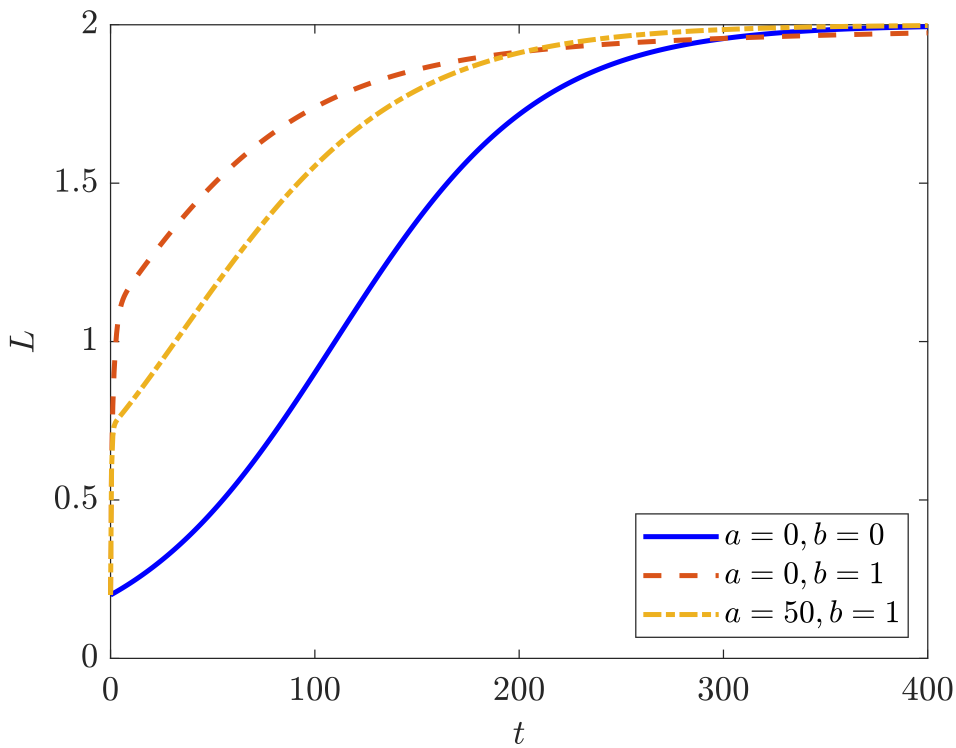
$a = 50, b = 1$


5
4
3
2
1
0
Y
0
100
200
300
400
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 50, b = 1$

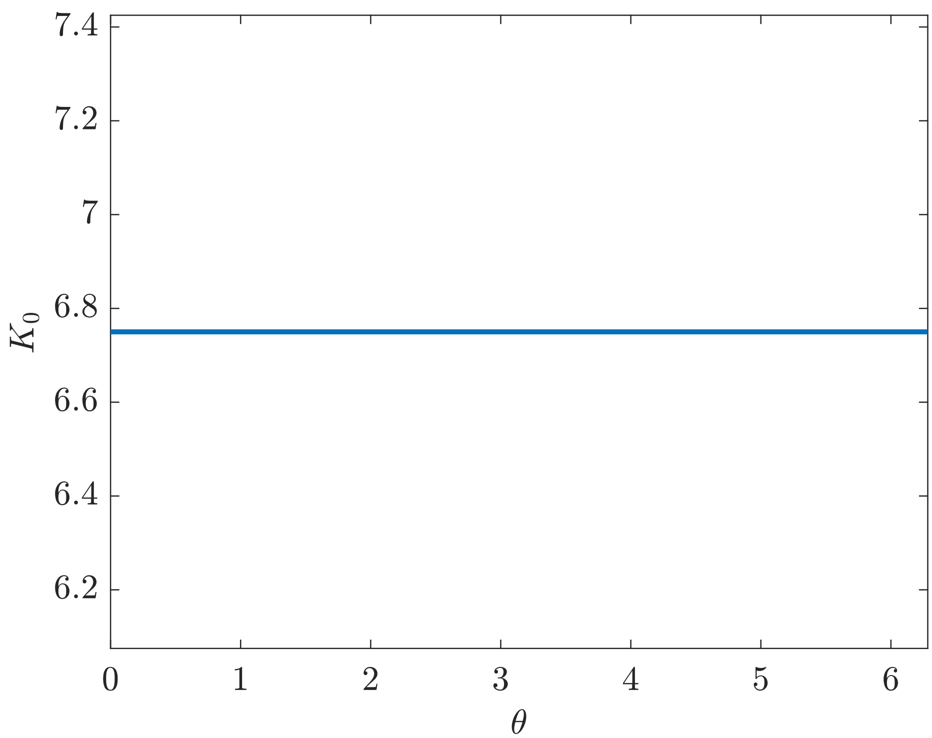


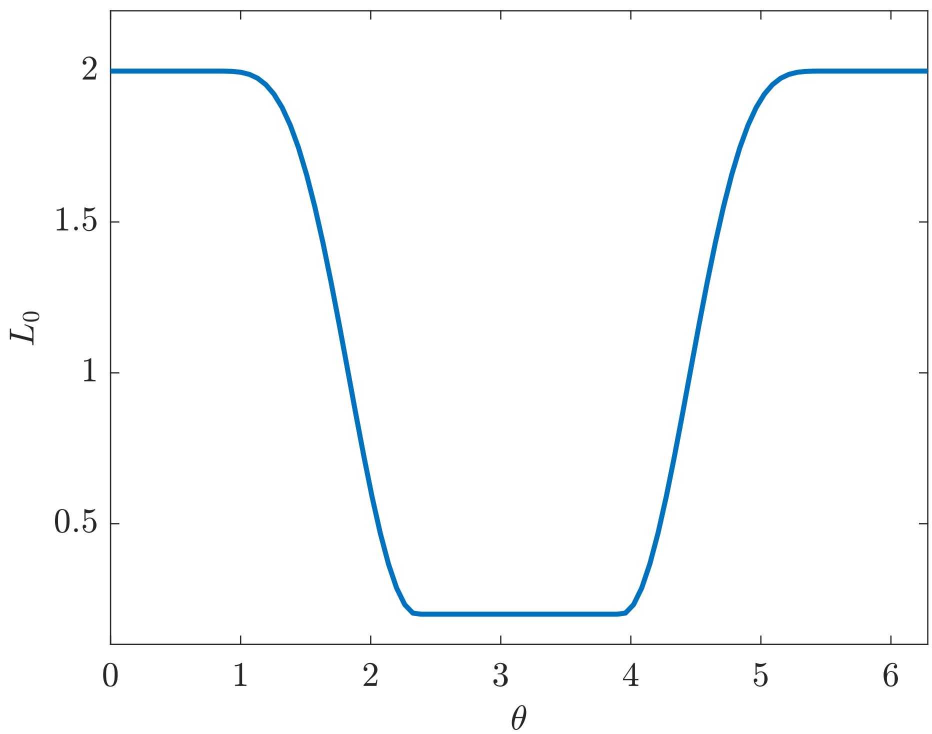
7
6
5
4
3
2
1
K
0
100
200
300
400
t
a = 0, b = 0
a = 0, b = 1
a = 50, b = 1

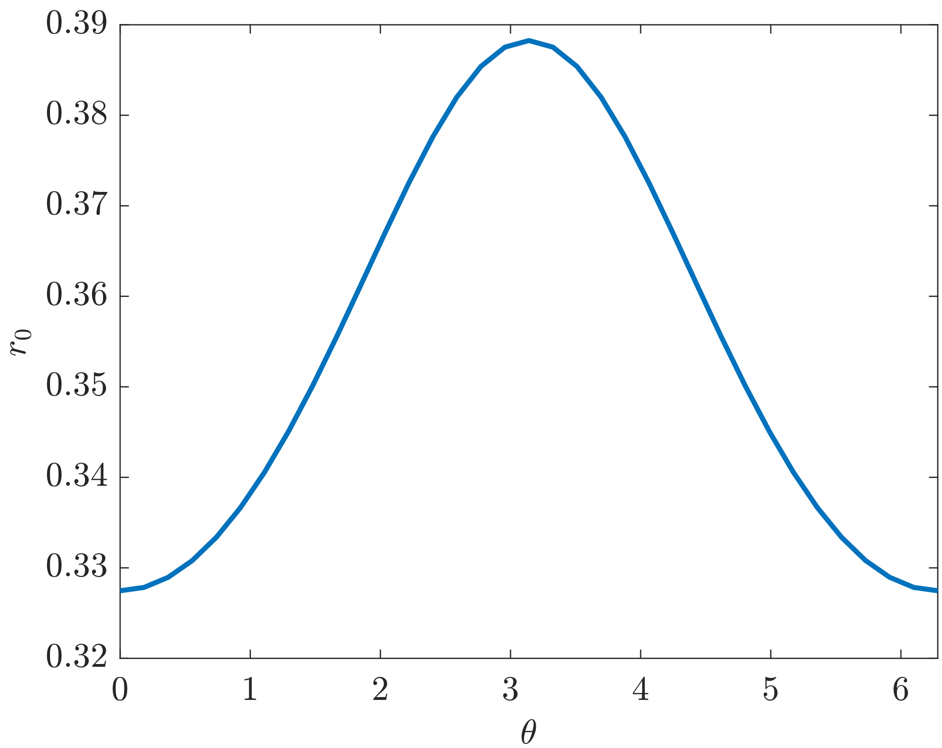

10
9
8
7
6
5
K
0
100
200
300
400
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 50, b = 1$



L
t
$a = 0, b = 0$
$a = 0, b = 1$
$a = 50, b = 1$


7.4
7.2
7
6.8
6.6
6.4
6.2
K_0
0
1
2
3
4
5
6
θ


2
1.5
1
0.5
L_0
0
1
2
3
4
5
6
θ


0.39
0.38
0.37
0.36
0.35
0.34
0.33
0.32
r_0
0
1
2
3
4
5
6
θ

Highlights

- A new model of economic growth with migration of capital and labor is proposed
- The effect of differences in returns on the migration of labor and capital is considered
- Capital and labor follow a non-linear (coupled) spatial dynamics
- Local non-linear stability of a spatially uniform equilibrium solution is proved
- The stability analysis is performed using the theory of abstract evolution problems