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On G^1 and G^2 Hermite interpolation by spatial Algebraic-Trigonometric Pythagorean Hodograph curves with polynomial parametric speed

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Abstract

In this paper we focus on the class of Algebraic-Trigonometric Pythagorean Hodograph curves (ATPH for short) that is characterized by a purely polynomial parametric speed. Within such a class of ATPH curves, we first construct interpolants to spatial G^1 Hermite data equipped with curvature values. With respect to the solutions proposed in [24], the G^1 Hermite ATPH interpolants we here propose are characterized by C^0 - and C^1 -continuous curvature plots. Secondly, we investigate the existence of ATPH interpolants to spatial G^2 Hermite data and show that solutions exist under some restrictions on the Hermite input data.

Keywords: Algebraic-Trigonometric curves; Pythagorean Hodograph; Polynomial parametric speed; G^1 and G^2 Hermite interpolation.

1. Introduction

The most known examples of non-polynomial Pythagorean Hodograph (PH for short) curves are certainly rational Pythagorean Hodograph curves [10]. However, in the last decade, other instances of non-polynomial PH curves have appeared, such as, the so-called Algebraic-Trigonometric PH curves (ATPH for short) [20, 13, 19, 5] and their hyperbolic counterpart [18, 6, 21]. Recently, a special subfamily of ATPH curves has been proposed by Wu and Yang [23, 24]. Their hodographs are obtained by multiplying a degree- d polynomial function ρ_d with the parametric representation of a spherical curve that identifies the unit tangent vector field of the ATPH. Thus, although such ATPH curves are defined in an algebraic-trigonometric space (like any other alternative ATPH), their parametric speed is purely polynomial. The last property is the main distinctive feature of this subclass of ATPH curves and gives great advantages in many situations arising in CAGD, CAD and computer graphics since it guarantees that the arc length of such curves can be computed exactly. This is indeed what happens for standard polynomial PH curves but, differently from them, these special ATPH curves may also better approximate spatial curves that lie on the surface of three-dimensional geometric shapes like spheres, ellipsoids, cylinders or cones (see, e.g., Figure 1 where a piecewise ATPH curve interpolating G^1 Hermite data is compared with a piecewise quintic PH curve interpolating C^1 Hermite data [9] as well as with a C^2 quintic PH B-spline curve [1, 2] interpolating the same points coming from a given trigonometric reference curve). Additionally, when the reconstruction is performed via ATPH curves with polynomial parametric speed, we don't have to play with any free parameters in order to get a visually-pleasing result, while a reasonable approximation by means of polynomial PH curves requires either a suitable manual setting of the variables that identify their degrees of freedom (e.g. angular variables or quaternion coefficients of the PH pre-image [9, 1, 2]) or an automatic setting that relies on some optimization strategies.

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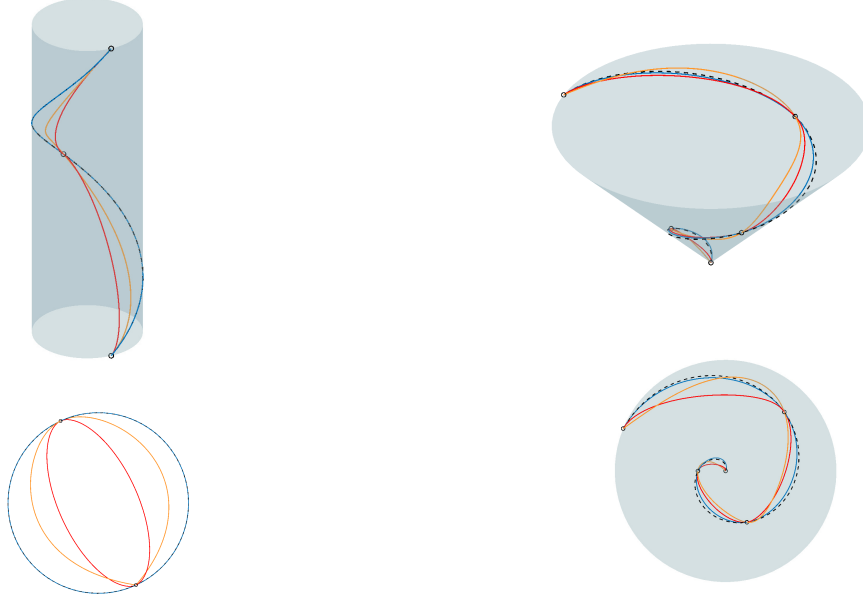


Figure 1: Comparison between a piecewise ATPH curve interpolating G^1 Hermite data (solid blue), a piecewise quintic PH curve interpolating C^1 Hermite data (solid red) and a C^2 quintic PH B-spline curve (solid orange) interpolating the same points coming from the given reference curve (dashed black). First row: 3D view. Second row: projection on the xy-plane.

To support a concrete use of such ATPH curves, one has to propose a way of constructing them according to provided specifications that meet practical needs. One of the most common methods is the so-called G^1 Hermite interpolation which consists in finding the ATPH curve joining two given points in the space with certain conditions on the tangent vectors at both ends. This problem has been already solved in [24]. In our work we contribute to the advancement of research in this area by showing how this subfamily of ATPH curves can be employed to solve interpolation problems of spatial G^1 Hermite data equipped with curvature values, as well as of spatial G^2 Hermite data. Such problems may be of interest in robot path planning, computer animation, path planning for unmanned or autonomous vehicles, and related applications.

The remainder of this manuscript is specifically organized as follows. In Section 2 we introduce preliminary notions and basic results required to deal with the construction of this special subclass of ATPH curves and, in particular, with their use in solving interpolation problems of spatial G^1 Hermite data. In Section 3 we focus our attention on the construction of the minimum degree polynomial ρ_d that allows us to obtain ATPH curves, with either C^0 - or C^1 -continuous curvature, that interpolate G^1 Hermite data. In Section 4 we consider G^2 Hermite data and identify the restrictions on the input data that must be taken into account to solve the Hermite interpolation problem via spatial ATPH curves with parametric speed of the lowest possible degree. Finally, in Section 5 we include numerical experiments aimed at showing how the newly proposed curves compare favourably with previously proposed ATPH interpolants to the same input data. Closing remarks are collected in Section 6.

2. Preliminaries on G^1 Hermite interpolation via spatial ATPH curves with polynomial parametric speed

In this Section we first recall the definition of a spatial ATPH curve with polynomial parametric speed and then we summarize the main results from the literature dealing with interpolation of spatial G^1 Hermite

data via polynomial, non-polynomial and in particular ATPH curves.

2.1. Preliminaries on spatial ATPH curves with polynomial parametric speed

Let

$$\mathbf{S}(\theta, \phi) = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ \sin \theta \end{pmatrix}, \quad \theta \in [-\pi/2, \pi/2], \phi \in [-\pi, \pi]$$

be the standard parametrization of the unit sphere, and let

$$\varphi(t) = \begin{pmatrix} a_1 t + a_0 \\ b_1 t + b_0 \end{pmatrix}, \quad t \in [0, 1]$$

be the usual parametrization of the straight line where the real coefficients a_0, a_1, b_0, b_1 are usually assumed to be such that

$$a_1 \neq 0, \quad a_1 + b_1 \neq 0, \quad a_1 - b_1 \neq 0, \quad a_1^2 + b_1^2 \neq 0. \quad (1)$$

The cases $a_1 = \pm b_1$ and $a_1 = 0$ will be discussed as special cases in this work.

Definition 1. A spatial ATPH curve with polynomial parametric speed is defined as

$$\begin{aligned} \mathbf{r} : [0, 1] &\rightarrow \mathbb{R}^3 \\ \xi &\mapsto \mathbf{r}(\xi) = \mathbf{r}_0 + \int_0^\xi \rho_d(t) \mathbf{T}(t) dt, \end{aligned} \quad (2)$$

where $\mathbf{r}_0$ is an arbitrary point in $\mathbb{R}^3$,

$$\rho_d(t) = \sum_{i=0}^d c_i t^i$$

is an arbitrary degree- d polynomial with real coefficients satisfying the conditions

$$c_0 \neq 0, \quad \sum_{i=0}^d c_i \neq 0,$$

and

$$\mathbf{T}(t) = \mathbf{S}(\varphi(t)) = \begin{pmatrix} \cos(a_1 t + a_0) \cos(b_1 t + b_0) \\ \cos(a_1 t + a_0) \sin(b_1 t + b_0) \\ \sin(a_1 t + a_0) \end{pmatrix} \quad (3)$$

is a curve on the unit sphere that provides the unit tangent vector field of $\mathbf{r}$.

Thus, the family of ATPH curves we are studying is defined by degree- d polynomials and trigonometric functions, and is characterized by unit tangent vectors represented by spherical curves. Recalling (2) and the fundamental theorem of integral calculus, the first derivative of $\mathbf{r}$ satisfies

$$\mathbf{r}'(\xi) = \rho_d(\xi) \mathbf{T}(\xi) \quad (4)$$

and

$$\|\mathbf{r}'(\xi)\| = |\rho_d(\xi)|. \quad (5)$$

Equation (4) allows us to state that, if $c_i, i = 0, \dots, d$ are such that $\rho_d(t) \neq 0$ for all $t \in [0, 1]$, then $\mathbf{r}'$ never vanishes and $\mathbf{r}$ is non-singular.

Differently, Equation (5) means that the norm of the hodograph is a polynomial, and this guarantees that the arc length of $\mathbf{r}$, which is given by

$$L(\xi) = \int_0^\xi |\rho_d(t)| dt,$$

can be computed exactly. Due to the fact that $\rho_d(t)$ is a degree- d polynomial, a special feature of these ATPH curves is that they have a purely polynomial parametric speed and arc length, exactly as it happens with standard PH curves [7].

From the definition of $\mathbf{r}$ and $\mathbf{T}$ we can work out the Cartesian coordinates of the ATPH curve, which read as

$$\mathbf{r}(\xi) = \mathbf{r}_0 + \begin{pmatrix} \frac{1}{2} \int_0^\xi \rho_d(t) \cos(A_1 t + A_0) dt + \frac{1}{2} \int_0^\xi \rho_d(t) \cos(B_1 t + B_0) dt \\ \frac{1}{2} \int_0^\xi \rho_d(t) \sin(A_1 t + A_0) dt - \frac{1}{2} \int_0^\xi \rho_d(t) \sin(B_1 t + B_0) dt \\ \int_0^\xi \rho_d(t) \sin(a_1 t + a_0) dt \end{pmatrix} \quad (6)$$

where

$$A_0 = a_0 + b_0, \quad A_1 = a_1 + b_1, \quad B_0 = a_0 - b_0, \quad B_1 = a_1 - b_1. \quad (7)$$

From the definition of $\mathbf{r}$ we can also obtain the following simplified expression for its curvature formula

$$\kappa(\xi) = \frac{\|\mathbf{T}(\xi) \times \mathbf{T}'(\xi)\|}{|\rho_d(\xi)|} = \frac{\|\mathbf{T}'(\xi)\|}{|\rho_d(\xi)|},$$

where the last equality follows in light of the fact that $\|\mathbf{T}(\xi)\| = 1$ and $\mathbf{T}(\xi) \perp \mathbf{T}'(\xi)$.

Remark 1. Equation (4) allows us to infer also the reproduction capabilities of this special class of ATPH curves. Precisely, since the first derivative of a polynomial PH curve never matches the expression in (4), we can claim that polynomial PH curves can never be represented as special ATPH curves. Moreover, if the Hermite data we use to build an ATPH curve are sampled from a reference curve whose first derivative satisfies (4), as it happens with the circular helix for example, then the ATPH interpolant will reproduce the reference curve exactly.

2.2. Known results on G^1 Hermite interpolation via spatial PH curves

The G^1 Hermite interpolation problem via spatial PH curves has been fully investigated in [16]. In that paper, which extends and completes the results previously published in [22, 12, 17], necessary and sufficient conditions for the existence of spatial PH cubics that solve the G^1 Hermite interpolation problem are provided. The analogous G^1 Hermite interpolation problem via spatial PH quintics has been more recently investigated in [8]. In that paper it is shown that a unique feature of polynomial PH curves of degree 5 is the ability to interpolate G^1 Hermite data (i.e., end points and tangents) with a specified total arc length.

Solutions to the G^1 Hermite interpolation problem via non-polynomial spatial PH curves have been also investigated. In [14] cubic rational PH curves are used to interpolate two points and two tangent directions. It is shown that the interpolants exist for all possible true spatial data configurations, while in case of polynomial PH curves a range of data for which the solutions don't exist is quite large [22, 12, 17, 16, 11]. Further, in [15] a closed-form solution to the problem of interpolating spatial G^1 Hermite data using rational PH curves with prescribed arc lengths is proposed.

With regard to ATPH curves, the G^1 Hermite interpolation problem has been already satisfactorily studied for planar Hermite data [23, 4]. However, much less has been done in the context of G^1 Hermite interpolation of spatial data [24]. In the latter it has been shown that $d = 2$ is the lowest possible degree of the polynomial $\rho_d(t)$ that allows the construction of an ATPH curve as in (2) which solves the G^1 Hermite interpolation problem

$$\mathbf{r}(0) = \mathbf{P}_1, \quad \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \mathbf{t}_1, \quad \mathbf{r}(1) = \mathbf{P}_2, \quad \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \mathbf{t}_2, \quad (8)$$

with $\mathbf{P}_1 \neq \mathbf{P}_2$ given points in $\mathbb{R}^3$ specified by the coordinates $(x_i, y_i, z_i)^T$, $i = 1, 2$, and $\mathbf{t}_1, \mathbf{t}_2$ given 3-dimensional unit vectors defined either as

$$\mathbf{t}_i = (t_{i1}, t_{i2}, t_{i3})^T, \quad i = 1, 2$$

or as

$$\mathbf{t}_i = (\cos \theta_i \cos \phi_i, \cos \theta_i \sin \phi_i, \sin \theta_i)^T = \mathbf{S}(\theta_i, \phi_i), \quad i = 1, 2 \quad (9)$$

depending on whether we use a Cartesian or spherical coordinate system. As illustrated in [24], the construction of the G^1 Hermite interpolant is performed with a two-step process.

(I) In the first step, the space curve $\mathbf{T}(\xi)$ lying on the unit sphere and interpolating $\mathbf{t}_1$ and $\mathbf{t}_2$ at $\xi = 0$ and $\xi = 1$, respectively, is computed. In other words, in this step the unknown coefficients a_0, a_1, b_0, b_1 in (3) are computed by exploiting the relationships

$$\begin{cases} \mathbf{S}(\theta_1, \phi_1) = \mathbf{t}_1 = \mathbf{T}(0) = \mathbf{S}(\varphi(0)) \\ \mathbf{S}(\theta_2, \phi_2) = \mathbf{t}_2 = \mathbf{T}(1) = \mathbf{S}(\varphi(1)) \end{cases}$$

which give

$$\begin{cases} a_0 = \theta_1 \\ a_1 = \theta_2 - \theta_1 \\ b_0 = \phi_1 \\ b_1 = \phi_2 - \phi_1. \end{cases} \quad (10)$$

Remark 2. In [24], in consequence of the assumptions in (1), the existence of the G^1 Hermite interpolant is restricted to the case

$$\theta_2 \neq \theta_1, \quad (\theta_2 - \theta_1) + (\phi_2 - \phi_1) \neq 0, \quad (\theta_2 - \theta_1) - (\phi_2 - \phi_1) \neq 0, \quad (\theta_2 - \theta_1)^2 + (\phi_2 - \phi_1)^2 \neq 0.$$

(II) Then, in the second step, the endpoint interpolation conditions are exploited to determine the unknown coefficients c_i , $i = 0, \dots, d$ of the polynomial ρ_d . As discussed in [24], the fulfillment of the condition $\mathbf{r}(0) = \mathbf{r}_0 = \mathbf{P}_1$ fixes the arbitrary point $\mathbf{r}_0$ to coincide with $\mathbf{P}_1$. The remaining condition $\mathbf{r}(1) = \mathbf{P}_2$ (which amounts to 3 scalar equations) fixes the minimum degree of the polynomial ρ_d to be 2.

3. On the computation of minimum degree polynomial functions that guarantee desirable properties of the ATPH curvature

This section is devoted to the construction of the minimum degree polynomial ρ_d (with $d \geq 2$) that allows us to obtain ATPH interpolants to G^1 Hermite data whose curvature $\kappa(\xi)$ fulfills some crucial properties in applications such as, e.g., C^0 and C^1 continuity.

We start by introducing two preliminary lemmas that are needed to work out the coefficient matrices and the known terms of the linear systems to be solved for computing the unknown coefficients of ρ_d in the above mentioned cases.

Lemma 1. Let $\mathbf{T} : [0, 1] \rightarrow \mathbb{R}^3$ be the unit tangent vector field of the spatial ATPH $\mathbf{r}(\xi)$, $\xi \in [0, 1]$, defined in (2). Then

$$\begin{aligned} \mathbf{T}'(0) &= \begin{pmatrix} -a_1 \sin(a_0) \cos(b_0) - b_1 \cos(a_0) \sin(b_0) \\ -a_1 \sin(a_0) \sin(b_0) + b_1 \cos(a_0) \cos(b_0) \\ a_1 \cos(a_0) \end{pmatrix}, \\ \mathbf{T}'(1) &= \begin{pmatrix} -a_1 \sin(a_1 + a_0) \cos(b_1 + b_0) - b_1 \cos(a_1 + a_0) \sin(b_1 + b_0) \\ -a_1 \sin(a_1 + a_0) \sin(b_1 + b_0) + b_1 \cos(a_1 + a_0) \cos(b_1 + b_0) \\ a_1 \cos(a_1 + a_0) \end{pmatrix} \end{aligned} \quad (11)$$

and

$$\|\mathbf{T}'(0)\| = \left(b_1^2 \cos^2(a_0) + a_1^2\right)^{\frac{1}{2}}, \quad \|\mathbf{T}'(1)\| = \left(b_1^2 \cos^2(a_0 + a_1) + a_1^2\right)^{\frac{1}{2}}, \quad (12)$$

where the coefficients a_0, a_1, b_0, b_1 are specified in (10).

Proof. By differentiating (3) we find

$$\mathbf{T}'(t) = \begin{pmatrix} -a_1 \sin(a_1 t + a_0) \cos(b_1 t + b_0) - b_1 \cos(a_1 t + a_0) \sin(b_1 t + b_0) \\ -a_1 \sin(a_1 t + a_0) \sin(b_1 t + b_0) + b_1 \cos(a_1 t + a_0) \cos(b_1 t + b_0) \\ a_1 \cos(a_1 t + a_0) \end{pmatrix},$$

so that

$$\|\mathbf{T}'(t)\| = \left(b_1^2 \cos^2(a_0 + a_1 t) + a_1^2\right)^{\frac{1}{2}}. \quad (13)$$

Hence, by evaluating the above expressions at $t = 0$ and $t = 1$, the claimed results follow straightforwardly. $\square$

Lemma 2. Let $\mathbf{T} : [0, 1] \rightarrow \mathbb{R}^3$ be the unit tangent vector field of the spatial ATPH $\mathbf{r}(\xi)$, $\xi \in [0, 1]$, defined in (2). Then

$$\begin{aligned} \mathbf{T}''(0) &= \begin{pmatrix} -(a_1^2 + b_1^2) \cos(a_0) \cos(b_0) + 2a_1 b_1 \sin(a_0) \sin(b_0) \\ -(a_1^2 + b_1^2) \cos(a_0) \sin(b_0) - 2a_1 b_1 \cos(b_0) \sin(a_0) \\ -a_1^2 \sin(a_0) \end{pmatrix}, \\ \mathbf{T}''(1) &= \begin{pmatrix} -(a_1^2 + b_1^2) \cos(a_1 + a_0) \cos(b_1 + b_0) + 2a_1 b_1 \sin(a_1 + a_0) \sin(b_1 + b_0) \\ -(a_1^2 + b_1^2) \cos(a_1 + a_0) \sin(b_1 + b_0) - 2a_1 b_1 \cos(b_1 + b_0) \sin(a_1 + a_0) \\ -a_1^2 \sin(a_1 + a_0) \end{pmatrix}, \end{aligned}$$

and

$$-\mathbf{T}'(0) \cdot \mathbf{T}''(0) = \frac{1}{2} a_1 b_1^2 \sin(2a_0), \quad -\mathbf{T}'(1) \cdot \mathbf{T}''(1) = \frac{1}{2} a_1 b_1^2 \sin(2a_0 + 2a_1) \quad (14)$$

where the coefficients a_0, a_1, b_0, b_1 are those specified in (10).

Proof. By differentiating (3) twice we find

$$\mathbf{T}''(t) = \begin{pmatrix} -(a_1^2 + b_1^2) \cos(a_1 t + a_0) \cos(b_1 t + b_0) + 2a_1 b_1 \sin(a_1 t + a_0) \sin(b_1 t + b_0) \\ -(a_1^2 + b_1^2) \cos(a_1 t + a_0) \sin(b_1 t + b_0) - 2a_1 b_1 \cos(b_1 t + b_0) \sin(a_1 t + a_0) \\ -a_1^2 \sin(a_1 t + a_0) \end{pmatrix}.$$

Thus, by evaluating at $t = 0$ and $t = 1$, the claimed expressions follow. Furthermore, recalling the expressions of $\mathbf{T}'(0)$ and $\mathbf{T}'(1)$ in (11), the results in (14) are obtained straightforwardly. $\square$

Proposition 1. Let the G^1 Hermite data be such that $a_1 \notin \{0, \pm b_1\}$. The lowest possible degree d of the polynomial $\rho_d(t)$ that allows the construction of a curvature continuous, piecewise ATPH curve whose pieces are defined as in (2) and solve the G^1 Hermite interpolation problem (8) with the additional conditions

$$\kappa(0) = \kappa_1, \quad \kappa(1) = \kappa_2 \quad (\kappa_1, \kappa_2 \neq 0) \quad (15)$$

is $d = 4$. Moreover, the constant term of $\rho_4(t)$ is

$$c_0 = \frac{u_0}{\kappa_1},$$

and the remaining four coefficients are obtained by solving the linear system

$$\begin{pmatrix} e_x & a_x & b_x & c_x \\ e_y & a_y & b_y & c_y \\ e_z & a_z & b_z & c_z \\ \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 \end{pmatrix} \begin{pmatrix} c_4 \\ c_3 \\ c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2(x_2 - x_1) - \frac{d_x u_0}{\kappa_1} \\ 2(y_2 - y_1) - \frac{d_y u_0}{\kappa_1} \\ (z_2 - z_1) - \frac{d_z u_0}{\kappa_1} \\ u_1 - \frac{\kappa_2 u_0}{\kappa_1} \end{pmatrix} \quad (16)$$

where

$$u_0 = \left(b_1^2 \cos^2(a_0) + a_1^2 \right)^{\frac{1}{2}}, \quad u_1 = \left(b_1^2 \cos^2(a_0 + a_1) + a_1^2 \right)^{\frac{1}{2}}, \quad (17)$$

$$e_x = \gamma_E(A_0, A_1) + \gamma_E(B_0, B_1), \quad e_y = \omega_E(A_0, A_1) - \omega_E(B_0, B_1), \quad e_z = \omega_E(a_0, a_1)$$

with

$$\begin{aligned} \gamma_E(\lambda, \mu) &= \frac{4(\mu^2 - 6)}{\mu^4} \cos(\lambda + \mu) + \frac{24 - 12\mu^2 + \mu^4}{\mu^5} \sin(\lambda + \mu) - \frac{24}{\mu^5} \sin \lambda, \\ \omega_E(\lambda, \mu) &= \frac{4(\mu^2 - 6)}{\mu^4} \sin(\lambda + \mu) - \frac{24 - 12\mu^2 + \mu^4}{\mu^5} \cos(\lambda + \mu) + \frac{24}{\mu^5} \cos \lambda, \end{aligned}$$

and a_0, a_1, b_0, b_1 specified in (10), A_0, A_1, B_0, B_1 in (7), and $a_x = A_x(1)$, $b_x = B_x(1)$, $c_x = C_x(1)$, $d_x = D_x(1)$ with $A_x(\xi), B_x(\xi), C_x(\xi), D_x(\xi)$ (as well as the analogous expressions for y and z) defined as in [3, Appendix A] (see also [24, pages 3-4]).

Proof. Due to the fact that $\mathbf{r}(0) = \mathbf{r}_0 = \mathbf{P}_1$, we can satisfy the first interpolation condition at the endpoint $\xi = 0$ by simply selecting $\mathbf{r}_0 = \mathbf{P}_1$. Thus, in order to fulfill the remaining conditions $\mathbf{r}(1) = \mathbf{P}_2$ and $\kappa(0) = \kappa_1$, $\kappa(1) = \kappa_2$ (which amount to 5 scalar equations) we need to work with a polynomial $\rho_d(t)$ of degree $d = 4$, and compute its coefficients accordingly. By replacing $\rho_4(t) = \sum_{i=0}^4 c_i t^i$ into (6), we can rewrite the expression of the curve as

$$\mathbf{r}(\xi) = \mathbf{r}_0 + \begin{pmatrix} \frac{1}{2} \left(E_x(\xi) c_4 + A_x(\xi) c_3 + B_x(\xi) c_2 + C_x(\xi) c_1 + D_x(\xi) c_0 \right) \\ \frac{1}{2} \left(E_y(\xi) c_4 + A_y(\xi) c_3 + B_y(\xi) c_2 + C_y(\xi) c_1 + D_y(\xi) c_0 \right) \\ E_z(\xi) c_4 + A_z(\xi) c_3 + B_z(\xi) c_2 + C_z(\xi) c_1 + D_z(\xi) c_0 \end{pmatrix} \quad (18)$$

where $A_x(\xi), B_x(\xi), C_x(\xi), D_x(\xi)$ (and the analogous expressions for y and z) are defined as in [3, Appendix A] (see also [24, pages 3-4]). To compute $E_x(\xi), E_y(\xi), E_z(\xi)$ we write

$$\begin{pmatrix} \frac{1}{2} E_x(\xi) \\ \frac{1}{2} E_y(\xi) \\ E_z(\xi) \end{pmatrix} = \int_0^\xi t^4 \mathbf{T}(t) dt$$

from which we obtain

$$\begin{aligned} E_x(\xi) &= \Gamma_E(A_0, A_1, \xi) + \Gamma_E(B_0, B_1, \xi), \\ E_y(\xi) &= \Omega_E(A_0, A_1, \xi) - \Omega_E(B_0, B_1, \xi), \\ E_z(\xi) &= \Omega_E(a_0, a_1, \xi) \end{aligned} \quad (19)$$

with

$$\begin{aligned} \Gamma_E(\lambda, \mu, \xi) &= \frac{4\xi(\mu^2 \xi^2 - 6)}{\mu^4} \cos(\lambda + \mu\xi) + \frac{24 - 12\mu^2 \xi^2 + \mu^4 \xi^4}{\mu^5} \sin(\lambda + \mu\xi) - \frac{24}{\mu^5} \sin \lambda, \\ \Omega_E(\lambda, \mu, \xi) &= \frac{4\xi(\mu^2 \xi^2 - 6)}{\mu^4} \sin(\lambda + \mu\xi) - \frac{24 - 12\mu^2 \xi^2 + \mu^4 \xi^4}{\mu^5} \cos(\lambda + \mu\xi) + \frac{24}{\mu^5} \cos \lambda. \end{aligned}$$

Note that, in light of the considered assumptions, we always have $a_1 \neq 0$ and $A_1 \neq 0$, $B_1 \neq 0$, so that $E_x(\xi), E_y(\xi), E_z(\xi)$ are well defined. Then, the five unknown coefficients of $\rho_4(t)$ are obtained by solving the linear system

$$\begin{pmatrix} E_x(1) & A_x(1) & B_x(1) & C_x(1) & D_x(1) \\ E_y(1) & A_y(1) & B_y(1) & C_y(1) & D_y(1) \\ E_z(1) & A_z(1) & B_z(1) & C_z(1) & D_z(1) \\ 0 & 0 & 0 & 0 & \kappa_1 \\ \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 \end{pmatrix} \begin{pmatrix} c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 2(x_2 - x_1) \\ 2(y_2 - y_1) \\ z_2 - z_1 \\ \|\mathbf{T}'(0)\| \\ \|\mathbf{T}'(1)\| \end{pmatrix}$$

that arises by considering the equations

$$\begin{aligned} \mathbf{r}(1) - \mathbf{r}(0) &= \mathbf{P}_2 - \mathbf{P}_1 \quad (\text{which incorporates three scalar equations}), \\ \kappa(0) &= \frac{\|\mathbf{T}'(0)\|}{\rho_4(0)} = \frac{\|\mathbf{T}'(0)\|}{c_0} = \kappa_1, \\ \kappa(1) &= \frac{\|\mathbf{T}'(1)\|}{\rho_4(1)} = \frac{\|\mathbf{T}'(1)\|}{c_0 + c_1 + c_2 + c_3 + c_4} = \kappa_2, \end{aligned}$$

where the expressions of $\|\mathbf{T}'(0)\|, \|\mathbf{T}'(1)\|$ are those given in (12) and here shortly denoted with u_0 and u_1 , respectively (see Eq (17)). Now, by noticing that $\gamma_E(\lambda, \mu) = \Gamma_E(\lambda, \mu, 1)$ and $\omega_E(\lambda, \mu) = \Omega_E(\lambda, \mu, 1)$, the expressions of $e_x = E_x(1)$, $e_y = E_y(1)$, $e_z = E_z(1)$ follow. Moreover, by observing that the fourth equation in the 5×5 linear system gives $c_0 = \frac{u_0}{\kappa_1}$, we are left with a 4×4 linear system in the unknowns c_4, c_3, c_2, c_1 . This concludes the proof. $\square$

Proposition 2. *Let the G^1 Hermite data be such that $a_1 \notin \{0, \pm b_1\}$. The lowest possible degree d of the polynomial $\rho_d(t)$ that allows the construction of a piecewise ATPH interpolant with a C^1 -continuous curvature, whose pieces are defined as in (2) and solve the G^1 Hermite interpolation problem (8) with the additional conditions*

$$\kappa(0) = \kappa_1, \quad \kappa(1) = \kappa_2, \quad \kappa'(0) = \eta_1, \quad \kappa'(1) = \eta_2 \quad (\kappa_1, \kappa_2 \neq 0) \quad (20)$$

is $d = 6$. Moreover, the coefficient of t and the constant term of $\rho_6(t)$ are

$$c_1 = -\frac{1}{\kappa_1} \left(\frac{\eta_1 u_0}{\kappa_1} + \frac{v_0}{u_0} \right), \quad c_0 = \frac{u_0}{\kappa_1},$$

while the remaining five coefficients are obtained by solving the linear system

$$\begin{pmatrix} g_x & f_x & e_x & a_x & b_x \\ g_y & f_y & e_y & a_y & b_y \\ g_z & f_z & e_z & a_z & b_z \\ \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 \\ h_1 & h_2 & h_3 & h_4 & h_5 \end{pmatrix} \begin{pmatrix} c_6 \\ c_5 \\ c_4 \\ c_3 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2(x_2 - x_1) - c_x c_1 - d_x c_0 \\ 2(y_2 - y_1) - c_y c_1 - d_y c_0 \\ (z_2 - z_1) - c_z c_1 - d_z c_0 \\ u_1 - \kappa_2(c_1 + c_0) \\ -\frac{\eta_2 u_1^3}{\kappa_2^2} - h_6 c_1 - v_1 c_0 \end{pmatrix} \quad (21)$$

where

$$\begin{aligned} h_i &= (7 - i) u_1^2 + v_1 \quad \text{for } i = 1, \dots, 6, \\ v_0 &= \frac{1}{2} a_1 b_1^2 \sin(2a_0), \quad v_1 = \frac{1}{2} a_1 b_1^2 \sin(2a_0 + 2a_1), \end{aligned} \quad (22)$$

$$\begin{aligned} g_x &= \gamma_G(A_0, A_1) + \gamma_G(B_0, B_1), & g_y &= \omega_G(A_0, A_1) - \omega_G(B_0, B_1), & g_z &= \omega_G(a_0, a_1), \\ f_x &= \gamma_F(A_0, A_1) + \gamma_F(B_0, B_1), & f_y &= \omega_F(A_0, A_1) - \omega_F(B_0, B_1), & f_z &= \omega_F(a_0, a_1) \end{aligned}$$

with

$$\begin{aligned}
\gamma_G(\lambda, \mu) &= \frac{6(120 - 20\mu^2 + \mu^4)}{\mu^6} \cos(\lambda + \mu) - \frac{720 - 360\mu^2 + 30\mu^4 - \mu^6}{\mu^7} \sin(\lambda + \mu) + \frac{720}{\mu^7} \sin \lambda, \\
\omega_G(\lambda, \mu) &= \frac{6(120 - 20\mu^2 + \mu^4)}{\mu^6} \sin(\lambda + \mu) + \frac{720 - 360\mu^2 + 30\mu^4 - \mu^6}{\mu^7} \cos(\lambda + \mu) - \frac{720}{\mu^7} \cos \lambda, \\
\gamma_F(\lambda, \mu) &= \frac{5(24 - 12\mu^2 + \mu^4)}{\mu^6} \cos(\lambda + \mu) + \frac{120 - 20\mu^2 + \mu^4}{\mu^5} \sin(\lambda + \mu) - \frac{120}{\mu^6} \cos \lambda, \\
\omega_F(\lambda, \mu) &= \frac{5(24 - 12\mu^2 + \mu^4)}{\mu^6} \sin(\lambda + \mu) - \frac{120 - 20\mu^2 + \mu^4}{\mu^5} \cos(\lambda + \mu) - \frac{120}{\mu^6} \sin \lambda,
\end{aligned}$$

whereas $a_0, a_1, b_0, b_1, A_0, A_1, B_0, B_1, u_0, u_1$ and e_x, d_x, c_x, b_x, a_x (as well as the analogous expressions for y and z) are the ones of Proposition 1.

Proof. First of all we observe that conditions (20) are two more than the ones previously considered in (15), thus the minimum degree of ρ is now increased to 6. There follows that the conditions to obtain a C^0 -continuous curvature now read as

$$\kappa(0) = \frac{\|\mathbf{T}'(0)\|}{c_0} = \kappa_1, \quad \kappa(1) = \frac{\|\mathbf{T}'(1)\|}{\sum_{i=0}^6 c_i} = \kappa_2.$$

In order to achieve a C^1 -continuous curvature, we would need to compute also

$$\kappa'(\xi) = \frac{d}{d\xi} \frac{\|\mathbf{T}'(\xi)\|}{|\rho_6(\xi)|}$$

and then impose the last two conditions in (20). Unfortunately, $\kappa'(\xi)$ yields a quadratic expression in the unknowns, and thus it cannot be used to generate the two missing linear equations needed to set up the linear system in the unknowns c_i , $i = 0, \dots, 6$. Thus, we exploit the observation that a function f is C^1 if and only if $\frac{1}{f}$ is C^1 , except at the points where f vanishes. As in our setting $f(\xi) := \kappa(\xi) = \frac{\|\mathbf{T}'(\xi)\|}{|\rho_6(\xi)|}$ and the parameter values which verify $|\rho_6(\xi)| = 0$ are undesirable since they identify singular points of $\mathbf{r}$ (see Eq. (5)) whereas $\|\mathbf{T}'(\xi)\| \neq 0$ for all $\xi \in [0, 1]$ if $a_1 \notin \{0, \pm b_1\}$ (see Eq. (13)), we can assume that this case doesn't occur and then we show that $1/\kappa(\xi)$ is C^1 . Thus, we start by computing

$$\left(\frac{1}{\kappa(\xi)} \right)' = \frac{\rho'_6(\xi)}{\|\mathbf{T}'(\xi)\|} - \frac{\mathbf{T}'(\xi) \cdot \mathbf{T}''(\xi)}{\|\mathbf{T}'(\xi)\|^3} \rho_6(\xi).$$

In light of the fact that

$$\left(\frac{1}{\kappa(\xi)} \right)' = -\frac{\kappa'(\xi)}{(\kappa(\xi))^2}$$

we thus arrive at

$$-\frac{\kappa'(\xi)}{(\kappa(\xi))^2} = \frac{\rho'_6(\xi)}{\|\mathbf{T}'(\xi)\|} - \frac{\mathbf{T}'(\xi) \cdot \mathbf{T}''(\xi)}{\|\mathbf{T}'(\xi)\|^3} \rho_6(\xi). \quad (23)$$

After evaluating (23) at $\xi = 0$ and $\xi = 1$ we get the conditions

$$\begin{aligned}
\frac{\rho'_6(0)}{\|\mathbf{T}'(0)\|} - \frac{\mathbf{T}'(0) \cdot \mathbf{T}''(0)}{\|\mathbf{T}'(0)\|^3} \rho_6(0) &= -\frac{\kappa'(0)}{(\kappa(0))^2}, \\
\frac{\rho'_6(1)}{\|\mathbf{T}'(1)\|} - \frac{\mathbf{T}'(1) \cdot \mathbf{T}''(1)}{\|\mathbf{T}'(1)\|^3} \rho_6(1) &= -\frac{\kappa'(1)}{(\kappa(1))^2}.
\end{aligned}$$

Now, exploiting that $\rho_6(0) = c_0$, $\rho_6(1) = \sum_{i=0}^6 c_i$, $\rho'_6(0) = c_1$, $\rho'_6(1) = \sum_{i=1}^6 i c_i$, and recalling (20), we get

$$\begin{aligned} \frac{c_1}{\|\mathbf{T}'(0)\|} - \frac{\mathbf{T}'(0) \cdot \mathbf{T}''(0)}{\|\mathbf{T}'(0)\|^3} c_0 &= -\frac{\eta_1}{\kappa_1^2} \\ \frac{1}{\|\mathbf{T}'(1)\|} \sum_{i=1}^6 i c_i - \frac{\mathbf{T}'(1) \cdot \mathbf{T}''(1)}{\|\mathbf{T}'(1)\|^3} \sum_{i=0}^6 c_i &= -\frac{\eta_2}{\kappa_2^2} \end{aligned}$$

or, equivalently,

$$\begin{aligned} c_1 \|\mathbf{T}'(0)\|^2 - \mathbf{T}'(0) \cdot \mathbf{T}''(0) c_0 &= -\frac{\eta_1}{\kappa_1^2} \|\mathbf{T}'(0)\|^3 \\ \|\mathbf{T}'(1)\|^2 \sum_{i=1}^6 i c_i - \mathbf{T}'(1) \cdot \mathbf{T}''(1) \sum_{i=0}^6 c_i &= -\frac{\eta_2}{\kappa_2^2} \|\mathbf{T}'(1)\|^3. \end{aligned}$$

So, the linear system in the unknowns c_i , $i = 0, \dots, 6$ that guarantees C^1 -continuity of the curvature is obtained by collecting the first four equations already used in Proposition 1 plus the last two just obtained, and can be written as

$$\begin{pmatrix} G_x(1) & F_x(1) & E_x(1) & A_x(1) & B_x(1) & C_x(1) & D_x(1) \\ G_y(1) & F_y(1) & E_y(1) & A_y(1) & B_y(1) & C_y(1) & D_y(1) \\ G_z(1) & F_z(1) & E_z(1) & A_z(1) & B_z(1) & C_z(1) & D_z(1) \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_1 \\ \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 \\ 0 & 0 & 0 & 0 & 0 & \|\mathbf{T}'(0)\|^2 & -\mathbf{T}'(0) \cdot \mathbf{T}''(0) \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & -\mathbf{T}'(1) \cdot \mathbf{T}''(1) \end{pmatrix} \begin{pmatrix} c_6 \\ c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 2(x_2 - x_1) \\ 2(y_2 - y_1) \\ z_2 - z_1 \\ \|\mathbf{T}'(0)\| \\ \|\mathbf{T}'(1)\| \\ -\frac{\eta_1}{\kappa_1^2} \|\mathbf{T}'(0)\|^3 \\ -\frac{\eta_2}{\kappa_2^2} \|\mathbf{T}'(1)\|^3 \end{pmatrix}$$

where

$$h_i = (7 - i) \|\mathbf{T}'(1)\|^2 - \mathbf{T}'(1) \cdot \mathbf{T}''(1) \quad \text{for } i = 1, \dots, 6,$$

$$\begin{aligned} G_x(\xi) &= \Gamma_G(A_0, A_1, \xi) + \Gamma_G(B_0, B_1, \xi), \\ G_y(\xi) &= \Omega_G(A_0, A_1, \xi) - \Omega_G(B_0, B_1, \xi), \\ G_z(\xi) &= \Omega_G(a_0, a_1, \xi) \end{aligned} \tag{24}$$

with

$$\begin{aligned} \Gamma_G(\lambda, \mu, \xi) &= \frac{6\xi(120 - 20\mu^2\xi^2 + \mu^4\xi^4)}{\mu^6} \cos(\lambda + \mu\xi) - \frac{720 - 360\mu^2\xi^2 + 30\mu^4\xi^4 - \mu^6\xi^6}{\mu^7} \sin(\lambda + \mu\xi) + \frac{720}{\mu^7} \sin \lambda, \\ \Omega_G(\lambda, \mu, \xi) &= \frac{6\xi(120 - 20\mu^2\xi^2 + \mu^4\xi^4)}{\mu^6} \sin(\lambda + \mu\xi) + \frac{720 - 360\mu^2\xi^2 + 30\mu^4\xi^4 - \mu^6\xi^6}{\mu^7} \cos(\lambda + \mu\xi) - \frac{720}{\mu^7} \cos \lambda \end{aligned}$$

and

$$\begin{aligned} F_x(\xi) &= \Gamma_F(A_0, A_1, \xi) + \Gamma_F(B_0, B_1, \xi), \\ F_y(\xi) &= \Omega_F(A_0, A_1, \xi) - \Omega_F(B_0, B_1, \xi), \\ F_z(\xi) &= \Omega_F(a_0, a_1, \xi) \end{aligned} \tag{25}$$

with

$$\begin{aligned}\Gamma_F(\lambda, \mu, \xi) &= \frac{5(24 - 12\mu^2\xi^2 + \mu^4\xi^4)}{\mu^6} \cos(\lambda + \mu\xi) + \frac{\xi(120 - 20\mu^2\xi^2 + \mu^4\xi^4)}{\mu^5} \sin(\lambda + \mu\xi) - \frac{120}{\mu^6} \cos \lambda, \\ \Omega_F(\lambda, \mu, \xi) &= \frac{5(24 - 12\mu^2\xi^2 + \mu^4\xi^4)}{\mu^6} \sin(\lambda + \mu\xi) - \frac{\xi(120 - 20\mu^2\xi^2 + \mu^4\xi^4)}{\mu^5} \cos(\lambda + \mu\xi) - \frac{120}{\mu^6} \sin \lambda.\end{aligned}$$

Now, by noticing that $\gamma_G(\lambda, \mu) = \Gamma_G(\lambda, \mu, 1)$ and $\omega_G(\lambda, \mu) = \Omega_G(\lambda, \mu, 1)$, the expressions of $g_x = G_x(1)$, $g_y = G_y(1)$, $g_z = G_z(1)$ follow. Analogously, by noticing that $\gamma_F(\lambda, \mu) = \Gamma_F(\lambda, \mu, 1)$ and $\omega_F(\lambda, \mu) = \Omega_F(\lambda, \mu, 1)$, the expressions of $f_x = F_x(1)$, $f_y = F_y(1)$, $f_z = F_z(1)$ are obtained. Then, recalling that $\|\mathbf{T}'(0)\|$, $\|\mathbf{T}'(1)\|$, $-\mathbf{T}'(0) \cdot \mathbf{T}''(0)$ and $-\mathbf{T}'(1) \cdot \mathbf{T}''(1)$ are abbreviated with u_0 , u_1 , v_0 and v_1 , respectively (see Eq. (17) and Eq. (22)), and observing that the fourth and the sixth equation of the linear system yield the expressions of c_0 and c_1 , respectively, the formulation of the reduced linear system in (21) is recovered. $\square$

Remark 3. The coefficient matrices of the linear systems in Propositions 1 and 2 rely on the expressions of $A_x(\xi), B_x(\xi), C_x(\xi), D_x(\xi)$ (and the analogous expressions for y and z) given in [24] and in [3, Appendix A], as well as on the expressions of $E_x(\xi), E_y(\xi), E_z(\xi)$ in (19), $F_x(\xi), F_y(\xi), F_z(\xi)$ in (25) and $G_x(\xi), G_y(\xi), G_z(\xi)$ in (24), respectively. All of them are valid under the assumption $a_1 \notin \{0, \pm b_1\}$.

Remark 4. If the G^1 Hermite data of Proposition 1 and Proposition 2 are such that $a_1 = -b_1 \neq 0$ (i.e., $A_1 = 0$) and $B_1 \neq 0$ or $a_1 = b_1 \neq 0$ (i.e., $B_1 = 0$) and $A_1 \neq 0$, the values of $A_x, B_x, C_x, D_x, E_x, F_x, G_x$ (and analogous expressions for y and z) must be computed by means of suitable formulas that are collected in [3, Appendix B] and [3, Appendix C], respectively.

Remark 5. If the G^1 Hermite data of Proposition 1 are such that $a_1 = 0$, $b_1 \neq 0$ and $a_0 \notin \{-\pi/2, 0, \pi/2\}$, the values of $A_x, B_x, C_x, D_x, E_x, F_x, G_x$ (and analogous expressions for y and z) must be computed by means of suitable formulas that are collected in [3, Appendix D].

4. On the existence of spatial G^2 Hermite ATPH interpolants with polynomial parametric speed

The G^2 Hermite interpolation problem consists in constructing a spatial ATPH curve $\mathbf{r}(\xi)$, $\xi \in [0, 1]$ that fulfills the conditions in (8) together with the additional conditions

$$\frac{\mathbf{r}''(0)}{\|\mathbf{r}''(0)\|} = \mathbf{n}_1, \quad \frac{\mathbf{r}''(1)}{\|\mathbf{r}''(1)\|} = \mathbf{n}_2 \quad (26)$$

where $\mathbf{n}_1, \mathbf{n}_2$ are given 3-dimensional unit vectors.

We start by introducing a remark that identifies the osculating plane of $\mathbf{r}$, followed by a preliminary lemma that allows us to establish the restrictions that must be taken into account when selecting the G^2 Hermite input data.

Remark 6. The unit normal and the unit binormal of a spatial ATPH curve $\mathbf{r}(\xi)$, $\xi \in [0, 1]$ are defined as

$$\mathbf{N}(\xi) = \frac{\mathbf{T}'(\xi)}{\|\mathbf{T}'(\xi)\|}$$

and

$$\mathbf{B}(\xi) = \frac{\mathbf{T}(\xi) \times \mathbf{T}'(\xi)}{\|\mathbf{T}(\xi) \times \mathbf{T}'(\xi)\|},$$

respectively. Thus, the osculating plane of a spatial ATPH curve (usually spanned by $\mathbf{T}$ and $\mathbf{N}$), can be also seen as the plane spanned by $\mathbf{T}$ and $\mathbf{T}'$.

Lemma 3. Let $\mathbf{r}(\xi)$, $\xi \in [0, 1]$ be the spatial ATPH curve defined in (2). Then, for all $\xi \in [0, 1]$, $\mathbf{r}''(\xi)$ belongs to the osculating plane $\text{span}(\mathbf{T}(\xi), \mathbf{T}'(\xi))$.

Proof. By differentiating (4) we obtain

$$\mathbf{r}''(\xi) = \rho_d'(\xi) \mathbf{T}(\xi) + \rho_d(\xi) \mathbf{T}'(\xi) \quad (27)$$

which implies the claimed result. $\square$

As a consequence of Lemma 3, it turns out that the constraints in (26) cannot be satisfied for any arbitrarily chosen 3-dimensional unit vectors $\mathbf{n}_1, \mathbf{n}_2$ provided as input. Indeed, the two conditions in (26) can be satisfied only if $\mathbf{n}_1 \in \text{span}(\mathbf{T}(0), \mathbf{T}'(0))$ and $\mathbf{n}_2 \in \text{span}(\mathbf{T}(1), \mathbf{T}'(1))$.

In the next proposition we focus on the construction of the lowest degree polynomial function ρ_d that allows us to obtain spatial ATPH interpolants to G^2 Hermite data.

Proposition 3. Let $\mathbf{P}_1 \neq \mathbf{P}_2$ be given points in $\mathbb{R}^3$ specified by the Cartesian coordinates $(x_i, y_i, z_i)^T$, $i = 1, 2$, and let $\mathbf{t}_1, \mathbf{t}_2$ be given 3-dimensional unit vectors described by the spherical coordinates in (9). Let also $\mathbf{n}_1$ and $\mathbf{n}_2$ be given 3-dimensional unit vectors belonging respectively to the osculating planes $\Pi_1 := \text{span}(\mathbf{T}(0), \mathbf{T}'(0))$ and $\Pi_2 := \text{span}(\mathbf{T}(1), \mathbf{T}'(1))$ that turn out to be univocally defined by the space curve $\mathbf{T}(\xi)$ lying on the unit sphere and interpolating $\mathbf{t}_1$ and $\mathbf{t}_2$ at $\xi = 0$ and $\xi = 1$. The lowest possible degree of the polynomial ρ_d that satisfies $\rho_d'(\xi) \neq 0$ for $\xi \in \{0, 1\}$ and allows the construction of an ATPH curve as in (2) which solves the G^2 Hermite interpolation problem (8)-(26), is $d = 4$. Moreover, the unknown coefficients of ρ_4 can be obtained by solving the linear system

$$\begin{pmatrix} e_x & a_x & b_x & c_x & d_x \\ e_y & a_y & b_y & c_y & d_y \\ e_z & a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & -\sin(\alpha_1) & u_0 \cos(\alpha_1) \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 \end{pmatrix} \begin{pmatrix} c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 2(x_2 - x_1) \\ 2(y_2 - y_1) \\ z_2 - z_1 \\ 0 \\ 0 \end{pmatrix} \quad (28)$$

where

$$\ell_i = u_1 \cos(\alpha_2) - (5 - i) \sin(\alpha_2), \quad \text{for } i = 1, \dots, 5,$$

$$\cos(\alpha_1) = \mathbf{n}_1 \cdot \mathbf{T}(0), \quad \sin(\alpha_1) = \mathbf{n}_1 \cdot \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|},$$

$$\cos(\alpha_2) = \mathbf{n}_2 \cdot \mathbf{T}(1), \quad \sin(\alpha_2) = \mathbf{n}_2 \cdot \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|},$$

and $u_0, u_1, e_x, d_x, c_x, b_x, a_x$ (as well as the analogous expressions for y and z) are the ones of Proposition 1.

Proof. Due to the fact that $\mathbf{r}(0) = \mathbf{r}_0 = \mathbf{P}_1$, we can fulfill the first interpolation condition at the endpoint $\xi = 0$ by simply selecting $\mathbf{r}_0 = \mathbf{P}_1$. Moreover, the interpolation of the unit tangent vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ is achieved by suitably constructing $\mathbf{T}(\xi)$ as recalled in Section 2. Now, in order to identify the minimum degree of the polynomial $\rho_d(t)$ that is needed to fulfill the remaining conditions, we must figure out how many scalar equations are left to satisfy. First, we observe that the condition $\mathbf{r}(1) = \mathbf{P}_2$ amounts to three scalar equations. Then, we show that the two remaining conditions (26) give rise to two scalar equations, so that we need to work with a polynomial $\rho_d(t)$ of degree $d = 4$. Indeed, since $\mathbf{n}_1 \in \text{span}(\mathbf{T}(0), \mathbf{T}'(0))$ and $\mathbf{T}(0) \perp \mathbf{T}'(0)$ as well as $\|\mathbf{n}_1\| = \|\mathbf{T}(0)\| = 1$, we can write

$$\mathbf{n}_1 = \cos(\alpha_1) \mathbf{T}(0) + \sin(\alpha_1) \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}. \quad (29)$$

Taking into account that, in light of (27), we also have

$$\mathbf{r}''(0) = \rho'_4(0) \mathbf{T}(0) + \rho_4(0) \|\mathbf{T}'(0)\| \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}, \quad (30)$$

we can interpret the first constraint in (26) as the collinearity of $\mathbf{r}''(0)$ and $\mathbf{n}_1$. Since, due to (29) and (30), the two vectors are both expressed in the orthonormal basis $\left\{ \mathbf{T}(0), \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|} \right\}$, they turn out to be collinear if

$$\cos(\alpha_1) \|\mathbf{T}'(0)\| \rho_4(0) - \sin(\alpha_1) \rho'_4(0) = 0. \quad (31)$$

Analogously, since $\mathbf{n}_2 \in \text{span}(\mathbf{T}(1), \mathbf{T}'(1))$ and $\mathbf{T}(1) \perp \mathbf{T}'(1)$ as well as $\|\mathbf{n}_2\| = \|\mathbf{T}(1)\| = 1$, we can write

$$\mathbf{n}_2 = \cos(\alpha_2) \mathbf{T}(1) + \sin(\alpha_2) \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|}$$

and, in light of (27), we also have

$$\mathbf{r}''(1) = \rho'_4(1) \mathbf{T}(1) + \rho_4(1) \|\mathbf{T}'(1)\| \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|}.$$

We can thus interpret the second constraint in (26) as the collinearity of $\mathbf{r}''(1)$ and $\mathbf{n}_2$ in the orthonormal basis $\left\{ \mathbf{T}(1), \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|} \right\}$, which gives rise to the condition

$$\cos(\alpha_2) \|\mathbf{T}'(1)\| \rho_4(1) - \sin(\alpha_2) \rho'_4(1) = 0. \quad (32)$$

Now, due to the fact that

$$\begin{aligned} \rho_4(0) &= c_0, & \rho_4(1) &= c_4 + c_3 + c_2 + c_1 + c_0, \\ \rho'_4(0) &= c_1, & \rho'_4(1) &= 4c_4 + 3c_3 + 2c_2 + c_1, \end{aligned}$$

we can finally obtain from (31)-(32) the pair of conditions

$$\begin{aligned} \cos(\alpha_1) \|\mathbf{T}'(0)\| c_0 - \sin(\alpha_1) c_1 &= 0, \\ \cos(\alpha_2) \|\mathbf{T}'(1)\| (c_4 + c_3 + c_2 + c_1 + c_0) - \sin(\alpha_2) (4c_4 + 3c_3 + 2c_2 + c_1) &= 0. \end{aligned}$$

These last two scalar equations, together with the three scalar equations originated from $\mathbf{r}(1) - \mathbf{r}(0) = \mathbf{P}_2 - \mathbf{P}_1$, lead to the linear system

$$\begin{pmatrix} e_x & a_x & b_x & c_x & d_x \\ e_y & a_y & b_y & c_y & d_y \\ e_z & a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & -\sin(\alpha_1) & \|\mathbf{T}'(0)\| \cos(\alpha_1) \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 \end{pmatrix} \begin{pmatrix} c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 2(x_2 - x_1) \\ 2(y_2 - y_1) \\ z_2 - z_1 \\ 0 \\ 0 \end{pmatrix}$$

where

$$\ell_i = \|\mathbf{T}'(1)\| \cos(\alpha_2) - (5 - i) \sin(\alpha_2), \quad \text{for } i = 1, \dots, 5.$$

Hence, recalling that the expressions of $\|\mathbf{T}'(0)\|$ and $\|\mathbf{T}'(1)\|$ are shortly denoted with u_0 and u_1 , respectively (see Eq. (17)), the proof is completed. $\square$

5. Numerical experiments and comparisons

In the first subsection we include some application examples of Proposition 1 and show how our spatial ATPH curves compare favourably with the G^1 Hermite ATPH interpolants proposed by Wu and Yang

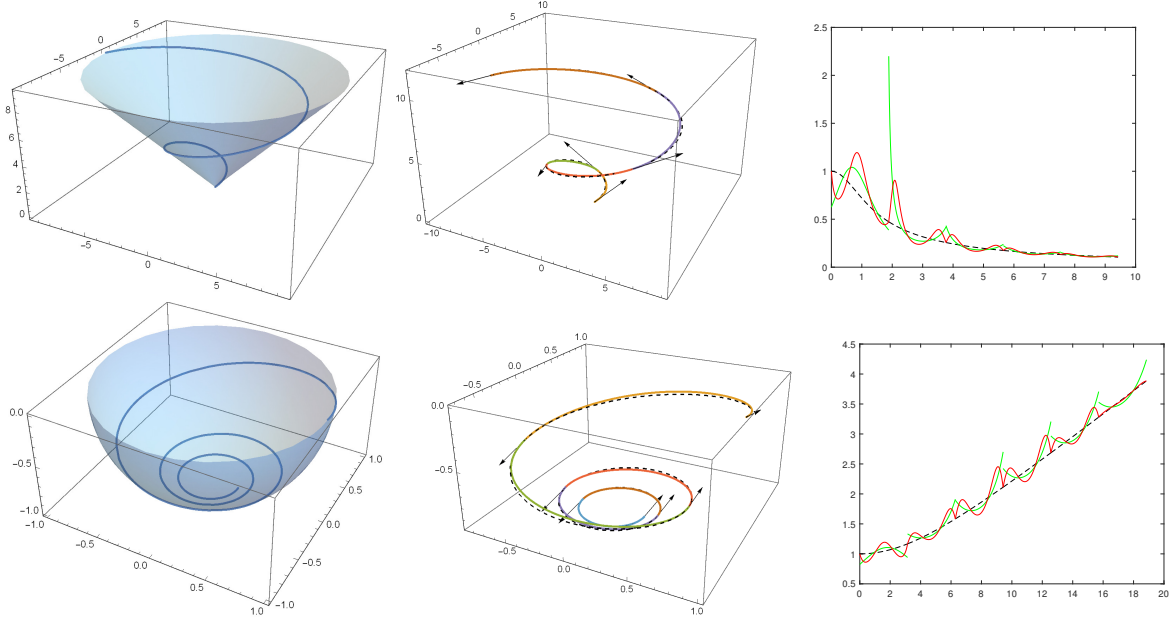


Figure 2: ATPH interpolants to G^1 Hermite data, obtained by applying the result in Proposition 1. In first column: trigonometric surface from which the data were sampled and reconstructed spatial ATPH curve; in second column: reference curve (dashed black) and ATPH interpolant (one color per segment); in third column: reference curvature (dashed black), curvature of Wu and Yang ATPH interpolant (solid green) and curvature of our ATPH interpolant (solid red). First row is related to dataset (a) and second row to dataset (b).

[24], both in terms of curvature behaviour and closer approximation to the reference curve. In order to show further improvements of the curvature behaviour of our ATPH interpolants, in the second subsection we include two application examples of Proposition 2. Finally, in the third subsection, we focus on the interpolation of G^2 Hermite data and illustrate two application examples of Proposition 3. For the sake of clarity, we point out that Proposition 3 cannot be applied to planar G^2 Hermite data since, in this special subcase, the matrix of the linear system in (28) turns out to be singular. In the same way, the linear systems (16) and (21), related to Proposition 1 and 2, are also singular when applied to planar input data.

5.1. G^1 Hermite interpolation: application examples of Proposition 1

First we consider spatial G^1 Hermite data which fulfill the condition $a_1 \notin \{0, \pm b_1\}$. In particular, we sample the G^1 Hermite data from:

- (a) the conical spiral with parametric representation $(t \cos(t), t \sin(t), t)$, $t \in [0, 3\pi]$, by considering as sampling parameter values $\frac{3\pi i}{5}$, $i = 0, \dots, 5$;
- (b) the spherical spiral with parametric representation $(\cos(t) \cos(\arctan(0.2t)), \sin(t) \cos(\arctan(0.2t)), -\sin(\arctan(0.2t)))$, $t \in [0, 6\pi]$, by considering as sampling parameter values πi , $i = 0, \dots, 6$.

Figure 2 shows that, for the input data in (a) and (b), Wu and Yang Hermite ATPH interpolant fails to guarantee continuity of curvature, which translates into a worse approximation of the reference curve.

Then we take spatial G^1 Hermite data such that $a_1 = 0$, $b_1 \neq 0$, $a_0 \notin \{-\pi/2, 0, \pi/2\}$. These turn out to be obtained by sampling

- (c) the circular helix with parametric representation $(\cos(t), \sin(t), t)$, $t \in [0, 2\pi]$, at the parameter values πi , $i = 0, 1, 2$.

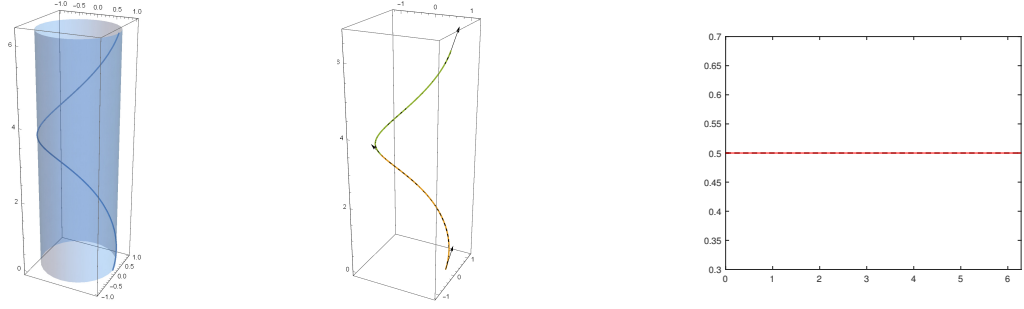


Figure 3: ATPH interpolant to G^1 Hermite data, obtained by applying the result in Proposition 1 with $a_1 = 0$, $b_1 \neq 0$, $a_0 \notin \{-\pi/2, 0, \pi/2\}$. In first column: trigonometric surface from which the data were sampled and reconstructed spatial ATPH curve; in second column: reference curve (dashed black) and our ATPH interpolant (one color per segment); in third column: reference curvature (dashed black) and curvature of our ATPH interpolant (solid red).

As Figure 3 shows, since the circular helix falls into the class of spatial trigonometric curves exactly reproduced by our ATPH model, the constant curvature of the piecewise Hermite ATPH interpolant perfectly coincides with that of the reference curve.

According to the authors, the fulfillment of the condition $a_1 = 0$ on several consecutive curve segments is so restrictive that it's difficult to conjecture the existence of many other trigonometric curves providing such a result.

5.2. G^1 Hermite interpolation: application examples of Proposition 2

Proposition 2 is illustrated by two examples, as can be seen from Figure 4. In particular, we sample the G^1 Hermite data from:

- (d) the ellipsoid curve with parametric representation $(4 \cos(t)^3 - \frac{1}{2} \cos(t), \frac{1}{2} \sin(t) - 4 \sin(t)^3, -\sin(2t))$, $t \in [0, 2\pi]$, at the parameter values $\frac{\pi i}{2}$, $i = 0, \dots, 4$;
- (e) the conical spiral with parametric representation $(t \cos(t), t \sin(t), t)$, $t \in [\pi/3, \pi]$, at the parameter values $\frac{(2i+5)\pi}{15}$, $i = 0, \dots, 5$.

Concerning the first example, the junction parameters $\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ of the curve segments are chosen in areas with high curvature variation. We can notice that this choice does not affect the reconstruction: the points are not singular and are correctly interpolated by our solution.

5.3. G^2 Hermite interpolation: application examples of Proposition 3

The result of Proposition 3 is tested on two different input data. The first dataset is:

- $\mathbf{P}_1 = (-3.0978, -0.0616, -0.153)^T$, $\mathbf{P}_2 = (-3.0692, 0.0209, 0.0551)^T$, $\mathbf{P}_3 = (-3.1542, 0.037, 0.0649)^T$,
 $\mathbf{P}_4 = (-3.1534, 0.0059, 0.0190)^T$;
- $\mathbf{t}_1 = (0.0291, 0.0017, 0.9996)^T$, $\mathbf{t}_2 = (-0.4015, 0.757, 0.5155)^T$, $\mathbf{t}_3 = (-0.5455, -0.7117, -0.4425)^T$,
 $\mathbf{t}_4 = (0.1093, -0.025, -0.9937)^T$;
- $\mathbf{n}_1 = (0.5981, 0.0715, -0.7982)^T$, $\mathbf{n}_2 = (-0.6798, -0.4369, -0.589)^T$, $\mathbf{n}_3 = (0.9382, -0.1619, -0.306)^T$,
 $\mathbf{n}_4 = (-0.8679, 0.4051, 0.2873)^T$.

The second dataset is:

- $\mathbf{P}_1 = (0.0209, 0.0025, 0.0013)^T$, $\mathbf{P}_2 = (0.2293, 0.3011, 0.2157)^T$, $\mathbf{P}_3 = (0.0838, -0.2609, 0.3791)^T$;
- $\mathbf{t}_1 = (0.9752, 0.1977, 0.0998)^T$, $\mathbf{t}_2 = (-0.0203, 0.6964, 0.7174)^T$, $\mathbf{t}_3 = (-0.07, 0.01, 0.9975)^T$;
- $\mathbf{n}_1 = (0.8007, 0.5333, 0.273)^T$, $\mathbf{n}_2 = (-0.7585, -0.1767, 0.6273)^T$, $\mathbf{n}_3 = (0.8878, -0.2609, 0.3791)^T$.

For each considered dataset, the resulting ATPH Hermite interpolant and the associated curvature plot are displayed in Figure 5.

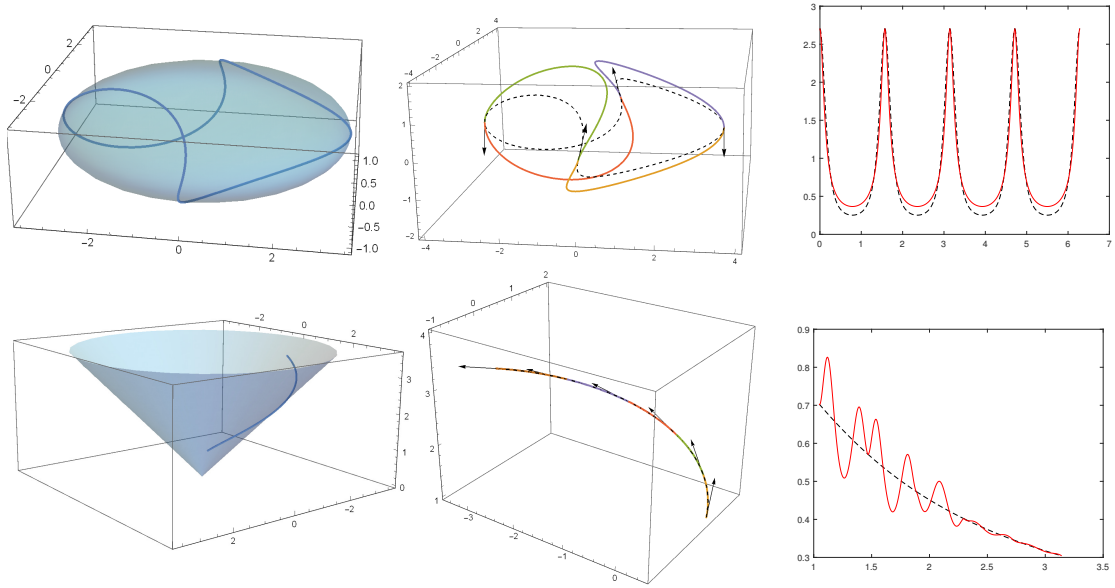


Figure 4: ATPH interpolants to G^1 Hermite data, obtained by applying the result in Proposition 2. In first column: trigonometric surface from which the data were sampled and reconstructed spatial ATPH curve; in second column: reference curve (dashed black) and our ATPH interpolant (one color per segment); in third column: reference curvature (dashed black) and curvature of our ATPH interpolant (solid red). First row is related to dataset (d) and second row to dataset (e).

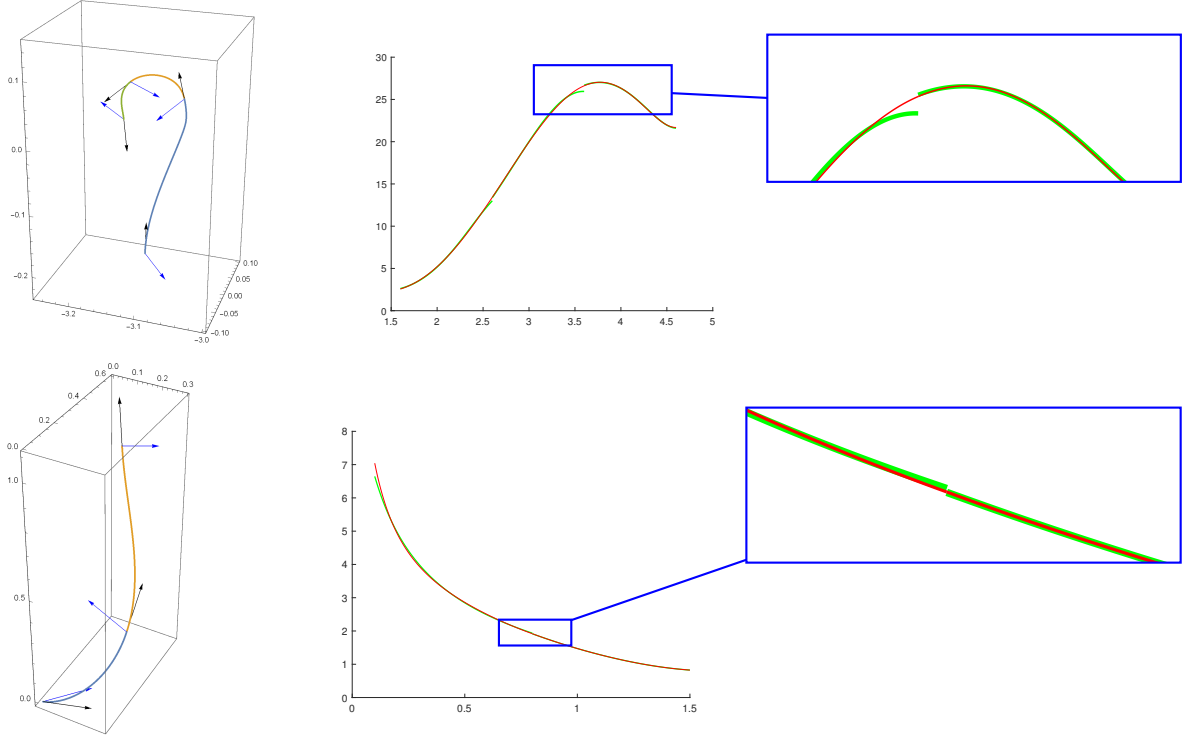


Figure 5: Left: ATPH interpolant to G^2 Hermite data, i.e., points $\{\mathbf{P}_i\}$ and unit vectors $\{\mathbf{t}_i\}$, $\{\mathbf{n}_i\}$. Right: curvature of the resulting ATPH interpolant (solid red) versus curvature of Wu and Yang ATPH interpolant (solid green).

6. Closing remarks

In this paper we have investigated how to solve the G^1 Hermite interpolation problem of both open and closed spatial datasets by means of ATPH curves with polynomial parametric speed. With respect to the solution proposed in [24], the spatial ATPH curves constructed in our paper have either C^0 - or C^1 -continuous curvature (depending if the degree of the associated polynomial function ρ_d is $d = 4$ or $d = 6$). As to the interpolation of spatial G^2 Hermite data, we have shown that the lowest degree solution is achieved by means of a polynomial function ρ_d with $d = 4$, but its existence requires some restrictions on the input data.

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