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A scheme for the game p -Laplacian and its application to Image Inpainting

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Abstract

We propose a new numerical scheme for the game p -Laplacian, based on a semi-Lagrangian approximation. We focus on the 2D version of the game p -Laplacian, with the aim to apply the new scheme in the context of image processing. Specifically, we want to solve the so-called inpainting problem, which consists in reconstructing one or more missing parts of an image using information taken from the known part. The numerical tests show the reliability of the proposed method and the advantages of taking a $p > 1$ in terms of execution time and accuracy.

Keywords: game p -Laplacian, Semi-Lagrangian scheme, Inpainting, Viscosity solution

1. Introduction

In this paper we address the following problem: Given two bounded open domains $D, \Omega \subset \mathbb{R}^2$ such that $D \subset \Omega$ with $\overline{\Omega} \cap \overline{D} = \emptyset$, and two functions $f : D \rightarrow \mathbb{R}$ and $F : \overline{\Omega} \setminus D \rightarrow \mathbb{R}$, the goal is to find a function $u : \overline{\Omega} \rightarrow \mathbb{R}$ solution of the following problem

$$\begin{cases} -\Delta_p^G u = f & \text{in } D, \\ u = F & \text{in } \overline{\Omega} \setminus D. \end{cases} \quad (1)$$

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6 The operator Δ_p^G denotes the game p -Laplacian and it has been intro-
7 duced for the first time in [25] to model a stochastic game. The interest in
8 such an operator derives from the observation that it can include in itself
9 various operators as particular cases (i.e. the operator in the Aronsson equa-
10 tion [3], the infinity Laplacian [26], the mean curvature motion operator [20],
11 or, in the case $p = 2$, a multiple of the ordinary Laplacian).

12 Our final aim is to deal with problem (1) with $f \equiv 0$ in the context of the
13 inpainting problem. Inpainting in image processing consists in reconstructing
14 one or more missing or damaged parts of an image using information taken
15 from the known part. A grayscale image is interpreted as a bounded function
16 $u : \bar{\Omega} \rightarrow [0, +\infty)$. Typically, Ω is a rectangular domain and $u(x)$ represents
17 the intensity of the gray level at the point x . Usually, the missing part
18 is called *inpainting domain* and it is denoted by D . Recently, the image
19 inpainting has been also applied as a decoding step for image compression,
20 see e.g. [19] and references therein for more details.

21 The numerical image inpainting methods can be roughly separated into
22 two categories: variational and non-variational ones. In both methods the
23 reconstructed image is obtained as solution of a partial differential equation
24 (PDE). The first approach is based on a energy minimization model and the
25 PDE is the Euler-Lagrange equation associated to the optimality condition.
26 In the non-variational approach, the PDE does not necessarily derives from
27 a variational principle. For a complete survey, we refer the interested reader
28 to the books by Aubert and Kornprobst and by Schönlieb [2, 27].

29 In the context of differential approaches for facing the inpainting problem,
30 the 1-Laplacian is used in order to propagate information in the directions of
31 the isophotes, i.e. the direction orthogonal to the gradient of the image, ∇u^\perp ,
32 see [5]. Anisotropic diffusion processes have been also considered. Among
33 these we mention [6], where the anisotropic process is applied to the whole
34 original image, with the purpose of minimizing the influence of noise on the
35 estimation of the direction of the isophotes arriving at the damaged portion
36 of the image. More recently, fourth order anisotropic diffusion processes have
37 been proposed, in order to achieve better accuracy with respect to second
38 order operator, see [21] and references therein.

39 Image inpainting was also addressed through the p -Laplacian in [28],
40 showing better results than the performance of the 1-Laplacian. Here, we
41 want to follow this idea by using the game p -Laplacian and explore the be-
42 havior for different values of p in terms of accuracy and execution time, as
43 we will see later in the section related to numerical tests. Note that the game

p -Laplacian has been already used in image processing (see e.g. [15] for its application on weighted graphs with applications in image inpainting and data clustering).

With the aim to present some recent advances in the numerical discretization of (1), here we propose a new scheme, applying it to the inpainting problem for improving results.

Numerical schemes for second order possibly degenerate equations have been presented by several authors. We focus our attention on semi-Lagrangian (SL) discretization, which have been shown to be particularly suitable for the numerical treatment of degenerate diffusions, see [10]. For a comprehensive introduction to SL scheme, we refer to the book by Falcone and Ferretti [17]. A SL scheme for the mean curvature motion has been proposed in [11]. High-order SL techniques to treat possibly degenerate advection-diffusion equations are analyzed in [18], whereas SL methods for diffusion equation with non linear reaction term are addressed in [7]. A SL scheme for the game p -Laplacian has been proposed and analyzed in [16]. Compared to this scheme, we propose a method that does not require a minimization procedure to find the direction of diffusion, which is instead obtained explicitly using the numerical gradient. For this reason, the scheme here proposed results to have a lower computational cost with respect to the SL scheme proposed in [16]. The scheme we propose is much more in the spirit of [11, 13], where the second order operator is approximated by means of directional second finite differences.

The theoretical convergence of the scheme is beyond the scope of this work and will be addressed in the future.

The paper is organized as follows: in Section 2 we recall the definition of the game p -Laplacian and how we can rewrite it as a convex combination of the ∞ -Laplacian and 1-Laplacian operators. In Section 3 we propose a new SL scheme, showing its performance in Section 4. In Section 5 we apply the SL scheme to the inpainting problem. The paper ends with final comments and future perspectives contained in Section 6.

2. Game p -Laplacian operator

Let us recall the definition of the so-called game p -Laplacian operator introduced by Peres and Sheffield [25], that is

$$\Delta_p^G u := \frac{1}{p} |\nabla u|^{2-p} \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \text{for } 1 < p < \infty. \quad (2)$$

78 With respect to the variational p -Laplacian operator, in the game version
 79 appears the multiplicative term $\frac{1}{p}|\nabla u|^{2-p}$ before the divergence. This term
 80 causes the game p -Laplacian to be singular at critical points ($\nabla u = 0$) for
 81 every $p \neq 2$, whereas the variational p -Laplacian is singular for any $1 < p < 2$.
 82 Both are degenerate for $p > 2$.

83 Well-posedness for this kind of problem should be understood in a weak
 84 sense, we refer to [24, 12, 4, 23] for the weak and viscosity theory framework.
 85 In [26], authors shows that problem (1), for the case of the infinity Laplacian,
 86 has a unique viscosity solution when F, f are uniformly continuous and f does
 87 not change sign, i.e. $\inf_{\bar{\Omega}} f > 0$ or $\sup_{\bar{\Omega}} f > 0$. See also [22] for an overview
 88 on the basic results of Tug-of-War games.

89 If u is a smooth function, by expanding the derivative in (2), we obtain

$$\Delta_p^G u = \frac{1}{p} \Delta_2 u + \frac{p-2}{p} |\nabla u|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}. \quad (3)$$

90 At that point, if we formally take the limit for $p \rightarrow \infty$, one can define the
 91 game ∞ -Laplacian operator as

$$\Delta_\infty^G u := |\nabla u|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}. \quad (4)$$

92 Finally, at the points where $|\nabla u| \neq 0$, the game ∞ -Laplacian can be viewed
 93 as the second derivative in the direction of ∇u , that is

$$\Delta_\infty^G u = \sigma_\infty(\nabla u)^T D^2 u \sigma_\infty(\nabla u), \quad (5)$$

94 where $D^2 u$ denotes the Hessian matrix, and $\sigma_\infty : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as

$$\sigma_\infty(a) := \frac{1}{|a|} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (6)$$

95 The game 1-Laplacian is defined as

$$\Delta_1^G u := \Delta_2 u - \Delta_\infty^G u. \quad (7)$$

96 If u is a smooth function, by using (3) and the definition (7), the game p -
 97 Laplacian can be expressed as a convex combination of the ∞ -Laplacian and
 98 1-Laplacian as

$$\Delta_p^G u = \frac{1}{p} \Delta_1^G u + \frac{1}{q} \Delta_\infty^G u, \quad (8)$$

99 with q such that $\frac{1}{p} + \frac{1}{q} = 1$. Note that the game 1-Laplacian, at the points
 100 where $|\nabla u| \neq 0$, can be viewed as the second derivative in the direction of
 101 $(\nabla u)^\perp$, that is

$$\Delta_1^G u := \sigma_1(\nabla u)^T D^2 u \sigma_1(\nabla u), \quad (9)$$

102 where $\sigma_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as

$$\sigma_1(a) = \frac{1}{|a|} \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}. \quad (10)$$

103 3. A new semi-Lagrangian scheme for the game p -Laplacian

104 A simple way to construct a semi-Lagrangian scheme for (1) consists in
 105 discretizing the second order operators by a directional second finite differ-
 106 ence. Supposing $\sigma \in \mathbb{R}^2$ is given, let us introduce a discretization parameter
 107 $\delta > 0$ and let us consider the following approximation:

$$\sigma^T D^2 u(x) \sigma \approx \frac{1}{\delta^2} (u(x + \delta \sigma) + u(x - \delta \sigma) - 2u(x)). \quad (11)$$

108 A similar approximation with σ replaced by $\sigma_\infty(\nabla u)$ and $\sigma_1(\nabla u)$ can be
 109 considered in order to approximate (5) and (9), respectively. Such approxi-
 110 mations are valid only in the non-singular case, i.e. when $\nabla u \neq 0$. By using
 111 (8) we derive, for x such that $\nabla u(x) \neq 0$, an approximation for (2) given by

$$\begin{aligned} \Delta u_p^G(x) \approx & \frac{1}{p\delta^2} (u(x + \delta \sigma_1(\nabla u(x))) + u(x - \delta \sigma_1(\nabla u(x))) - 2u(x)) + \\ & \frac{1}{q\delta^2} (u(x + \delta \sigma_\infty(\nabla u(x))) + u(x - \delta \sigma_\infty(\nabla u(x))) - 2u(x)). \end{aligned} \quad (12)$$

112 For simplicity, let us suppose Ω be the unit square, $\Omega := (0, 1) \times (0, 1)$. Given
 113 a integer N_h , we define a space discretization step $h = \frac{1}{N_h}$. Let us introduce
 114 a uniform grid $\mathcal{G}_h(\overline{\Omega}) = \{x_j = jh, j \in \{0, \dots, N_h\}^2\}$ and let us consider the
 115 following sets of indexes:

$$Q = \{j \in \mathbb{Z}^2 \text{ such that } x_j \in D\}, \quad Q_b = \{j \in \mathbb{Z}^2 \text{ such that } x_j \in \overline{\Omega} \setminus D\}. \quad (13)$$

116 For any $j \in Q$, we denote by $D_j[u]$ a centered finite difference approximation
 117 of $\nabla u(x_j)$ and we define two couples of discrete characteristics as

$$y_\infty^\pm(x_j) := x_j \pm \delta \sigma_\infty(D_j[u]), \quad y_1^\pm(x_j) := x_j \pm \delta \sigma_1(D_j[u]). \quad (14)$$

118 Since $|\sigma_\infty| = |\sigma_1| = 1$, for δ small enough, characteristics never leave from
 119 $\overline{\Omega}$. In the case the characteristics are points which belong to D , we need to
 120 introduce an interpolation operator in order to reconstruct the value of u at
 121 these points. Given a grid function $v : \mathcal{G}_h(\overline{\Omega}) \rightarrow \mathbb{R}$, we denote by $I[v] : \Omega \rightarrow \mathbb{R}$
 122 a piecewise polynomial interpolation of v in Ω . Note that, in the case $y_1^\pm(x_j)$
 123 or $y_\infty^\pm(x_j)$ lies in $\overline{\Omega} \setminus D$, there is no need to interpolate since the data F is given.
 124 The knowledge of F in $\overline{\Omega} \setminus D$ avoids extrapolation techniques, truncation or
 125 reflection of the characteristics, as done, respectively, in [7, 17, 9] in order to
 126 numerically treat boundary conditions in the context of SL schemes. Then,
 127 we define

$$\tilde{I}[v](x) := \begin{cases} I[v](x) & x \in D, \\ F(x) & x \in \overline{\Omega} \setminus D. \end{cases} \quad (15)$$

128 In order to deal with the singular case, we consider a finite difference ap-
 129 proximation of $\frac{1}{2}\Delta u = \Delta_2^G u(x)$ at the points x where $\nabla u(x) \approx 0$. This is
 130 in agreement with the definition of viscosity solution at points where the
 131 gradient vanishes, see Remark 2.1 in [16].

132 Summing up, we propose the following scheme to approximate (1). Find
 133 $v : \mathcal{G}_h(\overline{\Omega}) \rightarrow \mathbb{R}$ such that, for any $x_j \in \mathcal{G}_h(\overline{\Omega})$,

$$G_\rho(x_j, v) = 0, \quad (16)$$

134 where

$$G_\rho(x_j, v) := \begin{cases} S_\rho(x_j, v) - f_j & j \in Q, \\ v(x_j) - F(x_j) & j \in Q_b, \end{cases} \quad (17)$$

135 with $\rho := (h, \delta)$, and S_ρ defined as

$$\begin{aligned} S_\rho(x_j, v) := & \frac{1}{p\delta^2} \left(\tilde{I}[v](y_1^+(x_j)) + \tilde{I}[v](y_1^-(x_j)) \right) \\ & + \frac{1}{q\delta^2} \left(\tilde{I}[v](y_\infty^+(x_j)) + \tilde{I}[v](y_\infty^-(x_j)) \right) - \frac{2}{\delta^2} v_j, \end{aligned}$$

for $j \in Q$ such that $|D_j[u]| > Ch^s$, where $s > 0$ is a fixed parameter, and

$$S_\rho(x_j, v) := \frac{1}{2\delta^2} \left(\sum_{i \in \mathcal{D}(j)} v_i - 4v_j \right),$$

136 for $j \in Q$ such that $|D_j[v]| \leq Ch^s$, with $\mathcal{D}(j) = \{i \in Q \text{ such that } |i - j| = 1\}$.

137 4. Numerical Results

138 In this section we show the performance of the scheme in solving two
 139 problems for which the exact solution is known. We approximate heuristically
 140 the solution v of (16) by a fixed point iteration method based on a time
 141 marching approximation. Given $\Delta t > 0$ and an initial condition $v^0 : \mathcal{G}_h(\overline{\Omega}) \rightarrow$
 142 \mathbb{R} , we compute the sequence $(v^n)_{n \in \mathbb{N}}$ with $v^n : \mathcal{G}_h(\overline{\Omega}) \rightarrow \mathbb{R}$ by the following
 143 iterative scheme

$$\begin{cases} v_j^n = v_j^{n-1} + \Delta t S_\rho(x_j, v^{n-1}), & j \in Q, \\ v_j^n = F(x_j) & j \in Q_b. \end{cases} \quad (18)$$

144 Then the solution v of (16) is approximated as $v_j \simeq \lim_{n \rightarrow \infty} v_j^n$ for any
 145 $j \in Q \cup Q_b$.

The errors are obtained by comparing the numerical solution v^n with the exact solution u on the grid nodes using the following discrete norms

$$\|u(\cdot) - v^n(\cdot)\|_\infty := \max_{j \in Q} |u(x_j) - v_j^n|,$$

$$\|u(\cdot) - v^n(\cdot)\|_1 := h^2 \sum_{j \in Q} |u(x_j) - v_j^n|.$$

We denote by r_∞ , r_1 the corresponding rates. For the fictitious time iteration, we consider the following stopping criterion

$$\|v^{n+1} - v^n\|_1 \leq \varepsilon,$$

146 where $\varepsilon > 0$ is a given tolerance. Since we take a time step of size Δt and
 147 δ represents the size of a diffusion (mean squared displacement of Brownian
 148 walk in the direction of the gradient), δ should be proportional with $\sqrt{2\Delta t}$.
 149 We expect rate one of convergence, which will be confirmed by the simulations
 150 in the following two tests. The first problem is related to the case $p = \infty$
 151 and the solution is not smooth. The second problem has smooth solution
 152 and it is tested with $p = 1.2$ and $p = \infty$. We observe that in Test 1 a
 153 smaller time step have to be considered, in addition to a regularization of
 154 the discrete gradient, in order to obtain the expected convergence rate. The
 155 restriction on the time step is due to accuracy reasons and not to stability
 156 reasons, as point out also in [18]. Heuristically, supposing that the direction
 157 σ is computed exactly, for smooth solution and for the choice $\delta = O(\sqrt{\Delta t})$

the truncation error has order given by $\Delta t + \frac{h^2}{\Delta t}$ (the first term is due to the remainder of the Taylor expansion and the second term is due to the linear interpolation). Then, the choice $\Delta t = h$ would optimize the consistency order, but, at the same time, would produce a stencil of size 2δ which can lead to low accuracy, especially when the solution to be reconstructed has a high gradient or is, in general, not smooth. This motivates the choice of $\Delta t = h^2$ in Test 1, since the solution is non-differentiable, and $\Delta t = h$ in Test 2, where the solution is smooth. In Test 2, scheme (18) is therefore implemented without the usual parabolic CFL condition. This remark seem to confirm that the proposed scheme retains a main advantage of SL schemes, with respect to explicit difference schemes, [14]. In particular, the results in the next tests indicate that the scheme (18) does not need to be implicit in order to be absolute stable.

4.1. Test 1

We compute the solution of problem (1) on $D = (-1, 1) \times (-1, 1)$ with $f = 0$, $\bar{\Omega} = [-1.5, 1.5] \times [-1.5, 1.5]$, $p = \infty$ and

$$F(x) = |x_1|^{4/3} - |x_2|^{4/3}.$$

We recall that this classical benchmark has an explicit solution, which is $u(x) = |x_1|^{4/3} - |x_2|^{4/3}$, known as *Aronsson function*. The solution is only continuous and is not differentiable in $x_1 = 0$ and $x_2 = 0$. For this reason, we need to regularize the gradient as following

$$\tilde{D}_j[u] = \frac{1}{9} \left(\sum_{i \in \mathcal{D}^9(j)} D_i[u] \right)$$

with

$$\mathcal{D}^9(j) = \{i \in Q \text{ such that } \|i - j\|_\infty \leq 1, j \in Q\}$$

and $D_i[u]$ is computed by centered finite difference with step $2h$. We compute the errors in the ∞ and 1 norm, with parameters $\delta = 2\sqrt{2\Delta t}$, $\Delta t = h^2$, $C = 1$, $s = 0.5$, $v^0 = 0$, and $\varepsilon = 1e - 8$. In Table 1, we show errors, convergence rates, and number of iterations needed to verify the stopping criterion. Rates greater than one are obtained in both norms, except in the last refinement where a degradation of the rate with respect to the 1 norm is shown, compared to an almost two order visible for the previous refinements.

Table 1: Test1. Errors and rates for $p = \infty$.

h	$\ \cdot\ _\infty$	$\ \cdot\ _1$	r_∞	r_1	n
$1.00 \cdot 10^{-1}$	$7.65 \cdot 10^{-2}$	$3.35 \cdot 10^{-2}$			115
$5.00 \cdot 10^{-2}$	$3.06 \cdot 10^{-2}$	$1.12 \cdot 10^{-2}$	1.32	1.58	311
$2.50 \cdot 10^{-2}$	$1.21 \cdot 10^{-2}$	$3.08 \cdot 10^{-3}$	1.33	1.82	1055
$1.25 \cdot 10^{-2}$	$4.72 \cdot 10^{-3}$	$2.27 \cdot 10^{-3}$	1.35	0.44	3630

182 4.2. Test 2

We approximate the solution of problem (1) on $D = (-1, 1) \times (-1, 1)$ with $f = 1$, $\bar{\Omega} = [-1.5, 1.5] \times [-1.5, 1.5]$ and

$$F(x_1, x_2) = \frac{1 - x_1^2 - x_2^2}{2}.$$

183 This problem has an exact solution given by $u = F$ for any $p > 1$. We
 184 consider $p = 1.2$ and we compute the errors in the ∞ and 1 norm, with
 185 parameters $\delta = \sqrt{2\Delta t}$, $\Delta t = h$, $C = 0.1$, $s = 1$, $v^0 = 0$, and $\varepsilon = 1e - 5$. In
 186 Tables 2 and 3, we report errors, convergence rates, and number of iterations
 187 needed to verify the stopping criterion, for the choices $p = 1.2$ and $p = \infty$,
 188 respectively. Both tables show a rate of convergence mostly close to 1 in both
 189 norms. Note that, since this test has a smooth exact solution, much larger
 190 time step Δt is allowed.

Table 2: Test2. Errors and rates for $p = 1.2$.

h	$\ \cdot\ _\infty$	$\ \cdot\ _1$	r_∞	r_1	n
$1.00 \cdot 10^{-1}$	$1.03 \cdot 10^{-2}$	$1.89 \cdot 10^{-2}$			33
$5.00 \cdot 10^{-2}$	$5.28 \cdot 10^{-3}$	$9.69 \cdot 10^{-3}$	0.96	0.96	66
$2.50 \cdot 10^{-2}$	$2.75 \cdot 10^{-3}$	$5.17 \cdot 10^{-3}$	0.94	0.90	125
$1.25 \cdot 10^{-2}$	$1.35 \cdot 10^{-3}$	$2.58 \cdot 10^{-3}$	1.022	1.00	323

191 5. Application to the inpainting problem

192 In this Section we apply the new proposed SL scheme to the inpainting
 193 problem. The purpose is also to compare different choices of values for p ,
 194 considering both, a qualitative and a quantitative analysis of the results. As

Table 3: Test2. Errors and rates for $p = \infty$.

h	$\ \cdot\ _\infty$	$\ \cdot\ _1$	r_∞	r_1	n
$1.00 \cdot 10^{-1}$	$1.08 \cdot 10^{-2}$	$2.53 \cdot 10^{-2}$			78
$5.00 \cdot 10^{-2}$	$7.98 \cdot 10^{-3}$	$1.36 \cdot 10^{-2}$	0.43	0.89	139
$2.50 \cdot 10^{-2}$	$4.56 \cdot 10^{-3}$	$8.97 \cdot 10^{-3}$	0.80	0.60	262
$1.25 \cdot 10^{-2}$	$2.11 \cdot 10^{-3}$	$4.18 \cdot 10^{-3}$	1.11	1.10	498

Image Quality Metrics for a quantitative evaluation, we consider the following:

- The Mean Squared Error (MSE), defined as

$$MSE = \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N (I_{true}(i, j) - I_{approx}(i, j))^2 \quad (19)$$

where I_{true} represents the original image not damaged (if given), I_{approx} denotes the image resulting after the iterative process of our scheme, $M \times N$ indicates the size of the image.

- Peak Signal-to-Noise Ratio (PSNR) computed via the Matlab routine $peaksnr = psnr(I_{approx}, I_{true})$.

A greater PSNR value indicates better image quality. PSNR is the ratio between the maximum possible power of an image and the power of corrupting noise that affects the quality of its representation. In formula, we can define it as

$$PSNR = 10 \log_{10} \left(\frac{R^2}{MSE} \right) \quad (20)$$

where R is the maximum fluctuation in the input image data type (if the input image has a double-precision floating-point data type, then $R = 1$. If it has an 8-bit unsigned integer data type, $R = 255$, etc.).

- Structural Similarity Index Measure (SSIM) defined as follows:

$$SSIM = \frac{(2\mu_x\mu_y + c_1)(2\sigma_{xy} + c_2)}{(\mu_x^2 + \mu_y^2 + c_1)(\sigma_x^2 + \sigma_y^2 + c_2)}, \quad (21)$$

211 where μ_x is the average of I_{true} , μ_y is the average of I_{approx} , σ_x^2 is the
212 variance of I_{true} , σ_y^2 is the variance of I_{approx} , σ_{xy} is the covariance of
213 I_{true} and I_{approx} , c_1 and c_2 are constants that are proportional to the
214 dynamic range of the pixel values. SSIM is a perception-based model
215 which takes into account inter-dependencies between pixels as index of
216 information on the structure of the objects under observation. Greater
217 value of SSIM, closer to 1, corresponds to a better quality of the image
218 in terms of similarity with the reference image I_{true} . ($SSIM = 1$ means
219 that the two considered images are equal).

220 These Full-Reference Quality Metrics are the most common used in inpaint-
221 ing papers [1].

222 For all the tests illustrated in this section, we consider $f = 0$, and we fix the
223 following parameters: $h = \Delta t = 1$, $\delta = \sqrt{2\Delta t}$, $s = 1$.

224 5.1. Test 1: Robin Hood with lettering

225 We start the numerical experiments related to the inpainting problem
226 considering a real image representing a statue of Robin Hood (size 500×667),
227 visible in Figure 1. The missing part is behind the writings and the purpose
228 of this test is to compare the results obtained from the new proposed scheme
229 with respect to different choices of the parameter p . The results obtained
230 choosing $p = 1$ and $p = 1.2$ are visible in Figure 2, first row. The qualitative
231 comparison is enough in this case to conclude that the case with a bigger
232 value of p is better, as also stressed in the second row of the same figure,
233 where artifacts are visible in the results related to the case with $p = 1$ which
234 are not present in the case $p = 1.2$. Some of these artifacts are highlighted
235 inside the red circles and compared with the absence of them in the case
236 with $p = 1.2$ (zooming in for a better comparison is recommended). As
237 stopping rule, we fix a common number of iterations to be done, equal to
238 $itermax = 1000$.

239 5.2. Test 2: Sleeping dog

240 For this second test, we consider a real image of a sleeping dog (size
241 601×421), for which we know the original image (see Figure 3). We damaged
242 the original image I_{true} in two different ways, visible in Figure 4: adding three
243 small ellipses or three larger hearts. The results related to the first damaged
244 image, i.e. the sleeping dog with three ellipses, are visible in Figure 5. For
245 this numerical test, one needs a quantitative evaluation in addition to the

246 qualitative comparison done by the human visual perception. In fact, looking
247 at the results in Figure 5, it is almost impossible to distinguish which value
248 of p leads to the best performance. For this reason, we compute the Image
249 Quality Metrics introduced at the beginning of Subsect. 5.2. The errors are
250 reported into Table 4. Looking at Table 4, we can observe that comparable
251 results with a fixed common tolerance ($\varepsilon = 0.001$) are achieved with a much
252 smaller number of iterations choosing a $p > 1$ (193 using $p = 1.2$, and 85 with

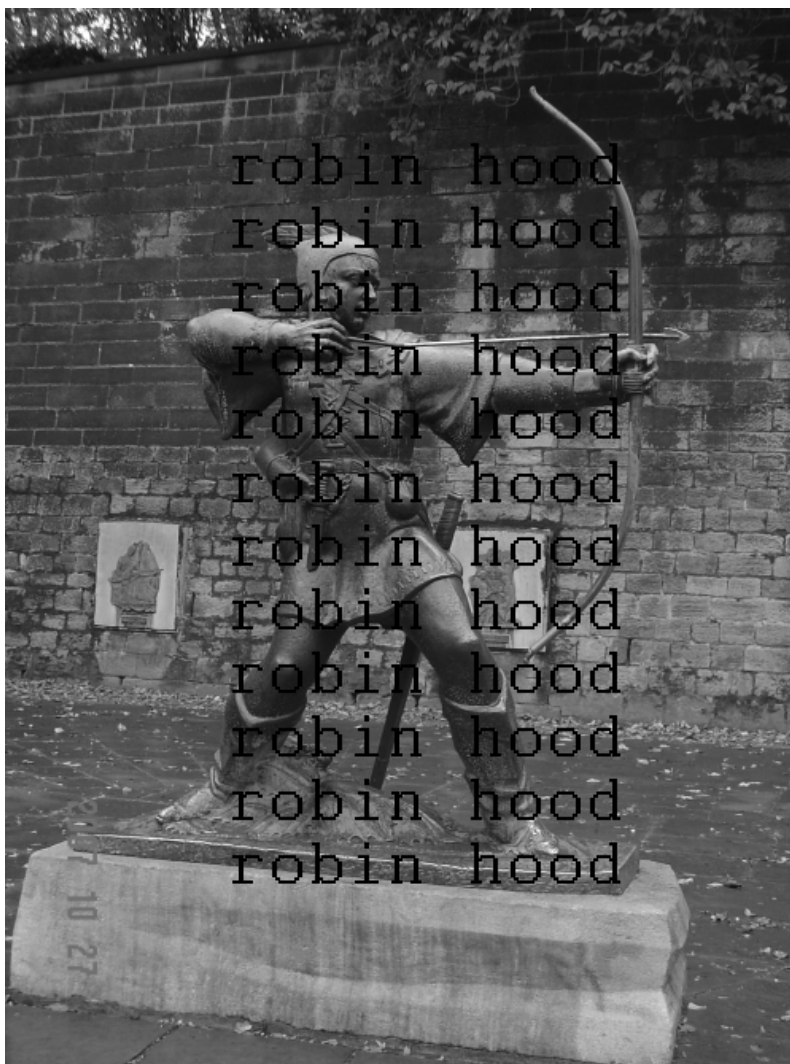


Figure 1: Test 1: Damaged real image of a statue of Robin Hood.

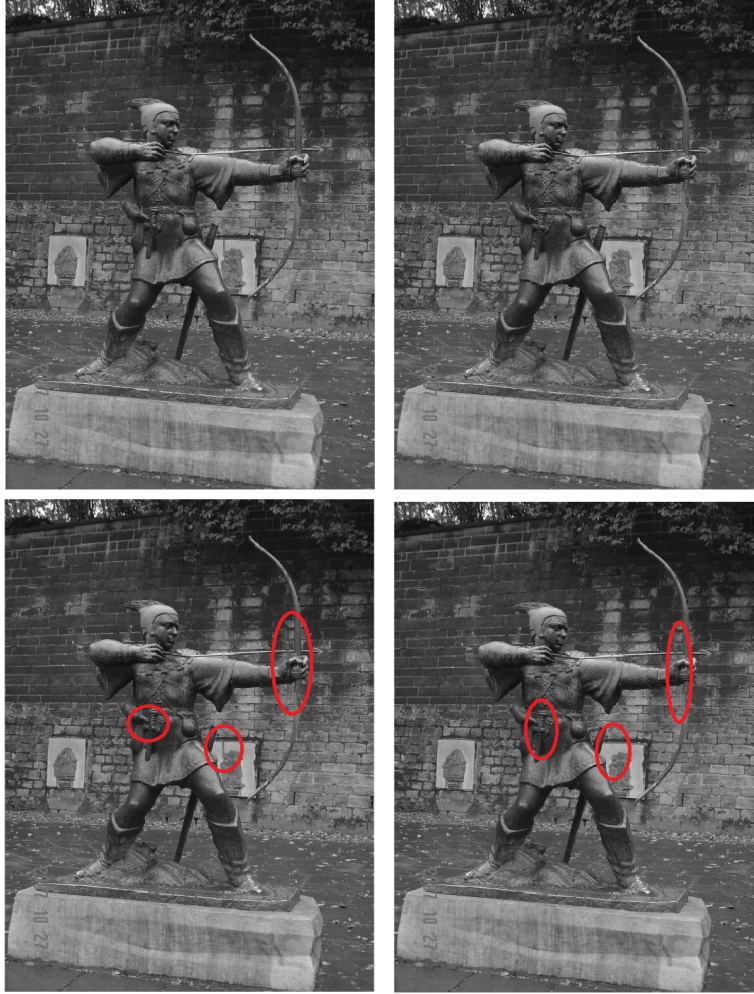


Figure 2: Test 1: First row: Restored images obtained with $p = 1$ (on the left) and $p = 1.2$ (on the right), and $itermax = 1000$. Second row: The same results visible in the first row with some artifacts highlighted in red related to the results with $p = 1$, which disappear with $p = 1.2$. Zooming in for a better visualization.

253 $p = 2.0$). Analyzing the errors reported in Table 4, we see that best results
 254 are achieved choosing $p = 1.2$, but all the errors related to the three cases
 255 are of the same order of magnitude. The big difference lies in the number of
 256 iterations.

257 Starting from the second damaged image, i.e. the sleeping dog with three
 258 hearts visible in Figure 4 on the right, we want to illustrate the behavior



Figure 3: Test 2: Original real image of a sleeping dog.

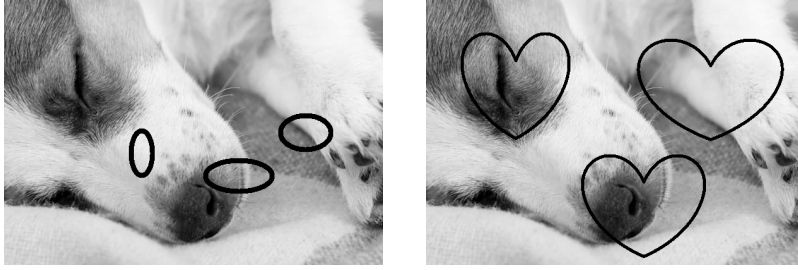


Figure 4: Test 2: Damaged images. On the left, three ellipses representing the missing part of the sleeping dog image. On the right, three hearts are the missing part.

Table 4: Test 2: Image Quality Metrics MSE, PSNR, and SSIM related to the results visible in Figure 5.

Game p -Laplacian	iter	MSE	PSNR	SSIM
$p = 1.0$	1000	3.34E-05	4.48E+01	9.95E-01
$p = 1.2$	193	2.56E-05	4.59E+01	9.95E-01
$p = 2.0$	85	2.89E-05	4.54E+01	9.95E-01

of the proposed scheme varying much more the value of p . We considered
 $p = 1$, $p = 1.2$, $p = 2$, $p = 5$, $p = 10$. The achieved results obtained



Figure 5: Test 2. From left to right: Restored images obtained with $p = 1$, $p = 1.2$, $p = 2$, respectively, starting from the sleeping dog with three ellipses. Tollerance $\varepsilon = 0.001$, $C = 0.01$. Zooming in for a better visualization.

261 with the chosen values of p are visible in Figure 6. The associated quality
 262 metrics are reported in Table 5. Looking at the table, we can observe that
 263 as p value increases, comparable accuracy is obtained with fewer iterations,
 264 which decreasing by increasing p . The MSE error decreases by increasing the
 265 value of p , whereas the PSNR and SSIM metrics are almost the same for all
 the p . The big advantage is, hence, in terms of the execution time.

Table 5: Test 2: Image Quality Metrics MSE, PSNR, and SSIM related to the results visible in Figure 6.

Game p -Laplacian	iter	MSE	PSNR	SSIM
$p = 1.0$	191	8.37E-05	4.08E+01	9.91E-01
$p = 1.2$	77	8.23E-05	4.08E+01	9.91E-01
$p = 2.0$	66	8.09E-05	4.09E+01	9.91E-01
$p = 5.0$	62	8.06E-05	4.09E+01	9.91E-01
$p = 10.0$	62	8.05E-05	4.09E+01	9.91E-01



Figure 6: Test 2. From left to right: Restored images obtained with $p = 1$, $p = 1.2$, $p = 2$, $p = 5$, $p = 10$, respectively, starting from the sleeping dog with three hearts. $\varepsilon = 0.001$, $C = 0.1$. Zooming in for a better visualization.

267 6. Conclusions and future perspectives

268 In this work we have presented a new semi-Lagrangian scheme for the
 269 game p -Laplacian equation, by observing that the game p -Laplacian oper-
 270 ator can be expressed as a convex combination of the ∞ -Laplacian and
 271 1-Laplacian. We have applied the new proposed scheme to the inpainting
 272 image problem, analyzing the results in terms of both qualitative and quan-
 273 titative accuracy. The numerical simulations have showed the behavior of
 274 the proposed scheme, with the advantage of considering larger values of p ,
 275 which allows one to obtain good accuracy in fewer iterations. In the future,
 276 we would like to investigate theoretically the numerical scheme introduced
 277 in this work and its properties, such as stability and convergence. More-

over, a high order extension of the scheme could be addressed, in particular implicit-explicit Runge-Kutta schemes for the fictitious time marching (see for instance [8]) coupled with a high order approximation of the p -Laplacian may be considered.

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