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# $C_0$ -sequentially equicontinuous semigroups <sup>\*</sup>

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April 22, 2018

## Abstract

We present and apply a theory of one parameter  $C_0$ -semigroups of linear operators in locally convex spaces. Replacing the notion of equicontinuity considered by the literature with the weaker notion of sequential equicontinuity, we prove the basic results of the classical theory of  $C_0$ -equicontinuous semigroups: we show that the semigroup is uniquely identified by its generator and we provide a generation theorem in the spirit of the celebrated Hille-Yosida theorem. Then, we particularize the theory in some functional spaces and identify two locally convex topologies that allow to gather under a unified framework various notions  $C_0$ -semigroup introduced by some authors to deal with Markov transition semigroups. Finally, we apply the results to transition semigroups associated to stochastic differential equations.

**Keywords:** One parameter semigroup, sequential equicontinuity, transition semigroup.

**AMS 2010 subject classification:** 46N30, 47D06, 47D07, 60J35.

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## 1 Introduction

The aim of this paper is to present and apply a notion of one parameter strongly continuous ( $C_0$ ) semigroups of linear operators in locally convex spaces based on the notion of sequential equicontinuity and following the spirit and the methods of the classical theory in Banach spaces.

The theory of  $C_0$ -semigroups was first stated in Banach spaces (a widespread presentation can be found in several monographs, e.g. [11, 19, 30]). The theory was extended to locally convex spaces by introducing the notions of  $C_0$ -equicontinuous semigroup ([35, Ch. IX]),  $C_0$ -quasi-equicontinuous semigroup ([6]),  $C_0$ -locally equicontinuous semigroup ([9, 23]), weakly integrable semigroup ([21, 22]). A mixed approach is the one followed by [25], which introduces the notion of bi-continuous semigroup: in a framework of Banach spaces, semigroups that are strongly continuous with respect to a weaker locally convex topology are considered.

In this paper we deal with semigroups of linear operators in locally convex spaces that are only *sequentially* continuous. The idea is due to the following key observation: the theory of  $C_0$ -(locally) equicontinuous semigroups can be developed, with appropriate adjustments, to semigroups of operators which are only  $C_0$ -(locally) *sequentially* equicontinuous (in the sense specified by Definition 3.1). On the other hand, as we will show by examples, the passage from equicontinuity to sequential equicontinuity is motivated and fruitful: as discussed in Remark 3.4 and shown by Example 6.4, in concrete applications, replacing equicontinuity with sequential equicontinuity might turn out to be much more convenient.

The main motivation that led us to consider sequential continuity is that it allows a convenient treatment of Markov transition semigroups. The employment of Markov transition semigroups to the study of partial differential equations through the use of stochastic representation formulas is the subject of a wide mathematical literature (here we only refer to [5] in finite and infinite dimension and to [8] in infinite dimension). Also, the regularizing properties of such semigroups is the core of a regularity theory for second order PDEs (see, e.g., [27]). Unfortunately, the framework of  $C_0$ -semigroup in Banach spaces is not always appropriate to treat such semigroups. Indeed, on Banach spaces of functions not vanishing at infinity, the  $C_0$ -property fails already in basic cases, such as the one-dimensional Ornstein-Uhlenbeck semigroup, when considering it in the space of bounded uniformly continuous real-valued functions  $(UC_b(\mathbb{R}), |\cdot|_\infty)$  (see, e.g., [4, Ex. 6.1] for a counterexample, or [7, Lemma 3.2], which implies this semigroup is strongly continuous in  $(UC_b(\mathbb{R}), |\cdot|_\infty)$  if and only if the drift of the associated stochastic differential equation vanishes). On the other hand, finding a locally convex topology on these spaces to frame Markov transition semigroups within the theory of  $C_0$  equicontinuous semigroups is not a simple task (see also the considerations of Remark 3.4). In the case of the Ornstein-Uhlenbeck semigroup, such approach is adopted by [17], dealing with the so called *mixed topology* in the space of continuous functions. Other authors have bypassed these difficulties by introducing some *ad hoc* notions of continuous semigroups in the space of uniformly continuous and bounded functions (*weakly continuous* semigroups [4],  *$\pi$ -continuous* semigroups [31]). A more general approach fitting Markov transitions semigroups is represented by the theory of *bi-continuous* semigroups, dealing in abstract (not limited to functional) spaces endowed both with a Banach norm and with a coarser locally

convex topology; the requirements of strongly continuity and of equicontinuity in the definition plays with both such topologies. Such theory is nowadays well-developed and basically restates the classical theory of strongly continuous semigroups in Banach spaces in all aspects, including generation, perturbation, and approximation (see [1, 2, 12, 13, 14, 15, 20, 25]). As we will show in Sections 4 and 5, the notion we propose contains all the aforementioned ones and therefore allows to develop a common theory containing the previous notions under a unified framework. We point out that our notion has been already employed in a recent paper [24] developing the issue of subordination in this context.

We end the introduction by describing in detail the contents of the paper. Section 2 contains notations that will hold throughout the paper.

In Section 3 we first provide and study the notions of sequential continuity of linear operators and sequential equicontinuity of families of linear operators on locally convex spaces. Then, we give the definition of  $C_0$ -sequentially (locally) equicontinuous semigroup in locally convex spaces. Next, we define the generator of the semigroup and the resolvent of the generator. In order to guarantee the existence of the resolvent, the theory is developed under Assumption 3.7, requiring the existence of the Laplace transform (3.4) as Riemann integral (see Remark 3.8). This assumption is immediately verified if the underlying space  $X$  is sequentially complete. Otherwise, the Laplace transform always exists in the (sequential) completion of  $X$  and then one should check that it lies in  $X$ , as we do in Proposition 5.9. The properties of generator and resolvent are stated through a series of results: their synthesis is represented by Theorem 3.12, stating that the semigroup is uniquely identified by its generator, and by Theorem 3.14, stating that the resolvent coincides with the Laplace transform. Then we provide a generation theorem (Theorem 3.18), characterizing, in the same spirit of the Hille-Yosida theorem, the linear operators generating  $C_0$ -sequentially equicontinuous semigroups. Finally, we provide some examples which illustrate our notion in relation to other notions of semigroup on locally convex spaces.

In Section 4, we show that the notion of bi-continuous semigroups can be seen as a specification of ours (Proposition 4.3).

Section 5 implements the theory of Section 3 in spaces of bounded Borel functions, continuous and bounded functions, or uniformly continuous and bounded functions defined on a metric space. The main aim of this section is to find and study appropriate locally convex topologies in these functional spaces allowing a comparison between our notion with the aforementioned other ones. We identify them in two topologies belonging to a class of locally convex topologies defined through the family of seminorms (5.5). We study these topological spaces through a series of results ending with Proposition 5.6. Then, we proceed with the desired comparison: in Subsections 5.2, 5.3, and 5.4, we show that the notions developed in [4], [31], and [17] to treat Markov transition semigroups can be reinterpreted in our framework.

Section 6 applies the results of to Markov transition semigroups. This is done, in Subsection 6.1, in the space of bounded continuous functions endowed with the topology  $\tau_{\mathcal{X}}$  defined in Section 5. Finally, in Subsection 6.2, we treat the case of Markov transition semigroups associated to stochastic differential equations in Hilbert spaces.

## 2 Notation

- (N1) Throughout the paper,  $X, Y$  will denote Hausdorff locally convex topological vector spaces.
- (N2) The topological dual of a topological vector space  $X$  is denoted by  $X^*$ .
- (N3) If  $X$  is a vector space and  $\Gamma$  is a vector space of linear functionals on  $X$  separating points in  $X$ , we denote by  $\sigma(X, \Gamma)$  the weakest locally convex topology on  $X$  making continuous the elements of  $\Gamma$ .

- (N4) The weak topology on the topological vector space  $X$  is denoted by  $\tau_w$ , i.e.,  $\tau_w := \sigma(X, X^*)$ .
- (N5) If  $X$  and  $Y$  are topological vector spaces, the space of continuous operators from  $X$  into  $Y$  is denoted by  $L(X, Y)$ , and the space of sequentially continuous operators from  $X$  into  $Y$  (see Definition 3.1) is denoted by  $\mathcal{L}_0(X, Y)$ . We also denote  $L(X) := L(X, X)$  and  $\mathcal{L}_0(X) := \mathcal{L}_0(X, X)$ .
- (N6) Given a locally convex topological vector space  $X$ , the symbol  $\mathcal{P}_X$  denotes a family of seminorm on  $X$  inducing the locally convex topology.
- (N7)  $E$  denotes a metric space;  $\mathcal{E} := \mathcal{B}(E)$  denotes the Borel  $\sigma$ -algebra of subsets of  $E$ .
- (N8) Given the metric space  $E$ ,  $\mathbf{ba}(E)$  denotes the space of finitely additive signed measures with bounded total variation on  $\mathcal{E}$ ,  $\mathbf{ca}(E)$  denotes the subspace of  $\mathbf{ba}(E)$  of countably additive finite measure, and  $\mathbf{ca}^+(E)$  denotes the subspace of  $\mathbf{ca}(E)$  of positive countably additive finite measures.
- (N9) Given the metric space  $E$ , we denote by  $B(x, r)$  the open ball centered at  $x \in E$  and with radius  $r$  and by  $B(x, r]$  the closed ball centered at  $x$  and with radius  $r$ .
- (N10) The common symbol  $\mathcal{S}(E)$  denotes indifferently one of the spaces  $B_b(E)$ ,  $C_b(E)$ ,  $UC_b(E)$ , i.e., respectively, the space of real-valued *bounded Borel / continuous and bounded / uniformly continuous and bounded* functions defined on  $E$ .
- (N11) On  $\mathcal{S}(E)$ , we consider the sup-norm  $\|f\|_\infty := \sup_{x \in E} |f(x)|$ , which makes it a Banach space. The topology on  $\mathcal{S}(E)$  induced by such norm is denoted by  $\tau_\infty$ .
- (N12) On  $\mathcal{S}(E)$ , the symbol  $\tau_{\mathcal{C}}$  denotes the topology of the uniform convergence on compact sets.
- (N13) By  $\mathcal{S}(E)_\infty^*$  we denote the topological dual of  $(\mathcal{S}(E), \|\cdot\|_\infty)$  and by  $\|\cdot\|_{\mathcal{S}(E)_\infty^*}$  the operator norm in  $\mathcal{S}(E)_\infty^*$ .

We make use of the conventions  $\inf \emptyset = +\infty$ ,  $\sup \emptyset = -\infty$ ,  $1/\infty = 0$ .

### 3 $C_0$ -sequentially equicontinuous semigroups

In this section, we introduce and investigate the notion of  $C_0$ -sequentially equicontinuous semigroups on locally convex topological vector spaces. Hereafter,  $X$  and  $Y$  denote Hausdorff locally convex topological vector spaces and  $\mathcal{P}_X, \mathcal{P}_Y$  denote families of seminorms inducing the topology on  $X, Y$ , respectively.

#### 3.1 Definition and preliminaries

We first recall the notion of sequential continuity for functions and define the notion of sequential equicontinuity for families of functions on topological spaces.

**Definition 3.1.** *Let  $A, B$  be Hausdorff topological spaces.*

- (i) *A function  $f: A \rightarrow B$  is said to be sequentially continuous if, for every sequence  $\{x_n\}_{n \in \mathbb{N}}$  converging to  $x$  in  $A$ , we have  $f(x_n) \rightarrow f(x)$  in  $B$ .*
- (ii) *If  $B$  is a vector space, a family of functions  $\mathcal{F} = \{f_i: A \rightarrow B\}_{i \in \mathcal{I}}$  is said to be sequentially equicontinuous if for every  $x \in A$ , for every sequence  $\{x_n\}_{n \in \mathbb{N}}$  converging to  $x$  in  $A$  and for every neighborhood  $U$  of 0 in  $B$ , there exists  $\bar{n} \in \mathbb{N}$  such that  $f_i(x_n) \in f_i(x) + U$  for every  $i \in \mathcal{I}$  and  $n \geq \bar{n}$ .*

If  $B$  is a locally convex topological vector space, then Definition 3.1(ii) is equivalent to

$$\{x_n\}_{n \in \mathbb{N}} \subset B, x_n \rightarrow x \text{ in } B \implies \lim_{n \rightarrow +\infty} \sup_{i \in \mathcal{I}} q(f_i(x_n) - f_i(x)) = 0, \quad \forall q \in \mathcal{P}_B, \quad (3.1)$$

where  $\mathcal{P}_B$  is a set of seminorms inducing the topology on  $B$ . The characterization of sequential continuity (3.1) will be very often used throughout the paper.

We define the vector space

$$\mathcal{L}_0(X, Y) := \{F: X \rightarrow Y \text{ s.t. } F \text{ is linear and sequentially continuous}\}.$$

We will use  $\mathcal{L}_0(X)$  to denote the space  $\mathcal{L}_0(X, X)$ . Clearly, we have the inclusion

$$L(X, Y) \subset \mathcal{L}_0(X, Y). \quad (3.2)$$

The proof of the following proposition can be found in [16, Prop. 3.5].

**Proposition 3.2.** *Let  $F \in \mathcal{L}_0(X, Y)$ . Then*

- (i)  $F$  is a bounded operator;
- (ii)  $F$  maps Cauchy sequences into Cauchy sequences.

We now introduce the notion of  $C_0$ -sequentially (locally) equicontinuous semigroups.

**Definition 3.3** ( $C_0$ -sequentially (locally) equicontinuous semigroup). *A family of linear operators (not necessarily continuous)*

$$T := \{T_t: X \rightarrow X\}_{t \in \mathbb{R}^+}$$

*is called a  $C_0$ -sequentially equicontinuous semigroup on  $X$  if the following properties hold.*

- (i) (*Semigroup property*)  $T_0 = I$  and  $T_{t+s} = T_t T_s$  for all  $t, s \geq 0$ .
- (ii) ( $C_0$ - or strong continuity property)  $\lim_{t \rightarrow 0^+} T_t x = x$ , for every  $x \in X$ .
- (iii) (*Sequential equicontinuity*)  $T$  is a sequentially equicontinuous family.

*The family  $T$  is said to be a  $C_0$ -sequentially locally equicontinuous semigroup if (iii) is replaced by (iii)' (*Sequential local equicontinuity*)  $\{T_t\}_{t \in [0, R]}$  is sequentially locally equicontinuous for every  $R > 0$ .*

**Remark 3.4.** *The notion of  $C_0$ -sequentially (locally) equicontinuous semigroup that we introduced is clearly a generalization of the notion of  $C_0$ -(locally) equicontinuous semigroup considered, e.g., in [35, Ch. IX], [23]. By Proposition A.2 the two notions coincide if  $X$  is metrizable. In order to motivate the introduction of  $C_0$ -sequentially equicontinuous semigroups, we stress two facts.*

- (1) *Even if a semigroup on a sequentially complete space is  $C_0$ -(locally) equicontinuous, proving this property might be harder than proving that it is only  $C_0$ -sequentially equicontinuous. For instance, in locally convex functional spaces with topologies defined by seminorms involving integrals, one can use integral convergence theorems for sequences of functions which do not hold for nets of functions.*
- (2) *The notion of  $C_0$ -sequentially equicontinuous semigroup is a genuine generalization of the notion of  $C_0$ -equicontinuous semigroup of [35], as shown by Example 3.26.*

As for  $C_0$ -semigroups in Banach spaces, given a  $C_0$ -sequentially locally equicontinuous semigroup  $T$ , we define

$$\mathcal{D}(A) := \left\{ x \in X : \exists \lim_{h \rightarrow 0^+} \frac{T_h x - x}{h} \in X \right\}.$$

Clearly,  $\mathcal{D}(A)$  is a linear subspace of  $X$ . Then, we define the linear operator  $A: \mathcal{D}(A) \rightarrow X$  as

$$Ax := \lim_{h \rightarrow 0^+} \frac{T_h x - x}{h}, \quad x \in \mathcal{D}(A),$$

and call it the *infinitesimal generator* of  $T$ .



**Proposition 3.5.** Let  $T := \{T_t : X \rightarrow X\}_{t \in \mathbb{R}^+}$  be a  $C_0$ -sequentially locally equicontinuous semigroup.

(i) For every  $x \in X$ , the function  $Tx : \mathbb{R}^+ \rightarrow X$ ,  $t \mapsto T_t x$ , is continuous.

(ii) If  $T$  is sequentially equicontinuous, then, for every  $x \in X$ , the function  $Tx : \mathbb{R}^+ \rightarrow X$ ,  $t \mapsto T_t x$ , is bounded.

*Proof.* (i) Let  $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$  be a sequence converging from the right (resp., from the left) to  $t \in \mathbb{R}$ . By Definition 3.3(i), we have, for every  $p \in \mathcal{P}_X$  and  $x \in X$ ,

$$p(T_{t_n} x - T_t x) = p(T_{t_n}(T_{t_n-t} x - x)) \quad (\text{resp., } p(T_{t_n} x - T_t x) = p(T_{t_n}(T_{t-t_n} x - x))).$$

By Definition 3.3(ii),  $\{T_{t_n-t} x - x\}_{n \in \mathbb{N}}$  (resp.  $\{T_{t-t_n} x - x\}_{n \in \mathbb{N}}$ ) converges to 0. Now conclude by using local sequential equicontinuity and (3.1).

(ii) This is provided by Proposition A.1(iii). ■

As well known, unlike the Banach space case, in locally convex spaces the passage from  $C_0$ -locally equicontinuous semigroups to  $C_0$ -equicontinuous semigroups through a renormalization with an exponential function cannot be obtained in general (see Examples 3.23 and 3.24 in Subsection 3.4). Nevertheless, we have the following partial result.

**Proposition 3.6.** Let  $\tau$  denote the locally convex topology on  $X$  and let  $|\cdot|_X$  be a norm on  $X$ . Assume that a set is  $\tau$ -bounded if and only if it is  $|\cdot|_X$ -bounded. Let  $T$  be a  $C_0$ -sequentially locally equicontinuous semigroup on  $(X, \tau)$ .

(i) If there exist  $\alpha \in \mathbb{R}$  and  $M \geq 1$  such that

$$|T_t|_{L((X, |\cdot|_X))} \leq M e^{\alpha t}, \quad \forall t \in \mathbb{R}^+, \quad (3.3)$$

then, for every  $\lambda > \alpha$ , the family  $\{e^{-\lambda t} T_t : (X, \tau) \rightarrow (X, \tau)\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup.

(ii) If  $(X, |\cdot|_X)$  is Banach, then there exist  $\alpha \in \mathbb{R}$  and  $M \geq 1$  such that (3.3) holds.

*Proof.* (i) Let  $\lambda > \alpha$  and let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence converging to 0 in  $(X, \tau)$ . Then  $\{x_n\}_{n \in \mathbb{N}}$  is bounded in  $(X, \tau)$ , thus, by assumption, also in  $(X, |\cdot|_X)$ . Set  $N := \sup_{n \in \mathbb{N}} |x_n|_X$  and let  $p \in \mathcal{P}_{(X, \tau)}$ . Then

$$\begin{aligned} \sup_{t \in \mathbb{R}^+} p(e^{-\lambda t} T_t x_n) &\leq \sup_{0 \leq t \leq s} p(e^{-\lambda t} T_t x_n) + \sup_{t > s} p(e^{-\lambda t} T_t x_n) \\ &\leq \sup_{0 \leq t \leq s} p(e^{-\lambda t} T_t x_n) + L_p e^{(\alpha - \lambda)s} M N, \end{aligned}$$

where  $L_p := \sup_{x \in X \setminus \{0\}} p(x)/|x|_X$  is finite, because  $|\cdot|_X$ -bounded sets are  $\tau$ -bounded. Now we can conclude by applying to the right hand side of the inequality above first the  $\limsup_{n \rightarrow +\infty}$  and considering that  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $(X, \tau)$ , then the  $\lim_{s \rightarrow +\infty}$  and taking into account that  $\lambda > \alpha$ .

(ii) By assumption, the bounded sets of  $(X, |\cdot|_X)$  coincide with the bounded sets of  $(X, \tau)$ . By Proposition 3.2(i), we then have  $\mathcal{L}_0((X, \tau)) \subset L((X, |\cdot|_X))$ . In particular  $T_t \in L((X, |\cdot|_X))$ , for all  $t \in \mathbb{R}^+$ . Now, by Proposition 3.5(i), the set  $\{T_t x\}_{t \in [0, t_0]}$  is compact in  $(X, \tau)$  for every  $x \in X$  and  $t_0 > 0$ , hence bounded. We can then apply the Banach-Steinhaus Theorem in  $(X, |\cdot|_X)$  and conclude that there exists  $M \geq 0$  such that  $|T_t|_{L((X, |\cdot|_X))} \leq M$  for all  $t \in [0, t_0]$ . The conclusion now follows in a standard way from the semigroup property. ■



From here on in this subsection and in Subsections 3.2-3.3, unless differently specified, we will deal with  $C_0$ -sequentially equicontinuous semigroups and, to simplify the exposition, we will adopt a standing notation for them and their generator, i.e.,

- $T = \{T_t\}_{t \in \mathbb{R}^+}$  denotes a  $C_0$ -sequentially equicontinuous semigroup;
- $A$  denotes the infinitesimal generator of  $T$ .

Also, unless differently specified, from here on in this subsection and in Subsections 3.2-3.3, we will assume the following

**Assumption 3.7.** *For every  $x \in X$  and  $\lambda > 0$ , there exists the generalized Riemann integral in  $X$ <sup>(1)</sup>*

$$R(\lambda)x := \int_0^{+\infty} e^{-\lambda t} T_t x dt. \quad (3.4)$$

**Remark 3.8.** *By Proposition 3.5, the generalized Riemann integral (3.4) always exists in the sequential completion of  $X$ . In particular, Assumption 3.7 is satisfied if  $X$  is sequentially complete.*

For every  $p \in \mathcal{P}_X$ , and every  $\lambda, \hat{\lambda} \in (0, +\infty)$ , we have the following inequalities, whose proof is straightforward, by triangular inequality and definition of Riemann integral, and by recalling Proposition 3.5:

$$p(R(\lambda)x - y) \leq \int_0^{+\infty} e^{-\lambda t} p(T_t x - \lambda y) dt, \quad \forall x, y \in X \quad (3.5)$$

$$p(R(\lambda)x - R(\hat{\lambda})x) \leq \int_0^{+\infty} |e^{-\lambda t} - e^{-\hat{\lambda} t}| p(T_t x) dt, \quad \forall x \in X. \quad (3.6)$$

**Proposition 3.9.** (i) *For every  $\lambda > 0$ , the operator  $R(\lambda) : X \rightarrow X$  is linear and sequentially continuous.*

(ii) *For every  $x \in X$ , the function  $(0, +\infty) \rightarrow X$ ,  $\lambda \mapsto R(\lambda)x$ , is continuous.*

*Proof.* (i) The linearity of  $R(\lambda)$  is clear. It remains to show its sequential continuity. Let  $\{x_n\}_{n \in \mathbb{N}} \subset X$  be a sequence convergent to 0. Then, for all  $p \in \mathcal{P}_X$ ,

$$\lim_{n \rightarrow +\infty} p(R(\lambda)x_n) \leq \lim_{n \rightarrow +\infty} \int_0^{+\infty} e^{-\lambda t} p(T_t x_n) dt = \lambda^{-1} \lim_{n \rightarrow +\infty} \sup_{t \in \mathbb{R}^+} p(T_t x_n) = 0$$

where the last limit is obtained by sequential equicontinuity and by recalling (3.1).

(ii) For  $p \in \mathcal{P}_X$ ,  $x \in X$ ,  $\lambda, \hat{\lambda} \in (0, +\infty)$ , by (3.6),

$$p(R(\lambda)x - R(\hat{\lambda})x) \leq \int_0^{+\infty} |e^{-\lambda t} - e^{-\hat{\lambda} t}| p(T_t x) dt \leq \sup_{r \in \mathbb{R}^+} p(T_r x) \int_0^{+\infty} |e^{-\lambda t} - e^{-\hat{\lambda} t}| dt.$$

The last integral converges to 0 as  $\lambda \rightarrow \hat{\lambda}$ , and we conclude as  $\sup_{r \in \mathbb{R}^+} p(T_r x) < +\infty$  by Proposition 3.5(ii). ■

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<sup>1</sup>That is, for every  $a \geq 0$ , the Riemann integral  $\int_0^a e^{-\lambda t} T_t x dt$  exists in  $X$ , and the limit  $\int_0^{+\infty} e^{-\lambda t} T_t x dt := \lim_{a \rightarrow +\infty} \int_0^a e^{-\lambda t} T_t x dt$  exists in  $X$ .

### 3.2 Generators of $C_0$ -sequentially equicontinuous semigroups

In this subsection we study the generator  $A$  of the  $C_0$ -sequentially equicontinuous semigroup  $T$ .

Recall that a subset  $U$  of a topological space  $Z$  is said to be sequentially dense in  $Z$  if, for every  $z \in Z$ , there exists a sequence  $\{u_n\}_{n \in \mathbb{N}} \subset U$  converging to  $z$  in  $Z$ . In such a case, it is clear that  $U$  is also dense in  $Z$ .

**Proposition 3.10.**  $\mathcal{D}(A)$  is sequentially dense in  $X$ .

*Proof.* Let  $\lambda > 0$  and set  $\psi_\lambda := \lambda R(\lambda) \in X$ . By (A.1),

$$T_h R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T_{h+t} x dt \in X, \quad \forall x \in X.$$

Then, following the proof of [35, p. 237, Theorem 1]<sup>(2)</sup>, we have

$$\frac{T_h \psi_\lambda x - \psi_\lambda x}{h} = \frac{e^{\lambda h} - 1}{h} \left( \psi_\lambda x - \lambda \int_0^h e^{-\lambda t} T_t x dt \right) - \frac{\lambda}{h} \int_0^h e^{-\lambda t} T_t x dt \in X, \quad \forall x \in X.$$

Passing to the limit for  $h \rightarrow 0^+$ , we obtain

$$\lim_{h \rightarrow 0^+} \frac{T_h \psi_\lambda x - \psi_\lambda x}{h} = \lambda(\psi_\lambda x - x) \in X, \quad \forall x \in X.$$

Then  $\psi_\lambda x \in \mathcal{D}(A)$  and

$$A \psi_\lambda x = \lambda(\psi_\lambda - I)x \in X, \quad \forall x \in X. \quad (3.7)$$

For future reference, we notice that this shows, in particular, that

$$\text{Im}(R(\lambda)) \subset \mathcal{D}(A). \quad (3.8)$$

Now we prove that

$$\lim_{\lambda \rightarrow +\infty} \psi_\lambda x = x \quad \forall x \in X, \quad (3.9)$$

which concludes the proof. By (3.5), we have

$$p(\psi_\lambda x - x) = \lambda p(R(\lambda)x - \lambda^{-1}x) \leq \int_0^{+\infty} \lambda e^{-\lambda t} p(T_t x - x) dt \quad \forall x \in X, \quad \forall p \in \mathcal{P}_X.$$

By Proposition 3.5(ii), we can apply the dominated convergence theorem to the last integral above when  $\lambda \rightarrow +\infty$ . Then we have

$$p(\psi_\lambda x - x) \rightarrow 0, \quad \forall x \in X, \quad \forall p \in \mathcal{P}_X,$$

and we obtain (3.9) by arbitrariness of  $p \in \mathcal{P}_X$ . ■

**Proposition 3.11.** Let  $x \in \mathcal{D}(A)$ . Then

- (i)  $T_t x \in \mathcal{D}(A)$  for all  $t \in \mathbb{R}^+$ ;
- (ii) the map  $Tx: \mathbb{R}^+ \rightarrow X$ ,  $t \mapsto T_t x$  is differentiable;
- (iii) the following identity holds

$$\frac{d}{dt} T_t x = A T_t x = T_t A x, \quad \forall t \in \mathbb{R}^+. \quad (3.10)$$

<sup>2</sup>In the cited result,  $X$  is assumed sequentially complete. However, the completeness of  $X$  is used in the proof only to define the integrals. In our case, existence for the integrals involved in the proof holds by assumption.

*Proof.* Let  $x \in \mathcal{D}(A)$ . Consider the function  $\Delta : \mathbb{R}^+ \rightarrow X$  defined by

$$\Delta(h) := \begin{cases} \frac{T_h - I}{h}x, & \text{if } h \neq 0 \\ \Delta(0) = Ax. \end{cases}$$

This function is continuous by definition of  $A$ . Then we have

$$T_t Ax = T_t \lim_{h \rightarrow 0^+} \Delta(h) = \lim_{h \rightarrow 0^+} T_t \Delta(h) = \lim_{h \rightarrow 0^+} \frac{T_h T_t x - T_t x}{h}, \quad \forall t \in \mathbb{R}^+,$$

which shows that (i) holds and that

$$T_t Ax = AT_t x, \quad \forall t \in \mathbb{R}^+.$$

The rest of the proof follows exactly as in [35, p. 239, Theorem 2]. ■

We are going to show that the infinitesimal generator identifies uniquely the semigroup  $T$ .

**Theorem 3.12.** *Let  $S$  be a  $C_0$ -sequentially equicontinuous semigroup on  $X$  with infinitesimal generator  $A_S$ . If  $A_S = A$  then  $S = T$ .*

*Proof.* For  $t > 0$  and  $x \in \mathcal{D}(A)$ , consider the function  $f : [0, t] \rightarrow X$ ,  $s \mapsto T_{t-s} S_s x$ . By Proposition 3.11 and Lemma A.4,  $f'(s) = 0$  for all  $s \in [0, t]$ , and then  $T_t x = f(0) = f(t) = S_t x$ . Since  $\mathcal{D}(A)$  is sequentially dense in  $X$  and the operators  $T_t, S_t$  are sequentially continuous, we have  $T_t x = S_t x$  for all  $x \in X$ , and we conclude by arbitrariness of  $t > 0$ . ■

**Definition 3.13.** *Let  $\mathcal{D}(C) \subset X$  be a linear subspace. For a linear operator  $C : \mathcal{D}(C) \rightarrow X$ , we define the spectrum  $\sigma_0(C)$  as the set of  $\lambda \in \mathbb{R}$  such that one of the following holds:*

- (i)  $\lambda - C$  is not one-to-one;
- (ii)  $\text{Im}(\lambda - C) \neq X$ ;
- (iii) there exists  $(\lambda - C)^{-1}$ , but it is not sequentially continuous.

We denote  $\rho_0(C) := \mathbb{R} \setminus \sigma_0(C)$ , and call it resolvent set of  $C$ . If  $\lambda \in \rho_0(C)$ , we denote by  $R(\lambda, C)$  the sequentially continuous inverse  $(\lambda - C)^{-1}$  of  $\lambda - C$ .

**Theorem 3.14.** *If  $\lambda > 0$ , then  $\lambda \in \rho_0(A)$  and  $R(\lambda, A) = R(\lambda)$ .*

*Proof.* Step 1. Here we show that  $\lambda - A$  is one-to-one for every  $\lambda > 0$ . Let  $x \in \mathcal{D}(A)$ . By Proposition 3.11, for any  $f \in X^*$ , the function  $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $t \mapsto f(e^{-\lambda t} T_t x)$  is differentiable, and  $F'(t) = f(e^{-\lambda t} T_t (A - \lambda)x)$ . If  $(A - \lambda)x = 0$ , then  $F$  is constant. By Proposition 3.5(ii),  $F(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , hence it must be  $F \equiv 0$ . Then  $f(x) = F(0) = 0$ . As  $f$  is arbitrary, we conclude that  $x = 0$  and, therefore, that  $\lambda - A$  is one-to-one.

Step 2. Here we show that  $\lambda - A$  is invertible and  $R(\lambda, A) = R(\lambda)$ , for every  $\lambda > 0$ . By (3.8) and (3.7),

$$(\lambda - A)R(\lambda) = I \tag{3.11}$$

which shows that  $\lambda - A$  is onto, and then invertible (by recalling also Step 1), and that  $(\lambda - A)^{-1} = R(\lambda)$ .

Step 3. The fact  $(\lambda - A)^{-1} \in \mathcal{L}_0(X)$  follows from Step 2 and Proposition 3.9(i). ■

**Corollary 3.15.** *The operator  $A$  is sequentially closed, i.e., its graph  $\text{Gr}(A)$  is sequentially closed in  $X \times X$ .*

*Proof.* Observe that  $(x, y) \in \text{Gr}(A)$  if and only if  $(x, x - y) \in \text{Gr}(I - A)$ , and hence if and only if  $(x - y, x) \in \text{Gr}(R(1, A))$ . As  $R(1, A) \in \mathcal{L}_0(X)$ , then its graph is sequentially closed in  $X \times X$ , and we conclude.  $\blacksquare$

**Corollary 3.16.**

- (i)  $AR(\lambda, A)x = \lambda R(\lambda, A)x - x$ , for all  $\lambda > 0$  and  $x \in X$ .
- (ii)  $R(\lambda, A)Ax = AR(\lambda, A)x$ , for all  $\lambda > 0$  and  $x \in \mathcal{D}(A)$ .
- (iii) (Resolvent equation) For every  $\lambda > 0$  and  $\mu > 0$ ,

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A). \quad (3.12)$$

- (iv) For every  $x \in X$ ,  $\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A)x = x$ .

*Proof.* (i) It follows from (3.11).

- (ii) By (i) and considering that  $x \in \mathcal{D}(A)$ , we can write

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x = \lambda R(\lambda, A)x - R(\lambda, A)(\lambda - A)x = R(\lambda, A)Ax.$$

- (iii) It follows from (i) by standard algebraic computations.
- (iv) This follows from (3.9) and from Theorem 3.14.  $\blacksquare$

**Remark 3.17.** The computations involved in the proof of Corollary 3.16(iii) require only that  $A : \mathcal{D}(A) \subset X \rightarrow X$  is a linear operator and  $\lambda, \mu \in \rho_0(A)$ .

### 3.3 Generation of $C_0$ -sequentially equicontinuous semigroups

The aim of this subsection is to prove the following generation theorem for  $C_0$ -sequentially equicontinuous semigroups, in the spirit of the Hille-Yosida theorem stated for  $C_0$ -semigroups in Banach spaces. In order to implement the classical arguments (with slight variations due to our “sequential continuity” setting), and, more precisely, in order to define the Yosida approximation, we need the sequential completeness of the space  $X$ .

**Theorem 3.18.** Let  $\hat{A} : \mathcal{D}(\hat{A}) \subset X \rightarrow X$  be a linear operator. Consider the following two statements.

- (i)  $\hat{A}$  is the infinitesimal generator of a  $C_0$ -sequentially equicontinuous semigroup  $\hat{T}$  on  $X$ .
- (ii)  $\hat{A}$  is a sequentially closed linear operator,  $\mathcal{D}(\hat{A})$  is sequentially dense in  $X$ , and there exists a sequence  $\{\lambda_n\}_{n \in \mathbb{N}} \subset \rho_0(\hat{A})$ , with  $\lambda_n \rightarrow +\infty$ , such that the family  $\left\{ (\lambda_n R(\lambda_n, \hat{A}))^m \right\}_{n, m \in \mathbb{N}}$  is sequentially equicontinuous.

Then (i)  $\Rightarrow$  (ii). If  $X$  is sequentially complete, then (ii)  $\Rightarrow$  (i).

In order to prove Theorem 3.18, we first need to introduce a locally convex topology on the space  $\mathcal{L}_0(X)$ , as follows. Let  $\mathbf{B}$  be the set of all bounded subsets of  $X$ . By Proposition 3.2(i), the quantity

$$\rho_{q, D}(F) := \sup_{x \in D} q(Fx) \quad (3.13)$$

is finite for all  $F \in \mathcal{L}_0(X)$ ,  $D \in \mathbf{B}$ , and  $q \in \mathcal{P}_X$ . Given  $D \in \mathbf{B}$  and  $q \in \mathcal{P}_X$ , (3.13) defines a seminorm in the space  $\mathcal{L}_0(X)$ . We denote by  $\mathcal{L}_{0, b}(X)$  the space  $\mathcal{L}_0(X)$  endowed with the locally convex vector topology  $\tau_b$  induced by the family of seminorms  $\{\rho_{q, D}\}_{q \in \mathcal{P}_X, D \in \mathbf{B}}$ :

$$\mathcal{L}_{0, b}(X) := (\mathcal{L}_0(X), \tau_b).$$

It is clear that  $\tau_b$  does not depend on the choice of the family  $\mathcal{P}_X$  inducing the topology of  $X$ . Since  $\mathbf{B}$  contains all singletons  $\{x\}_{x \in X}$  and  $X$  is Hausdorff, also  $\mathcal{L}_{0,b}(X)$  is Hausdorff. The proof of the following proposition can be found in [16, Prop. 3.9].

**Proposition 3.19.** (i) *If  $X$  is complete, then  $\mathcal{L}_{0,b}(X)$  is complete.*

(ii) *If  $X$  is sequentially complete, then  $\mathcal{L}_{0,b}(X)$  is sequentially complete.*

**Lemma 3.20.** *Let  $\lambda_1, \dots, \lambda_j$  be strictly positive real numbers. Then*

$$p \left( \left( \prod_{i=1}^j \lambda_i R(\lambda_i, A) \right) x \right) \leq \sup_{t \in \mathbb{R}^+} p(T_t x), \quad \forall p \in \mathcal{P}_X, \quad \forall x \in X.$$

*Proof.* We can assume  $j = 1$ , since the general case follows immediately by recursion and by Proposition A.3. For  $\lambda > 0$  and  $x \in X$ , by Theorem 3.14 we have

$$p(R(\lambda, A)x) \leq \left( \int_0^{+\infty} e^{-\lambda t} dt \right) \sup_{t \in \mathbb{R}^+} p(T_t x) = \lambda^{-1} \sup_{t \in \mathbb{R}^+} p(T_t x).$$

■

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be an analytic function of the form  $f(t) = \sum_{n=0}^{+\infty} a_n t^n$ , with  $t \in \mathbb{R}$ . We consider the formal series

$$f_B(t) := \sum_{n=0}^{+\infty} a_n t^n B^n$$

to define analytic functions  $f_B$  with values in  $\mathcal{L}_0(X)$ , associated with the real valued analytic function  $f$  and the operator  $B \in \mathcal{L}_0(X)$ . The study of this object is deferred to the Appendix.

**Notation 3.21.** *We denote  $e^{tB} := f_B(t)$  when  $f(t) = e^t$ .*

**Proof of Theorem 3.18.** (i)  $\Rightarrow$  (ii). The fact that  $\hat{A}$  is a sequentially closed linear operator was proved in Corollary 3.15. The fact that  $\mathcal{D}(\hat{A})$  is sequentially dense in  $X$  was proved in Proposition 3.10. The remaining facts follow by Lemma 3.20 and Theorem 3.14.

(ii)  $\Rightarrow$  (i). We split this part of the proof in several steps.

Step 1. Let  $\{\lambda_n\}_{n \in \mathbb{N}} \subset \rho_0(\hat{A})$  be a sequence as in (ii). For  $n \in \mathbb{N}$ , define  $J_{\lambda_n} := \lambda_n R(\lambda_n, \hat{A})$ . Observe that, for all  $x \in \mathcal{D}(\hat{A})$ , it is  $(J_{\lambda_n} - I)x = R(\lambda_n, \hat{A})\hat{A}x$ . By assumption, the family  $\{J_{\lambda_n}\}_{n \in \mathbb{N}}$  is sequentially equicontinuous, and then, for every  $x \in \mathcal{D}(\hat{A})$  and  $p \in \mathcal{P}_X$ ,

$$\lim_{n \rightarrow +\infty} p(J_{\lambda_n} x - x) = \lim_{n \rightarrow +\infty} p(R(\lambda_n, \hat{A})\hat{A}x) \leq \lim_{n \rightarrow +\infty} \lambda_n^{-1} \left( \sup_{k \in \mathbb{N}} p(J_k \hat{A}x) \right) = 0. \quad (3.14)$$

Now let  $x \in X$ . By assumption, there exists a sequence  $\{x_k\}_{k \in \mathbb{N}}$  in  $\mathcal{D}(\hat{A})$  converging to  $x$  in  $X$ . We have

$$p(J_{\lambda_n} x - x) \leq p(x - x_k) + p(J_{\lambda_n} x_k - x_k) + p(J_{\lambda_n} (x - x_k)), \quad \forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \forall p \in \mathcal{P}_X.$$

By taking first the lim sup in  $n$  and then the limit as  $k \rightarrow +\infty$  in the inequality above, and recalling (3.14) and the sequential equicontinuity of  $\{J_{\lambda_n}\}_{n \in \mathbb{N}}$ , we conclude

$$\lim_{n \rightarrow +\infty} J_{\lambda_n} x = x, \quad \forall x \in X. \quad (3.15)$$

Step 2. Here we show that, for  $t \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ ,  $T_t^{(n)} := e^{t\hat{A}J_{\lambda_n}}$  is well-defined as a convergent series in  $\mathcal{L}_{0,b}(X)$ , and that  $\{T_t^{(n)}\}_{t \in \mathbb{R}^+, n \in \mathbb{N}}$  is sequentially equicontinuous. Taking into account that

$\hat{A}J_{\lambda_n} = \lambda_n(J_{\lambda_n} - I)$ , we have (as formal sums)  $T_t^{(n)} = e^{t\hat{A}J_{\lambda_n}} = e^{t\lambda_n(J_{\lambda_n} - I)}$ . Since  $\{J_{\lambda_n}^k\}_{k \in \mathbb{N}}$  is assumed to be sequentially equicontinuous, by Proposition A.9(i),  $T_t^{(n)}$  is well-defined as a convergent series in  $\mathcal{L}_{0,b}(X)$ , and

$$T_t^{(n)} = e^{-t\lambda_n I} e^{t\lambda_n J_{\lambda_n}}. \quad (3.16)$$

Hence, using Proposition A.9(ii), the family  $\{T_t^{(n)}\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially locally equicontinuous semigroup for each fixed  $n \in \mathbb{N}$ . On the other hand, by (3.16) and by Lemma A.6, we have

$$\sup_{n \in \mathbb{N}} p\left(T_t^{(n)}x\right) = \sup_{n \in \mathbb{N}} \left( e^{-t\lambda_n} p\left(e^{t\lambda_n J_{\lambda_n}}x\right) \right) \leq \sup_{n, k \in \mathbb{N}} p\left(J_{\lambda_n}^k x\right), \quad \forall p \in \mathcal{P}_X, \forall x \in X.$$

As, by assumption,  $\{J_{\lambda_n}^k\}_{n, k \in \mathbb{N}}$  is sequentially equicontinuous, this shows that  $\{T_t^{(n)}\}_{t \in \mathbb{R}^+, n \in \mathbb{N}}$  is sequentially equicontinuous.

Step 3. Here we show that the sequence  $\{T_t^{(n)}x\}_{n \in \mathbb{N}}$  is Cauchy for every  $t \in \mathbb{R}^+$  and  $x \in \mathcal{D}(\hat{A})$ . First note that, since the family  $\{R(\lambda_n, \hat{A})\}_{n \in \mathbb{N}}$  is a commutative set (see (3.12) and Remark 3.17), also the family  $\{J_{\lambda_n}\}_{n \in \mathbb{N}}$  is a commutative set. Then  $\lambda_m(J_{\lambda_m} - I)$  commutes with every  $J_{\lambda_n}$ . Since the sum defining  $T_t^{(m)}$  is convergent in  $\mathcal{L}_{0,b}(X)$ , we have  $T_t^{(m)}J_{\lambda_n} = J_{\lambda_n}T_t^{(m)}$  and  $T_t^{(m)}T_s^{(n)} = T_s^{(n)}T_t^{(m)}$  for every  $m, n \in \mathbb{N}$ ,  $t, s \in \mathbb{R}^+$ . By Lemma A.4 and by the commutativity just noticed, if  $x \in X$  and  $t \in \mathbb{R}^+$ , the map  $F: [0, t] \rightarrow X$ ,  $s \mapsto T_{t-s}^{(n)}T_s^{(m)}x$ , is differentiable and

$$T_t^{(m)}x - T_t^{(n)}x = \int_0^t F'(s)ds = \int_0^t T_{t-s}^{(n)}T_s^{(m)}\hat{A}(J_{\lambda_m} - J_{\lambda_n})x ds,$$

where the integral is well-defined by sequential completeness of  $X$ . We notice that  $J_{\lambda_n}\hat{A} = \hat{A}J_{\lambda_n}$  on  $\mathcal{D}(\hat{A})$ . Then, from the equality above we deduce

$$p\left(T_t^{(m)}x - T_t^{(n)}x\right) \leq \int_0^t p\left(T_{t-s}^{(n)}T_s^{(m)}(J_{\lambda_m} - J_{\lambda_n})\hat{A}x\right) ds, \quad \forall x \in \mathcal{D}(\hat{A}), \forall p \in \mathcal{P}_X,$$

and then

$$\sup_{t \in [0, \hat{t}]} p\left(T_t^{(m)}x - T_t^{(n)}x\right) \leq \hat{t} \sup_{t, s \in [0, \hat{t}]} p\left(T_{t-s}^{(n)}T_s^{(m)}(J_{\lambda_m} - J_{\lambda_n})\hat{A}x\right), \quad \forall \hat{t} > 0, \forall x \in \mathcal{D}(\hat{A}), \forall p \in \mathcal{P}_X. \quad (3.17)$$

Now observe that, by Proposition A.1(i) and Step 2, the family  $\{T_t^{(m)}T_s^{(n)}\}_{t, s \in \mathbb{R}^+, m, n \in \mathbb{N}}$  is sequentially equicontinuous, and then the term on the right-hand side of (3.17) goes to 0 as  $n, m \rightarrow +\infty$ , because of (3.15). Hence, the sequence  $\{T_t^{(n)}x\}_{n \in \mathbb{N}}$  is Cauchy for every  $t \in \mathbb{R}$  and  $x \in \mathcal{D}(\hat{A})$ .

Step 4. By Step 3 and by sequential completeness of  $X$ , we conclude that there exists in  $X$

$$\hat{T}_t x := \lim_{n \rightarrow +\infty} T_t^{(n)}x, \quad \forall t \in \mathbb{R}^+, \forall x \in \mathcal{D}(\hat{A}). \quad (3.18)$$

Moreover, by (3.17), the limit (3.18) is uniform in  $t \in [0, \hat{t}]$ , for every  $\hat{t} > 0$ .

Step 5. We extend the result of Step 4, stated for  $x \in \mathcal{D}(\hat{A})$ , to all  $x \in X$ . Let  $\hat{t} > 0$  and let  $\{x_k\}_{k \in \mathbb{N}} \subset D(\hat{A})$  be a sequence converging to  $x$  in  $X$ . We can write

$$T_t^{(m)}x - T_t^{(n)}x = \left(T_t^{(m)} - T_t^{(n)}\right)(x - x_k) + \left(T_t^{(m)} - T_t^{(n)}\right)x_k, \quad \forall t \in [0, \hat{t}], \forall m, n, k \in \mathbb{N}.$$

Then, using Step 4, we have, uniformly for  $t \in [0, \hat{t}]$ ,

$$\begin{aligned} \limsup_{n, m \rightarrow +\infty} \sup_{t \in [0, \hat{t}]} p\left(T_t^{(m)}x - T_t^{(n)}x\right) &\leq \limsup_{n, m \rightarrow +\infty} \sup_{t \in [0, \hat{t}]} p\left(\left(T_t^{(m)} - T_t^{(n)}\right)(x - x_k)\right) \\ &\leq \sup_{\substack{n, m \in \mathbb{N} \\ t \in [0, \hat{t}]}} p\left(\left(T_t^{(m)} - T_t^{(n)}\right)(x - x_k)\right), \quad \forall k \in \mathbb{N}, \forall p \in \mathcal{P}_X. \end{aligned}$$

The last term goes to 0 as  $k \rightarrow +\infty$ , because of sequential equicontinuity of the family  $\{T_t^{(n)}\}_{n \in \mathbb{N}, t \in \mathbb{R}^+}$  (Step 2).

Hence, recalling that  $\mathcal{D}(\hat{A})$  is sequentially dense in  $X$ , we have proved that there exists in  $X$ , uniformly for  $t \in [0, \hat{t}]$ ,

$$\hat{T}_t x := \lim_{n \rightarrow +\infty} T_t^{(n)} x, \quad \forall x \in X. \quad (3.19)$$

Step 6. We show that the family  $\hat{T} = \{\hat{T}_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup on  $X$ . First we notice that, as by Step 5 the limit in (3.19) defining  $\hat{T}_t x$  is uniform for  $t \in [0, \hat{t}]$ , for every  $\hat{t} > 0$ , then the function  $\mathbb{R}^+ \rightarrow X$ ,  $t \mapsto \hat{T}_t x$ , is continuous. In particular,  $\hat{T}_t x \rightarrow \hat{T}_0 x$  as  $t \rightarrow 0^+$  for every  $x \in X$ . Moreover,  $\hat{T}_0 = I$  as  $T_0^{(n)} = I$  for each  $n \in \mathbb{N}$ . The linearity of  $\hat{T}_t$  and the semigroup property come from the same properties holding for every  $T_t^{(n)}$ . It remains to show that the family  $\hat{T}$  is sequentially equicontinuous. This comes from sequential equicontinuity of the family  $\{T_t^{(n)}\}_{n \in \mathbb{N}, t \in \mathbb{R}^+}$  (Step 2), and from the estimate

$$p(\hat{T}_t x) \leq p(\hat{T}_t x - T_t^{(n)} x) + p(T_t^{(n)} x) \leq p(\hat{T}_t x - T_t^{(n)} x) + \sup_{\substack{t \in \mathbb{R}^+ \\ n \in \mathbb{N}}} p(T_t^{(n)} x) \quad \forall t \in \mathbb{R}^+, \forall n \in \mathbb{N},$$

by taking first the limit as  $n \rightarrow +\infty$  and then the supremum over  $t$ .

Step 7. To conclude the proof, we only need to show that the infinitesimal generator of  $\hat{T}$  is  $\hat{A}$ . Let  $p \in \mathcal{P}_X$  and  $x \in \mathcal{D}(\hat{A})$ . By applying Proposition 3.11 to  $T^{(n)}$ , we can write

$$\hat{T}_t x - x = \lim_{n \rightarrow +\infty} (T_t^{(n)} x - x) = \lim_{n \rightarrow +\infty} \int_0^t T_s^{(n)} \hat{A} J_{\lambda_n} x ds,$$

where the integral on the right-hand side exists because of sequential completeness of  $X$  and of continuity of the integrand function, and where the latter equality is obtained, as usual, by pairing the two members of the equality with functionals  $\Lambda \in X^*$  and by using (A.1).

Now we wish to exchange the limit with the integral. This is possible, as, by Step 2, Step 5, and (3.15), we have

$$\lim_{n \rightarrow +\infty} T_t^{(n)} J_{\lambda_n} \hat{A} x = \hat{T}_t \hat{A} x \quad \text{uniformly for } t \text{ over compact sets.}$$

Then

$$\hat{T}_t x - x = \int_0^t \lim_{n \rightarrow +\infty} T_s^{(n)} \hat{A} J_{\lambda_n} x ds = \int_0^t \hat{T}_s \hat{A} x ds.$$

Dividing by  $t$  and letting  $t \rightarrow 0^+$ , we conclude that  $x \in \mathcal{D}(\tilde{A})$ , where  $\tilde{A}$  is the infinitesimal generator of  $\hat{T}$ , and that  $\tilde{A} = \hat{A}$  on  $\mathcal{D}(\hat{A})$ . But, by assumption, for some  $\lambda_n > 0$ , the operator  $\lambda_n - \hat{A}$  is one-to-one and full-range. By Theorem 3.14, the same thing holds true for  $\lambda_n - \tilde{A}$ . Then we conclude  $\mathcal{D}(\tilde{A}) = \mathcal{D}(\hat{A})$  and  $\tilde{A} = \hat{A}$ .  $\blacksquare$

**Remark 3.22.** Let  $X$  be a Banach space with norm  $|\cdot|_X$  and let  $\tau$  be a sequentially complete locally convex topology on  $X$  such that the  $\tau$ -bounded sets are the  $|\cdot|_X$ -bounded sets. Then, by Proposition 3.2(i), we have  $\mathcal{L}_0((X, \tau)) \subset L((X, |\cdot|_X))$ . Let  $\hat{T}$  be a  $C_0$ -sequentially equicontinuous semigroup on  $(X, \tau)$  with infinitesimal generator  $\hat{A}$ . By referring to the notation of the proof of Theorem 3.18, we make the following observations.

(1) Since  $R(\lambda_n, \hat{A}) \in \mathcal{L}_0((X, \tau)) \subset L((X, |\cdot|_X))$ , then the Yosida approximations  $\{T^{(n)}\}_{n \in \mathbb{N}}$ , approximating  $\hat{T}$  according to (3.19), are equicontinuous semigroups on the Banach space  $(X, |\cdot|_X)$ .



(2) The fact that  $\{(\lambda_n R(\lambda_n, \hat{A}))^m\}_{n,m \in \mathbb{N}}$  is sequentially equicontinuous implies that such a family is uniformly bounded in  $L((X, |\cdot|_X))$ . Indeed, as the unit ball  $B$  in  $(X, |\cdot|_X)$  is bounded in  $(X, \tau)$ , by Proposition A.1(iii) the set  $\{(\lambda_n R(\lambda_n, \hat{A}))^m x\}_{n,m \in \mathbb{N}, x \in B}$  is bounded in  $(X, \tau)$ . Hence, it is also bounded in  $(X, |\cdot|_X)$ , as we are assuming that the bounded sets are the same in both the topologies. As a consequence, by recalling the Hille-Yosida theorem for  $C_0$ -semigroups in Banach spaces, we have that  $\hat{T}$  is also a  $C_0$ -semigroup in the Banach space  $(X, |\cdot|_X)$  if and only if  $D(\hat{A})$  is norm dense in  $X$ .

### 3.4 Examples and counterexamples

In this subsection we provide examples to clarify some features of the notion of  $C_0$ -sequentially (locally) equicontinuous semigroup.

First, with respect to the case of  $C_0$ -semigroups on Banach spaces, we notice two relevant basic implications that we loose when dealing with strong continuity and (sequential) local equicontinuity in locally convex spaces. The first one is related to the growth rate of the orbits of the semigroup, and consequently to the possibility to define the Laplace transform. The fact that  $T$  is a  $C_0$ -locally (sequentially) equicontinuous semigroup does not imply, in general, the existence of  $\alpha > 0$  such that  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ - (sequentially) locally equicontinuous semigroup. We give two examples.

**Example 3.23.** Consider the vector space  $X := C(\mathbb{R})$ , endowed with the topology of the uniform convergence on compact sets, which makes  $X$  a Fréchet space. Define  $T_t: X \rightarrow X$  by

$$T_t \varphi(s) := e^{st} \varphi(s) \quad \forall s \in \mathbb{R}, \forall t \in \mathbb{R}^+, \forall \varphi \in X.$$

One verifies that  $T = \{T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $X$  (actually, locally equicontinuous, by Proposition A.2). On the other hand, for whatever  $\alpha > 0$ , the family  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$  is not sequentially equicontinuous. Indeed, one has that  $\{e^{-\alpha t} T_t f\}_{t \in \mathbb{R}^+}$  is unbounded in  $X$  for every  $f$  not identically zero on  $(\alpha, +\infty)$ .

**Example 3.24.** Another classical example is given in [23]. Let  $X$  be as in Example 3.23, with the same topology. For  $t \in \mathbb{R}^+$ , we define  $T := \{T_t\}_{t \in \mathbb{R}^+}$  by

$$T_t: X \rightarrow X, \varphi \mapsto \varphi(t + \cdot).$$

Then  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $X$  (equivalently,  $T$  is a  $C_0$ -locally equicontinuous semigroup, by Proposition A.2), but there does not exist any  $\alpha > 0$  such that  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$  is equicontinuous.

The second relevant difference with respect to  $C_0$ -semigroups in Banach spaces is that the strong continuity does not imply, in general, the sequential local equicontinuity. The following example shows that Definition 3.3(iii') in general cannot be derived by Definition 3.3(i)-(ii), even if Definition 3.3(ii) is strengthened by requiring the continuity of  $\mathbb{R}^+ \rightarrow X, t \mapsto T_t x, x \in X$ .

**Example 3.25.** Let  $X := C(\mathbb{R})$  be endowed with the topology of the pointwise convergence. Define the semigroup  $T := \{T_t\}_{t \in \mathbb{R}^+}$  by

$$T_t: X \rightarrow X, \varphi \mapsto \varphi(t + \cdot).$$

Then  $T_t \in \mathcal{L}_0(X)$  for all  $t \in \mathbb{R}^+$ . It is clear that, for every  $\varphi \in C(\mathbb{R})$ , the map  $\mathbb{R}^+ \rightarrow X, t \mapsto T_t \varphi$ , is continuous. Nevertheless, for each  $\hat{t} > 0$  we can find a sequence  $\{\varphi_n\}_{n \in \mathbb{N}} \subset C(\mathbb{R})$  of functions converging pointwise to 0 and such that

$$\liminf_{n \rightarrow +\infty} \sup_{t \in [0, \hat{t}]} |(T_t \varphi_n)(0)| = \liminf_{n \rightarrow +\infty} \sup_{t \in [0, \hat{t}]} |\varphi_n(t)| > 0.$$

Hence,  $T$  is not a  $C_0$ -sequentially locally equicontinuous semigroup. We observe that the same conclusion holds true if we restrict the action of  $T$  to the space  $C_b(\mathbb{R})$ .

Referring to Remark 3.4(2), we provide the following example <sup>(3)</sup>.

**Example 3.26.** Consider the Banach space  $\ell^1$ , with its usual norm  $\|\mathbf{x}\|_1 = \sum_{k=0}^{+\infty} |x_k|$ , where  $\mathbf{x} := \{x_k\}_{k \in \mathbb{N}} \in \ell^1$ , and denote by  $\tau_1$  and  $\tau_w$  the  $|\cdot|_1$ -topology and the weak topology respectively. Define  $Z := \ell^1 \times \ell^1$  and endow it with the product topology  $\tau_w \otimes \tau_1$ . Let

$$B: Z \rightarrow Z, (x_1, x_2) \mapsto (x_1, x_1).$$

We recall that  $\ell^1$  enjoys Schur's property (weak convergent sequences are strong convergent; see [10, p. 85]). As a consequence, we have that  $Z$  is sequentially complete and  $B \in \mathcal{L}_0(Z)$ . On the other hand, as  $\tau_w$  is strictly weaker than  $\tau_1$ , we have  $B \notin L(Z)$ . By induction, we see that  $(I - B)^n = (I - B)$  for each  $n \geq 1$ , and then  $\{(I - B)^n\}_{n \in \mathbb{N}}$  is a family of sequentially equicontinuous operators. By Proposition A.9, if we define  $T_t := e^{t(B-I)}$  for  $t \in \mathbb{R}^+$ , then  $T := \{T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup on  $Z$ . Actually, we have  $e^{t(B-I)} = e^{-t}(I - B) + B$ . However, if  $t > 0$ , the operators  $e^{t(B-I)} = e^{-t}I + (1 - e^{-t})B$  are not continuous on  $Z$ .

## 4 Relationship with *bi-continuous* semigroups

In this subsection we establish a comparison of our notion of  $C_0$ -sequentially equicontinuous semigroup with the notion of *bi-continuous* semigroup developed in [25, 26]. The latter requires to deal with Banach spaces as underlying spaces.

**Definition 4.1.** Let  $(X, |\cdot|_X)$  be a Banach space and let  $X^*$  be its topological dual. A linear subspace  $\Gamma \subset X^*$  is called *norming* for  $(X, |\cdot|_X)$  if  $\|x\|_X = \sup_{\gamma \in \Gamma, |\gamma|_{X^*} \leq 1} |\gamma(x)|$ , for every  $x \in X$ .

We recall the definition of bi-continuous semigroup as given in [26, Def. 3] and [25, Def. 1.3].

**Definition 4.2.** Let  $(X, |\cdot|_X)$  be a Banach space with topological dual  $X^*$ . Let  $\tau$  be a Hausdorff locally convex topology on  $X$  with the following properties.

- (i) The space  $(X, \tau)$  is sequentially complete on  $|\cdot|_X$ -bounded sets.
- (ii)  $\tau$  is weaker than the topology induced by the norm  $|\cdot|_X$ .
- (iii) The topological dual of  $(X, \tau)$  is norming for  $(X, |\cdot|_X)$ .

A family of linear operators  $T = \{T_t: X \rightarrow X\}_{t \in \mathbb{R}^+} \subset L((X, |\cdot|_X))$  is called a *bi-continuous* semigroup with respect to  $\tau$  and of type  $\alpha \in \mathbb{R}$  if the following conditions hold:

- (iv)  $T_0 = I$  and  $T_t T_s = T_{t+s}$  for every  $t, s \in \mathbb{R}^+$ ;
- (v) for some  $M \geq 0$ ,  $\|T_t\|_{L((X, |\cdot|_X))} \leq M e^{\alpha t}$ , for every  $t \in \mathbb{R}^+$ ;
- (vi)  $T$  is strongly  $\tau$ -continuous, i.e., the map  $\mathbb{R}^+ \rightarrow (X, \tau)$ ,  $t \mapsto T_t x$  is continuous for every  $x \in X$ ;
- (vii)  $T$  is locally bi-continuous, i.e., for every  $|\cdot|_X$ -bounded sequence  $\{x_n\}_{n \in \mathbb{N}} \subset X$   $\tau$ -convergent to  $x \in X$  and every  $\hat{t} > 0$ , we have

$$\lim_{n \rightarrow +\infty} T_t x_n = T_t x \quad \text{in } (X, \tau), \text{ uniformly in } t \in [0, \hat{t}].$$

<sup>3</sup>Example 3.26 could seem a bit artificial and *ad hoc*. In the next section we will provide another more meaningful example by a very simple Markov transition semigroup (Example 6.4).

The following proposition shows that the notion of bi-continuous semigroup is a specification of our notion of  $C_0$ -sequentially locally equicontinuous semigroup in sequentially complete spaces. Indeed, given a bi-continuous semigroup on a Banach space  $(X, |\cdot|_X)$  with respect to a topology  $\tau$ , one can define a locally convex sequentially complete topology  $\tau' \supset \tau$  and see the bi-continuous semigroup as a  $C_0$ -sequentially locally equicontinuous semigroup on  $(X, \tau')$ .

**Proposition 4.3.** *Let  $\{T_t\}_{t \in \mathbb{R}^+}$  be a bi-continuous semigroup on  $X$  with respect to  $\tau$  and of type  $\alpha$ . Then there exists a locally convex topology  $\tau'$  with the following properties:*

- (i)  $\tau \subset \tau'$  and  $\tau'$  is weaker than the  $|\cdot|_X$ -topology;
- (ii) a sequence converges in  $\tau'$  if and only if it is  $|\cdot|_X$ -bounded and convergent in  $\tau$ ;
- (iii)  $(X, \tau')$  is sequentially complete;
- (iv)  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup in  $(X, \tau')$ ; moreover, for every  $\lambda > \alpha$ ,  $\{e^{-\lambda t} T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup on  $(X, \tau')$  satisfying Assumption 3.7.

To prove Proposition 4.3 we need the following

**Lemma 4.4.** *Let  $(X, |\cdot|_X)$  be a Banach space and let  $\Gamma \subset X^*$  be norming for  $(X, |\cdot|_X)$  and closed with respect to the operator norm  $|\cdot|_{X^*}$ . Then  $B \subset X$  is  $\sigma(X, \Gamma)$ -bounded if and only if it is  $|\cdot|_X$ -bounded.*

*Proof.* As  $\sigma(X, \Gamma)$  is weaker than the  $|\cdot|_X$ -topology, clearly  $|\cdot|_X$ -bounded sets are also  $\sigma(X, \Gamma)$ -bounded. Conversely, let  $B \subset X$  be  $\sigma(X, \Gamma)$ -bounded and consider the family of continuous functionals

$$\{\Lambda_b : \Gamma \rightarrow \mathbb{R}, \gamma \mapsto \gamma(b)\}_{b \in B},$$

By assumption,  $\sup_{b \in B} |\gamma(b)| < +\infty$  for every  $\gamma \in \Gamma$ . The Banach-Steinhaus theorem applied in the Banach space  $(\Gamma, |\cdot|_{X^*})$  yields

$$M := \sup_{b \in B} \sup_{\gamma \in \Gamma, |\gamma|_{X^*} \leq 1} |\gamma(b)| < +\infty.$$

Then, since  $\Gamma$  is norming for  $(X, |\cdot|_X)$ , we have

$$|b|_X = \sup_{\gamma \in \Gamma, |\gamma|_{X^*} \leq 1} |\gamma(b)| \leq M < +\infty \quad \forall b \in B,$$

and then  $B$  is  $|\cdot|_X$ -bounded. ■

**Proof of Proposition 4.3.** Denote by  $X^*$  the topological dual of  $(X, |\cdot|_X)$ , and let  $\mathcal{P}_X$  be a set of seminorms on  $X$  inducing  $\tau$ . Denote by  $\Gamma$  the dual of  $(X, \tau)$ . On  $X$ , define the seminorms

$$q_{p, \gamma}(x) = p(x) + |\gamma(x)|, \quad p \in \mathcal{P}_X, \gamma \in \bar{\Gamma},$$

where  $\bar{\Gamma}$  is the closure of  $\Gamma$  with respect to the norm  $|\cdot|_{X^*}$ . Let  $\tau'$  be the locally convex topology induced by the family of seminorms  $\{q_{p, \gamma}\}_{p \in \mathcal{P}_X, \gamma \in \bar{\Gamma}}$ .

(i) Clearly  $\tau \subset \tau'$  and  $\tau'$  is weaker than the  $|\cdot|_X$ -topology.

(ii) As  $\tau \subset \tau'$ , the  $\tau'$ -convergent sequences are  $\tau$ -convergent. Moreover, as  $\Gamma$  is norming,  $\bar{\Gamma}$  is norming too. Then, by Lemma 4.4, every  $\sigma(X, \bar{\Gamma})$ -bounded set is  $|\cdot|_X$ -bounded. In particular, every convergent sequence in  $\tau'$  is  $|\cdot|_X$ -bounded.

Conversely, consider a sequence  $\{x_n\}_{n \in \mathbb{N}} \subset X$  which is  $\tau$ -convergent to 0 in  $X$  and  $|\cdot|_X$ -bounded by a constant  $M > 0$ . To show that  $x_n \xrightarrow{\tau'} 0$ , we only need to show that  $\gamma(x_n) \rightarrow 0$  for every  $\gamma \in \bar{\Gamma}$ .

For that, notice first that the convergence to 0 with respect to  $\tau$  implies the convergence  $\gamma(x_n) \rightarrow 0$  for every  $\gamma \in \Gamma$ . Take now  $\gamma \in \bar{\Gamma}$  and a sequence  $\{\gamma_k\}_{k \in \mathbb{N}} \subset \Gamma$  converging to  $\gamma$  with respect to  $|\cdot|_{X^*}$ . Then the estimate

$$|\gamma(x_n - x)| \leq M|\gamma - \gamma_k|_{X^*} + |\gamma_k(x_n)| \quad \forall n, k \in \mathbb{N},$$

yields

$$\limsup_{n \rightarrow +\infty} |\gamma(x_n)| \leq M|\gamma - \gamma_k|_{X^*} \quad \forall k \in \mathbb{N}.$$

Since  $\gamma_k \rightarrow \gamma$  with respect to  $|\cdot|_{X^*}$  when  $k \rightarrow +\infty$ , we now conclude that sequence  $\{x_n\}_{n \in \mathbb{N}}$  converges to 0 also with respect to  $\tau'$ .

(iii) A Cauchy sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $(X, \tau')$  is  $\tau'$ -bounded. By Lemma 4.4, it is  $|\cdot|_X$ -bounded. Clearly,  $\{x_n\}_{n \in \mathbb{N}}$  is also  $\tau$ -Cauchy. Then, by Definition 4.2(i),  $\{x_n\}_{n \in \mathbb{N}}$  converges to some  $x$  in  $(X, \tau)$ . Since the sequence is  $|\cdot|_X$ -bounded, by (ii) the convergence takes place also in  $\tau'$ . This proves that  $(X, \tau')$  is sequentially complete.

(iv) We start by proving that  $\{T_t\}_{t \in \mathbb{R}^+}$  is a sequentially locally equicontinuous family of operators in the space  $(X, \tau')$ . Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence  $\tau'$ -convergent to 0. By (ii),  $\{x_n\}_{n \in \mathbb{N}}$  is  $|\cdot|_X$ -bounded and  $\tau$ -convergent to 0. By Definition 4.2(vii)

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, \hat{t}]} p(T_t x_n) = 0, \quad \forall p \in \mathcal{P}_X, \forall \hat{t} > 0. \quad (4.1)$$

Assume now, by contradiction, that there exist  $R > 0$ ,  $p \in \mathcal{P}_X$ ,  $\gamma \in \bar{\Gamma}$ , and  $\varepsilon > 0$ , such that

$$\limsup_{n \rightarrow +\infty} \sup_{t \in [0, R]} q_{p, \gamma}(T_t x_n) > \varepsilon.$$

Then, due to (4.1), there exist a sequence  $\{t_n\}_{n \in \mathbb{N}} \subset [0, R]$  convergent to some  $t \in [0, R]$  and a subsequence of  $\{x_n\}_{n \in \mathbb{N}}$ , still denoted by  $\{x_n\}_{n \in \mathbb{N}}$ , such that

$$|\gamma(T_{t_n} x_n)| \geq \varepsilon \quad \forall n \in \mathbb{N}. \quad (4.2)$$

By Definition 4.2(v), the family  $\{T_t\}_{t \in [0, R]}$  is uniformly bounded in the operator norm. Then, by recalling that  $\{x_n\}_{n \in \mathbb{N}}$  is  $|\cdot|_X$ -bounded, we have

$$\hat{M} := \sup_{n \in \mathbb{N}} |T_{t_n} x_n|_X < +\infty.$$

Let  $\hat{\gamma} \in \Gamma$  be such that  $|\hat{\gamma} - \gamma|_{X^*} \leq \varepsilon/(2\hat{M})$ . Then

$$\limsup_{n \rightarrow +\infty} |\gamma(T_{t_n} x_n)| \leq \frac{\varepsilon}{2} + \limsup_{n \rightarrow +\infty} |\hat{\gamma}(T_{t_n} x_n)| = \frac{\varepsilon}{2}, \quad (4.3)$$

where the last equality is due to (4.1) and to the fact that  $\hat{\gamma} \in \Gamma = (X, \tau)^*$ . But (4.3) contradicts (4.2). The fact that  $T$  is strongly continuous with respect to  $\tau'$  follows from (ii) and from Definition 4.2(v)-(vi).

Finally, by Definition 4.2(v) we can apply Proposition 3.6(i) and conclude that  $\{e^{-\lambda t} T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup on  $(X, \tau')$  for every  $\lambda > \alpha$ . Due to part (iii), such a semigroup satisfies Assumption 3.7 (recall Remark 3.8).  $\blacksquare$

## 5 Relationship with semigroups on functional spaces

The aim of this section is to develop the theory of the previous section in some specific functional spaces. Throughout the rest of the paper,  $E$  will denote a metric space,  $\mathcal{E}$  will denote the associated Borel  $\sigma$ -algebra, and  $\mathcal{S}(E)$  will denote one of the spaces  $UC_b(E)$ ,  $C_b(E)$ ,  $B_b(E)$ . We recall

that  $(\mathcal{S}(E), |\cdot|_\infty)$ , where  $|\cdot|_\infty$  is the usual sup-norm, is a Banach space. For simplicity of notation, we denote by  $\mathcal{S}(E)_\infty^*$  the dual of  $(\mathcal{S}(E), |\cdot|_\infty)$  and by  $|\cdot|_{\mathcal{S}(E)_\infty^*}$  the operator norm in  $\mathcal{S}(E)_\infty^*$ .

We are going to define on  $\mathcal{S}(E)$  two particular locally convex topologies. The motivation for introducing such topologies is that they allow to frame under a general unified viewpoint some of the approaches used in the literature of Markov transition semigroups. In particular, we are able to cover the following types of semigroups.

1. Weakly continuous semigroups, introduced in [4] for the space  $UC_b(E)$  with  $E$  separable Hilbert space (an overview can also be found in [5, Appendix B], with  $E$  separable Banach space).
2.  $\pi$ -semigroups, introduced in [31] for the space  $UC_b(E)$ , with  $E$  separable metric space.
3.  $C_0$ -locally equicontinuous semigroups with respect to the so called mixed topology in the space  $C_b(E)$ , considered by [17], with  $E$  separable Hilbert space.

### 5.1 A family of locally convex topologies on $\mathcal{S}(E)$

In the following, by  $\mathbf{ba}(E)$  we denote the space of finitely additive signed measures on  $(E, \mathcal{E})$  with bounded total variation. The space  $\mathbf{ba}(E)$  is Banach when endowed with the norm  $|\cdot|_1$  given by the total variation and is canonically identified with  $(B_b(E)_\infty^*, |\cdot|_{B_b(E)_\infty^*})$  (see [3, Theorem 14.4]) through the isometry

$$\Phi: (\mathbf{ba}(E), |\cdot|_1) \rightarrow (B_b(E)_\infty^*, |\cdot|_{B_b(E)_\infty^*}), \mu \mapsto \Phi_\mu, \quad (5.1)$$

where

$$\Phi_\mu(f) := \int_E f d\mu \quad \forall f \in B_b(E), \quad (5.2)$$

with  $\int_E f d\mu$  interpreted in the Darboux sense (see [3, Sec. 11.2]).

We denote by  $\mathbf{ca}(E)$  the space of elements of  $\mathbf{ba}(E)$  that are countably additive. The space  $(\mathbf{ca}(E), |\cdot|_1)$  is Banach as well. If  $\mu \in \mathbf{ca}(E)$ , then the Darboux integral in (5.2) coincides with the Lebesgue integral.

For future reference, we recall the following result (see [29, Th. 5.9, p. 39]).

**Lemma 5.1.** *Let  $\nu \in \mathbf{ca}(E)$  be such that  $\int_E f d\nu = 0$  for all  $f \in UC_b(E)$ . Then  $\nu = 0$ .*

**Proposition 5.2.** *The space  $(\mathbf{ca}(E), |\cdot|_1)$  is isometrically embedded into  $(\mathcal{S}(E)_\infty^*, |\cdot|_{\mathcal{S}(E)_\infty^*})$  by*

$$\Phi: \mathbf{ca}(E) \rightarrow \mathcal{S}(E)_\infty^*, \mu \mapsto \Phi_\mu, \quad (5.3)$$

where

$$\Phi_\mu(f) := \int_E f d\mu, \quad \forall f \in \mathcal{S}(E). \quad (5.4)$$

*Proof.* It is clear that  $\Phi$  is linear.

Let  $\mu \in \mathbf{ca}(E)$ . As  $|\Phi_\mu(f)| \leq |f|_\infty |\mu|_1$  for every  $f \in \mathcal{S}(E)$ , then  $\Phi_\mu \in \mathcal{S}(E)_\infty^*$  and  $|\Phi_\mu|_{\mathcal{S}(E)_\infty^*} \leq |\mu|_1$ . To show that  $\Phi$  is an isometry it remains to show that  $|\Phi_\mu|_{\mathcal{S}(E)_\infty^*} \geq |\mu|_1$ . Let  $\mu = \mu^+ - \mu^-$  be the Jordan decomposition of  $\mu$ , and let  $C^+ := \text{supp}(\mu^+)$ ,  $C^- := \text{supp}(\mu^-)$ . Let  $\varepsilon > 0$ . Then we can find a closed set  $C_\varepsilon^+ \subset C^+$  such that  $\mu^+(C^+ \setminus C_\varepsilon^+) < \varepsilon$ , and  $d(C_\varepsilon^+, C^-) > 0$ . Let  $f$  be defined by

$$f(x) := \frac{d(x, C^-) - d(x, C_\varepsilon^+)}{d(x, C^-) + d(x, C_\varepsilon^+)} \quad \forall x \in E.$$

Then  $f \in UC_b(E)$ ,  $f \equiv 1$  on  $C_\varepsilon^+$ ,  $f \equiv -1$  on  $C^-$ , and  $|f|_\infty = 1$ . Therefore,

$$\int_E f d\mu = \int_{C_\varepsilon^+} f d\mu^+ + \int_{C^+ \setminus C_\varepsilon^+} f d\mu^+ - \int_{C^-} f d\mu^- \geq \mu^+(C_\varepsilon^+) - \varepsilon + \mu^-(C^-) \geq |\mu|_1 - 2\varepsilon.$$

Then  $|\Phi_\mu|_{\mathcal{S}(E)_\infty^*} \geq |\mu|_1 - 2\varepsilon$ . We conclude by arbitrariness of  $\varepsilon$ .  $\blacksquare$

Let  $\mathbf{P}$  be a set of non-empty parts of  $E$  such that  $E = \bigcup_{P \in \mathbf{P}} P$ . For every  $P \in \mathbf{P}$  and every  $\mu \in \mathbf{ca}(E)$ , let us introduce the seminorm

$$p_{P,\mu}(f) := [f]_P + \left| \int_E f d\mu \right|, \quad \forall f \in \mathcal{S}(E), \quad (5.5)$$

where

$$[f]_P := \sup_{x \in P} |f(x)|.$$

Denote by  $\tau_{\mathbf{P}}$  the locally convex topology on  $\mathcal{S}(E)$  induced by the family of seminorms

$$\{p_{P,\mu} : P \in \mathbf{P}, \mu \in \mathbf{ca}(E)\}.$$

Since  $E = \bigcup_{P \in \mathbf{P}} P$ ,  $\tau_{\mathbf{P}}$  is Hausdorff.

Let us denote by  $\tau_\infty$  the topology induced by the norm  $|\cdot|_\infty$  on  $\mathcal{S}(E)$ . Since the functional  $\Phi_\mu$  defined in (5.4) is  $\tau_{\mathbf{P}}$ -continuous for every  $\mu \in \mathbf{ca}(E)$ , and since  $p_{P,\mu}$  is  $\tau_\infty$ -continuous for every  $P \in \mathbf{P}$  and every  $\mu \in \mathbf{ca}(E)$ , we have the inclusions

$$\sigma(\mathcal{S}(E), \mathbf{ca}(E)) \subset \tau_{\mathbf{P}} \subset \tau_\infty. \quad (5.6)$$

Observe that, when  $\mathbf{P}$  contains only finite parts of  $E$ , then  $\tau_{\mathbf{P}} = \sigma(\mathcal{S}(E), \mathbf{ca}(E))$ , because  $\mathbf{ca}(E)$  contains all Dirac measures. The opposite case is when  $E \in \mathbf{P}$ , and then  $\tau_{\mathbf{P}} = \tau_\infty$ .

**Proposition 5.3.** *Let  $B \subset \mathcal{S}(E)$ . The following are equivalent.*

- (i)  $B$  is  $\sigma(\mathcal{S}(E), \mathbf{ca}(E))$ -bounded.
- (ii)  $B$  is  $\tau_{\mathbf{P}}$ -bounded.
- (iii)  $B$  is  $\tau_\infty$ -bounded.

*Proof.* By (5.6), it is sufficient to prove that (i)  $\Rightarrow$  (iii). Let  $B$  be  $\sigma(\mathcal{S}(E), \mathbf{ca}(E))$ -bounded. By Proposition 5.2,  $\mathbf{ca}(E)$  is closed in  $\mathcal{S}(E)_\infty^*$ . Moreover, since  $\mathbf{ca}(E)$  contains the Dirac measures, it is norming. Then we conclude by applying Lemma 4.4.  $\blacksquare$

We now focus on the following two cases:

- (a)  $\mathbf{P}$  is the set of all finite subsets of  $E$ , and then  $\tau_{\mathbf{P}} = \sigma(\mathcal{S}(E), \mathbf{ca}(E))$ ;
- (b)  $\mathbf{P}$  is the set of all non-empty compact subsets of  $E$ ; in this case, we denote  $\tau_{\mathbf{P}}$  by  $\tau_{\mathcal{K}}$ , i.e.,

$$\tau_{\mathcal{K}} := \text{l.c. topology on } \mathcal{S}(E) \text{ generated by } \{p_{K,\mu} : K \subset E \text{ compact}, \mu \in \mathbf{ca}(E)\}. \quad (5.7)$$

The relationship between  $\tau_\infty$ ,  $\tau_{\mathcal{K}}$ , and  $\tau_{\mathcal{C}}$ , where  $\tau_{\mathcal{C}}$  denotes the topology on  $\mathcal{S}(E)$  defined by the uniform convergence on compact sets of  $E$ , induced by the family of seminorms

$$\{p_K = [\cdot]_K : K \text{ non-empty compact subset of } E\},$$

is investigated in [16, Sec. 4.1]. In particular it is shown ([16, Prop. 4.6, Prop. 4.12]) that the (obvious) inclusions  $\tau_{\mathcal{C}} \subset \tau_{\mathcal{K}} \subset \tau_\infty$  are equalities if and only if  $E$  is compact. Also the proofs of the following results, which will be used afterwards, can be found in [16, Sec. 4.1].

**Proposition 5.4.** *The following statements hold.*

(i) *If a net  $\{f_i\}_{i \in \mathcal{I}}$  is bounded and convergent to  $f$  in  $(\mathcal{S}(E), \tau_{\mathcal{K}})$ , then*

$$\sup_{i \in \mathcal{I}} \|f_i\|_{\infty} < +\infty \quad \text{and} \quad \lim_i f_i = f \text{ in } (\mathcal{S}(E), \tau_{\mathcal{G}}).$$

*If either  $\mathcal{I} = \mathbb{N}$  or  $E$  is homeomorphic to a Borel subset of a Polish space, then also the converse holds true.*

(ii) *If a net  $\{f_i\}_{i \in \mathcal{I}}$  is bounded and Cauchy in  $(\mathcal{S}(E), \tau_{\mathcal{K}})$ , then*

$$\sup_{i \in \mathcal{I}} \|f_i\|_{\infty} < +\infty \quad \text{and} \quad \{f_i\}_i \text{ is Cauchy in } (\mathcal{S}(E), \tau_{\mathcal{G}}).$$

*If either  $\mathcal{I} = \mathbb{N}$  or  $E$  is homeomorphic to a Borel subset of a Polish space, then also the converse holds true.*

**Proposition 5.5.** *The following statements hold.*

(i) *If a net  $\{f_i\}_{i \in \mathcal{I}}$  is bounded and convergent to  $f$  in  $(\mathcal{S}(E), \sigma(\mathcal{S}(E), \mathbf{ca}(E)))$ , then*

$$\sup_{i \in \mathcal{I}} \|f_i\|_{\infty} < +\infty \quad \text{and} \quad \lim_i f_i = f \text{ pointwise.}$$

*If  $\mathcal{I} = \mathbb{N}$  then also the converse holds true.*

(ii) *If a net  $\{f_i\}_{i \in \mathcal{I}}$  is bounded and Cauchy in  $(\mathcal{S}(E), \sigma(\mathcal{S}(E), \mathbf{ca}(E)))$ , then*

$$\sup_{i \in \mathcal{I}} \|f_i\|_{\infty} < +\infty \quad \text{and} \quad \{f_i(x)\}_i \text{ is Cauchy for every } x \in E.$$

*If  $\mathcal{I} = \mathbb{N}$  then also the converse holds true.*

**Proposition 5.6.** *The following statements hold.*

(i)  $(B_b(E), \sigma(B_b(E), \mathbf{ca}(E)))$  and  $(B_b(E), \tau_{\mathcal{K}})$  are sequentially complete.

(ii)  $C_b(E)$  is  $\tau_{\mathcal{K}}$ -closed in  $B_b(E)$  (hence, by (i),  $(C_b(E), \tau_{\mathcal{K}})$  is sequentially complete).

(iii) *If  $E$  is homeomorphic to a Borel subset of a Polish space, then  $UC_b(E)$  is dense in  $(C_b(E), \tau_{\mathcal{K}})$ .*

## 5.2 Relationship with weakly continuous semigroups

In this subsection we first recall the notions of  $\mathcal{K}$ -convergence and of weakly continuous semigroup in the space  $UC_b(E)$ , introduced and studied first in [4, 5] in the case of  $E$  separable Banach space<sup>4</sup>. So, throughout this subsection  $E$  is assumed to be a separable Banach space. We will show that every weakly continuous semigroup is a  $C_0$ -sequentially locally equicontinuous semigroup and, up to a renormalization, a  $C_0$ -sequentially equicontinuous semigroup on  $(UC_b(E), \tau_{\mathcal{K}})$  (Proposition 5.9).

The notion of  $\mathcal{K}$ -convergence was introduced in [4, 5] for sequences. We recall it in its natural extension to nets. A net of functions  $\{f_i\}_{i \in \mathcal{I}} \subset UC_b(E)$  is said  $\mathcal{K}$ -convergent to  $f \in UC_b(E)$  if it is  $\|\cdot\|_{\infty}$ -bounded and if  $\{f_i\}_{i \in \mathcal{I}}$  converges to  $f$  uniformly on compact sets of  $E$ , i.e.,

$$\left\{ \begin{array}{l} \sup_{i \in \mathcal{I}} \|f_i\|_{\infty} < +\infty \\ \lim_i [f_i - f]_K = 0 \quad \text{for every non-empty compact } K \subset E. \end{array} \right. \quad (5.8)$$

---

<sup>4</sup>In order to avoid misunderstanding, we stress that [5] uses the notation  $C_b(E)$  to denote the space of uniformly continuous bounded functions on  $E$ , i.e. our space  $UC_b(E)$ . Also we notice that the separability of  $E$  is not needed here for our discussion.



In such a case, we write  $f_t \xrightarrow{\mathcal{K}} f$ . If  $E$  is separable, in view of Proposition 5.4(i), the convergence (5.8) is equivalent to the convergence with respect to the locally convex topology  $\tau_{\mathcal{K}}$ . In this sense,  $\tau_{\mathcal{K}}$  is a natural vector topology to treat weakly continuous semigroups (whose definition is recalled below) within the framework of  $C_0$ -sequentially locally equicontinuous semigroups.

**Definition 5.7.** A weakly continuous semigroup on  $UC_b(E)$  is a family  $T = \{T_t\}_{t \in \mathbb{R}^+}$  of bounded linear operators on  $(UC_b(E), |\cdot|_{\infty})$  satisfying the following conditions.

(P1)  $T_0 = I$  and  $T_t T_s = T_{t+s}$  for  $t, s \in \mathbb{R}^+$ .

(P2) There exist  $M \geq 1$  and  $\alpha \in \mathbb{R}$  such that  $|T_t f|_{\infty} \leq M e^{\alpha t} |f|_{\infty}$  for every  $t \in \mathbb{R}^+$ ,  $f \in UC_b(E)$ .

(P3) For every  $f \in UC_b(E)$  and every  $\hat{t} > 0$ , the family of functions  $\{T_t f : E \rightarrow \mathbb{R}\}_{t \in [0, \hat{t}]}$  is equi-uniformly continuous, i.e., there exists a modulus of continuity  $w$  (depending on  $\hat{t}$ ) such that

$$\sup_{t \in [0, \hat{t}]} |T_t f(\xi) - T_t f(\xi')| \leq w(|\xi - \xi'|_E), \quad \forall \xi, \xi' \in E. \quad (5.9)$$

(P4) For every  $f \in UC_b(E)$ , we have  $T_t f \xrightarrow{\mathcal{K}} f$  as  $t \rightarrow 0^+$ ; in view of (P2) the latter convergence is equivalent to

$$\lim_{t \rightarrow 0^+} [T_t f - f]_K = 0 \text{ for every non-empty compact } K \subset E. \quad (5.10)$$

(P5) If  $f_n \xrightarrow{\mathcal{K}} f$ , then  $T_t f_n \xrightarrow{\mathcal{K}} T_t f$  uniformly in  $t \in [0, \hat{t}]$  for every  $\hat{t} > 0$ ; in view of (P2), the latter convergence is equivalent to

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, \hat{t}]} [T_t f_n - T_t f]_K = 0 \text{ for every non-empty compact } K \subset E, \quad \forall \hat{t} \in \mathbb{R}^+. \quad (5.11)$$

**Lemma 5.8.** Let  $C \subset X$  be sequentially closed, convex, and containing the origin, let  $\hat{t} > 0$ , and let  $x \in X$ . If  $T_t x \in C$  for all  $t \in [0, \hat{t}]$ , then

$$\int_0^{\hat{t}} e^{-\lambda t} T_t x dt \in \frac{1}{\lambda} C, \quad \forall \lambda > 0. \quad (5.12)$$

If  $T_t x \in C$  for all  $t \in \mathbb{R}^+$  then,

$$R(\lambda)x \in \frac{1}{\lambda} C, \quad \forall \lambda > 0. \quad (5.13)$$

*Proof.* We prove the first claim, as the second one is a straightforward consequence of it because of the sequential completeness of  $C$ . Let  $\hat{t} > 0$ . The Riemann integral in (5.12) is the limit of a sequence of Riemann sums  $\{\sigma(\pi^k)\}_{k \in \mathbb{N}}$  of the form

$$\sigma(\pi^k) = \sum_{i=1}^{m_k} e^{-\lambda t_i^k} (t_i^k - t_{i-1}^k) T_{t_i^k} x,$$

with  $\pi^k := \{0 = t_0^k < t_1^k < \dots < t_{m_k}^k = \hat{t}\}$  and  $|\pi^k| \rightarrow 0$  as  $k \rightarrow +\infty$ , where  $|\pi^k| := \sup\{|t_i - t_{i-1}| : i = 1, \dots, m_k\}$ . Then, by sequential closedness of  $C$ , we are reduced to show that  $\sigma(\pi^k) \in \frac{1}{\lambda} C$  for every  $k \in \mathbb{N}$ . Denote

$$\alpha_k := \sum_{i=1}^{m_k} e^{-\lambda t_i^k} (t_i^k - t_{i-1}^k), \quad \forall k \in \mathbb{N}.$$

Then

$$0 < \alpha_k < \int_0^{+\infty} e^{-\lambda t} dt = \lambda^{-1}, \quad \forall k \in \mathbb{N}.$$

As  $\sigma(\pi^k)/\alpha_k$  is a convex combination of the elements  $\{T_{t_i^k} x\}_{i=1, \dots, m_k}$ , which belong to  $C$  by assumption, recalling that  $C$  is convex and contains the origin, we conclude  $\sigma(\pi^k) \in \alpha_k C \subset \frac{1}{\lambda} C$ , for every  $k \in \mathbb{N}$ , and the proof is complete.  $\blacksquare$

**Proposition 5.9.** *Let  $T := \{T_t\}_{t \in \mathbb{R}^+}$  be a weakly continuous semigroup on  $UC_b(E)$ . Then  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $(UC_b(E), \tau_{\mathcal{K}})$  and, for every  $\lambda > \alpha$  (where  $\alpha$  is as in (P2)),  $\{e^{-\lambda t} T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup on  $(UC_b(E), \tau_{\mathcal{K}})$  satisfying Assumption 3.7.*

*Conversely, if  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $(UC_b(E), \tau_{\mathcal{K}})$  satisfying (P3), then  $T$  is a weakly continuous semigroup on  $UC_b(E)$ .*

*Proof.* Let  $f \in UC_b(E)$ . By (P4) and by Proposition 5.4(i),  $T_t f \rightarrow f$  in  $(UC_b(E), \tau_{\mathcal{K}})$  when  $t \rightarrow 0^+$ . This shows the strong continuity of  $T$  in  $(UC_b(E), \tau_{\mathcal{K}})$ .

Now let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence converging to 0 in  $(UC_b(E), \tau_{\mathcal{K}})$  and let  $\hat{t} \in \mathbb{R}^+$ . By Proposition 5.4(i), it follows that  $f_n \xrightarrow{\mathcal{K}} 0$ . By (P5) we then have  $T_t f_n \xrightarrow{\mathcal{K}} 0$  uniformly in  $t \in [0, \hat{t}]$ . Using again Proposition 5.4(i), we conclude that  $T$  is locally sequentially equicontinuous in  $(UC_b(E), \tau_{\mathcal{K}})$ .

By (P2) and by Proposition 5.3, we can apply Proposition 3.6(i) to  $T$  and conclude that  $\{e^{-\lambda t} T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup on  $(UC_b(E), \tau_{\mathcal{K}})$ .

We finally show that, for  $\lambda > \alpha$ ,  $\{e^{-\lambda t} T_t\}_{t \in \mathbb{R}^+}$  satisfies Assumption 3.7. Let  $\alpha < \lambda' < \lambda$  and  $f \in UC_b(E)$ . By Proposition 5.6,  $(C_b(E), \tau_{\mathcal{K}})$  is sequentially complete. By Proposition 3.5, the map

$$\mathbb{R}^+ \rightarrow (UC_b(E), \tau_{\mathcal{K}}), \quad t \mapsto e^{-\lambda' t} T_t f,$$

is continuous and bounded. It then follows that the Riemann integral  $R(\lambda)f$  exists in  $C_b(E)$ . We show that  $R(\lambda)f \in UC_b(E)$ . Since the Dirac measures are contained in  $(C_b(E), \tau_{\mathcal{K}})^*$ , by Proposition A.3 we have

$$R(\lambda)f(\xi) = \int_0^{+\infty} e^{-\lambda t} T_t f(\xi) dt \quad \forall \xi \in E.$$

On the other hand, by (P2), for every  $\varepsilon > 0$  there exists  $\hat{t} \in \mathbb{R}^+$  such that

$$\sup_{\xi \in E} \int_{\hat{t}}^{+\infty} e^{-\lambda t} T_t f(\xi) dt < \varepsilon.$$

Hence, to prove that  $R(\lambda)f \in UC_b(E)$ , it suffices to show that, for every  $\hat{t} \in \mathbb{R}^+$ ,

$$\int_0^{\hat{t}} e^{-\lambda t} T_t f dt \in UC_b(E). \quad (5.14)$$

Let us define the set

$$C := \left\{ g \in C_b(E) : \sup_{\xi, \xi' \in E} |g(\xi) - g(\xi')| \leq w(|\xi - \xi'|_E) \right\},$$

where  $w$  is as in (5.9). Clearly  $C$  is a subset of  $UC_b(E)$ , it is convex, it contains the origin, and is closed in  $(C_b(E), \tau_{\mathcal{K}})$ . By (5.9),  $\{e^{-\lambda' t} T_t f\}_{t \in [0, \hat{t}]} \subset C$ . Hence, we conclude by Lemma 5.8 that

$$\int_0^{\hat{t}} e^{-\lambda t} T_t f dt \in \frac{1}{\lambda - \lambda'} C \quad \forall \lambda > \lambda',$$

which shows (5.14), concluding the proof of the first part of the proposition.

Now let  $T$  be a  $C_0$ -sequentially locally equicontinuous on  $(UC_b(E), \tau_{\mathcal{K}})$  satisfying (P3). We only need to show that  $T$  verifies (P2), (P4), and (P5). Now, (P2) follows from Proposition 5.3 and Proposition 3.6, whereas (P4) comes once again by Proposition 5.4(i). Finally, (P5) is due to Proposition 5.4(i) and to sequential local equicontinuity of  $T$ .  $\blacksquare$

### 5.3 Relationship with $\pi$ -semigroups

In this subsection we provide a connection between the notion of  $\pi$ -semigroups in  $UC_b(E)$  introduced in [31] and bounded  $C_0$ -sequentially continuous semigroups (see Definition 5.11) in the space  $(UC_b(E), \sigma(UC_b(E), \mathbf{ca}(E)))$  <sup>(5)</sup>. We recall that the assumption  $E$  Banach space was standing only in the latter subsection, and that in the present subsection we restore the assumption that  $E$  is a generic metric space. We start by recalling the definition of  $\pi$ -semigroup in  $UC_b(E)$ .

**Definition 5.10.** A  $\pi$ -semigroup on  $UC_b(E)$  is a family  $T = \{T_t\}_{t \in \mathbb{R}^+}$  of bounded linear operators on  $(UC_b(E), |\cdot|_\infty)$  satisfying the following conditions.

(P1)  $T_0 = I$  and  $T_t T_s = T_{t+s}$  for  $t, s \in \mathbb{R}^+$ .

(P2) There exist  $M \geq 1$ ,  $\alpha \in \mathbb{R}$  such that  $|T_t f|_\infty \leq M e^{\alpha t} |f|_\infty$  for every  $t \in \mathbb{R}^+$ ,  $f \in UC_b(E)$ .

(P3) For each  $\xi \in E$  and  $f \in UC_b(E)$ , the map  $\mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $t \mapsto T_t f(\xi)$  is continuous.

(P4) If a sequence  $\{f_n\}_{n \in \mathbb{N}} \subset UC_b(E)$  is such that

$$\sup_{n \in \mathbb{N}} |f_n|_\infty < +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} f_n = f \quad \text{pointwise,}$$

then, for every  $t \in \mathbb{R}^+$ ,

$$\lim_{n \rightarrow +\infty} T_t f_n = T_t f \quad \text{pointwise.}$$

**Definition 5.11.** Let  $X$  be a Hausdorff locally convex space. Let  $T := \{T_t\}_{t \in \mathbb{R}^+} \subset \mathcal{L}_0(X)$  be a family of sequentially continuous linear operators. We say that  $T$  is a bounded  $C_0$ -sequentially continuous semigroup if

(i)  $T_0 = I$  and  $T_{t+s} = T_t T_s$  for all  $t, s \in \mathbb{R}^+$ ;

(ii) for each  $x \in X$ , the map  $\mathbb{R}^+ \rightarrow X$ ,  $t \mapsto T_t x$ , is continuous and bounded.

By recalling Proposition 3.5, we see that Definition 3.3 is stronger than Definition 5.11. Let  $T$  be a bounded  $C_0$ -sequentially continuous semigroup on  $X$  and let us assume that, for every  $x \in X$ , the Riemann integral

$$R(\lambda)x := \int_0^{+\infty} e^{-\lambda t} T_t x dt, \tag{5.15}$$

(which exists in the completion of  $X$ , by Definition 5.11(ii)) belongs to  $X$  (this happens, for example, if  $X$  is sequentially complete). Then, by an inspection of the proofs one easily sees that the following results still hold: Proposition 3.6(ii); Proposition A.3; Proposition 3.9(ii); Proposition 3.10; Proposition 3.11; Theorem 3.14, except for the conclusion  $(\lambda - A)^{-1} \in \mathcal{L}_0(X)$ ; Corollary 3.16.

To summarize, if the Laplace transform (5.15) of a bounded  $C_0$ -sequentially continuous semigroup is well-defined, then the domain  $\mathcal{D}(A)$  of the generator  $A$  is sequentially dense in  $X$  and  $\lambda - A$  is one-one and onto for every  $\lambda > 0$ .

We outline, however, that without the sequential local equicontinuity of  $T$  the proof of Lemma A.4 does not work and consequently the proof of Theorem 3.12 does not work.

**Proposition 5.12.**  $T$  is a  $\pi$ -semigroup in  $UC_b(E)$  if and only if  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$  is a bounded  $C_0$ -sequentially continuous semigroup in  $(UC_b(E), \sigma(UC_b(E), \mathbf{ca}(E)))$ .

<sup>5</sup>Also in this case, in order to avoid misunderstanding, we stress that [31] uses the notation  $C_b(E)$  to denote the space of uniformly continuous bounded functions on  $E$ , i.e., our space  $UC_b(E)$ . We also notice that in [31] the metric space  $E$  is assumed to be separable, but, for our discussion, this is not needed.

*Proof.* Let us denote  $\sigma := \sigma(UC_b(E), \mathbf{ca}(E))$ . Let  $T$  be a  $\pi$ -semigroup in  $UC_b(E)$ . By Definition 5.10(P2),(P4) and Proposition 5.5(i), we have  $\{e^{-at}T_t\}_{t \in \mathbb{R}^+} \subset \mathcal{L}_0((UC_b(E), \sigma))$ . By Definition 5.10(P2),(P3) and by Proposition 5.5(i), the map  $\mathbb{R}^+ \rightarrow (UC_b(E), \sigma)$ ,  $t \mapsto e^{-at}T_t f$  is continuous for every  $f \in UC_b(E)$ . Moreover, by Definition 5.10(P2) and by Proposition 5.3, it is also bounded. This shows that  $\{e^{-at}T_t\}_{t \in \mathbb{R}^+}$  is a bounded  $C_0$ -sequentially continuous semigroup in  $(UC_b(E), \sigma)$ .

Conversely, let  $\{e^{-at}T_t\}_{t \in \mathbb{R}^+}$  be a bounded  $C_0$ -sequentially continuous semigroup in  $(UC_b(E), \sigma)$ . By Proposition 5.3, for every  $f \in UC_b(E)$  the family  $\{e^{-at}T_t f\}_{t \in \mathbb{R}^+}$  is bounded in  $(UC_b(E), |\cdot|_\infty)$ . By the Banach-Steinhaus theorem we conclude that there exists  $M > 0$  such that

$$|e^{-at}T_t|_{L((UC_b(E), |\cdot|_\infty))} \leq M \quad \forall t \in \mathbb{R}^+,$$

which provides  $T \subset L((UC_b(E), |\cdot|_\infty))$  and (P2). Then, (P3) is implied by the fact that the map  $\mathbb{R}^+ \rightarrow (UC_b(E), \sigma)$ ,  $t \mapsto e^{-at}T_t f$ , is continuous and that Dirac measures are contained in  $\sigma$ . Finally, (P4) is due to the assumption  $\{e^{-at}T_t\}_{t \in \mathbb{R}^+} \subset \mathcal{L}_0((UC_b(E), \sigma))$  and to Proposition 5.5(i).  $\blacksquare$

**Remark 5.13.** *As observed, if the Laplace transform (5.15) of a bounded  $C_0$ -sequentially continuous semigroup in  $(UC_b(E), \sigma(UC_b(E), \mathbf{ca}(E)))$  is well-defined, several results stated for  $C_0$ -sequentially equicontinuous semigroups still hold. Nevertheless, some other important results, as the generation theorem, or the fact that two semigroups with the same generator are equal, cannot be proved for bounded  $C_0$ -sequentially continuous semigroups within the approach of the previous sections. Due to Proposition 5.12, this is reflected in the fact that, as far as we know, such results are not available in the literature for  $\pi$ -semigroups.*

#### 5.4 Relationship with locally equicontinuous semigroups with respect to the mixed topology

When  $E$  is a separable Hilbert space, in [17] the so called *mixed topology* (introduced in [34]) is employed in the space  $C_b(E)$  to frame a class of Markov transition semigroups within the theory of  $C_0$ -locally equicontinuous semigroups. The same topology, but in the more general case of  $E$  separable Banach space, is used in [18] to deal with Markov transition semigroups associated with the Ornstein-Uhlenbeck process in Banach spaces.

In this subsection, we assume that  $E$  is a separable Banach space and we briefly precise what is the relation between the mixed topology and  $\tau_{\mathcal{X}}$  in the space  $C_b(E)$ , and between  $C_0$ -locally equicontinuous semigroups with respect to the mixed topology and  $C_0$ -sequentially locally equicontinuous semigroups with respect to  $\tau_{\mathcal{X}}$ .

The mixed topology on  $C_b(E)$ , denoted by  $\tau_{\mathcal{M}}$ , can be defined by seminorms as follows. Let  $\mathbf{K} := \{K_n\}_{n \in \mathbb{N}}$  be a sequence of compact subsets of  $E$ , and let  $\mathbf{a} := \{a_n\}_{n \in \mathbb{N}}$  be a sequence of strictly positive real numbers such that  $a_n \rightarrow 0$ . Define

$$p_{\mathbf{K}, \mathbf{a}}(f) = \sup_{n \in \mathbb{N}} \{a_n [f]_{K_n}\} \quad \forall f \in C_b(E). \quad (5.16)$$

Then  $p_{\mathbf{K}, \mathbf{a}}$  is a seminorm and  $\tau_{\mathcal{M}}$  is defined as the locally convex topology induced by the family of seminorms  $p_{\mathbf{K}, \mathbf{a}}$ , when  $\mathbf{K}$  ranges on the set of sequences of compact subsets of  $E$  and  $\mathbf{a}$  ranges on the set of sequences of strictly positive real numbers converging to 0.

It can be proved (see [33, Theorem 2.4]), that  $\tau_{\mathcal{M}}$  is the finest locally convex topology on  $C_b(E)$  such that a net  $\{f_i\}_{i \in \mathcal{I}}$  is bounded in the uniform norm and converges to  $f$  in  $\tau_{\mathcal{M}}$  if and only if it is  $\mathcal{X}$ -convergent, i.e., if and only if (5.8) is verified.

By Proposition 5.4(i), every sequence convergent in  $\tau_{\mathcal{X}}$  is bounded and convergent uniformly on compact sets, and then it is convergent in  $\tau_{\mathcal{M}}$ . In [16, Prop. 4,23] it is also shown that  $\tau_{\mathcal{X}} \subset \tau_{\mathcal{M}}$ . So, we immediately obtain the following

**Proposition 5.14.** *A semigroup  $T$  is  $C_0$ -sequentially (locally) equicontinuous in  $(C_b(E), \tau_{\mathcal{M}})$  if and only if it is  $C_0$ -sequentially (locally) equicontinuous in  $(C_b(E), \tau_{\mathcal{X}})$ .*

## 6 Application to Markov transition semigroups

In this section we apply our results to Markov transition semigroups in spaces of bounded and continuous functions.

### 6.1 Transition semigroups in $(C_b(E), \tau_{\mathcal{X}})$

Let  $\mu := \{\mu_t(\xi, \cdot)\}_{\substack{t \in \mathbb{R}^+ \\ \xi \in E}}$  be a subset of  $\mathbf{ca}^+(E)$  and consider the following assumptions.

**Assumption 6.1.** *The family  $\mu := \{\mu_t(\xi, \cdot)\}_{\substack{t \in \mathbb{R}^+ \\ \xi \in E}} \subset \mathbf{ca}^+(E)$  has the following properties.*

(i) *The family  $\mu$  is bounded in  $\mathbf{ca}^+(E)$  and  $p_0(\xi, \Gamma) = \mathbf{1}_{\Gamma}(\xi)$  for every  $\xi \in E$  and every  $\Gamma \in \mathcal{E}$ .*

(ii) *For every  $f \in C_b(E)$  and  $t \in \mathbb{R}^+$ , the map*

$$E \rightarrow \mathbb{R}, \xi \mapsto \int_E f(\xi') \mu_t(\xi, d\xi') \quad (6.1)$$

*is continuous.*

(iii) *For every  $f \in C_b(E)$ , every  $t, s \in \mathbb{R}^+$ , and every  $\xi \in E$ ,*

$$\int_E f(\xi') \mu_{t+s}(\xi, d\xi') = \int_E \left( \int_E f(\xi'') \mu_t(\xi', d\xi'') \right) \mu_s(\xi, d\xi').$$

(iv) *For every  $\hat{t} > 0$  and every compact  $K \subset E$ , the family  $\{\mu_t(\xi, \cdot) : t \in [0, \hat{t}], \xi \in K\}$  is tight, i.e., for every  $\varepsilon > 0$ , there exists a compact set  $K_0 \subset E$  such that*

$$\mu_t(\xi, K_0) > \mu_t(\xi, E) - \varepsilon \quad \forall t \in [0, \hat{t}], \forall \xi \in K.$$

(v) *For every  $r > 0$  and every non-empty compact  $K \subset E$ ,*

$$\limsup_{t \rightarrow 0^+} \sup_{\xi \in K} |\mu_t(\xi, B(\xi, r)) - 1| = 0, \quad (6.2)$$

*where  $B(\xi, r)$  denotes the open ball  $B(\xi, r) := \{\xi' \in E : d(\xi, \xi') < r\}$ .*

We observe that in Assumption 6.1 it is not required that  $p_t(\xi, E) = 1$  for every  $t \in \mathbb{R}^+$ ,  $\xi \in E$ , i.e., the family  $\mu$  is not necessarily a probability kernel in  $(E, \mathcal{E})$ . Assumptions 6.1(ii),(iii) can be rephrased by saying that

$$T_t : C_b(E) \rightarrow C_b(E), f \mapsto \int_E f(\xi) \mu_t(\cdot, d\xi)$$

is well defined for all  $t \in \mathbb{R}^+$  and  $T := \{T_t\}_{t \in \mathbb{R}^+}$  is a transition semigroup in  $C_b(E)$ . If  $\mu$  is a probability kernel, then  $T$  is a Markov transition semigroup.

**Proposition 6.2.** *Let Assumption 6.1 holds and let  $T := \{T_t\}_{t \in \mathbb{R}^+}$  be defined as in (6.1). Then  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $(C_b(E), \tau_{\mathcal{X}})$ . Moreover, for every  $\alpha > 0$ , the normalized semigroup  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup on  $(C_b(E), \tau_{\mathcal{X}})$  satisfying Assumption 3.7.*

*Proof.* Assumptions 6.1(i),(ii),(iii) imply that  $T$  maps  $C_b(E)$  into itself and that it is a semigroup. We show that the  $C_0$ -property holds, i.e.,  $\lim_{t \rightarrow 0^+} T_t f = f$  in  $(C_b(E), \tau_{\mathcal{X}})$  for every  $f \in C_b(E)$ . Let  $M := \sup_{\substack{t \in \mathbb{R}^+ \\ \xi \in E}} |\mu_t(\xi, E)|$ . By Assumption 6.1(i),  $M < +\infty$  and

$$|T_t f|_{\infty} \leq M |f|_{\infty} \quad \forall f \in C_b(E), \quad \forall t \in \mathbb{R}^+. \quad (6.3)$$

Let  $f \in C_b(E)$ . By (6.3) and by Proposition 5.4(i), in order to show that  $\lim_{t \rightarrow 0^+} T_t f = f$  in  $(C_b(E), \tau_{\mathcal{X}})$ , it is sufficient to show that  $\lim_{t \rightarrow 0^+} [T_t f - f]_K = 0$ , for every  $K \subset E$  non-empty compact. Let  $K \subset E$  be such a set. We claim that

$$\lim_{t \rightarrow 0^+} \sup_{\xi \in K} |\mu_t(\xi, E) - 1| = 0. \quad (6.4)$$

Indeed, let  $\varepsilon$  and  $K_0$  as in Assumption 6.1(iv), when  $\hat{t} = 1$ , and let  $r := \sup_{(\xi, \xi') \in K \times K_0} d(\xi, \xi') + 1$ . Then  $K_0 \subset B(\xi, r)$  for every  $\xi \in K$ . For  $t \in [0, 1]$  and  $\xi \in K$ , we have

$$\begin{aligned} |\mu_t(\xi, E) - 1| &\leq |\mu_t(\xi, E \setminus B(\xi, r))| + |\mu_t(\xi, B(\xi, r)) - 1| \\ &\leq |\mu_t(\xi, E \setminus K_0)| + |\mu_t(\xi, B(\xi, r)) - 1| \\ &\leq \varepsilon + |\mu_t(\xi, B(\xi, r)) - 1|. \end{aligned}$$

By taking the supremum over  $x \in K$ , by passing to the limit as  $t \rightarrow 0^+$ , by using (6.2), and by arbitrariness of  $\varepsilon$ , we obtain (6.4). In particular, (6.4) implies

$$\lim_{t \rightarrow 0^+} \sup_{\xi \in K} |f(\xi) - \mu_t(\xi, E)f(\xi)| = 0, \quad (6.5)$$

and then  $T_t f \rightarrow f$  in  $\tau_{\mathcal{X}}$  as  $t \rightarrow 0^+$  if and only if

$$\lim_{t \rightarrow 0^+} \sup_{\xi \in K} |T_t f(\xi) - \mu_t(\xi, E)f(\xi)| = 0. \quad (6.6)$$

Again, let  $\varepsilon > 0$  and  $K_0$  be as in Assumption 6.1(iv), when  $\hat{t} = 1$ . Let  $w$  be a modulus of continuity for  $f|_{K_0}$ . For  $\delta > 0$ ,  $t \in [0, 1]$ , and  $\xi \in K$ , we write

$$\begin{aligned} |T_t f(\xi) - \mu_t(\xi, E)f(\xi)| &\leq \int_E |f(\xi') - f(\xi)| \mu_t(\xi, d\xi') = \int_{K_0 \cap B(\xi, \delta)} |f(\xi') - f(\xi)| \mu_t(\xi, d\xi') \\ &\quad + \int_{K_0 \cap B(\xi, \delta)^c} |f(\xi') - f(\xi)| \mu_t(\xi, d\xi') + \int_{K_0^c} |f(\xi') - f(\xi)| \mu_t(\xi, d\xi') \\ &\leq w(\delta) + 2|f|_{\infty} (\mu_t(\xi, B(\xi, \delta)^c) + \varepsilon). \end{aligned}$$

We then obtain

$$\sup_{\xi \in K} |T_t f(\xi) - \mu_t(\xi, E)f(\xi)| \leq w(\delta) + 2|f|_{\infty} \left( \sup_{\xi \in K} \mu_t(\xi, B(\xi, \delta)^c) + \varepsilon \right) \quad \forall \delta > 0, \quad \forall t \in [0, 1], \quad \forall \xi \in K.$$

By passing to the limit as  $t \rightarrow 0^+$ , by (6.2), by (6.4), and by arbitrariness of  $\delta$  and  $\varepsilon$ , we obtain (6.6).

We now show that  $\{T_t\}_{t \in [0, \hat{t}]}$  is sequentially equicontinuous for every  $\hat{t} > 0$ . Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence converging to 0 in  $(C_b(E), \tau_{\mathcal{X}})$  and let  $\hat{t} > 0$ . By Proposition 5.4(i),  $\{|f_n|_{\infty}\}_{n \in \mathbb{N}}$  is bounded by some  $b > 0$ . Then, by (6.3),  $\{T_t f_n\}_{t \in \mathbb{R}^+, n \in \mathbb{N}}$  is bounded. To show that  $T_t f_n \rightarrow 0$  in  $(C_b(E), \tau_{\mathcal{X}})$ , uniformly for  $t \in [0, \hat{t}]$ , it is then sufficient to show that

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, \hat{t}]} [T_t f_n]_K = 0 \quad \forall K \subset E \text{ non-empty compact.}$$



Let  $\varepsilon > 0$  and  $K_0$  be as in Assumption 6.1(iv), when  $\hat{t} = 1$ . Then, for  $t \in [0, \hat{t}]$ ,  $\xi \in K$ ,  $n \in \mathbb{N}$ , we have

$$|T_t f_n(\xi)| \leq \int_{K_0} |f_n(\xi')| \mu_t(\xi, d\xi') + \int_{K_0^c} |f_n(\xi')| \mu_t(\xi, d\xi') \leq M[f_n]_{K_0} + b\varepsilon.$$

Since  $[f_n]_{K_0} \rightarrow 0$  as  $n \rightarrow +\infty$ , by arbitrariness of  $\varepsilon$  we conclude  $\sup_{t \in [0, \hat{t}]} [T_t f_n]_K \rightarrow 0$  as  $n \rightarrow +\infty$ . This concludes the proof that  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $(C_b(E), \tau_{\mathcal{X}})$ . Next, by Proposition 5.3 and by (6.3), we can apply Proposition 3.6 and obtain that, for every  $\alpha > 0$ ,  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$  is  $C_0$ -sequentially locally equicontinuous semigroup on  $(C_b(E), \tau_{\mathcal{X}})$ . Finally, by Remark 3.8 and Proposition 5.6(ii), we conclude that Assumption 3.7 holds true for  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$ .  $\blacksquare$

## 6.2 Markov transition semigroups associated to stochastic differential equations

Propositions 6.2 has a straightforward application to transition functions associated to mild solutions of stochastic differential equations in Hilbert spaces. Let  $(U, |\cdot|_U)$ ,  $(H, |\cdot|_H)$  be separable Hilbert spaces, let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$  be a complete filtered probability space, let  $Q$  be a positive self-adjoint operator, and let  $W^Q$  be a  $U$ -valued  $Q$ -Wiener process defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$  (see [8, Ch. 4]). Denote by  $L_2(U_0, H)$  the space of Hilbert-Schmidt operators from  $U_0 := Q^{1/2}(U)$ <sup>(6)</sup> into  $H$ , let  $A$  be the generator of a strongly continuous semigroup  $\{S_A(t)\}_{t \in \mathbb{R}^+}$  in  $(H, |\cdot|_H)$ , and let  $F: H \rightarrow H$ ,  $B: H \rightarrow L_2(U_0, H)$ . Then, under suitable assumptions on the coefficients  $F$  and  $B$  (e.g., [8, p. 187, Hypotehsis 7.1]), for every  $\xi \in H$ , the SDE in the space  $H$

$$\begin{cases} dX(t) = AX(t) + F(X(t))dt + B(X(t))dW^Q(t) & t \in (0, T] \\ X(0) = \xi, \end{cases} \quad (6.7)$$

admits a unique (up to undistinguishability) mild solution  $X(\cdot, \xi)$  with continuous trajectories (see [8, p. 188, Theorem 7.2]), that is, there exists a unique  $H$ -valued process  $X(\cdot, \xi)$  with continuous trajectories satisfying the integral equation

$$X(t, \xi) = S_A(t)\xi + \int_0^t S_A(t-s)F(X(s, \xi))ds + \int_0^t S_A(t-s)B(X(s, \xi))dW^Q(s) \quad \forall t \in \mathbb{R}^+. \quad (6.8)$$

By standard estimates (see, e.g., [8, p. 188, Theorem 7.2]<sup>(7)</sup>), for every  $p \geq 2$  we have, for some  $K_p > 0$  and  $\hat{\alpha}_p \in \mathbb{R}$ ,

$$\mathbb{E}[|X(t, \xi)|_H^p] \leq K_p e^{\hat{\alpha}_p t} (1 + |\xi|_H^p) \quad \forall (t, \xi) \in \mathbb{R}^+ \times H. \quad (6.9)$$

Moreover, by [8, p. 235, Theorem 9.1],

$$(t, \xi) \mapsto X(t, \xi) \text{ is stochastically continuous.} \quad (6.10)$$

**Proposition 6.3.** *Let [8, Hypothesis 7.1] hold and let  $X(\cdot, \xi)$  be the mild solution to (6.7). Define*

$$T_t f(\xi) := \mathbb{E}[f(X(t, \xi))] \quad \forall f \in C_b(H) \forall \xi \in H, \forall t \in \mathbb{R}^+. \quad (6.11)$$

*Then  $T := \{T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially locally equicontinuous semigroup in  $(C_b(H), \tau_{\mathcal{X}})$ . Moreover,  $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially equicontinuous semigroup in  $(C_b(H), \tau_{\mathcal{X}})$  for every  $\alpha > 0$ .*

<sup>6</sup>The scalar product on  $U_0$  is defined by  $\langle u, v \rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v \rangle_H$ .

<sup>7</sup>The constant in that estimate can be taken exponential in time, because the SDE is autonomous.



*Proof.* Define

$$\mu_t(\xi, \Gamma) := \mathbb{P}(X(t, \xi) \in \Gamma) \quad \forall t \in \mathbb{R}^+, \forall \xi \in H, \forall \Gamma \in \mathcal{B}(H). \quad (6.12)$$

We show that we can apply Proposition 6.2 with the family  $\boldsymbol{\mu} := \{\mu_t(\xi, \cdot)\}_{\substack{t \in \mathbb{R}^+ \\ \xi \in H}}$  given by (6.12).

The condition of Assumption 6.1(i) is clearly verified. The condition of Assumption 6.1(ii) is consequence of (6.10). The condition of Assumption 6.1(iii) is verified by [8, p. 249, Corollaries 9.15 and 9.16].

Now we verify the condition of Assumption 6.1(iv). Let  $\hat{t} > 0$  and let  $K \subset E$  compact. By (6.10) the map

$$\mathbb{R}^+ \times H \rightarrow (\mathbf{ca}(H), \sigma(\mathbf{ca}(H), C_b(H))), \quad (t, \xi) \mapsto \mu(\xi, \cdot)$$

is continuous. Then the family of probability measures  $\{\mu_t(\xi, \cdot)\}_{(t, \xi) \in [0, \hat{t}] \times H}$  is  $\sigma(\mathbf{ca}(H), C_b(H))$ -compact. Hence, by [3, p. 519, Theorem 15.22], it is tight.

We finally verify the condition of Assumption 6.1(v). Let  $r > 0$ , let  $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$  be a sequence converging to 0, and let  $\{\xi_n\}_{n \in \mathbb{N}}$  be sequence converging to  $\xi$  in  $H$ . By (6.10) and recalling that  $X(0, \xi) = \xi$ , we get

$$\lim_{n \rightarrow +\infty} \mu_{t_n}(\xi_n, B(\xi_n, r)) = \lim_{n \rightarrow +\infty} \mathbb{P}(|\xi_n - X(t_n, \xi_n)|_H < r) = 0.$$

By arbitrariness of the sequences  $\{t_n\}_{n \in \mathbb{N}}$ ,  $\{\xi_n\}_{n \in \mathbb{N}}$  and of  $\xi$ , this implies the condition of Assumption 6.1(v).  $\blacksquare$

We provide now a very basic example of a  $C_0$ -sequentially equicontinuous semigroup which is not  $C_0$ -equicontinuous.

**Example 6.4.** Let  $H$  be a (non-trivial) separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . Let  $Q \in L(H)$  be a positive self-adjoint trace-class operator and let  $W^Q$  be a  $Q$ -Wiener process in  $H$  on some filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$  (see [8, Ch. 4]). Let  $T = \{T_t\}_{t \in \mathbb{R}^+}$  be defined by

$$T_t f(\xi) := \mathbb{E}[f(\xi + W_t^Q)] = \int_H f(\xi') \mu_t(\xi, d\xi') \quad \forall f \in C_b(H), \forall \xi \in H, \forall t \in \mathbb{R}^+,$$

where  $\mu_t(\xi, \cdot)$  denotes the law of  $\xi + W_t^Q$ . Then, by Proposition 6.3,  $T$  is a  $C_0$ -sequentially locally equicontinuous semigroup in  $(C_b(H), \tau_{\mathcal{X}})$ . We claim that  $T$  is not locally equicontinuous. Indeed, if  $T$  was locally equicontinuous, for any fixed  $\hat{t} > 0$ , there should exist  $L > 0$ , a compact set  $K \subset H$ , and  $\eta_1, \dots, \eta_n \in \mathbf{ca}(H)$  such that

$$\sup_{t \in [0, \hat{t}]} |T_t f(0)| \leq L \left( [f]_K + \sum_{i=1}^n \left| \int_H f d\eta_i \right| \right) \quad \forall f \in C_b(H). \quad (6.13)$$

Let  $v \in H \setminus \{0\}$  and let  $a := \max_{h \in K} |\langle v, h \rangle|$ . Then, denoting by  $\lambda_t$  the pushforward measure of  $\mu_t(0, \cdot)$  through the application  $\langle v, \cdot \rangle$  (i.e., the law of the real-valued random variable  $\langle v, W_t^Q \rangle$ ), and by  $\nu_i$ ,  $i = 1, \dots, n$ , the pushforward measure of  $\eta_i$  through the same application, inequality (6.13) provides, in particular,

$$\sup_{t \in [0, \hat{t}]} \left| \int_a^{+\infty} g d\lambda_t \right| \leq L \sum_{i=1}^n \left| \int_a^{+\infty} g d\nu_i \right|, \quad \forall g \in C_{0,b}([a, +\infty)), \quad (6.14)$$

where  $C_{0,b}([a, +\infty))$  is the space of bounded continuous functions  $f$  on  $[a, +\infty)$  such that  $f(a) = 0$ . Then, by [32, p. 63, Lemma 3.9], every  $\lambda_t$  restricted to  $(a, +\infty)$  must be a linear combination of the measures  $\nu_1, \dots, \nu_n$  restricted to  $(a, +\infty)$ . In particular, choosing an arbitrary sequence  $0 < t_1 < \dots < t_n < t_{n+1} \leq \hat{t}$ , the family  $\{\lambda_{t_i}|_{(a, +\infty)}\}_{i=1, \dots, n+1}$  is linearly dependent. This is not possible, as they are restrictions of nondegenerate Gaussian measures having all different variances.

## A Appendix

### FAMILIES OF SEQUENTIALLY EQUICONTINUOUS FUNCTIONS.

**Proposition A.1.** For  $n \in \mathbb{N}$  and  $i = 1, \dots, n$ , let  $\mathcal{F}^{(i)} = \{F_{l_i}^{(i)} : X \rightarrow X\}_{l_i \in \mathcal{I}_i}$  be a family of sequentially equicontinuous linear operators. Then the following hold.

- (i) The family  $\mathcal{F} = \{F_{l_1}^{(1)} F_{l_2}^{(2)} \dots F_{l_n}^{(n)} : X \rightarrow X\}_{l_1 \in \mathcal{I}_1, \dots, l_n \in \mathcal{I}_n}$  is sequentially equicontinuous.
- (ii) The family  $\mathcal{F} = \{F_{l_1}^{(1)} + F_{l_2}^{(2)} + \dots + F_{l_n}^{(n)} : X \rightarrow Y\}_{l_1 \in \mathcal{I}_1, \dots, l_n \in \mathcal{I}_n}$  is sequentially equicontinuous.
- (iii) The family  $\mathcal{F}$  is equibounded, i.e., if  $D$  is a bounded subset of  $X$ , then  $\{F_{l_i}^{(i)} x\}_{\substack{l_i \in \mathcal{I}_i, \\ i=1, \dots, n \\ x \in D}}$  is bounded in  $X$ .

*Proof.* See [16, Prop. 3.10]. ■

The following proposition clarifies when the notion of sequential equicontinuity for a family of linear operators is equivalent to the notion of equicontinuity.

**Proposition A.2.** Let  $\mathcal{F} := \{F_l : X \rightarrow X\}_{l \in \mathcal{I}}$  be a family of linear operators. If  $\mathcal{F} \subset L(X)$  is equicontinuous, then  $\mathcal{F} \subset \mathcal{L}_0(X)$  and  $\mathcal{F}$  is sequentially equicontinuous.

Conversely, if  $X$  is metrizable and  $\mathcal{F} \subset \mathcal{L}_0(X)$  is sequentially equicontinuous, then  $\mathcal{F} \subset L(X)$  and  $\mathcal{F}$  is equicontinuous.

*Proof.* The first statement being obvious, we will only show the second one.

Assume that  $\mathcal{F}$  is sequentially equicontinuous and that  $X$  is metrizable. Since  $X$  is metrizable, we have  $\mathcal{L}_0(X) = L(X)$ . Assume, by contradiction, that  $\mathcal{F}$  is not equicontinuous. Since the topology of  $X$  is induced by a countable family of seminorms  $\{p_n\}_{n \in \mathbb{N}}$  (see [28, Th. 3.35, p. 77]), it then follows that there exist a continuous seminorm  $q$  on  $X$  and sequences  $\{x_n\}_{n \in \mathbb{N}} \subset X$ ,  $\{l_n\}_{n \in \mathbb{N}} \subset \mathcal{I}$  such that

$$\sup_{k=1, \dots, n} p_k(x_n) < \frac{1}{n}, \quad q(F_{l_n} x_n) > 1, \quad \forall n \in \mathbb{N}.$$

But then

$$\lim_{n \rightarrow +\infty} x_n = 0 \quad \text{and} \quad \liminf_{n \rightarrow +\infty} \left( \sup_{l \in \mathcal{I}} q(F_l x_n) \right) \geq \liminf_{n \rightarrow +\infty} q(F_{l_n} x_n) \geq 1,$$

which implies that  $\mathcal{F}$  is not sequentially equicontinuous, getting a contradiction and concluding the proof. ■

### TWO TECHNICAL RESULTS.

**Proposition A.3.** Let Assumption 3.7 holds. Let  $T \subset \mathcal{L}_0(X)$  be a  $C_0$ -sequentially equicontinuous semigroup. If  $L \in \mathcal{L}_0(X, Y)$ , then  $\mathbb{R}^+ \rightarrow Y, x \mapsto LT_t x$  is continuous and bounded. Moreover, for every  $x \in X$ , every  $a \geq 0$ , and every  $\lambda > 0$ ,

$$L \int_0^a e^{-\lambda t} T_t x dt = \int_0^a e^{-\lambda t} L T_t x dt \quad \text{and} \quad L \int_0^{+\infty} e^{-\lambda t} T_t x dt = \int_0^{+\infty} e^{-\lambda t} L T_t dt, \quad (\text{A.1})$$

where the Riemann integrals on the right-hand side of the equalities exist in  $Y$ .

*Proof.* Continuity of the map  $\mathbb{R}^+ \rightarrow X$ ,  $t \mapsto LT_t x$ , follows from sequential continuity of  $L$  and from Proposition 3.5(i). By Proposition 3.5(ii), we have that  $\{T_t x\}_{t \in \mathbb{R}^+}$  is bounded, for all  $x \in X$ . From Proposition 3.2(i), it then follows that  $\{LT_t x\}_{t \in \mathbb{R}^+}$  is bounded.

Let  $\{\pi^k\}_{k \in \mathbb{N}}$  be a sequence of partitions of  $[0, a] \subset \mathbb{R}^+$  of the form  $\pi^k := \{0 = t_0^k < t_1^k < \dots < t_{n_k}^k = a\}$ , with  $|\pi^k| \rightarrow 0$  as  $k \rightarrow +\infty$ , where  $|\pi^k| := \sup\{|t_{i+1}^k - t_i^k| : i = 0, \dots, n_k - 1\}$ . Then, by recalling Assumption 3.7 and by continuity of  $\mathbb{R}^+ \rightarrow X$ ,  $t \mapsto T_t x$ , we have in  $Y$

$$\int_0^a e^{-\lambda t} T_t x dt = \lim_{k \rightarrow +\infty} \sum_{i=0}^{n_k-1} e^{-\lambda t_i^k} T_{t_i^k} x (t_{i+1}^k - t_i^k).$$

By sequential continuity of  $L$  we then have

$$L \int_0^a e^{-\lambda t} T_t x dt = \lim_{k \rightarrow +\infty} \sum_{i=0}^{n_k-1} e^{-\lambda t_i^k} L T_{t_i^k} x (t_{i+1}^k - t_i^k). \quad (\text{A.2})$$

Since  $\mathbb{R}^+ \rightarrow X$ ,  $t \mapsto LT_t x$  is continuous, equality (A.2) entails that  $\mathbb{R}^+ \rightarrow X$ ,  $t \mapsto e^{-\lambda t} LT_t x$  is Riemann integrable and that the first equality of (A.1) holds true.

The second equality of (A.1) follows from the first one and from sequential continuity of  $L$ , by letting  $a \rightarrow +\infty$ .  $\blacksquare$

**Lemma A.4.** *Let  $0 \leq a < b$ ,  $f, g: (a, b) \rightarrow \mathcal{L}(X)$ ,  $t_0 \in (a, b)$ , and  $x \in X$ . Assume that*

- (i) *the family  $\{f(t)\}_{t \in [a', b']}$  is sequentially equicontinuous, for every  $a < a' < b' < b$ ;*
- (ii)  *$g(\cdot)x: (a, b) \rightarrow X$  is differentiable at  $t_0$ ;*
- (iii)  *$f(\cdot)g(t_0)x: (a, b) \rightarrow X$  is differentiable at  $t_0$ .*

*Then there exists the derivative of  $f(\cdot)g(\cdot)x: (a, b) \rightarrow X$  at  $t = t_0$  and*

$$\frac{d}{dt} [f(t)g(t)x] \Big|_{t=t_0} = \frac{d}{dt} [f(t)g(t_0)x] \Big|_{t=t_0} + f(t_0) \frac{d}{dt} [g(t)x] \Big|_{t=t_0}.$$

*Proof.* For  $h \in \mathbb{R} \setminus \{0\}$  such that  $[t_0 - |h|, t_0 + |h|] \subset (a, b)$ , write

$$\begin{aligned} f(t_0 + h)g(t_0 + h)x - f(t_0)g(t_0)x &= f(t_0 + h) \left( g(t_0 + h) - g(t_0) - h \frac{d}{dt} [g(t)x] \Big|_{t=t_0} \right) \\ &\quad + h f(t_0 + h) \frac{d}{dt} [g(t)x] \Big|_{t=t_0} + (f(t_0 + h) - f(t_0))g(t_0)x \\ &=: I_1(h) + I_2(h) + I_3(h). \end{aligned}$$

Letting  $h \rightarrow 0$ , we have  $h^{-1}I_2(h) \rightarrow f(t_0) \frac{d}{dt} [g(t)x] \Big|_{t=t_0} x$  and  $h^{-1}I_3(h) \rightarrow \frac{d}{dt} [f(t)g(t_0)x] \Big|_{t=t_0}$ . Moreover,

$$p(h^{-1}I_1(h)) \leq \sup_{s \in [t_0 - |h|, t_0 + |h|]} p \left( f(s) \left( \frac{g(t_0 + h) - g(t_0)}{h} - \frac{d}{dt} [g(t)x] \Big|_{t=t_0} \right) x \right), \quad \forall p \in \mathcal{P}_X,$$

and the member at the right-hand side of the inequality above tends to 0 as  $h \rightarrow 0$ , because of sequential local equicontinuity of the family  $\{f(s)\}_{s \in (a, b)}$  (part (i) of the assumptions) and because of differentiability of  $g(\cdot)x$  in  $t_0$ .  $\blacksquare$

**ANALYTIC  $\mathcal{L}_0(X)$ -VALUED FUNCTIONS.**

**Proposition A.5.** *Let  $X$  be sequentially complete and let  $B \in \mathcal{L}_0(X)$ . Assume that the family  $\{B^n : X \rightarrow X\}_{n \in \mathbb{N}}$  is sequentially equicontinuous. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an analytic function of the form  $f(t) = \sum_{n=0}^{+\infty} a_n t^n$ , with  $t \in \mathbb{R}$ . Then the following hold.*

(i) *The series*

$$f_B(t) := \sum_{n=0}^{+\infty} a_n t^n B^n \quad (\text{A.3})$$

*converges in  $\mathcal{L}_{0,b}(X)$  uniformly for  $t$  on compact sets of  $\mathbb{R}$ .*

(ii) *The function  $f_B : \mathbb{R} \rightarrow \mathcal{L}_{0,b}(X)$ ,  $t \mapsto f_B(t)$  is continuous.*

(iii) *The family  $\{f_B(t)\}_{t \in [-r,r]}$  is sequentially equicontinuous for every  $r > 0$ .*

*Proof.* (i) For  $0 \leq n \leq m$ ,  $p \in \mathcal{P}_X$ ,  $D \subset X$  bounded,  $r > 0$ ,  $x \in D$ ,  $t \in [-r, r]$ , we write

$$p \left( \sum_{k=n}^m a_k t^k B^k x \right) \leq \sum_{k=n}^m |a_k| |t|^k p(B^k x) \leq \left( \sum_{k=n}^{+\infty} |a_k| r^k \right) \sup_{i \in \mathbb{N}} p(B^i x) \leq \left( \sum_{k=n}^{+\infty} |a_k| r^k \right) \sup_{y \in \bigcup_{i \in \mathbb{N}} B^i D} p(y). \quad (\text{A.4})$$

Observe that, by Proposition A.1(iii), the supremum appearing in last term of (A.4) is finite. Then

$$\sup_{t \in [-r,r]} \rho_{p,D} \left( \sum_{k=n}^m a_k t^k B^k \right) \leq \left( \sum_{k=n}^{+\infty} |a_k| r^k \right) \sup_{y \in \bigcup_{i \geq 0} B^i D} p(y) \quad \forall n \in \mathbb{N} \quad (\text{A.5})$$

shows that the sequence of the partials sums of (A.3) is Cauchy in  $\mathcal{L}_{0,b}(X)$ , uniformly for  $t \in [-r, r]$ , and then, by Proposition 3.19(ii), the sum is convergent, uniformly for  $t \in [-r, r]$ .

(ii) This follows from convergence of the partial sums in the space  $C([-r, r], \mathcal{L}_{0,b}(X))$  endowed with the compact-open topology, as shown above.

(iii) By continuity of  $p$ , estimate (A.4) shows that

$$\sup_{t \in [-r,r]} p(f_B(t)x) = \sup_{t \in [-r,r]} \lim_{n \rightarrow +\infty} p \left( \sum_{k=0}^n a_k t^k B^k x \right) \leq \left( \sum_{k=0}^{+\infty} |a_k| r^k \right) \sup_{i \in \mathbb{N}} p(B^i x) \quad \forall x \in X,$$

which provides the sequential equicontinuity of  $\{f_B(t)\}_{t \in [-r,r]}$ . ■

**Lemma A.6.** *Let  $X$  be sequentially complete. Let  $B, C \in \mathcal{L}_0(X)$  be such that  $\{B^n\}_{n \in \mathbb{N}}$  and  $\{C^n\}_{n \in \mathbb{N}}$  are sequentially equicontinuous. Let  $f(t) = \sum_{n=0}^{+\infty} a_n t^n$ ,  $g(t) = \sum_{n=0}^{+\infty} b_n t^n$  be analytic functions defined on  $\mathbb{R}$ . Then*

$$p(f_B(t)g_C(s)x) \leq \left( \sum_{n=0}^{+\infty} |a_n| |t|^n \right) \left( \sum_{n=0}^{+\infty} |b_n| |s|^n \right) \sup_{i,j \in \mathbb{N}} p(B^i C^j x), \quad \forall p \in \mathcal{P}_X, \forall x \in X, \forall t, s \in \mathbb{R}, \quad (\text{A.6})$$

*and the family  $\{f_B(t)g_C(s)\}_{t,s \in [-r,r]}$  is sequentially equicontinuous for every  $r > 0$ .*

*Proof.* By Proposition A.5 and by recalling that every partial sum  $\sum_{i=0}^n a_i t^i B^i$  is sequentially continuous, we can write

$$p(f_B(t)g_C(s)x) = \lim_{n \rightarrow +\infty} \lim_{m \rightarrow +\infty} p \left( \left( \sum_{i=0}^n a_i t^i B^i \right) \left( \sum_{j=0}^m b_j s^j C^j \right) x \right) \quad \forall p \in \mathcal{P}_X, \forall x \in X, \forall t, s \in \mathbb{R}.$$

Then, we obtain (A.6) by the properties of the seminorms. The sequential equicontinuity of the family  $\{f_B(t)g_C(s)\}_{t,s \in [-r,r]}$  comes from (A.6) and Proposition A.1(i). ■

**Proposition A.7.** *Let  $X$  be sequentially complete. Let  $B, C, f, g$ , as in Lemma A.6. We have the following:*

(i)  $(f + g)_B = f_B + g_B$  and  $(fg)_B = f_B g_B$ ;

(ii) if  $BC = CB$ , then  $f_B(t)g_C(s) = g_C(s)f_B(t)$ , for every  $t, s \in \mathbb{R}$ , and  $\{f_B(t)g_C(s)\}_{t,s \in [-r,r]}$  is sequentially equicontinuous for every  $r > 0$ .

*Proof.* The proof follows by algebraic computations on the partial sums and then passing to the limit.  $\blacksquare$

**Lemma A.8.** *The map*

$$\mathcal{L}_{0,b}(X) \times \mathcal{L}_{0,b}(X) \rightarrow \mathcal{L}_{0,b}(X), \quad (F, G) \mapsto FG,$$

*is sequentially continuous.*

*Proof.* Let  $(F, G) \in \mathcal{L}_0(X) \times \mathcal{L}_0(X)$ , and let  $D \subset X$  be bounded. Let  $\{(F_n, G_n)\}_{n \in \mathbb{N}}$  be a sequence converging to  $(F, G)$  in  $\mathcal{L}_{0,b}(X) \times \mathcal{L}_{0,b}(X)$ . Consider the set  $D' := \bigcup_{n \in \mathbb{N}} G_n D$ . We have

$$\sup_{n \in \mathbb{N}} \sup_{x \in D} q(G_n x) \leq \sup_{n \in \mathbb{N}} \sup_{x \in D} q((G_n - G)x) + \sup_{x \in D} q(Gx) \quad \forall q \in \mathcal{P}_X.$$

On the other hand,  $G_n \rightarrow G$  yields

$$\sup_{n \in \mathbb{N}} \sup_{x \in D} q((G_n - G)x) = \sup_{n \in \mathbb{N}} \rho_{q,D}(G_n - G) < +\infty, \quad \forall q \in \mathcal{P}_X.$$

Then, combining with Proposition 3.2(i), we conclude that  $D'$  is bounded.

Now fix  $q \in \mathcal{P}_X$ . For every  $n \in \mathbb{N}$ , we can write

$$\rho_{q,D}((FG - F_n G_n)) \leq \rho_{q,D}(F(G - G_n)) + \rho_{q,D}((F - F_n)G_n) \leq \rho_{q,D}(F(G - G_n)) + \rho_{q,D'}(F - F_n).$$

Now  $\lim_{n \rightarrow +\infty} \rho_{q,D'}(F - F_n) = 0$ , because  $D' \in \mathbf{B}$  and  $F_n \rightarrow F$  in  $\mathcal{L}_{0,b}(X)$ . Hence we conclude if we show  $\lim_{n \rightarrow +\infty} \rho_{q,D}(F(G - G_n)) = 0$ . Assume, by contradiction, that there exist  $\varepsilon > 0$ ,  $\{x_k\}_{k \in \mathbb{N}} \subset D$ , and a subsequence  $\{G_{n_k}\}_{k \in \mathbb{N}}$ , such that

$$q(F(G - G_{n_k})x_k) \geq \varepsilon \quad \forall k \in \mathbb{N}. \tag{A.7}$$

Since

$$\lim_{n \rightarrow +\infty} q'((G - G_{n_k})x_k) \leq \lim_{n \rightarrow +\infty} \rho_{q',D}(G - G_{n_k}) = 0 \quad \forall q' \in \mathcal{P}_X,$$

then  $\{z_k := (G - G_{n_k})x_k\}_{k \in \mathbb{N}}$  is a sequence converging to 0 in  $X$ . By sequential continuity of  $F$ , we have  $\lim_{k \rightarrow +\infty} q(Fz_k) = 0$ , contradicting (A.7) and concluding the proof.  $\blacksquare$

**Proposition A.9.** *Let  $X$  be sequentially complete.*

(i) *Let  $B, C \in \mathcal{L}_0(X)$  be such that  $BC = CB$ , and assume that the families  $\{B^n\}_{n \in \mathbb{N}}$  and  $\{C^n\}_{n \in \mathbb{N}}$  are sequentially equicontinuous. Then, for every  $t, s \in \mathbb{R}$ ,*

(a) *the sum  $e^{tB+sC} := \sum_{n=0}^{+\infty} \frac{(tB+sC)^n}{n!}$  converges in  $\mathcal{L}_{0,b}(X)$ ;*

(b)  $e^{tB+sC} = e^{tB} e^{sC} = e^{sC} e^{tB}$ ;

(c) *the family  $\{e^{tB+sC}\}_{t,s \in [-r,r]}$  is sequentially equicontinuous for every  $r > 0$ .*

(ii) *Let  $B \in \mathcal{L}_0(X)$  be such that the family  $\{B^n\}_{n \in \mathbb{N}}$  is sequentially equicontinuous. Then  $\{e^{tB}\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially locally equicontinuous semigroup on  $X$  with infinitesimal generator  $B$ .*

*Proof.* (i) Let  $r > 0$  and  $t \in [-r, r]$ . By standard computations, we have

$$\sum_{i=0}^n \frac{(B+C)^i}{i!} t^i = \left( \sum_{i=0}^n \frac{B^i}{i!} t^i \right) \left( \sum_{i=0}^n \frac{C^i}{i!} t^i \right) - \sum_{i=0}^n \frac{B^i}{i!} t^i \left( \sum_{k=n-i+1}^n \frac{C^k}{k!} t^k \right). \quad (\text{A.8})$$

Let  $D \subset X$  be a bounded set. For  $x \in D$  and  $p \in \mathcal{P}_X$ , we have

$$\begin{aligned} p \left( \sum_{i=0}^n \frac{B^i}{i!} t^i \left( \sum_{k=n-i+1}^n \frac{C^k x}{k!} t^k \right) \right) &\leq \sum_{i=0}^n \sum_{k=n-i+1}^n \frac{1}{i!k!} r^{i+k} p(B^i C^k x) \\ &\leq \left( \sum_{i=0}^n \sum_{k=n-i+1}^n \frac{1}{i!k!} r^{i+k} \right) \sup_{i,k \in \mathbb{N}} \rho_{p,D}(B^i C^k). \end{aligned}$$

By Proposition A.1(i), the family  $\{B^i C^k\}_{i,k \in \mathbb{N}}$  is sequentially equicontinuous. Hence, by Proposition A.1(iii), we have  $\sup_{i,k \in \mathbb{N}} \rho_{p,D}(B^i C^k) < +\infty$ . Moreover, Lebesgue's dominated convergence theorem applied in discrete spaces yields

$$\lim_{n \rightarrow +\infty} \sum_{i=0}^n \sum_{k=n-i+1}^n \frac{1}{i!k!} r^{i+k} = 0.$$

So, we conclude

$$\lim_{n \rightarrow +\infty} \rho_{p,D} \left( \sum_{i=0}^n \frac{B^i}{i!} t^i \left( \sum_{k=n-i+1}^n \frac{C^k}{k!} t^k \right) \right) = 0. \quad (\text{A.9})$$

On the other hand, by Lemma A.8,

$$\lim_{n \rightarrow +\infty} \left( \sum_{i=0}^n \frac{B^i}{i!} t^i \right) \left( \sum_{i=0}^n \frac{C^i}{i!} t^i \right) = \lim_{n \rightarrow +\infty} \left( \sum_{i=0}^n \frac{B^i}{i!} t^i \right) \lim_{n \rightarrow +\infty} \left( \sum_{i=0}^n \frac{C^i}{i!} t^i \right) = e^{tB} e^{tC}, \quad (\text{A.10})$$

where the limits are taken in the space  $\mathcal{L}_{0,b}(X)$ . By (A.8), (A.9) and (A.10), we obtain

$$\lim_{n \rightarrow +\infty} \sum_{i=0}^n \frac{(B+C)^i}{i!} t^i = e^{tB} e^{tC}, \quad (\text{A.11})$$

with the limit taken in  $\mathcal{L}_{0,b}(X)$ .

Now, let  $t \neq 0$  and  $|s| \leq |t|$ <sup>8</sup>. Then  $\{\binom{s}{t} C\}_{n \in \mathbb{N}}$  is sequentially equicontinuous. By replacing  $C$  by  $\binom{s}{t} C$  in (A.11), we have

$$\lim_{n \rightarrow +\infty} \sum_{i=0}^n \frac{(tB + sC)^i}{i!} = e^{tB} e^{\binom{s}{t} C t} = e^{tB} e^{sC}, \quad (\text{A.12})$$

where the limits are in  $\mathcal{L}_{0,b}(X)$ . So we have proved (a). Properties (b) and (c) now follow from (A.12) and from Proposition A.7(ii).

(ii) First we notice that  $e^{0B} = I$  by definition. The semigroup property for  $\{e^{tB}\}_{t \in \mathbb{R}^+}$  is given by (i), which also provides the sequential local equicontinuity. Proposition A.5 provides the continuity of the map  $\mathbb{R}^+ \rightarrow X$ ,  $t \mapsto e^{tB} x$ , for every  $x \in X$ . Hence, we have proved that  $\{e^{tB}\}_{t \in \mathbb{R}^+}$  is a  $C_0$ -sequentially locally equicontinuous semigroup. It remains to show that the infinitesimal generator is  $B$ . For  $h > 0$ , define  $f(t; h) := e^{ht} - 1 - ht$ . By applying (A.6) to the map  $\mathbb{R} \rightarrow \mathbb{R}$ ,  $t \mapsto f(t; h)$ , with  $B$  in place of  $B$ , and with  $C = I$  and  $g \equiv 1$ , we obtain

$$p \left( \frac{e^{hB} - I}{h} x - Bx \right) = h^{-1} p(f_B(1; h)) \leq h^{-1} f(1; h) \sup_{n \in \mathbb{N}} p(B^n x)$$

and the last term converges to 0 as  $h \rightarrow 0^+$ , because of sequential equicontinuity of  $\{B^n\}_{n \in \mathbb{N}}$ . This shows that the domain of the generator is the whole space  $X$  and that the generator is  $B$ . ■

<sup>8</sup>If  $|t| < |s|$ , we can exchange the role of  $B$  and  $C$ , by symmetry of the sums appearing in (A.12).

## References

- [1] A. A. Albanese, L. Lorenzi, and V. Manco. Mean ergodic theorems for bi-continuous semigroups. *Semigroup Forum*, 82(1):141–171, 2011.
- [2] Angela A. Albanese and Elisabetta Mangino. Trotter-Kato theorems for bi-continuous semigroups and applications to Feller semigroups. *J. Math. Anal. Appl.*, 289(2):477–492, 2004.
- [3] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis*. Springer, 3<sup>rd</sup> edition, 2006.
- [4] S. Cerrai. A Hille-Yosida theorem for weakly continuous semigroups. *Semigroup Forum*, 49(3):349–367, 1994.
- [5] S. Cerrai. *Second order PDE's in finite and infinite dimension: a probabilistic approach*. Springer Verlag, 2001.
- [6] Y. H. Choe.  $C_0$ -semigroups on a locally convex space. *Journal of Mathematical Analysis and Applications*, 106:293–320, 1985.
- [7] G. Da Prato and A. Lunardi. On the Ornstein-Uhlenbeck operator in spaces of continuous functions. *J. Funct. Anal.*, 131(1):94–114, 1995.
- [8] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2nd edition, 2014.
- [9] B. Dembart. On the theory of semigroups of operators on locally convex spaces. *Journal of Functional Analysis*, 16:123–160, 1974.
- [10] J. Diestel. *Sequences and Series in Banach Spaces*. Springer, 1984.
- [11] K. J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194. Springer, 1999.
- [12] Abdelhadi Es-Sarhir and Bálint Farkas. Positivity of perturbed Ornstein-Uhlenbeck semigroups on  $C_b(H)$ . *Semigroup Forum*, 70(2):208–224, 2005.
- [13] Abdelhadi Es-Sarhir and Bálint Farkas. Perturbation for a class of transition semigroups on the Hölder space  $C_{b,\text{loc}}^\theta(H)$ . *J. Math. Anal. Appl.*, 315(2):666–685, 2006.
- [14] Bálint Farkas. Perturbations of bi-continuous semigroups. *Studia Math.*, 161(2):147–161, 2004.
- [15] Bálint Farkas. Perturbations of bi-continuous semigroups with applications to transition semigroups on  $C_b(H)$ . *Semigroup Forum*, 68(1):87–107, 2004.
- [16] S. Federico and M. Rosestolato.  $C_0$ -sequentially equicontinuous semigroup: theory and applications. ArXiv 1512.04589.
- [17] B. Goldys and M. Kocan. Diffusion semigroups in spaces of continuous functions with mixed topology. *J. Differential Equations*, 173(1):17–39, 2001.
- [18] B. Goldys and J. M. A. M. Van Neerven. Transition semigroups of Banach space-valued Ornstein-Uhlenbeck process. *Acta Applicandae Mathematicae*, 76:283–330, 2003.



- [19] E. Hille and R. S. Phillips. *Functional Analysis and Semigroups*, volume 31 of *Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, 1957.
- [20] Patricio Jara. Rational approximation schemes for bi-continuous semigroups. *J. Math. Anal. Appl.*, 344(2):956–968, 2008.
- [21] B. Jefferies. Weakly integrable semigroups on locally convex spaces. *Journal of Functional Analysis*, 66(3):347–364, 1986.
- [22] B. Jefferies. The generation of weakly integrable semigroups. *Journal of Functional analysis*, 73(1):195–215, 1987.
- [23] T. Komura. Semigroups of operators in locally convex spaces. *Journal of Functional Analysis*, 2(3):258–296, 1968.
- [24] K. Kruse, J. Meichsner, and Seifert C. Subordination for sequentially equicontinuous equibounded  $C_0$ -semigroups. ArXiv 1802.05059v1.
- [25] F. Kühnemund. *Bi-Continuous Semigroups on Spaces with Two Topologies: Theory and Applications*. PhD thesis, Eberhard-Karls-Universität Tübingen, 2001.
- [26] F. Kühnemund. A Hille-Yosida theorem for bi-continuous semigroups. *Semigroup Forum*, 67(2):205–225, 2003.
- [27] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995.
- [28] M. S. Osborne. *Locally convex spaces*. Springer, 2014.
- [29] K. R. Parthasarathy. *Probability measures on metric spaces*. Probability and Mathematical Statistics, No. 3. Academic Press Inc., New York, 1967.
- [30] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [31] E. Priola. On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions. *Studia Math.*, 136(3):271–295, 1999.
- [32] W. Rudin. *Functional Analysis*. McGraw-Hill, 2<sup>nd</sup> edition, 1991.
- [33] F. D. Sentiilles. Bounded continuous functions on a completely regular space. *Trans. Amer. Math. Soc.*, 168:311–336, 1972.
- [34] A. Wiweger. Linear spaces with mixed topology. *Studia Math.*, 20:47–68, 1961.
- [35] K. Yosida. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, sixth edition, 1980.