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# The problem of detecting nonlinearity in time series generated by a state-dependent autoregressive model. A simulation study

Fabio Gobbi\*

## Abstract

The aim of the paper is to try to measure, through a Monte Carlo experiment, nonlinearity in time series generated by a strictly stationary and uniformly ergodic state-dependent autoregressive process. The model under study is intrinsically nonlinear but the choice of parameters strongly impacts on the type of serial dependence making its identification complicated. For this reason, the paper exploits two statistical tests of independence and linearity in order to select the parameter values which ensure the joint rejection of both hypothesis. After that, our study uses two measures of nonlinear dependence in time series recently introduced in the literature, the auto-distance correlation function and the autodependence function, in order to identify nonlinearity induced by the proposed model.

Mathematics Subject Classification (2010): 62M10, 65C05

JEL classification: C6

**Keywords:** nonlinear time series, independence and linearity tests, auto-distance correlation, autodependogram.

## 1 Introduction

In a recent paper Gobbi and Mulinacci (2019) have studied a state-dependent first-order autoregressive model (SDAR(1)) in which the autoregressive coefficient is specified as a function of the lagged variable and of a set of parameters,  $Y_t = \alpha + \psi(Y_{t-1}; \gamma)Y_{t-1} + \xi_t$ . The authors stated under what hypotheses the process  $(Y_t)_t$  is strictly stationary, uniformly ergodic and can be consistently estimated using the quasi-maximum likelihood technique (QML) showing how these conditions are related to the functional form of the coefficient  $\psi$  but

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also to the interval of possible values assumed by the parameters  $\gamma$ . Since the SDAR(1) model is intrinsically nonlinear, from a practical point of view the most relevant aspect is the model's ability to generate time series with nonlinear serial dependence that is easily recognizable. However, as we shall see, nonlinearity can be "hidden" even when the values of the parameters are adequately chosen. In this paper we present a Monte Carlo study to detect nonlinearity in time series generated from a SDAR(1) model once the functional form of  $\psi$  is established. The question if a time series arises nonlinearity in its serial dependence is crucial in economics and econometrics. Therefore, it is interesting to test whether or not a single economic time series appears to be generated by a linear model against the alternative that it is generated by a nonlinear model. In fact, nonlinearity seems to be adequate in explaining many properties observed in empirical time series such as volatility clustering, asymmetric cycles, extreme value dependence among others. Furthermore, the study of serial dependence is often a preliminary step carried out before modelling the data generating process in order to select the most adequate model; for example, is a common practice in finance to study serial dependence on returns of log-prices or exchange rates, see among others Bera and Robinson (1989), Booth Teppo and Yli-Olli (1994) and Harris and Küçüközmen (2001).

Inspired by these considerations, in the last few years two measures of nonlinearity in time series have been introduced into the literature: the distance correlation function and the autodependence function. The first measure was introduced by Székely et al. (2007) and discussed in Székely and Rizzo (2009), Gretton et al. (2009) and Remillard (2009). The authors propose the distance correlation to capture and test nonlinear dependence between two samples generated by an i.i.d. process whereas there are few works on how to extend a distance correlation methodology to the analysis of the temporal dependence in univariate time series. Among these we mention Zhou (2012), Dueck et al. (2014) and Davis et al. (2016). In particular, Zhou (2012) extends the concept of distance correlation to the serial dependence in time series and provides a corresponding measure, the auto-distance correlation function (ADCF), to explore and test nonlinear dependence structures in time series whose crucial feature is that it equals zero if and only if the measured time series components are independent. Moreover, the author studies the asymptotic behaviour of ADCF at a fixed lag order. Psitsillou and Fokianos (2016) provide and discuss the implementation of ADCF methodology with the R package *dCovTS*.

The second measure that we use in this work is the autodependence function (ADF) introduced by Bagnato et al. (2012). The authors graphically analyse the serial dependence between the observed time series and the lagged series. In this way, they fill a theoretical void represented by the fact that, differently from the linear case where the autocorrelogram provides a graphical resource to investigate serial correlation, in the case of general serial dependence we do not have an analogous of this graphical device. Only few attempts are known

in literature as in Genest and Remillard (2004), Bagnato and Punzo (2010) and Fisher and Switzer (1985). The first two papers suggest two diagrams allowing for a visual inspection of the subsets of lags leading to a possible rejection of the null hypothesis of serial independence. On the other hand, Fisher and Switzer (1985) present a graphical method to investigate possible dependence but in a sample of bivariate measurements. Differently, Bagnato et al. (2012) propose a graphical representation, called autodependogram, that can be really considered as the analogous of the autocorrelogram when the dependence structure to be analysed is the distributive one. The proposal applies the  $\chi^2$ -tset to pairs of lagged variables. Bagnato et al. (2015) provide an R package (SDD) to implement the ADF methodology.

As a preliminary step in our study, we concentrate in investigating independence and linearity in serial dependence of time series. A large number of independence and linearity tests are available in the literature. The interested reader can consult, among others, Granger and Andersen (1978), Granger et al. (2004), Dufour et al. (1982), McLeod and Li (1983), Brock et al. (1987), Brock et al. (1996), Takala and Viren (1996) and LeBaron (1997). In this work we use two different types of independence and linearity tests. The first test of independence we perform is introduced in Fokianos and Pitsillou (2017) who consider a portmanteau type statistic directly based on ADCF. The test is based on the null that the observed time series is generated by an i.i.d. process. The second test of linearity was introduced by White (1989) and thoroughly studied in Lee et al. (1993) and consists in a neural network test for neglected nonlinearity. More specifically, this test is a Lagrange multiplier test that statistically determines whether adding ‘hidden units’ to the linear network would be advantageous. The null hypothesis in this case is that data process is linear in mean based on the definition that the same authors report and discuss. Thanks to both tests we will show for which parameter values of our simulated models the hypotheses of independence (in the first case) and linearity (in the second case) can be simultaneously rejected, thus highlighting the possibility that the serial dependence is of nonlinear type.

The paper is organised as follows. In section 2 we introduce the SDAR(1) from a theoretical point of view. Section 3 describes the adopted models for the simulation experiment, reports and comments the results. Section 4 concludes.

## 2 The SDAR(1) model

Let  $(Y_t)_{t \in \mathbb{N}}$  be a stochastic process defined on a complete probability space  $(\mathbb{R}^\infty, \mathfrak{B}(\mathbb{R}^\infty), \mathbb{P})$ . We say that  $(Y_t)_t$  is a SDAR(1) process if it satisfies the following specification

$$\begin{cases} Y_t = \alpha + \psi(Y_{t-1}; \gamma)Y_{t-1} + \xi_t, & t \geq 1, \\ \xi_t \sim i.i.d., \end{cases} \quad (1)$$

where  $Y_0 = 0$   $\mathbb{P}$ -a.s. and  $\psi(Y_{t-1}; \gamma)$  is a measurable function of the lagged variable  $Y_{t-1}$  that depends on a  $p$ -dimensional vector of parameters  $\gamma$ . The sequence of error terms,  $(\xi_t)_t$ , is i.i.d. with gaussian distribution characterized by zero mean and finite standard deviation  $\sigma$ . Gobbi and Mulinacci (2019) have proved that under suitable conditions on the function  $\psi(y; \gamma)$  the SDAR(1) process  $(Y_t)_t$  is strictly stationary and uniformly ergodic. Moreover, the authors present and discuss the asymptotic properties of the QML estimator of the model parameter vector  $\theta = (\alpha, \gamma, \sigma)$ . As we can imagine, the conditions imposed concern the functional form of  $\psi$  and the range of values that the set of parameters may assume. For completeness, we report the required assumptions as reported in Gobbi and Mulinacci (2019).

- **a1.**  $\psi$  is differentiable with respect to  $y$  and  $|\psi(y; \gamma)| + \left| y \frac{d}{dy} \psi(y; \gamma) \right| \leq K < 1 \ \forall y \in \mathbb{R}$
- **a2.**  $\psi(y; \gamma)y$  is uniformly bounded in  $y$ .
- **a3.** First and second-order partial derivatives with respect to the parameters of the persistence function  $\psi$  are continuous and uniformly bounded in  $y$ .
- **a4.**  $|\psi^{\gamma_k}(y; \gamma)y| \leq C$  uniformly on  $\mathbb{R} \times \Theta$ , for all  $k$  and  $|\psi^{\gamma_k \gamma_j}(y; \gamma)y| \leq D$  uniformly on  $\mathbb{R} \times \Theta$ , for all  $k, j$ , where  $\Theta$  is the compact parameter space.

Notice that the SDAR(1) model is intrinsically nonlinear and the dynamics depends on the specification of the function  $\psi(y; \gamma)$ . However it is necessary to observe that the structure of the model is of the type "coefficient times lagged variable", very similar to the linear AR(1). Therefore, nonlinearity induced by the functional coefficient  $\psi$  can be confused with the linearity that comes from the AR(1) structure of the model.

### 3 Simulation design

The aim of the simulation experiment is to detect nonlinearity generated by 8 different specifications of the SDAR(1) model characterized primarily by a different choice of the function  $\psi$ . Indeed, in this simulation experiment we also consider an extension of the model in which the sequence of errors can be i.i.d. with Student's  $t$  distribution with 5 degrees of freedom. In this way we can appreciate the impact of a distribution characterized by heavy tails on the serial dependence. The explicit form of the simulated models (which is denoted M1-M8) is the following

$$M1 - M4 : \begin{cases} Y_t = \alpha + e^{-(\gamma_0 + \gamma_1 Y_{t-1}^{2r_1})} Y_{t-1} + \xi_t, \\ \xi_t \stackrel{i.i.d.}{\sim} N(0, \sigma) \quad \text{or} \quad \xi_t \stackrel{i.i.d.}{\sim} t_{(5)}, \\ r = 1, 2 \end{cases}$$

$$M5 - M8 : \begin{cases} Y_t = \alpha + \frac{1}{\gamma_0 + \gamma_1 Y_{t-1}^{2r}} Y_{t-1} + \xi_t, \\ \xi_t \stackrel{i.i.d.}{\sim} N(0, \sigma) \quad \text{or} \quad \xi_t \stackrel{i.i.d.}{\sim} t_{(5)}, \\ r = 1, 2 \end{cases} .$$

Clearly, both specifications of  $\psi$  satisfy assumptions **a1-a4** presented in the previous section. In particular, as shown in Gobbi and Mulinacci (2019), the parameter  $\gamma_1$  has no impact on the properties of stationarity, ergodicity and efficiency of QML estimates, which, on the contrary, depend on  $\gamma_0$  and  $r$ . For this reason and in order to make consistent our simulation study,  $\gamma_0$  will have a fixed value depending on the model considered. More precisely, we set two values of  $r$  ( $r = 1, 2$ ) and based on them we fix the value of  $\gamma_0$  according to the condition found in Gobbi and Mulinacci (2019). For models M1-M4 we have  $\gamma_0 > \ln(2r) - 1 + \frac{1}{2r}$ , whereas for models M5-M8 we have  $\gamma_0 > \frac{(1+2r)^2}{8r}$ . With this conditions in mind, we set  $\gamma_0 = 0.2$  for models M1-M2,  $\gamma_0 = 0.42$  for models M3-M4,  $\gamma_0 = 1.13$  for models M5-M6 and finally  $\gamma_0 = 1.57$  for models M7-M8. To make our simulations not too expansive in terms of calculation time, we set a discrete set of possible values of  $\gamma_1$ . Table 1 summarizes the adopted models and specifies the parameters values.

It is the case to notice that models differ in the amount of nonlinearity in the serial dependence induced by the specification of  $\psi$ . In all specifications proposed the parameter of nonlinearity is  $\gamma_1$ , in the sense that, the presence of nonlinearity in the serial dependence is closely connected to the value of  $\gamma_1$ . Moreover, notice that if  $\gamma_1 = 0$  all models considered in table 1 are equivalent to a standard AR(1) process of equation  $Y_t = \alpha + \phi Y_{t-1} + \xi_t$ , where  $\phi = e^{-\gamma_0}$  for models M1-M4 and  $\phi = \frac{1}{\gamma_0}$  for models M5-M8. It is clear that in this case the serial dependence can only be linear. On the contrary, when  $\gamma_1$  assumes very large values (theoretically when  $\gamma_1 \rightarrow +\infty$ ) all models under study are equivalent to an i.i.d. sequence.

As anticipated above, with our simulation study we are interested in investigating how the parameter  $\gamma_1$  affects nonlinearity in simulated time series. The study consider a sample size  $n = 500$  and it is based on  $S = 5000$  replications.

### 3.1 Independence and linearity tests

In this section we compute test statistics of two different independence and linearity tests: the independence test introduced in Fokianos and Pitsillou (2017) and the White neural network test considered in White (1989) and in Lee et al. (1993). The interested reader can find the explicit expressions of the test statistic and their asymptotic behaviour in the mentioned papers. To perform the two tests on simulated trajectories we use the R environment. The R functions necessary to compute the independence and linearity tests are 'UnivTest' from the package 'dCovTS' and 'wnnTest' from the package 'fNonlinear' respectively. The first independence test of Fokianos and Pitsillou (2017) (from now on ADCFind) is a portmanteau type statistic based on

Identifier	$\psi(y; \gamma_0, \gamma_1)$	Error distribution	Parameters
M1	$r = 1, e^{-(\gamma_0 + \gamma_1 y^2)}$	$\xi_t \sim N(0, 1)$	$\gamma_0 = 0.2, \gamma_1 \in \{0, 0.1, 0.2, \dots, 2.5\}$
M2	$r = 1, e^{-(\gamma_0 + \gamma_1 y^2)}$	$\xi_t \sim t_{(5)}$	$\gamma_0 = 0.2, \gamma_1 \in \{0, 0.1, 0.2, \dots, 2.5\}$
M3	$r = 2, e^{-(\gamma_0 + \gamma_1 y^4)}$	$\xi_t \sim N(0, 1)$	$\gamma_0 = 0.42, \gamma_1 \in \{0, 0.1, 0.2, \dots, 2.5\}$
M4	$r = 2, e^{-(\gamma_0 + \gamma_1 y^4)}$	$\xi_t \sim t_{(5)}$	$\gamma_0 = 0.42, \gamma_1 \in \{0, 0.1, 0.2, \dots, 2.5\}$
M5	$r = 1, \frac{1}{\gamma_0 + \gamma_1 y^2}$	$\xi_t \sim N(0, 1)$	$\gamma_0 = 1.13, \gamma_1 \in \{0, 0.2, 0.4, \dots, 5\}$
M6	$r = 1, \frac{1}{\gamma_0 + \gamma_1 y^2}$	$\xi_t \sim t_{(5)}$	$\gamma_0 = 1.13, \gamma_1 \in \{0, 0.2, 0.4, \dots, 5\}$
M7	$r = 2, \frac{1}{\gamma_0 + \gamma_1 y^4}$	$\xi_t \sim N(0, 1)$	$\gamma_0 = 1.57, \gamma_1 \in \{0, 0.2, 0.4, \dots, 5\}$
M8	$r = 2, \frac{1}{\gamma_0 + \gamma_1 y^4}$	$\xi_t \sim t_{(5)}$	$\gamma_0 = 1.57, \gamma_1 \in \{0, 0.2, 0.4, \dots, 5\}$

Table 1: Adopted models.

ADCF which tests the null hypothesis that the data generating process  $(Y_t)_t$  is i.i.d., whereas the second test of linearity of White (1989) (from now on Wnn) requires to precise about the meaning of the word linearity. In fact, as specified in Lee et al. (1993), the focus is on a property best described as linearity in conditional mean. In our autoregressive framework we can define that our process  $(Y_t)_t$  is linear in mean conditional on  $Y_{t-1}$ , if

$$\mathbb{P}(\mathbb{E}[Y_t|Y_{t-1}] = \phi_0 + \phi_1 Y_{t-1}) = 1, \quad \text{for some } (\phi_0, \phi_1) \in \mathbb{R}^2.$$

This is the null hypothesis of the Wnn test. The alternative of interest is that  $(Y_t)_t$ , is not linear in mean conditional on  $Y_{t-1}$ , so that

$$\mathbb{P}(\mathbb{E}[Y_t|Y_{t-1}] = \phi_0 + \phi_1 Y_{t-1}) < 1, \quad \text{for all } (\phi_0, \phi_1) \in \mathbb{R}^2.$$

When the alternative is true, a linear model is said to suffer from neglected nonlinearity. Notice that the specifications of the SDAR(1) model considered in table 1 satisfy the null hypothesis if  $\gamma_1 = 0$  or if  $\gamma_1 \rightarrow +\infty$ . In the first case, models M1-M4 are linear in mean since  $\phi_0 = \alpha$  and  $\phi_1 = e^{-\gamma_0}$  whereas models M5-M8 are linear in mean since  $\phi_0 = \alpha$  and  $\phi_1 = \frac{1}{\gamma_0}$ . In the second case, all models considered in this study are equivalent to an i.i.d. process which satisfies the definition of linearity in mean by taking  $\phi_0 = \alpha$  and  $\phi_1 = 0$ .

Results of the Monte Carlo simulation are displayed in figures 1 and 2, which report the average of simulated  $p$ -values as a function of  $\gamma_1$  relative to ADCFind test and to Wnn test. Figure 1 takes into account the first four models under consideration, M1-M4. We are interested in identifying which intervals of values of  $\gamma_1$  lead to the rejection of both null hypothesis simultaneously. Consider the upper panel of the figure 1. We can see that the dynamics of  $p$ -values is affected by the value of  $r$  and by the distribution of the errors. If we consider models M1 and M2 (for which  $r = 1$ ) the null hypothesis of independence is rejected for  $\gamma_1 \in [0, 1.3]$  if errors are normally distributed and for all  $\gamma_1 \in [0, 1]$  if errors are Student's  $t$  distributed. For

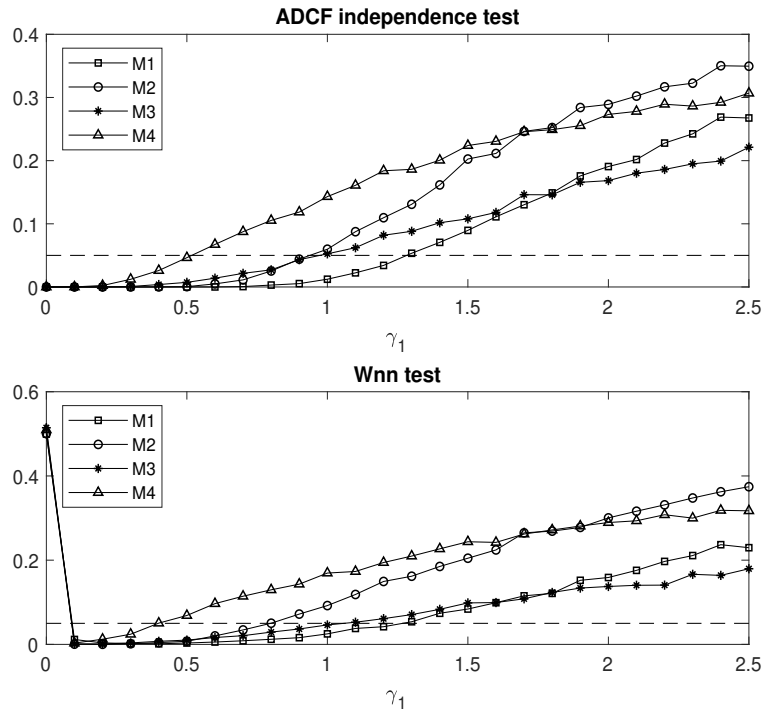


Figure 1: Average of simulated  $p$ -values as a function of  $\gamma_1$  for ADCFind test (upper panel) and for Wnn test of linearity (lower panel) relative to models M1-M4. The horizontal dotted line is the threshold of 5%.



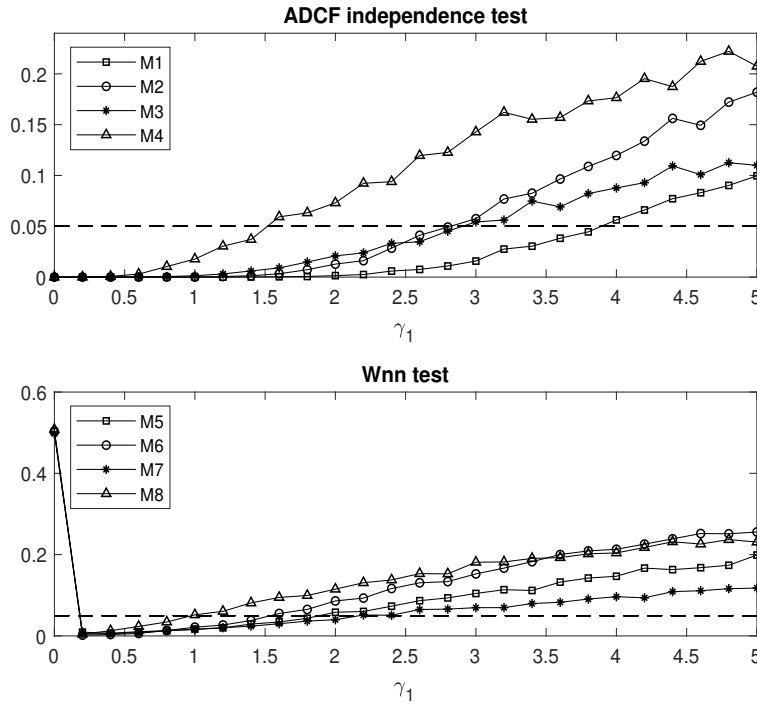


Figure 2: Average of simulated  $p$ -values as a function of  $\gamma_1$  for ADCFind test (upper panel) and for Wnn test of linearity (lower panel) relative to models M5-M8. The horizontal dotted line is the threshold of 5%.

model M3 and M4, where we are set  $r = 2$ , the range of values of the parameter shrinks,  $\gamma_1 \in [0, 1.1]$  for M3 and  $\gamma_1 \in [0, 0.4]$  for M4. This leads to at least three considerations. The first value of  $\gamma_1$  beyond which we cannot reject the hypothesis of independence is relatively low for all considered models M1-M4. In particular, for M4, it seems that only if  $\gamma_1$  is within  $[0, 0.4]$  the model yields non i.i.d. time series. A second consideration concerns the role of the error. Both for M1 and for M3, in which the error term has a gaussian distribution, the range of values of  $\gamma_1$  for which  $p$ -values does not exceed 5% (we reject the null) is wider than in the case of models M2 and M4 where the error term is  $t_{(5)}$  distributed. Finally, a third observation may be the most significant for the rest of our study. In fact, except for the case  $\gamma_1 = 0$  for which all considered models are AR(1) and so they point out serial correlation, the ranges of  $\gamma_1$  considered above potentially intercept a nonlinearity in simulated time series. This belief is reinforced by the observation of the lower panel of figure 1 where we report the average of simulated  $p$ -values of the White neural network linearity test. As expected, in this case if we set  $\gamma_1 = 0$  the Wnn test leads to the acceptance of the null hypothesis of linearity in mean for all adopted models M1-M4. On the other hand, the rejection of the null (the model is not linear in mean) occurs in a range of values of  $\gamma_1$  different for each model but coherent with the rejection intervals already identified for the independence test based on ADCF. For example, for model M1 we observe that if  $\gamma_1 \in (0, 1.2]$  the Wnn test rejects the hypothesis of linearity. On the other hand, if  $\gamma_1 > 1.2$  the Wnn test seems to signal that M1 generates i.i.d. time series which are linear in mean by construction. Similar considerations apply to the remaining models M2-M4. Actually, the two tests should be jointly considered in order to identify the intersections between the simulated rejection intervals in the two tests. In this way we highlight at least three ranges of interest based on two bounds of  $\gamma_1$ , say  $\gamma_1^a$  and  $\gamma_1^b$ . So, if  $\gamma_1 \in (0, \gamma_1^a)$  then both null hypotheses are rejected and therefore we can say that, according on the adopted tests, the model yields time series with nonlinear serial dependence. On the other hand, if  $\gamma_1 \in (\gamma_1^a, \gamma_1^b)$ , then only IndTest leads to the rejection of the null hypothesis while Wnn test leads to its acceptance, therefore we can conclude the model generates time series which are not i.i.d. but are linear in mean. Finally, if  $\gamma_1 > \gamma_1^b$  both hypotheses of linearity and independence are accepted, and therefore the model generates i.i.d time series.

Figure 2 shows the same results relative to models M5-M8. In this case we can observe that the range of possible values of  $\gamma_1$  suggesting the rejection of independence is wider than in models M1-M4. Considering both tests, we notice, for example, that for M5 choosing a  $\gamma_1$  lesser than 2 we can jointly reject the hypotheses of independence and linearity in mean. For the remaining models M6-M8, the maximum value of  $\gamma_1$  which leads to the rejection of the null is, respectively, 1.8, 1.8 and 1.4. Table 2 summarizes the independence and linearity bounds for all models considered.

	Identifier							
Bounds	M1	M2	M3	M4	M5	M6	M7	M8
$\gamma_1^a$	1.1	0.8	0.9	0.4	2	1.6	2	1
$\gamma_1^b$	1.3	1	1	0.5	3.8	3	3	1.5

Table 2: Bounds for intervals of  $\gamma_1$ : if  $\gamma_1 \in (0, \gamma_1^a)$  both null hypothesis are rejected, if  $\gamma_1 \in (\gamma_1^a, \gamma_1^b)$  the null hypothesis of independence is rejected whereas the null hypothesis of linearity is accepted, if  $\gamma_1 > \gamma_1^b$  both hypothesis are accepted.

### 3.2 Detecting nonlinearity

Motivated by the above considerations about the rejection of hypotheses of independence and linearity, we can try to detect nonlinearity in the simulated time series using two measures recently introduced in the literature: the auto-distance correlation function and the autodependence function. In the appendix we report and briefly discuss how both measures works with time series. For a detailed discussion of properties of the two measures the interested reader can consult Zhou (2012) for ADCF and Bagnato et al. (2012) for ADF. It is interesting to notice that both ADCF and ADF are capable of exploring every form of departure from independence, therefore they represent an effective measure of nonlinearity if accurately compared with the autocorrelation function (Acf). In fact, in the linear case Acf, ADCF and ADF provide similar information (see, i.e., Bagnato et al. (2012) for the MA(1) model), whereas in the nonlinear case only ADCF and ADF should indicate serial dependence (at a fixed lag) in time series and Acf should not be significantly different from zero. For example, figure 3 reports the simulated average values of the three considered measures in a linear case represented by model M1 setting  $\gamma_1 = 0$ . We clearly see that Acf, ADCF and ADF behave in a similar way remaining significantly different from zero for 11 or 12 lags, as we theoretically expect. In practice, in this case the linear serial dependence is detected in a similar way from Acf, ADCF and ADF. On the contrary, if we simulate, as in Bagnato et al. (2012), a quadratic MA(3) model of equation  $Y_t = 0.8\xi_{t-3}^2 + \xi_t$  where  $\xi_t \stackrel{i.i.d.}{\sim} N(0,1)$ , and we compute the simulated average values of Acf, ADCF and ADF relative to the first 30 lags, we get the results in figure 4. As expected, ADCF and ADF signal a spike at lag 3 indicating dependence whereas Acf is never significantly different from zero suggesting the absence of linearity. It should be noted that both ADCF and ADF are very effective in testing the adequacy of GARCH or ARMA models, as shown in Zhou (2012) and Bagnato et al. (2012), through the analysis of serial dependence of the residuals. In particular, neither ADCF nor ADF provide guidance about the type of nonlinearity but both allow the identification of possible lags relatively to which it is necessary to proceed with a further analysis.

The results that we are going to comment in this section will show that, although all the models proposed are intrinsically nonlinear, it may not be easy

to detect nonlinearity in time series. In fact, we simulate 5000 paths of 500 points from models M1-M8 with parameters identified by independence and linearity tests performed in the previous section. In other words,  $\gamma_1$  is set equal to a value which ensures the simultaneous rejection of hypothesis of independence and linearity (table 2).

The analysis of the results concerning the simulated average values of Acf, ADCF and ADF (figures 5-12) leads us to the following considerations.

- Model M1 (figure 5). Consider the case  $\gamma_1 = 0.1$  (case (a) in the figure). Despite both hypothesis of independence and linearity are rejected, the model is characterized by a non negligible linearity in the serial dependence. In fact, Acf is significantly different from zero for the first three lags, exactly as for ADCF, whereas ADF is significant for the first two lags. If we consider the cases  $\gamma_1 = 0.6$  and  $\gamma_1 = 1.1$  we cannot appreciate any different behavior.
- Model M2 (figure 6). Here, we observe that the case  $\gamma_1 = 0.9$  (column (c)) meets our expectations. Correctly, Acf is never significantly different from zero, suggesting the absence of linearity in the serial dependence, while ADCF and ADF are both significantly different from zero to the first lag. Therefore, in this case, the presence of nonlinear dependence in time series generated by this model is clearly evident. Recall that both ADCF and ADF suggest the presence of nonlinearity but not the percentage of the maximum dependence as it happens for Acf. On the contrary, in the remaining cases ( $\gamma_1 = 0.1$  and  $\gamma_1 = 0.4$ ) no differences can be found in the dynamics of Acf, ADCF and ADF.
- Model M3 (figure 7). We can see a behavior very similar to the model M2 with the difference that in the latter case the value of Acf is barely significant even at the first lag of the case (c). Note that M2 and M3 differ in the choice of  $r$  and in the distribution of errors.
- Model M4 (figure 8). Nonlinearity is correctly detected in the case where  $\gamma_1 = 0.5$  (column (c)). This model is characterized by  $r = 2$  and  $\xi_t$  distributed as  $t_{(5)}$ . Compared to the previous models M1-M3 when  $\gamma_1$  is small only the first lag is significant. Linearity in serial dependence is reduced only to the first lag but does not disappear as expected.
- Model M5 (figure 9). In this case, although the functional form of  $\psi$  has changed, we do not notice acceptable differences in the dynamics of Acf, ADCF and ADF that suggest us to detect nonlinearity in serial dependence. However, it should be noted from now on and for the remaining models analyzed, that ADF will be significant always and only at the first lag suggesting serial dependence only in this case.
- Model M6 (figure 10). Confirming what was observed in the previous case, it seems that  $\gamma_1$  does not affect the behavior of ADF while the dynamics of Acf and ADCF seem perfectly overlapping.
- Models M7 and M8 (figures 11 and 12). We can repeat the same considerations made for models M5 and M6.

At this point a general consideration is mandatory but not simple. However, at least a couple of conclusions can be set. Out of 24 simulated models only in two cases our expectations are realized: for model M2 with  $\gamma_1 = 0.9$  and model M4 with  $\gamma_1 = 0.5$ , Acf is never significantly different from zero, indicating the absence of linearity in serial dependence, whereas ADCF and ADF are both significant at the first lag, suggesting that time series are nonlinear. Both models are characterized by an autoregressive coefficient of exponential type and by a error distribution of Student's  $t$  type. On the other hand, no other case seems consistent with the independence and linearity tests discussed in the previous section. The presence of linearity in the serial dependence cannot be excluded as it is significantly different from zero at least at the first lag. With a picturesque statement we could say that the dependence is "dirty" with a residual linearity that Acf cannot exclude. At the same time, it is clear that nonlinearity is effectively detecting only the first lag in most cases (and this is particularly evident for ADF). In essence,  $Y_{t-1}$  does not transfer serial dependence on lags greater than 1, unlike what happens for example in an AR(1) model where the functional form involves only  $Y_{t-1}$  but the linear dependence is characterized by a power decay (figure 3).

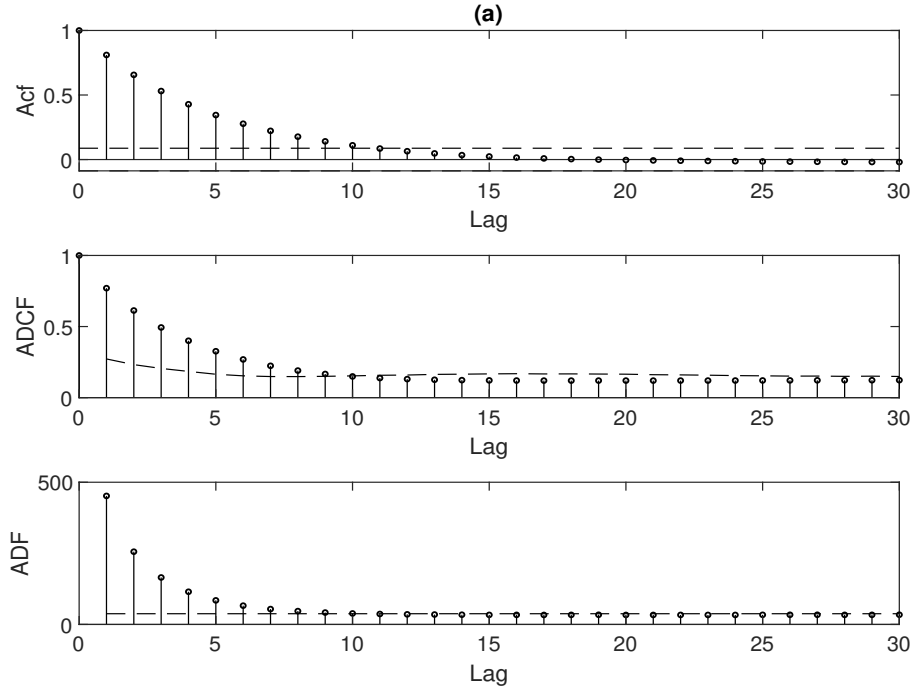


Figure 3: Plots of simulated Acf, ADCF and ADF in the linear case (model M1 with  $\gamma_1 = 0$ ). The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

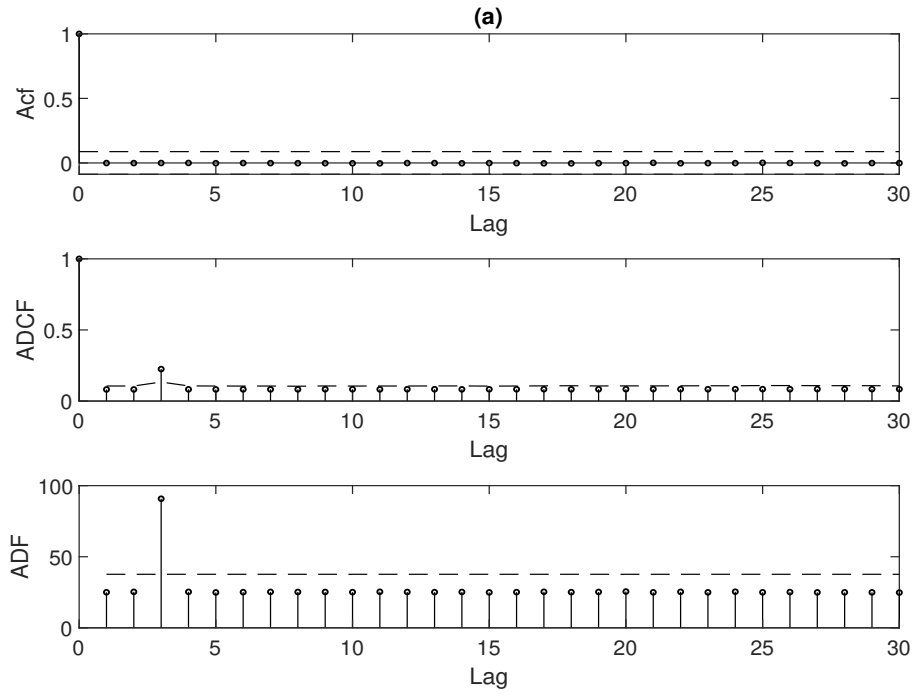


Figure 4: Plots of simulated Acf, ADCF and ADF for a quadratic MA(3). The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

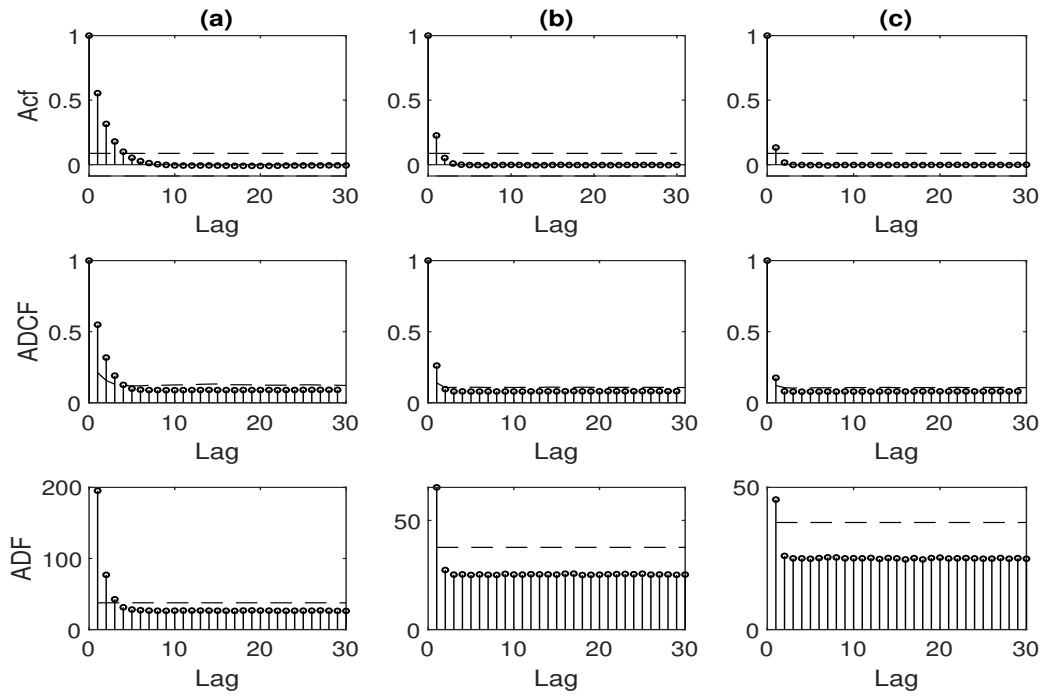


Figure 5: Plots of simulated Acf, ADCF and ADF relative to model M1. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.1$ , (b)  $\gamma_1 = 0.6$ , (c)  $\gamma_1 = 1.1$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

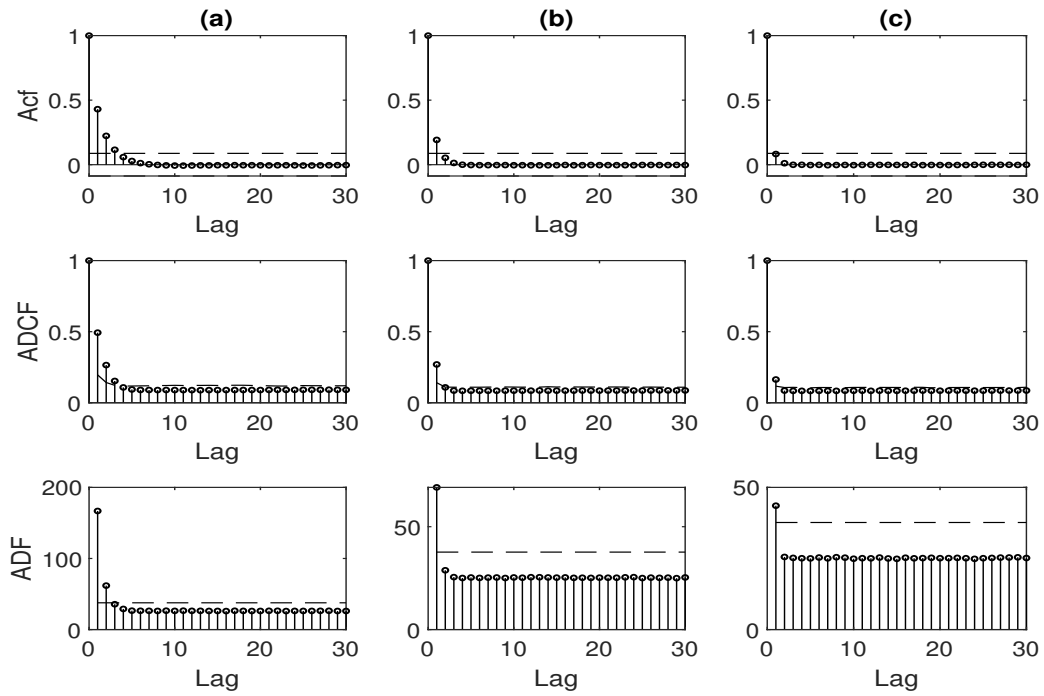


Figure 6: Plots of simulated Acf, ADCF and ADF relative to model M2. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.1$ , (b)  $\gamma_1 = 0.4$ , (c)  $\gamma_1 = 0.9$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.



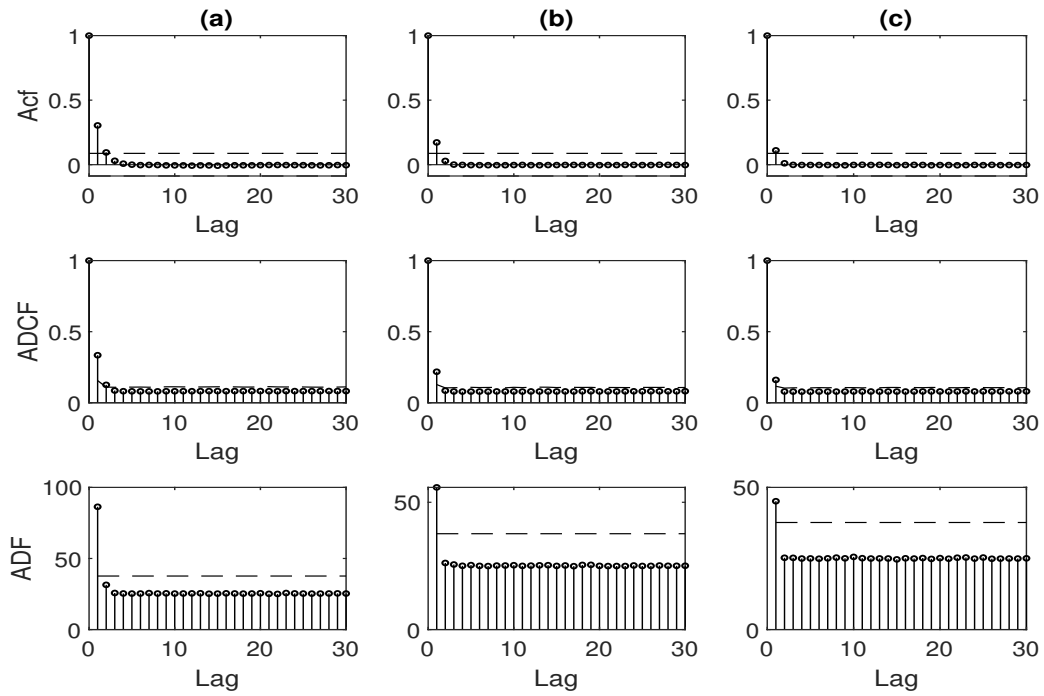


Figure 7: Plots of simulated Acf, ADCF and ADF relative to model M3. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.1$ , (b)  $\gamma_1 = 0.4$ , (c)  $\gamma_1 = 0.9$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

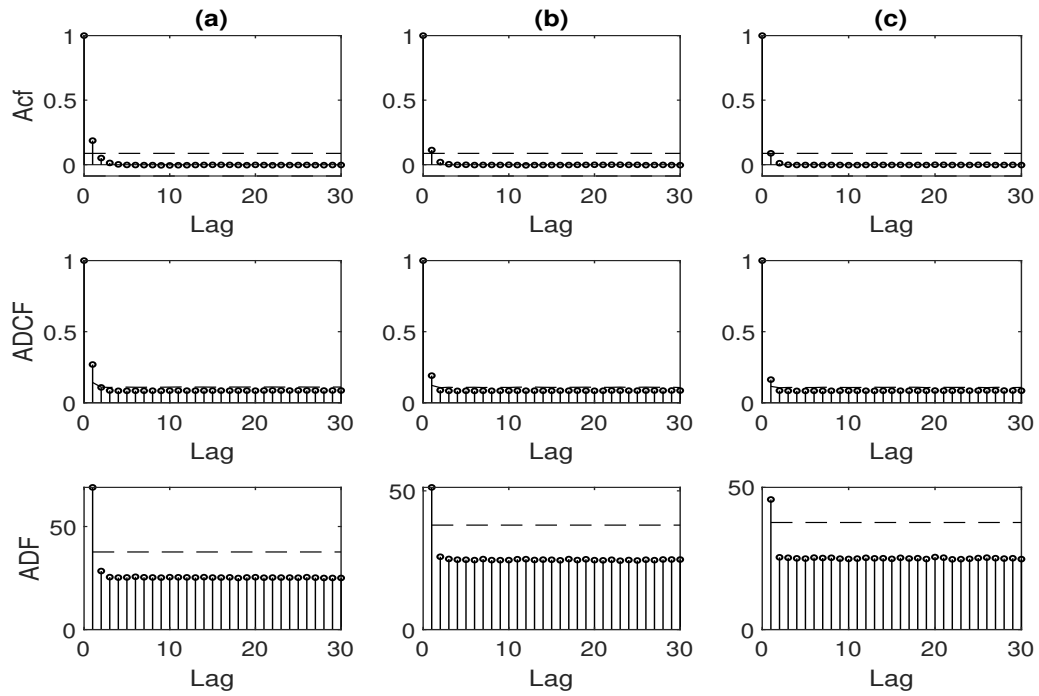


Figure 8: Plots of simulated Acf, ADCF and ADF relative to model M4. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.1$ , (b)  $\gamma_1 = 0.3$ , (c)  $\gamma_1 = 0.5$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

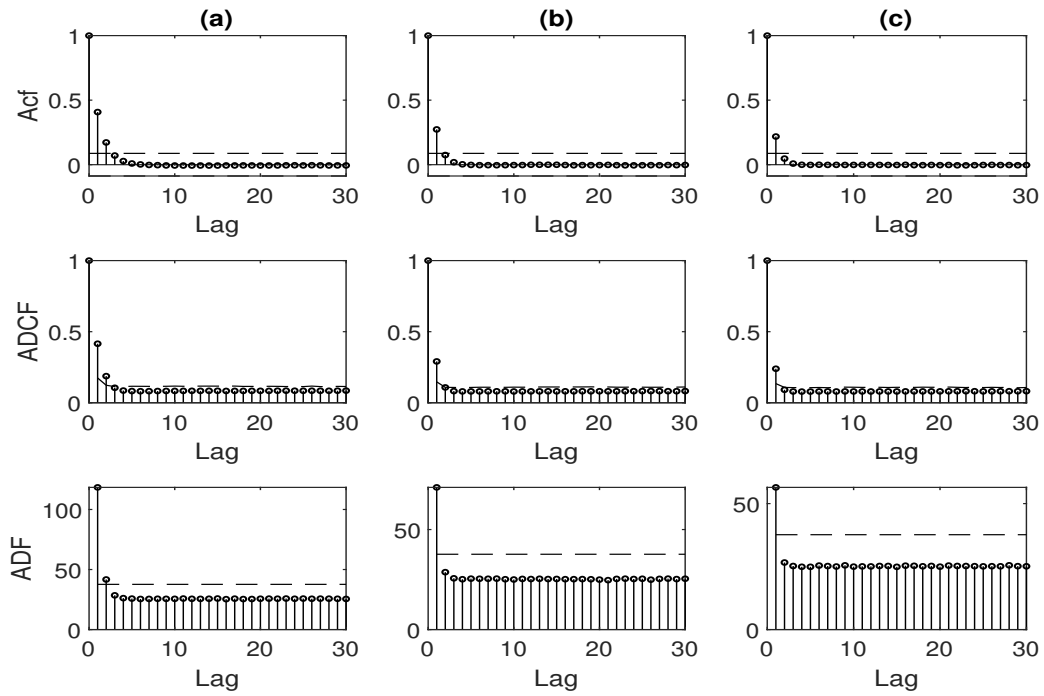


Figure 9: Plots of simulated Acf, ADCF and ADF relative to model M5. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.5$ , (b)  $\gamma_1 = 1.2$ , (c)  $\gamma_1 = 1.8$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

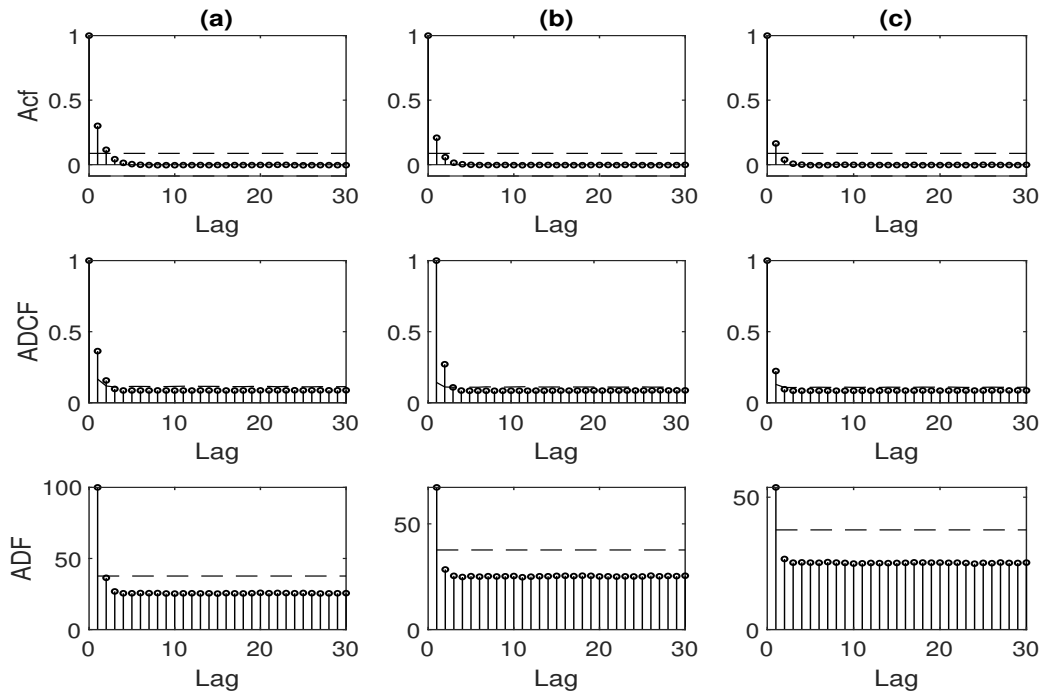


Figure 10: Plots of simulated Acf, ADCF and ADF relative to model M6. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.5$ , (b)  $\gamma_1 = 1$ , (c)  $\gamma_1 = 1.5$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

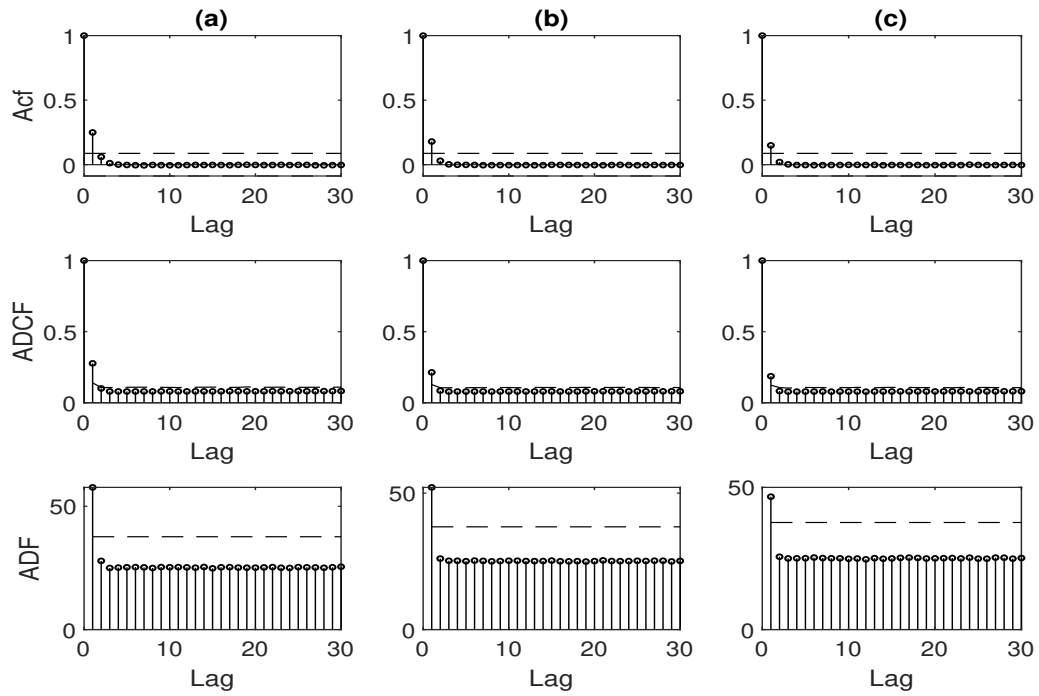


Figure 11: Plots of simulated Acf, ADCF and ADF relative to model M6. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.5$ , (b)  $\gamma_1 = 1.2$ , (c)  $\gamma_1 = 1.8$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

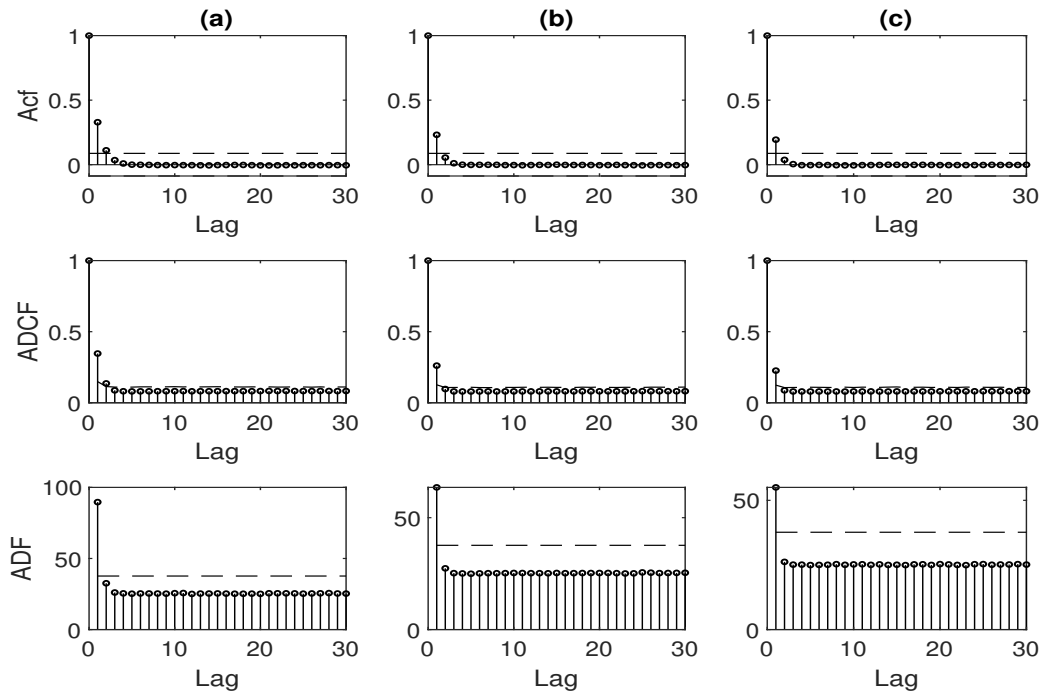


Figure 12: Plots of simulated Acf, ADCF and ADF relative to model M6. From top to bottom, the plots are referred to a fixed value of  $\gamma_1$ . (a)  $\gamma_1 = 0.2$ , (b)  $\gamma_1 = 0.6$ , (c)  $\gamma_1 = 1$ . The solid vertical lines are average values of Acf, ADCF and ADF and the dotted horizontal lines or curves are the critical values at 5% level.

## 4 Concluding remarks

The simulation study conducted in this paper allows us to conclude that is often not easy to recognize the nonlinear serial dependence in a time series even if it is generated by an intrinsically nonlinear autoregressive model as our SDAR(1) model proposed in this work. The key point reached in this simulation study is that, often, the nonlinearity is not clearly predominant on the linearity even when it should be by construction. As a consequence, the measures adopted to isolate nonlinearity do not allow to distinguish the two kinds of dependence. Future efforts can be devoted to understanding if there exists a different specification of  $\psi$  that allows to detect nonlinearity using the auto-distance correlation and the autodependence function just used in this work.

## 5 Appendix

We report here the definitions of auto-distance correlation and autodependence function which are used through the paper.

### 5.1 Auto-distance correlation

Let  $(Y_t)_t$  be a strictly stationary time series. Define the distance correlation function as a function of the bivariate marginal characteristic functions (c.f.) of  $(Y_t, Y_{t+h})$ ,  $h \in \mathbb{Z}$ . The joint c.f. of the pair  $(Y_t, Y_{t+h})$  is  $f_{t,t+h}(u, v) = \mathbb{E} [e^{i(uY_t + vY_{t+h})}]$ , where  $(u, v) \in \mathbb{R}^2$  and  $i^2 = -1$ . The marginal c.f. of  $Y_t$  and  $Y_{t+h}$  are  $f_t(u) = f_{t,t+h}(u, 0)$  and  $f_{t+h}(v) = f_{t,t+h}(0, v)$  respectively. It is well known that  $Y_t$  and  $Y_{t+h}$  are independent if and only if  $f_{t,t+h}(u, v) = f_t(u)f_{t+h}(v)$  for all  $(u, v) \in \mathbb{R}^2$ . The auto-distance covariance function (ADCV) between  $Y_t$  and  $Y_{t+h}$  is defined as

$$v_y(h) = \int_{\mathbb{R}^2} |f_{t,t+h}(u, v) - f_t(u)f_{t+h}(v)|^2 \frac{1}{\pi^2 u^2 v^2} du dv, \quad h = 0, \pm 1, \pm 2, \dots$$

whereas the auto-distance correlation (ADCF), intensively used in this paper, is given by

$$r_y(h) = \frac{v_y(h)}{v_y(0)}, \quad h = 0, \pm 1, \pm 2, \dots$$

if  $v_y(0) \neq 0$  and zero otherwise. The auto-distance correlation satisfies  $0 \leq r_y(h) \leq 1$  for all  $h$  and  $r_y(h) = 0$  if and only if  $Y_t$  and  $Y_{t+h}$  are independent.

Following Zhou (2012), given an observed time series  $(y_1, \dots, y_n)$  from  $(Y_t)_t$ , let  $a_{ts} = |y_t - y_s|$  and  $b_{ts} = |y_{t+h} - y_{s+h}|$ . Define

$$\bar{a}_{\cdot} = \frac{\sum_{s=1}^{n-h} a_{ts}}{n-h}, \quad \bar{a}_{\cdot s} = \frac{\sum_{t=1}^{n-h} a_{ts}}{n-h}, \quad \bar{a}_{\cdot} = \frac{\sum_{t,s=1}^{n-h} a_{ts}}{(n-h)^2},$$

and  $A_{ts} = a_{ts} - \bar{a}_t - \bar{a}_{\cdot s} + \bar{a}_{\cdot\cdot}$ . Define  $b_{ts}$ ,  $\bar{b}_t$ ,  $\bar{b}_{\cdot s}$ ,  $\bar{b}$  and  $B_{ts}$  similarly. The empirical ADCV is

$$\hat{v}_y(h) = \frac{1}{(n-h)^2} \sum_{t,s=1}^{n-h} A_{ts} B_{ts},$$

and the empirical ADCF will be  $\hat{r}_y(h) = \frac{\hat{v}_y(h)}{\hat{v}_y(0)}$ . The R package (dCovTS) used in this paper to implement the ADCF methodology is introduced in Pitsillou and Fokianos (2016).

## 5.2 Autodependogram

The method consider the problem of studying the generic dependence of lag  $h$  by using the well-known and general  $\chi^2$ -statistic of independence. Following Bagnato et al. (2012), let  $(Y_t)_t$  be a strictly stationary stochastic process and let  $(y_1, \dots, y_n)$  be an observed time series from  $(Y_t)_t$ . Define  $(C_u^{(h)})_{u=1}^k$  and  $(D_v^{(h)})_{v=1}^k$  two generic sets of  $k$  adjacent intervals such that, fixed  $h$ ,  $y_t \in \cup_{u=1}^k C_u^{(h)}$  for all  $t = 1, \dots, n-h$  and  $y_t \in \cup_{v=1}^k D_v^{(h)}$  for all  $t = h+1, \dots, n$ . Starting from these sets of intervals all the  $n^{(h)} = n-h$  couples  $(y_1, y_{1+h}), \dots, (y_{n-h}, y_n)$  can be grouped to obtain a contingency table where the observed frequencies are

$$n_{uv}^{(h)} = \#\{(y_t, y_{t+h}) : (y_t, y_{t+h}) \in C_u^{(h)} \times D_v^{(h)}, t = 1, \dots, n-h\}, \quad u, v = 1, \dots, k.$$

A discussion on how to choose the partitions  $(C_u^{(h)})_{u=1}^k$  and  $(D_v^{(h)})_{v=1}^k$  is available in Bagnato et al. (2012). So, the  $\chi^2$ -statistic that can be used to test the dependence of lag  $h$  is given by

$$\hat{\delta}_h = \sum_{u=1}^k \sum_{v=1}^k \frac{|n_{uv}^{(h)} - \hat{n}_{uv}^{(h)}|^2}{\hat{n}_{uv}^{(h)}},$$

where  $\hat{n}_{uv}^{(h)} = \frac{n_u^{(h)} n_v^{(h)}}{n^{(h)}}$  are the theoretical frequencies under the null hypothesis of independence of lag  $h$ . Bagnato et al. (2012) report that under this condition  $\hat{\delta}_h$  is asymptotically distributed as a  $\chi^2$  distribution with  $(k-1)^2$  degrees of freedom. The diagram obtained by plotting  $\hat{\delta}_h$  as a function of  $h$  is called autodependogram. The R package (SDD) used in this paper to implement the ADF methodology is introduced and discussed in Bagnato et al. (2015).

## References

- [1] Bagnato L., Punzo A., (2010): "On the use of  $\chi^2$ -test to check serial dependence", *Statistica & Applicazioni*, VIII(1), 57-74.
- [2] Bagnato L., Punzo A., Nicolis O. (2012): "The autodependogram: a graphical device to investigate serial dependencies", *Journal of Time Series Analysis*, 33, 233-254.



- [3] Bagnato L., De Capitani L., Mazza A., Punzo A. (2015): "SDD: An R Package for Serial Dependence Diagrams", *Journal of Statistical Software*, 64(2), 1-19.
- [4] Bera A., Robinson P.M. (1989): "Tests for serial dependence and other specification analysis in models of markets in disequilibrium", *Journal of Business & Economic Statistics*, 7(3), 343-352.
- [5] Booth Teppo G.G., Yli-Olli P. (1994): "Nonlinear dependence in Finnish stock returns", *European Journal of Operational Research*, 74(2), 273-283.
- [6] Brock W.A., Dechert W.D., Sheinkmann J.A. (1987): "A test of independence based on the correlation dimension", SSRN no. 8702, Department Of Economics, University of Wisconsin, Madison.
- [7] Brock W.A., Dechert W.D., Sheinkmann J.A., LeBaron B. (1996): "A test of independence based on the correlation dimension", *Econometric Reviews*, 15, 187-235.
- [8] Davis R.A., Matsui M., Mikosh T., Wom P. (2018): "Applications of distance correlation to time series", *Bernoulli*, 24(4A), 3087-3116.
- [9] Dueck J., Edelmann D., Gneiting T., Richards D. (2014): "The affinely invariant distance correlation", *Bernoulli*, 20, 2305-2330.
- [10] Dufour J.M., Lapage Y., Zeidan H. (1982): "Nonparametric testing for time series: a bibliography", *Canadian Journal of Statistics*, 10(1), 1-38.
- [11] Fisher N.I., Switzer P. (1985): "Chi-plots for assessing dependence", *Biometrika*, 72(2), 253-265.
- [12] Fokianos K., Pitsillou M. (2017): "Consistent testing for pairwise dependence in time series", *Technometrics*, 59, 262-270.
- [13] Genest C., Remillard B. (2004): "Test of independence and randomness based on the empirical copula process", *Test*, 13(2), 335-369.
- [14] Gobbi F., Mulinacci S. (2019): "State-Dependent Autoregressive Model for Nonlinear Time Series: Stationarity, Ergodicity and Estimation Methods". Available at SSRN: <https://ssrn.com/abstract=3431614> or <http://dx.doi.org/10.2139/ssrn.3431614>
- [15] Granger, C.W.J., Anderson A.P. (1978): "An introduction to bilinear time series models", (Vandenhoech und Ruprecht, Gottingen).
- [16] Granger C.W., Maasoumi E., Racine J. (2004): "A dependence metric for possibly nonlinear processes", *Journal of Time Series Analysis*, 25(5), 649-669.
- [17] Gretton A., Fukumizu K., Sriperumbudur B.K. (2009): "Discussion of: Brownian distance covariance", *The Annals of Applied Statistics*, 3, 1285-1294.
- [18] Harris R.D.F., Küçüközmen C.C. (2001): "Linear and nonlinear dependence in Turkish equity returns and its consequences for financial risk management", *European Journal of Operational Research*, 134(3), 481-492.

- [19] LeBaron B. (1997): A fast algorithm for the BDS statistic", *Studies in Nonlinear Dynamics and Econometrics*, 2, 53-59.
- [20] Lee T.H., White H. Granger C.W. (1993): Testing for neglected nonlinearity in time series models", *Journal of Econometrics*, 56, 269-290.
- [21] McLeod, A.I., Li W.K. (1983): "Diagnostic checking ARMA time series models using squared residual autocorrelations", *Journal of Time Series Analysis* 4, 169-176.
- [22] Pitsillou M., Fokianos K. (2016): "dCovTS: Distance Covariance/Correlation for Time Series", *The R Journal*, 8(2), 324-340.
- [23] Remillard B. (2009): "Discussion of: Brownian distance covariance", *The Annals of Applied Statistics*, 3, 1295-1298.
- [24] Szekely G.J., Rizzo M.L. (2009): "Brownian distance covariance (with discussions)", *Annals of Applied Statistics*, 3, 1236-1308.
- [25] Szekely G.J., Rizzo M.L., Bokirov N.K. (2007): "Measuring and testing dependence by correlation of distances", *The Annals of Statistics*, 35, 2769-2794.
- [26] Takala K., Viren M. (1996): "Chaos and nonlinear dynamics in financial and nonfinancial time series: Evidence from Finland", *European Journal of Operational Research*, 93(1), 155-172.
- [27] White H. (1989): "An additional hidden unit test for neglected nonlinearity in multilayer feedforward networks", in: Proceedings of the international joint conference on neural networks, Washington, DC (IEEE Press, New York, NY) II: 451-455.
- [28] Zhou Z. (2012): "Measuring nonlinear dependence in time series, a distance correlation approach", *Journal of Time Series Analysis*, 33, 438-457.