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A Priori Estimates for the  $\infty$ -Laplacian Relative to Vector Fields

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

*Published Version:*

Ferrari, F., Manfredi, J.J. (2023). A Priori Estimates for the  $\infty$ -Laplacian Relative to Vector Fields. *FUNKCIALAJ EKVACIOJ*, 66(1), 45-70 [10.1619/fesi.66.45].

*Availability:*

This version is available at: <https://hdl.handle.net/11585/925249> since: 2023-05-12

*Published:*

DOI: <http://doi.org/10.1619/fesi.66.45>

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(Article begins on next page)

This is the final peer-reviewed accepted manuscript of:

**Fausto Ferrari, Juan J. Manfredi, A Priori Estimates for the  $\infty$ -Laplacian Relative to Vector Fields, Funkcialaj Ekvacioj, 2023, Volume 66, Issue 1, Pages 45-70**

The final published version is available online at  
<https://dx.doi.org/10.1619/fesi.66.45>

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# A PRIORI ESTIMATES FOR THE $\infty$ -LAPLACIAN RELATIVE TO VECTOR FIELDS

FAUSTO FERRARI AND JUAN J. MANFREDI

*Dedicated to our friend Bruno Franchi*

ABSTRACT. In this paper we prove a priori Hölder and Lipschitz regularity estimates for viscosity solutions equations governed by the inhomogeneous infinite Laplace operator relative to a frame of vector fields.

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## 1. INTRODUCTION

The main results in this manuscript are the a priori local Hölder and Lipschitz continuity of viscosity solutions to the problem

$$(1.1) \quad \sum_{i,j=1}^n X_i X_j u(x) X_i u(x) X_j u(x) = f(x, u(x), X_1 u(x), \dots, X_n u(x)),$$

where  $f$  is a real valued continuous functions and  $X_1, X_2, \dots, X_n$  are linearly independent smooth vector fields in  $\mathbb{R}^n$ .

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*Date:* October 18, 2022.

1991 *Mathematics Subject Classification.* 35J94, 35R03, 53B20.

*Key words and phrases.* Infinity-Laplacian, viscosity solutions, vector fields.

F. F. was partially supported by INDAM-GNAMPA 2020 project: *Metodi di viscosità e applicazioni a problemi non lineari con debole ellitticità.*

We write

$$D_{\mathfrak{X}}u = \sum_{i=1}^n X_i u(x) X_i$$

for the gradient of the function  $u$  relative to the frame of vector fields  $\mathfrak{X} = \{X_1, X_2, \dots, X_n\}$ . We consider  $\mathbb{R}^n$  as a Riemannian manifold with a metric induced by the frame  $\mathfrak{X}$ . This frame determines a Riemannian metric  $g$  by requiring that  $\mathfrak{X}(x) = \{X_1(x), X_2(x), \dots, X_n(x)\}$  is an orthonormal basis for the metric  $g_x$  in the tangent space to  $\mathbb{R}^n$  at  $x$  (which we identify with  $\mathbb{R}^n$ ); that is, we have

$$g_x(X_i(x), X_j(x)) = \delta_{ij} \text{ for } i, j = 1 \dots n.$$

Write

$$(1.2) \quad X_i(x) = \sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j}$$

for smooth functions  $a_{ij}(x)$ . Denote by  $\mathbb{A}(x)$  the matrix whose  $(i, j)$ -entry is  $a_{ij}(x)$ . We always assume that  $\det(\mathbb{A}(x)) \neq 0$ . Let  $\mathbb{G}(x)$  denote the matrix of  $g_x$  with respect to the Euclidean coordinates. We then have

$$(1.3) \quad \mathbb{G}(x) = (\mathbb{A}^t(x) \mathbb{A}(x))^{-1}.$$

We can write equation (1.1) as

$$(1.4) \quad \Delta_{\mathfrak{X}, \infty} u = \left\langle (D_{\mathfrak{X}}^2 u)^* D_{\mathfrak{X}} u, D_{\mathfrak{X}} u \right\rangle_g = f(x, u(x), D_{\mathfrak{X}} u(x))$$

is the  $\infty$ -Laplacian relative to the frame  $\mathfrak{X}$ , where  $g$  is the Riemannian metric determined by  $\mathfrak{X}$ .

We use the notation  $d(x, y)$  for the Riemannian distance determined by  $g$ . For a point  $x \in \mathbb{R}^n$  the injectivity radius is  $i(x) > 0$ . The metric ball centered at  $x$  with radius  $r > 0$  is denoted by  $B_r(x)$ . The gradient of a smooth function  $u: \mathbb{R}^n \mapsto \mathbb{R}$  relative to  $\mathfrak{X}$  agrees with the Riemannian gradient of the function  $u$  (see Lemma 2.4 below). The  $\mathfrak{X}$ -second derivative matrix  $D_{\mathfrak{X}}^2 u$  is an  $n \times n$  matrix, not necessarily symmetric, with entries  $X_i(X_j(u))$ . We will consider its symmetrization

$$(D_{\mathfrak{X}}^2 u)^* = \frac{D_{\mathfrak{X}}^2 u + (D_{\mathfrak{X}}^2 u)^t}{2}$$

and note that  $(D_{\mathfrak{X}}^2 u)^*$  is, in general, different from  $\text{Hess}(u)$  the Riemannian Hessian of the function  $u$ . See Example 2.5 below.

Our starting point is the fact that the function  $u(x) = d(x_0, x)$ , which is smooth in the set  $B_{i(x_0)}(x_0) \setminus \{x_0\}$ , satisfies the eikonal equation

$$(1.5) \quad |D_{\mathfrak{X}} u|_g = 1,$$

and it is  $\infty$ -harmonic

$$(1.6) \quad \Delta_{\mathfrak{X}, \infty} u = \left\langle (D_{\mathfrak{X}}^2 u)^* D_{\mathfrak{X}} u, D_{\mathfrak{X}} u \right\rangle_g = 0.$$

See Proposition 2.7 below. For more information about distances and infinity-Laplacians see [BDM09, Bie10].

We shall also use the fact that  $(x, y) \mapsto d^2(x, y)$  is locally smooth, Proposition 2.8. Thus, functions of the distance are available as test functions for the viscosity formulation of (1.1) that we describe next.

**Definition 1.1.** An upper semi-continuous function  $u$  is a viscosity subsolution of (1.6) in a domain  $\Omega \subset \mathbb{R}^n$  if whenever  $\phi \in C^2(\Omega)$  touches  $u$  from above at a point  $x_0 \in \Omega$  we have

$$\Delta_{\mathfrak{X},\infty}\phi(x_0) \geq f(x_0, \phi(x_0), D_{\mathfrak{X}}\phi(x_0)).$$

A lower semi-continuous function  $v$  is a viscosity supersolution of (1.6) in a domain  $\Omega$  if whenever  $\phi \in C^2(\Omega)$  touches  $v$  from below at a point  $x_0 \in \Omega$  we have

$$\Delta_{\mathfrak{X},\infty}\phi(x_0) \leq f(x_0, \phi(x_0), D_{\mathfrak{X}}\phi(x_0)).$$

Recall that  $\phi$  touches  $u$  from above at  $x_0$  means  $\phi(x) \leq u(x)$  for all  $x$  in a neighborhood of  $x_0$  and  $\phi(x_0) = u(x_0)$ . To define  $\phi$  touches  $u$  from below at  $x_0$  just reverse the inequality.

A viscosity solution is both a super- and a subsolution. Our main results are the following:

**Theorem 1.2.** Let  $\Omega \subset \mathbb{R}^n$  be a domain and  $f: \Omega \mapsto \mathbb{R}$  be a continuous function. Let  $u$  be a viscosity solution of the inhomogeneous  $\infty$ -Laplace equation

$$(1.7) \quad \Delta_{\mathfrak{X},\infty}u(x) = f(x) \text{ in } \Omega$$

Then, the function  $u$  is locally Lipschitz continuous. More precisely, for all  $x_0 \in \Omega$  such that  $B_{2i(x_0)}(x_0) \subset \Omega$  we have

$$(1.8) \quad |u(x) - u(y)| \leq L d(x, y),$$

for  $x, y \in B_{i(x_0)/4}(0)$ , where  $L$  depends only on  $\|f\|_{L^\infty(B_{2i(0)})}$ ,  $\|u\|_{L^\infty(B_{2i(0)})}$  and the infimum of the injectivity radius on the compact set  $\overline{B_{i(x_0)}(x_0)}$ .

**Theorem 1.3.** Let  $\Omega \subset \mathbb{R}^n$  be a domain and  $f: \Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$  be a continuous function satisfying the condition

$$(1.9) \quad |f(x, p, \xi)| \leq C_2 |\xi|_g^\beta + C_3,$$

where  $0 \leq \beta < 4$  and  $C_2, C_3$  are nonnegative constants. Let  $u$  be a viscosity solution of the inhomogeneous  $\infty$ -Laplace equation

$$\Delta_{\mathfrak{X},\infty}u(x) = f(x, u(x), D_{\mathfrak{X}}u(x)) \text{ in } \Omega.$$

Then, the function  $u$  is locally Hölder continuous with exponent  $\alpha < \min\{1, \frac{4-\beta}{3}\}$ . More precisely, for  $x_0 \in \Omega$  such that  $B_{2i(x_0)}(x_0) \subset \Omega$  and  $x, y \in B_{i(x_0)/4}(x_0)$  we have

$$(1.10) \quad |u(x) - u(y)| \leq L_1 d(x, y)^\alpha,$$

where  $L_1$  depends only on  $\|u\|_{L^\infty(B_{i(x_0)}(x_0))}$ , the constants  $C_2$  and  $C_3$ , the exponent  $\beta$ , the injectivity radius  $i(x_0)$ , and a constant  $C(\overline{B_{2i(x_0)}(x_0)}, g)$  depending only on the metric  $g$  and the compact set  $\overline{B_{2i(x_0)}(x_0)}$ .

In the Euclidean case, where  $X_i = \partial_{x_i}$ , Theorem 1.2 was proven by Lindgren in [Lin14]. In the Riemannian case, Theorem 1.2 was proven by Lu, Miao, and Zhu in [LMZ19]. They consider the equation

$$(1.11) \quad \langle D\langle A(x)Du(x), Du(x) \rangle, A(x)Du(x) \rangle = f(x),$$

where  $A \in C^1$  and  $f$  is continuous. Their proof is based on using the Hamilton-Jacobi equation  $\langle A(x), p \rangle + \lambda u = 1$  to approximate the intrinsic metric associated to  $A(x)$ . It turns out that equations (1.11) and (1.7) are the same equation since we have

$$\left\langle (D_{\mathfrak{X}}^2 u)^* D_{\mathfrak{X}} u, D_{\mathfrak{X}} u \right\rangle_g = \langle D\langle A(x)Du(x), Du(x) \rangle, A(x)Du(x) \rangle$$

when we take  $A(x) = \mathbb{A}^t(x)\mathbb{A}(x)$ . Note that  $A(x) = \mathbb{G}(x)^{-1}$ , where  $\mathbb{G}(x)$  is the matrix of the metric  $g_x$ , see (1.3).

Our proof of Theorem 1.2 follows by using directly properties of the Riemannian metric that we discuss in Section §3 below. In particular we establish the analog of the Euclidean formula  $\Delta_\infty |x|^\alpha = 4\alpha^3(\alpha - 1)|x|^{3\alpha-4}$  for a general Riemannian metric

$$\langle (D_{\mathfrak{X}}^2 d^\alpha)^* \cdot D_{\mathfrak{X}} d^\alpha, D_{\mathfrak{X}} d^\alpha \rangle_g = 4\alpha^3(\alpha - 1)d^{3\alpha-4}$$

whenever  $x \mapsto d(x, y)$  is smooth, see Lemma 3.2 below. Another important result in [LMZ19] is the everywhere differentiability of the solutions when  $f \in C^1$ . In the Euclidean case Lindgren [Lin14] extended the result of Evans and Smart [ES11b] to the non-homogeneous case by establishing an almost-monotonicity property of incremental quotients to obtain the linear approximation property and the everywhere differentiability. In the Riemannian case Lu, Miao and Zhou again use Hamilton-Jacobi equations to establish their result.

Our proof of Theorem 1.3 is an adaptation of the standard penalization argument with several challenges posed by the non-commutativity of the vector fields in the frame. This is the Crandall-Ishii-Lions method for regularity of viscosity solutions (see for example [IL90, Cra97]). The authors found particularly useful the reading of [Ish95] and [IS13] as well. About such approach, there are many contributions in literature. Among them, we wish to recall the following works [BGI18], [BGL17], [FV20b], [FG22], where the regularity of viscosity solutions of truncated operators has been studied. Moreover, always in the frame of a degenerate situation, but in a non-commutative structures, we point out the results contained in [Fer20], [FV20a] and [Gof20]. We develop several properties of the second derivatives of the metric in Section §3 to double the variables and use an adapted theorem of sums. A key estimate is a bound for the symmetrized second derivatives of the distance, Lemma 3.5 below, that we obtain from the eikonal equation. Note that we allow for a general first order term  $f(x, u, D_{\mathfrak{X}}u)$  but that we only get Hölder estimates.

In addition to the blow-up and duality estimates in the homogeneous case in [ES11a] and [ES11b], we would like to mention [LW08], where the inhomogeneous  $\infty$ -Laplacian was treated from the PDE point of view, [AS12] for a finite differences treatment, and [PSSW09] for a tug-of-war interpretation. Sharp estimates for the Sobolev derivative of  $|\nabla u|^\alpha$  for solutions of (1.7) in the Euclidean plane  $\mathbb{R}^2$  are obtained in [KZZ19] when  $f$  is continuous, non vanishing, and of bounded variation.

A representative example is the Riemannian Heisenberg group, where the frame  $\mathfrak{X} = \{X, Y, Z\}$  is given by the left invariant vector fields in  $\mathbb{R}^3$  with respect to the Heisenberg group operation  $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - y'x))$ . These vector fields are  $X = \partial_x - \frac{1}{2}y\partial_z$ ,  $Y = \partial_y + \frac{1}{2}x\partial_z$ , and  $Z = \partial_z$ . The Levi-Civita connection (computed in Chapter 2 of [CDPT07]) is determined by the equations

$$\begin{aligned} \nabla_X X &= \nabla_Y Y = \nabla_Z Z = 0, \\ \nabla_X Y &= \frac{1}{2}Z, & \nabla_Y X &= -\frac{1}{2}Z, \\ \nabla_Z X &= \nabla_X Z = -\frac{1}{2}Y, & \text{and} \\ \nabla_Z Y &= \nabla_Y Z = \frac{1}{2}X. \end{aligned}$$

The matrix of  $Hess(u)$  with respect to basis  $\{X, Y, Z\}$  is then

$$\begin{pmatrix} XXu & XYu - \frac{1}{2}Zu & XZu + \frac{1}{2}Yu \\ YXu + \frac{1}{2}Zu & YYu & YZu - \frac{1}{2}Xu \\ ZXu + \frac{1}{2}Yu & ZYu - \frac{1}{2}Xu & ZZu \end{pmatrix},$$

which differs from  $(D_{\mathfrak{X}}^2 u)^*$  in the  $(1, 3), (2, 3), (3, 1)$  and  $(3, 2)$  entries.

Nevertheless, in this particular case we still have that the Riemannian  $\infty$ -Laplacian

$$(1.12) \quad \Delta_{g, \infty} u = \langle Hess(u) \mathfrak{X}u, \mathfrak{X}u \rangle_g$$

agrees with the frame  $\infty$ -Laplacian

$$\Delta_{\mathfrak{X}, \infty} u = \langle (D_{\mathfrak{X}}^2 u)^* \mathfrak{X}u, \mathfrak{X}u \rangle_g,$$

as a direct calculation shows. Therefore, Theorems 1.2 and 1.3 also hold for (1.12) in the Riemannian Heisenberg case.

The plan of the paper is as follows: in Section §2 we present the details of our set-up. In section §3 we present the proof of bound for the symmetrized second derivatives of the distance. Some facts about viscosity solutions and frames are in Section §4. The proof of the main results Theorems 1.2 and 1.3 are in Sections §5 and §6 respectively.

ACKNOWLEDGEMENT: We thank the anonymous referee for bringing to our attention the reference [LMZ19] and for several suggestions that have improved the readability of the paper.

## 2. PRELIMINAIRES

In  $\mathbb{R}^n$  the function  $u(x) = |x - x_0|$  satisfies both the eikonal equation  $|\nabla u| = 1$  and the  $\infty$ -Laplace equation  $\Delta_{\infty}(u) = \langle D^2 u \nabla u, \nabla u \rangle = 0$  in  $\mathbb{R}^n \setminus \{x_0\}$ . A similar phenomena occurs for the case of Riemannian and sub-Riemannian manifolds, where the function  $u(x) = d(x, x_0)$  satisfies the eikonal equation and the infinity-Laplace equation whenever it is smooth, see Proposition 2.7 below.

We consider the case where the manifold is  $\mathbb{R}^n$  endowed with a Riemannian metric induced by a frame  $\mathfrak{X} = \{X_1, X_2, \dots, X_n\}$ ; that is,  $\mathfrak{X}$  is a collection of  $n$  linearly independent vector fields in  $\mathbb{R}^n$ .

We first write down an appropriate Taylor theorem adapted to the frame  $\mathfrak{X}$ . For this, we will use exponential coordinates as done in [NSW85]. Fix a point  $p \in \mathbb{R}^n$  and let  $t = (t_1, t_2, \dots, t_n)$  denote a vector close to zero. We define the flow exponential based at  $p$  of  $t$ , denoted by  $\Theta_p(t)$ , as follows. Let  $\gamma$  be the unique solution to the system of ordinary differential equations

$$\gamma'(s) = \sum_{i=1}^n t_i X_i(\gamma(s))$$

satisfying the initial condition  $\gamma(0) = p$ . We set  $\Theta_p(t) = \gamma(1)$  and note this is defined in a neighborhood of zero.

Applying the one-dimensional Taylor's formula to  $u(\gamma(s))$  we get

**Lemma 2.1.** ([NSW85]) *Let  $u$  be a smooth function in a neighborhood of  $p$ . We have:*

$$u(\Theta_p(t)) = u(p) + \langle D_{\mathfrak{X}} u(p), t \rangle + \frac{1}{2} \langle (D_{\mathfrak{X}}^2 u(p))^* t, t \rangle + o(|t|^2)$$

as  $t \rightarrow 0$ .

If instead of the flow exponential based at  $p$  we use the Riemannian exponential  $\text{Exp}_p(t)$  we have

**Lemma 2.2.** *Let  $u$  be a smooth function in a neighborhood of  $p$ . We have:*

$$u(\text{Exp}_p(t)) = u(p) + \langle D_{\mathfrak{X}}u(p), t \rangle + \frac{1}{2} \langle \text{Hess}(u)(p)t, t \rangle + o(|t|^2)$$

as  $t \rightarrow 0$ .

For the proof, see for example Chapter 8 in [GQ20]. Applying Lemma 2.1 to the coordinate functions we obtain:

**Lemma 2.3.** *Write  $\Theta_p(t) = (\Theta_p^1(t), \Theta_p^2(t), \dots, \Theta_p^n(t))$ . Note that we can think of  $X_i(x)$  as the  $i$ -th row of  $\mathbb{A}(x)$ . Similarly  $D\Theta_p^k(0)$  is the  $k$ -column of  $\mathbb{A}(p)$  so that*

$$D\Theta_p(0) = \mathbb{A}(p).$$

In particular, the mapping  $t \mapsto \Theta_p(t)$  is a diffeomorphism taking a neighborhood of 0 into a neighborhood of  $p$ .

For vector fields  $Y = \sum_{i=1}^n y_i X_i$  and  $Z = \sum_{i=1}^n z_i X_i$  we have

$$\langle Y, Z \rangle_g = \sum_{i=1}^n y_i z_i.$$

Writing  $X$  and  $Y$  in Euclidean coordinates  $Y = \sum_{i=1}^n \bar{y}_i \partial_{x_i}$  and  $Z = \sum_{i=1}^n \bar{z}_i \partial_{x_i}$  we get

$$\langle Y, Z \rangle_g = \sum_{i,j=1}^n \bar{y}_i \bar{z}_j \langle \partial_{x_i}, \partial_{x_j} \rangle_g = \sum_{i,j=1}^n \bar{y}_i \bar{z}_j \mathbb{G}_{ij} = \sum_{i,j=1}^n \bar{y}_i \bar{z}_j (\mathbb{A}^t \mathbb{A})_{ij}^{-1}.$$

Conclude that

$$\langle Y, Z \rangle_g = \langle (\mathbb{A}^t \mathbb{A})^{-1} Y, Z \rangle = \langle (\mathbb{A}^{-1})^t Y, (\mathbb{A}^{-1})^t Z \rangle$$

and

$$\langle \mathbb{A}^t Y, \mathbb{A}^t Z \rangle_g = \langle Y, Z \rangle.$$

**Lemma 2.4.** *Let  $u: \mathbb{R}^n \mapsto \mathbb{R}$  be a smooth function. Then, the Riemannian gradient of  $u$  relative to the metric  $g$  is the vector field  $D_{\mathfrak{X}}u = \sum_{j=1}^n X_j(u) X_j$  with length*

$$|D_{\mathfrak{X}}u|_g = \langle D_{\mathfrak{X}}u, D_{\mathfrak{X}}u \rangle_g^{1/2} = \left( \sum_{i=1}^n (X_i u)^2 \right)^{1/2}.$$

*Proof.* The Riemannian gradient is the vector field given by the expression

$$\sum_{i,j=1}^n \mathbb{G}^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} = \sum_{i,j=1}^n (\mathbb{A}^t \mathbb{A})_{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} = \sum_{j=1}^n X_j(f) X_j.$$

□



**Example 2.5.** Note that it is not true, in general, that the Riemannian Hessian of a function  $u$  given by  $\text{Hess}(u)(V, W) = VWu - \nabla_V Wu$ , where  $V$  and  $W$  are arbitrary vector fields, equals the symmetrized second derivatives relative to the frame  $(D_{\mathfrak{X}}^2 u)^*$ . Here  $\nabla$  denotes the Levi-Civita connection. Consider the Riemannian Heisenberg group  $\mathbb{H}$  with left invariant vector fields in  $\mathbb{R}^3$  given by  $X = \partial_x - \frac{1}{2}y\partial_z$ ,  $Y = \partial_y + \frac{1}{2}x\partial_z$ , and  $Z = \partial_z$ . The Levi-Civita connection (computed in Chapter 2 of [CDPT07]) is determined by the equations

$$\begin{aligned} \nabla_X X &= \nabla_Y Y = \nabla_Z Z = 0, \\ \nabla_X Y &= \frac{1}{2}Z, & \nabla_Y X &= -\frac{1}{2}Z, \\ \nabla_Z X &= \nabla_X Z = -\frac{1}{2}Y, & \text{and} \\ \nabla_Z Y &= \nabla_Y Z = \frac{1}{2}X. \end{aligned}$$

The matrix of  $\text{Hess}(u)$  with respect to basis  $\{X, Y, Z\}$  is then

$$\begin{pmatrix} XXu & XYu - \frac{1}{2}Zu & XZu + \frac{1}{2}Yu \\ YXu + \frac{1}{2}Zu & YYu & YZu - \frac{1}{2}Xu \\ ZXu + \frac{1}{2}Yu & ZYu - \frac{1}{2}Xu & ZZu \end{pmatrix},$$

which differs from  $(D_{\mathfrak{X}}^2 u)^*$  in the  $(1, 3), (2, 3), (3, 1)$  and  $(3, 2)$  entries.

**Remark 2.6.** The mapping

$$t \mapsto \Theta_p(t)$$

is the flow exponential that agrees with the Lie group exponential when the frame  $\mathfrak{X}$  happens to be a basis for a Lie algebra of an  $n$ -dimensional Lie group.

Associated to the Riemannian metric  $g$  we also have the Riemannian exponential  $t \mapsto \text{Exp}_p(t)$  defined using geodesics. Both are diffeomorphisms in a neighborhood of 0. Lemma 2.4 shows that they agree up to first order since the Riemannian gradient equals the frame gradient (the linear terms in the Taylor development are the same).

Note that for the Riemannian Heisenberg group the flow exponential mapping is the group multiplication

$$\Theta_p(t) = p \cdot \Theta_0(t) = (x + t_1, y + t_2, z + t_3 + (1/2)(xt_2 - yt_1)).$$

Taking into account the explicit formula for the Riemannian exponential  $\text{Exp}_p(t)$  in the Riemannian Heisenberg group (see [BN16]) we conclude that  $\Theta_p(t)$  and  $\text{Exp}_p(t)$  are different mappings.

On the other hand, the flow exponential agrees with the Riemannian exponential in the case of Lie groups equipped with a bi-invariant metric, see Chapter 21 in [GQ20] or Chapter 2 in [AB15]. Compact Lie groups, like  $SO(n)$ , admit a bi-invariant metric. In fact a connected Lie group admits a bi-invariant metric if and only if it is isomorphic to the product of a compact group and an abelian group (Lemma 7.5 in [Mil76].)

**Proposition 2.7.** Fix  $x_0 \in \mathbb{R}^n$  and consider the function  $u(x) = d(x_0, x)$ . This function is smooth in the set  $B_{i(x_0)}(x_0) \setminus \{x_0\}$ , it satisfies the eikonal equation

$$(2.1) \quad |D_{\mathfrak{X}} u|_g = 1,$$

and it is  $\infty$ -harmonic

$$(2.2) \quad \Delta_{\mathfrak{X}, \infty} u = \left\langle (D_{\mathfrak{X}}^2 u)^* D_{\mathfrak{X}} u, D_{\mathfrak{X}} u \right\rangle_g = 0$$

*Proof.* Recall that  $d(x, x_0)$  is smooth in  $B_{i(x_0)}(x_0) \setminus \{x_0\}$  (see Chapter 6 in [Lee18] for example). The fact that  $d(x_0, x)$  satisfies (2.1) and (2.2) is also well-known (see Corollary 4.12 in [DMV13]).  $\square$

**Proposition 2.8.** *Fix  $x_0 \in \mathbb{R}^n$ . The function  $(x, y) \mapsto d^2(x, y)$  is smooth in  $B_{i(x_0)}(x_0) \times B_{i(x_0)}(x_0)$ .*

*Proof.* See Chapter 6 in [Lee18].  $\square$

We conclude that given a compact subset  $K \subset \mathbb{R}^n$ , there exists a constant  $C_0(K) > 0$  such that the function  $v_y(x) = d^2(x, y)$  satisfies

$$(2.3) \quad |D_{\mathfrak{X}}^2 v_y(x)|_g \leq C_0(K),$$

whenever  $x, y \in B_{i(x_0)/2}(x_0)$  for all  $x_0 \in K$ .

**Proposition 2.9.** *Given a compact subset  $K \subset \mathbb{R}^n$ , there exists a constant  $C_1(K) > 0$  such that the function  $u_y(x) = d(x, y)$  satisfies*

$$(2.4) \quad |D_{\mathfrak{X}}^2 u_y(x)|_g \leq C_1(K) \frac{1}{d(x, y)},$$

whenever  $x, y \in B_{i(x_0)/2}(x_0)$  for all  $x_0 \in K$ .

*Proof.* For  $y \in B_{i(x_0)/2}(x_0)$  and  $x \neq y$  we have

$$\begin{aligned} X_i(x) (X_j(x) d^2(x, y)) &= X_i(x) (2d(x, y) X_j(x) (d(x, y))) \\ &= 2X_i(x) (d(x, y)) X_j(x) (d(x, y)) + 2d(x, y) X_i(x) (X_j(x) (d(x, y))), \end{aligned}$$

from which we deduce

$$\begin{aligned} |X_i(x) (X_j(x) (d(x, y)))| &\leq \frac{C_0}{2d(x, y)} + \frac{|X_i(x) (d(x, y))| |X_j(x) (d(x, y))|}{d(x, y)} \\ &\leq \frac{C_0/2+1}{d(x, y)}. \end{aligned}$$

We can then take  $C_1(K) = n^2(C_0(K)/2 + 1)$ .  $\square$

### 3. SECOND DERIVATIVES OF THE METRIC

In this section we work in a region where the function of two variables  $(x, y) \mapsto d(x, y)$  is smooth. This is the case when  $x$  and  $y$  are in the ball  $B_{i(z)}(z)$  for some point  $z$  and  $x \neq y$ . Our starting point is that for fixed  $y$  the function  $x \mapsto d(x, y)$  satisfies the eikonal equation in a punctured neighborhood of  $y$

$$(3.1) \quad \sum_{i=1}^n (X_i^x d(x, y))^2 = 1,$$

where we have written  $X_i^x$  to indicate that the vector field  $X_i$  is acting on the  $x$  variable. See Proposition 2.7 above. Similarly, for a fixed  $x$  the function  $y \mapsto d(x, y)$  satisfies the eikonal equation in a punctured neighborhood of  $x$

$$(3.2) \quad \sum_{i=1}^n (X_i^y d(x, y))^2 = 1,$$

where we have written  $X_i^y$  to indicate that the vector field  $X_i$  is acting on the  $y$  variable. Next we apply  $X_j^x$  and  $X_j^y$  to both (3.1) and (3.2) obtaining the following result whose proof is a straightforward computation.

**Lemma 3.1.** *For  $j = 1, \dots, n$  we have*

$$\begin{aligned} \sum_{i=1}^n X_i^x d X_j^x X_i^x d &= 0 \quad , \quad \sum_{i=1}^n X_i^y d X_j^x X_i^y d = 0, \\ \sum_{i=1}^n X_i^x d X_j^y X_i^x d &= 0 \quad , \quad \sum_{i=1}^n X_i^y d X_j^y X_i^y d = 0. \end{aligned}$$

We introduce the following  $n \times n$  matrices of second derivatives:

$$\begin{aligned} (D_{\mathfrak{X}}^{2,x} u)_{ij} &= X_i^x X_j^x u \quad , \quad (D_{\mathfrak{X}}^{2,x,y} u)_{ij} = X_i^x X_j^y u \\ (D_{\mathfrak{X}}^{2,y,x} u)_{ij} &= X_i^y X_j^x u \quad , \quad (D_{\mathfrak{X}}^{2,y} u)_{ij} = X_i^y X_j^y u. \end{aligned}$$

With this notation, recalling Lemma 3.1, we obtain

$$(3.3) \quad \begin{aligned} D_{\mathfrak{X}}^{2,x} d \cdot D_{\mathfrak{X}}^x d &= 0 \quad , \quad D_{\mathfrak{X}}^{2,x,y} d \cdot D_{\mathfrak{X}}^y d = 0, \\ D_{\mathfrak{X}}^{2,y,x} d \cdot D_{\mathfrak{X}}^x d &= 0 \quad , \quad D_{\mathfrak{X}}^{2,y} d \cdot D_{\mathfrak{X}}^y d = 0. \end{aligned}$$

To keep the notation simpler we also denote by  $\mathfrak{Z}$  the frame  $\mathfrak{X} \otimes \mathfrak{X}$  in  $\mathbb{R}^n \times \mathbb{R}^n$  obtaining by considering two copies of  $\mathfrak{X}$ .

The  $\mathfrak{Z}$ -gradient of a function  $u(x, y)$  in the variables  $(x, y)$  is the  $2n \times 1$  vector field

$$D_{\mathfrak{Z}} u = \begin{pmatrix} D_{\mathfrak{X}}^x u \\ D_{\mathfrak{X}}^y u \end{pmatrix}.$$

Note that  $|D_{\mathfrak{Z}} d|_g = \sqrt{|D_{\mathfrak{X}}^x d|_g^2 + |D_{\mathfrak{X}}^y d|_g^2} = \sqrt{2}$ . The second derivative of  $u(x, y)$  is given by the  $2n \times 2n$  matrix

$$D_{\mathfrak{Z}}^2 u = \begin{pmatrix} D_{\mathfrak{X}}^{2,x} u & D_{\mathfrak{X}}^{2,x,y} u \\ D_{\mathfrak{X}}^{2,y,x} u & D_{\mathfrak{X}}^{2,y} u \end{pmatrix}.$$

From the identities (3.3) it follows that

$$(3.4) \quad D_{\mathfrak{Z}}^2 d \cdot D_{\mathfrak{Z}} d = 0$$

and, similarly for the symmetrized second derivatives, we obtain

$$(3.5) \quad \langle (D_{\mathfrak{Z}}^2 d)^* \cdot D_{\mathfrak{Z}} d, D_{\mathfrak{Z}} d \rangle_g = 0.$$

Since we have  $D_{\mathfrak{Z}} d^\alpha = \alpha d^{\alpha-1} D_{\mathfrak{Z}} d$  and  $D_{\mathfrak{Z}}^2 d^\alpha = \alpha d^{\alpha-1} D_{\mathfrak{Z}}^2 d + \alpha(\alpha-1) d^{\alpha-2} (D_{\mathfrak{Z}} d \otimes D_{\mathfrak{Z}} d)$  we get

$$\langle D_{\mathfrak{Z}}^2 d^\alpha \cdot D_{\mathfrak{Z}} d^\alpha, D_{\mathfrak{Z}} d^\alpha \rangle_g = \alpha^3 (\alpha-1) d^{3\alpha-4} \langle (D_{\mathfrak{Z}} d \otimes D_{\mathfrak{Z}} d) \cdot D_{\mathfrak{Z}} d, D_{\mathfrak{Z}} d \rangle_g$$

and  $\langle (D_{\mathfrak{Z}} d \otimes D_{\mathfrak{Z}} d) \cdot D_{\mathfrak{Z}} d, D_{\mathfrak{Z}} d \rangle_g = |D_{\mathfrak{Z}} d|_g^4 = 4$ . Summarizing, we have proved the following lemma.

**Lemma 3.2.**

$$\begin{aligned} \langle D_{\mathfrak{Z}}^2 d^\alpha \cdot D_{\mathfrak{Z}} d^\alpha, D_{\mathfrak{Z}} d^\alpha \rangle_g &= 4\alpha^3 (\alpha-1) d^{3\alpha-4}, \\ \langle (D_{\mathfrak{Z}}^2 d^\alpha)^* \cdot D_{\mathfrak{Z}} d^\alpha, D_{\mathfrak{Z}} d^\alpha \rangle_g &= 4\alpha^3 (\alpha-1) d^{3\alpha-4}. \end{aligned}$$

Choosing  $\alpha = 4/3$  we obtain

**Lemma 3.3.**

$$\Delta_{3,\infty} d^{\frac{4}{3}} = \left(\frac{4}{3}\right)^4.$$

The following identity follows easily from the fact that  $D_{\mathfrak{X}}^x d$  and  $D_{\mathfrak{X}}^y d$  are unit vectors

$$(3.6) \quad (D_3 d \otimes D_3 d)^2 = 2(D_3 d \otimes D_3 d).$$

**Lemma 3.4.**

$$\langle (D_3^2 d^\alpha)^2 \cdot D_3 d^\alpha, D_3 d^\alpha \rangle_g = 8\alpha^4(\alpha - 1)^2 d^{4\alpha-6}.$$

*Proof.* Let us compute  $(D_3^2 d^\alpha)^2$ :

$$\begin{aligned} (D_3^2 d^\alpha)^2 &= (\alpha d^{\alpha-1} D_3^2 d + \alpha(\alpha-1) d^{\alpha-2} (D_3 d \otimes D_3 d))^2 \\ &= \alpha^2 d^{2\alpha-2} (D_3^2 d)^2 + \alpha^2(\alpha-1) d^{2\alpha-3} D_3^2 d (D_3 d \otimes D_3 d) \\ &\quad + \alpha^2(\alpha-1) d^{2\alpha-3} (D_3 d \otimes D_3 d) D_3^2 d + \alpha^2(\alpha-1)^2 d^{2\alpha-4} (D_3 d \otimes D_3 d)^2. \end{aligned}$$

In the expression  $\langle (D_3^2 d^\alpha)^2 \cdot D_3 d^\alpha, D_3 d^\alpha \rangle$  there are four terms. The first and third terms vanish because of (3.4). The second term also vanishes since  $D_3^2 d (D_3 d \otimes D_3 d) = 0$  by (3.3). We are left with the fourth term

$$\begin{aligned} \alpha^2(\alpha-1)^2 \langle (D_3 d \otimes D_3 d)^2 \cdot D_3 d^\alpha, D_3 d^\alpha \rangle_g &= 2\alpha^2(\alpha-1)^2 d^{2\alpha-4} \langle (D_3 d \otimes D_3 d) \cdot D_3 d^\alpha, D_3 d^\alpha \rangle_g \\ &= 2\alpha^4(\alpha-1)^2 d^{4\alpha-6} \langle (D_3 d \otimes D_3 d) \cdot D_3 d, D_3 d \rangle_g \\ &= 8\alpha^4(\alpha-1)^2 d^{4\alpha-6}. \end{aligned}$$

□

We record the identity we get taking  $\alpha = 3/2$ , although we will not need it in the rest of the paper,

$$(3.7) \quad \langle (D_3^2 d^{3/2})^2 \cdot D_3 d^{3/2}, D_3 d^{3/2} \rangle_g = \frac{81}{8}.$$

We will also need to control a similar term with the symmetrized second derivatives. We first consider  $\langle ((D_3^2 d)^*)^2 \cdot D_3 d, D_3 d \rangle_g$ .

**Lemma 3.5.** *Given a compact set  $K \subset \mathbb{R}^n$  we can find a constant  $C(K, \mathfrak{X})$  depending on  $K$  and the frame  $\mathfrak{X}$  so that*

$$0 \leq \langle ((D_3^2 d)^*)^2 \cdot D_3 d, D_3 d \rangle_g \leq C(K, \mathfrak{X}).$$

*Proof.* The proof only uses basic properties of commutators of vector fields. Let us compute

$$\begin{aligned}
\langle ((D_3^2 d)^*)^2 \cdot D_3 d, D_3 d \rangle_g &= \langle (D_3^2 d)^* \cdot D_3 d, (D_3^2 d)^* \cdot D_3 d \rangle_g \\
&= \sum_{i=1}^{2n} ((D_3^2 d)^* \cdot D_3 d)_i^2 \\
&= \sum_{i=1}^{2n} \left( \sum_{k=1}^{2n} ((D_3^2 d)^*)_{ik} (D_3 d)_k \right)^2 \\
&= \sum_{i=1}^{2n} \left( \sum_{k=1}^{2n} \left( \frac{X_i X_k d + X_k X_i d}{2} \right) X_k d \right)^2 \\
&= \sum_{i=1}^{2n} \left( \sum_{k=1}^{2n} \left( X_i X_k d - \frac{[X_i, X_k] d}{2} \right) X_k d \right)^2 \\
&= \sum_{i=1}^{2n} \left( \sum_{k=1}^{2n} X_i X_k d X_k d - \frac{[X_i, X_k] d}{2} X_k d \right)^2 \\
&= \sum_{i=1}^{2n} \left( \sum_{k=1}^n \frac{[X_i, X_k] d}{2} X_k d \right)^2 \\
&\leq C(K, \mathfrak{X}),
\end{aligned}$$

where we have used equation (3.4) in the penultimate line and the fact that  $d$  is Lipschitz in the last line.  $\square$

We have  $(D_3^2 d^\alpha)^* = \alpha d^{\alpha-1} (D_3^2 d)^* + \alpha(\alpha-1) d^{\alpha-2} (D_3 d \otimes D_3 d)$  so that

$$\begin{aligned}
((D_3^2 d^\alpha)^*)^2 &= \alpha^2 d^{2\alpha-2} ((D_3^2 d)^*)^2 \\
&\quad + \alpha^2 (\alpha-1) d^{2\alpha-3} (D_3^2 d)^* (D_3 d \otimes D_3 d) \\
&\quad + \alpha^2 (\alpha-1) d^{2\alpha-3} (D_3 d \otimes D_3 d) (D_3^2 d)^* \\
&\quad + \alpha^2 (\alpha-1)^2 d^{2\alpha-4} (D_3 d \otimes D_3 d)^2.
\end{aligned}$$

Next, we observe that by (3.4) we have

$$\langle (D_3^2 d)^* (D_3 d \otimes D_3 d) D_3 d, D_3 d \rangle_g = 0$$

and by (3.3) and (3.4) we also have

$$\langle (D_3 d \otimes D_3 d) (D_3^2 d)^* D_3 d, D_3 d \rangle_g = 0.$$

Using (3.6) we conclude that

$$\begin{aligned}
\langle ((D_3^2 d^\alpha)^*)^2 D_3 d, D_3 d \rangle_g &= \alpha^2 d^{2\alpha-2} \langle ((D_3^2 d)^*)^2 D_3 d, D_3 d \rangle_g + 2\alpha^2 (\alpha-1)^2 \langle (D_3 d \otimes D_3 d) D_3 d, D_3 d \rangle_g \\
&= \alpha^2 d^{2\alpha-2} \langle ((D_3^2 d)^*)^2 D_3 d, D_3 d \rangle_g + 8\alpha^2 (\alpha-1)^2 d^{2\alpha-4}.
\end{aligned}$$

Hence, we can conclude this section with the following result whose proof immediately follows from the previous equality.

**Lemma 3.6.** *Given a compact set  $K \subset \mathbb{R}^n$  we can find a constant  $c_0 = C(K, \mathfrak{X})$  depending on  $K$  and the frame  $\mathfrak{X}$  so that*

$$0 \leq \langle ((D_3^2 d^\alpha)^*)^2 \cdot D_3 d, D_3 d \rangle_g \leq c_0 \alpha^2 d^{2\alpha-2} + 8\alpha^2 (\alpha-1)^2 d^{2\alpha-4}$$

and

$$0 \leq \langle ((D_3^2 d^\alpha)^*)^2 \cdot D_3 d^\alpha, D_3 d^\alpha \rangle_g \leq c_0 \alpha^4 d^{4\alpha-4} + 8\alpha^4 (\alpha-1)^2 d^{2\alpha-6}.$$

#### 4. VISCOSITY SOLUTIONS AND FRAMES

We are studying viscosity solutions of the equation

$$(4.1) \quad \Delta_{\mathfrak{X}, \infty} u(x) = f(x, u(x), D_{\mathfrak{X}} u(x))$$

where  $f$  is a continuous function satisfying the growth condition (1.9). We assume that  $u$  is a viscosity solution as in Definition 1.1.

We can use jets adapted to the frame  $\mathfrak{X}$  to characterize viscosity sub and supersolutions. To define second order superjets of an upper-semicontinuous function  $u$ , consider smooth functions  $\varphi$  touching  $u$  from above at a point  $x_0$ . The second-order super-jet of the upper-semicontinuous function  $u$  at the point  $x_0$  is the set

$$K_{\mathfrak{X}}^{2,+}(u, x_0) = \left\{ (D_{\mathfrak{X}} \varphi(x_0), (D_{\mathfrak{X}}^2 \varphi(x_0))^*) : \varphi \in C^2 \text{ in a neighborhood of } x_0, \varphi(x_0) = u(x_0), \right. \\ \left. \varphi(x) \geq u(x) \text{ in a neighborhood of } x_0 \right\}.$$

For each function  $\varphi \in C^2$  and a point  $x_0$  we write

$$(4.2) \quad \begin{aligned} \eta &= D_{\mathfrak{X}} \varphi(x_0) = (X_1 \varphi(x_0), X_2 \varphi(x_0), \dots, X_n \varphi(x_0)) \\ A_{ij} &= (D_{\mathfrak{X}}^2 \varphi(x_0))^* = \frac{1}{2} (X_i(X_j(\varphi))(x_0) + X_j(X_i(\varphi))(x_0)). \end{aligned}$$

This representation clearly depends on the frame  $\mathfrak{X}$ . Using the Taylor theorem (Lemma 2.1) for  $\varphi$  and the fact that  $\varphi$  touches  $u$  from above at  $x_0$  we get

$$(4.3) \quad u(\Theta_{x_0}(t)) \leq u(x_0) + \langle \eta, t \rangle + \frac{1}{2} \langle X t, t \rangle + o(|t|^2), \text{ as } t \rightarrow 0.$$

We may also consider  $J_{\mathfrak{X}}^{2,+}(u, x_0)$  defined as the collections of pairs  $(\eta, X)$  such that (4.3) holds. Denoting by  $J^{2,+}(v, t)$  the standard Euclidean superjets we also get from (4.3) the equivalence

$$(4.4) \quad (\eta, X) \in J_{\mathfrak{X}}^{2,+}(u, x_0) \iff (\eta, X) \in J^{2,+}(u \circ \Theta_{x_0}, 0)$$

Using the identification given by (4.2) it is clear that

$$K_{\mathfrak{X}}^{2,+}(u, x_0) \subset J_{\mathfrak{X}}^{2,+}(u, x_0).$$

In fact, we have equality. This is the analogue of the Crandall-Ishii Lemma of [Cra97] that was extended to vector fields in [BBM05]:

**Lemma 4.1.**

$$K_{\mathfrak{X}}^{2,+}(u, x_0) = J_{\mathfrak{X}}^{2,+}(u, x_0).$$

We define second order subjets  $J_{\mathfrak{X}}^{2,-}(u, x_0)$  similarly. We are in position to introduce the equivalent definition of viscosity solution based on jets to our  $\infty$ -Laplace equation.

**Definition 4.2.** An upper semi-continuous function  $u$  is a viscosity subsolution of (4.1) in a domain  $\Omega \subset \mathbb{R}^n$  if whenever  $(\eta, X) \in J_{\mathfrak{X}}^{2,+}(u, x_0)$  for  $x_0 \in \Omega$  we have

$$\langle X \cdot \eta, \eta \rangle_g \geq f(x_0, u(x_0), \eta).$$

A lower semi-continuous function  $v$  is a viscosity supersolution of (4.1) in a domain  $\Omega$  whenever  $(\eta, X) \in J_{\mathfrak{X}}^{2,-}(v, x_0)$  for  $x_0 \in \Omega$  we have

$$\langle X \cdot \eta, \eta \rangle_g \leq f(x_0, v(x_0), \eta).$$

We shall need the Euclidean Theorem of Sums (see [CIL92]) that we state for functions defined on  $D = B_1(0)$  the Euclidean ball of radius 1 centered at the origin.

**Theorem 4.3.** *Let  $u$  be upper-semicontinuous and  $v$  be lower-semicontinuous functions in  $B_1$ . Let  $\phi \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$  and suppose that there is a point  $(\hat{x}, \hat{y}) \in B_1 \times B_1$  such that*

$$u(\hat{x}) - v(\hat{y}) - \phi(\hat{x}, \hat{y}) = \max_{(x,y) \in \overline{B_1} \times \overline{B_1}} (u(x) - v(y) - \phi(x, y)).$$

*Then for each  $\mu > 0$  there are symmetric matrices  $X_\mu$  and  $Y_\mu$  such that*

$$(D_x \phi(\hat{x}, \hat{y}), X_\mu) \in \overline{J}^{2,+}(u, \hat{x}), \quad (-D_y \phi(\hat{x}, \hat{y}), Y_\mu) \in \overline{J}^{2,-}(v, \hat{y}),$$

*and we have the estimate*

$$-(\mu + \|D^2 \phi(\hat{x}, \hat{y})\|) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \leq \begin{pmatrix} X_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} \leq D^2 \phi(\hat{x}, \hat{y}) + \frac{1}{\mu} (D^2 \phi(\hat{x}, \hat{y}))^2.$$

## 5. LIPSCHITZ ESTIMATE: PROOF OF THEOREM 1.2

Let  $u$  be a viscosity solution of the equation

$$(5.1) \quad \Delta_{\mathfrak{X}, \infty} u(x) = f(x)$$

in a domain  $\Omega \subset \mathbb{R}^n$ . We shall assume that  $B_{2i(x_0)} \subset \Omega$ . The strategy of the proof taken from [Lin14] is to reduce the problem to the case when  $f \geq 0$ , so that  $u$  is a viscosity subsolution. It then follows from a comparison with the distance function that  $u$  is Lipschitz.

We add a new variable  $x_{n+1}$  and a new vector field  $X_{n+1} = \frac{\partial}{\partial x_{n+1}}$ . Consider the function

$$v(x_1, \dots, x_n, x_{n+1}) = u(x_1, \dots, x_n) + c|x_{n+1}|^{4/3},$$

where  $c$  is constant and the extended frame  $\mathcal{Y} = \{X_1, \dots, X_n, X_{n+1}\}$ . This frames induces a Riemannian metric  $h$  that satisfies

$$\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle_h = \langle (\xi_1, \xi_2) \rangle_g + \eta_1 \eta_2$$

for  $\xi_1, \xi_2 \in \mathbb{R}^n$  and  $\eta_1, \eta_2 \in \mathbb{R}$ . In the smooth case we have the identity

$$(5.2) \quad \begin{aligned} \langle D_{\mathcal{Y}}^2 v D_{\mathcal{Y}} v, D_{\mathcal{Y}} v \rangle_h &= \langle D_{\mathfrak{X}}^2 u D_{\mathfrak{X}} u, D_{\mathfrak{X}} u \rangle_g + (X^{n+1} X^{n+1} v)(X^{n+1} v)(X^{n+1} v) \\ &= f + c^3 \left(\frac{4}{3}\right)^3 \end{aligned}$$

In fact, this is also true in the viscosity sense. If a function  $u$  is a viscosity solution of (5.1), the extended function  $v$  is a viscosity solution of (5.2), see Chapter 10 in [Lin16]. Thus, we can assume that  $f \geq 0$  by taking an appropriate constant  $c$  depending only on  $\|f\|_{L^\infty(B_{i(x_0)}(x_0))}$ .

Therefore, we may assume that  $u$  is a subsolution of the  $\infty$ -Laplacian relative to the frame  $\mathcal{X}$ . Consider the functions  $w(y) = u(x) - u(x_0)$  and  $z(y) = M_r \frac{d(x_0, y)}{r}$  on the ball  $B_r(x_0)$  for  $r < i(x_0)/2$ , where

$$M_r = \sup\{w(x) : d(x_0, x) = r\}.$$

We compare these functions in the punctured ball  $B_r(x_0) \setminus \{x_0\}$ , where  $w$  is  $\infty$ -subharmonic and  $u$  is  $\infty$ -harmonic. We see that  $w \leq z$  on  $\partial B_r(x_0) \setminus \{x_0\}$  and thus in  $B_r(x_0)$  by the comparison principle. We conclude that

$$u(x) - u(x_0) \leq M_r \frac{d(x_0, x)}{r}$$

for all  $x \in B_r(x_0)$ . The constant  $M_r$  depends only on the  $L^\infty$ -norm of  $u$  on  $\overline{B_{i(x_0)/2}(x_0)}$ . Using a similar argument for  $-u$  we get

$$\frac{|u(x) - u(x_0)|}{d(x, x_0)} \leq \frac{M_{i(x_0)/4}}{i(x_0)/4} \leq \frac{4 \|u\|_{L^\infty(\overline{B_{i(x_0)/2}(x_0)})}}{i(x_0)}$$

for all  $x \in B_{i(x_0)/4}$ . We deduce the following bound of the local Lipschitz constant at  $x_0$

$$\text{Lip } u(x_0) = \lim_{r \rightarrow 0^+} \sup_{y \in B_r \setminus \{x_0\}} \frac{|u(x) - u(x_0)|}{d(x, x_0)} \leq \frac{4 \|u\|_{L^\infty(B_{i(x_0)}(x_0))}}{i(x_0)}.$$

By compactness we have  $\kappa(x_0) = \inf\{i(y) : y \in B_{i(x_0)}(x_0)\} > 0$ . Thus, for all  $y \in B_{i(x_0)}(x_0)$  we obtain

$$\text{Lip } u(y) \leq \frac{4 \|u\|_{L^\infty(B_{2i(0)}(x_0))}}{\kappa(x_0)}.$$

Therefore, we obtain

$$\text{ess sup}_{y \in B_{i(x_0)}(x_0)} \text{Lip } u(y) \leq \frac{4 \|u\|_{L^\infty(B_{2i(0)}(x_0))}}{\kappa(x_0)}.$$

From Theorem 4.7 in [DMV13] we deduce that

$$|D_{\mathfrak{X}} u(y)|_g \leq \frac{4 \|u\|_{L^\infty(B_{2i(0)}(x_0))}}{\kappa(x_0)}$$

for a.e.  $y$ , from which it follows that we can take

$$L = \frac{4 \|u\|_{L^\infty(B_{2i(0)}(x_0))}}{\kappa(x_0)}.$$

## 6. HÖLDER ESTIMATE: PROOF OF THEOREM 1.3

For  $\alpha \in (0, 1)$ , positive constants  $L$  and  $A$  to be determined later, and  $z \in B_{i(x_0)/4}$  consider the penalization function

$$G(x, y) = L d^\alpha(x, y) + A d^2(x, z).$$

Suppose that

$$(6.1) \quad u(\hat{x}) - u(\hat{y}) - G(\hat{x}, \hat{y}) = \sup \{u(x) - u(y) - G(x, y) : (x, y) \in \overline{B_{i(x_0)}} \times \overline{B_{i(x_0)}}\} = \theta > 0.$$

We will show that (6.1) leads to a contradiction for specific choices of  $L$  and  $A$  depending only on  $\|u\|_{L^\infty(B_{i(x_0)})}$ ,  $\|f\|_{L^\infty(B_{i(x_0)})}$ , and  $C(\overline{B_{2i(x_0)}}, g)$  when  $\alpha \in (0, 1)$ . When (6.1) does not hold we have

$$u(x) - u(y) \leq L d^\alpha(x, y) + A d^2(x, z), \text{ for } x, y \in B_{i(x_0)}.$$

Letting  $x = z$  we get the theorem.



Let us now assume that (6.1) holds. Since  $G(x, y) \geq 0$  we must have  $\hat{x} \neq \hat{y}$ . In what follows we temporarily omit the center  $x_0$  of the balls under consideration.

**Claim 6.1.** *For  $A \geq 8 \frac{\|u\|_{L^\infty(B_{i(x_0)})}}{i(x_0)^2}$  we have  $\hat{x} \in B_{(3/4)i(x_0)}$ .*

Suppose  $\hat{x} \notin B_{(3/4)i(x_0)}$ , then  $d(\hat{x}, z) \geq (1/2)i(x_0)$  so that we get

$$0 < \theta = u(\hat{x}) - u(\hat{y}) - L d^\alpha(\hat{x}, \hat{y}) - A d^2(\hat{x}, z),$$

and

$$0 \leq 2\|u\|_{L^\infty(B_1)} - L d^\alpha(\hat{x}, \hat{y}) - \frac{A}{4}.$$

This implies  $A < 8 \frac{\|u\|_{L^\infty(B_{i(x_0)})}}{i(x_0)^2}$ .

**From now on we take**  $A = 8 \frac{\|u\|_{L^\infty(B_{i(x_0)})}}{i(x_0)^2}$ .

**Claim 6.2.** *For  $L \geq 16\|u\|_{L^\infty(B_{i(x_0)})}$  we have  $\hat{y} \in B_{(7/8)i(x_0)}$ .*

If  $\hat{y} \notin B_{(7/8)i(x_0)}$ , we have  $d(\hat{y}, x_0) \geq (7/8)i(x_0)$  so that  $d(\hat{x}, \hat{y}) \geq (1/8)i(x_0)$ . From the inequality

$$0 < \theta = u(\hat{x}) - u(\hat{y}) - L d^\alpha(\hat{x}, \hat{y}) - A d^2(\hat{x}, z)$$

we obtain

$$0 < 2\|u\|_{L^\infty(B_{i(x_0)})} - L d^\alpha(\hat{x}, \hat{y}).$$

This implies

$$L < \frac{2\|u\|_{L^\infty(B_{i(x_0)})}}{d^\alpha(\hat{x}, \hat{y})} < \frac{2\|u\|_{L^\infty(B_{i(x_0)})}}{((1/8)i(x_0))^\alpha} = 28^\alpha \frac{\|u\|_{L^\infty(B_{i(x_0)})}}{i(x_0)^\alpha} < 16 \frac{\|u\|_{L^\infty(B_{i(x_0)})}}{i(x_0)^\alpha}.$$

**From now on we take**  $L \geq L_0 = 16 \frac{\|u\|_{L^\infty(B_{i(x_0)})}}{i(x_0)^\alpha}$ .

Therefore we can assume that  $u(x) - u(y) - G(x, y)$  has an interior positive maximum at the point  $(\hat{x}, \hat{y})$  for our choices of  $A$  and  $L \geq L_0$ . Note that we always have

$$L d^\alpha(\hat{x}, \hat{y}) + A d^2(\hat{x}, z) \leq 2\|u\|_{L^\infty(B_{i(x_0)})},$$

and that the point  $(\hat{x}, \hat{y})$  where the maximum is achieved depends on  $L, A, \alpha, z$  and  $u$ . The function  $u$  and the values of  $A, \alpha$  and  $z$  will remain fixed in the our arguments below. We will eventually let  $L \rightarrow \infty$ . From now on we will denote the point of maximum

$$(x_L, y_L),$$

where of course the subindex  $L$  denotes the dependence on  $L$ . In particular, we have

$$(6.2) \quad L d^\alpha(x_L, y_L) \leq 2\|u\|_{L^\infty(B_{i(x_0)})},$$

so that we have

$$(6.3) \quad \lim_{L \rightarrow \infty} d(x_L, y_L) = 0.$$

By selecting a sequence  $L_m \rightarrow \infty$  we conclude the existence of a point  $x^* \in B_{(3/4)i(x_0)}$  such that

$$x^* = \lim_{m \rightarrow \infty} x_{L_m} = \lim_{m \rightarrow \infty} y_{L_m}.$$

We will omit the subindex  $m$  and write just  $L$  for  $L_m$ . Note that we may assume that  $x_L$  and  $y_L$  are in the ball  $B_{i(x^*)/4}(x^*)$  for  $L$  large enough.

Consider next the flow exponentials  $s \mapsto \Theta_{x_L}(s)$  and  $t \mapsto \Theta_{y_L}(t)$  defined in a neighborhood of zero. The function  $u(x) - u(y) - G(x, y)$  has a positive local maximum at  $(x_L, y_L)$  if and only if the function

$$u(\Theta_{x_L}(s)) - u(\Theta_{y_L}(t)) - G(\Theta_{x_L}(s), \Theta_{y_L}(t))$$

has a positive local maximum at  $(0, 0)$ .

From the equivalence (4.4) we note the  $\mathfrak{Z}$  second order sub and superjets of the function  $L d^\alpha(x_L, y_L)$  at the point  $(x_L, y_L)$  are the same as the Euclidean second order sub and superjets of the function

$$\phi(s, t) = L d^\alpha(\Theta_{x_L}(s), \Theta_{y_L}(t))$$

so that  $G(\Theta_{x_L}(s), \Theta_{y_L}(t)) = \phi(s, t) + A d^2(\Theta_{x_L}(s), z)$ .

Next we write  $w(x) = u(x) - A d^2(x, z)$  so that

$$\begin{aligned} u(x) - u(y) - G(x, y) &= u(x) - A d^2(x, z) - u(y) - L d^\alpha(x, y) \\ &= w(x) - u(y) - d^\alpha(x, y). \end{aligned}$$

We are now ready to apply the Theorem of Sums 4.3 to the difference

$$w(\Theta_{x_L}(s)) - u(\Theta_{y_L}(t)) - \phi(s, t)$$

at the point  $(0, 0)$ . For each  $\mu > 0$ , there exists symmetric  $n \times n$  matrices  $X_\mu$  and  $Y_\mu$  so that

$$(D_s \phi(0, 0), X_\mu) \in \overline{J}^{2,+}(w \circ \Theta_{x_L}, 0), \quad (-D_t \phi(0, 0), Y_\mu) \in \overline{J}^{2,-}(u \circ \Theta_{y_L}, 0),$$

and we have the estimate

$$(6.4) \quad -(\mu + \|D^2 \phi(0, 0)\|) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \leq \begin{pmatrix} X_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} \leq D^2 \phi(0, 0) + \frac{1}{\mu} (D^2 \phi(0, 0))^2.$$

Using the equivalence (4.4) we translate back to the frame sub and super jets and set:

$$(6.5) \quad \begin{aligned} \xi_L &= D_s \phi(0, 0) = L D_{\mathfrak{X}}^x d^\alpha(x_L, y_L), \\ \eta_L &= D_t \phi(0, 0) = L D_{\mathfrak{X}}^y d^\alpha(x_L, y_L), \\ (\xi_L, X_\mu) &\in \overline{J}^{2,+}(w, x_L), \\ (\eta_L, Y_\mu) &\in \overline{J}^{2,-}(u, y_L). \end{aligned}$$

The second order Taylor expansion of  $\phi(s, t)$  at the point  $(0, 0)$  using the equivalence (4.4) can be written as

$$\phi(s, t) = \langle (\xi_L, \eta_L), (s, t) \rangle + \frac{1}{2} \langle L D_{\mathfrak{Z}}^2 d^\alpha(x_L, y_L) \cdot (s, t), (s, t) \rangle + o(|s|^2 + |t|^2) \text{ as } s, t \rightarrow 0,$$

from which it follows that

$$(6.6) \quad D^2 \phi(0, 0) = L (D_{\mathfrak{Z}}^2 d^\alpha(x_L, y_L))^* = M_L.$$

Note that the matrix  $D_{\mathfrak{Z}}^2 d^\alpha(x_L, y_L)$  is not symmetric in general, so we must symmetrize it. We can rewrite the third line in (6.5) as

$$(6.7) \quad (\xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z), X_\mu + A (D_{\mathfrak{X}}^{2,x} d^2(x_L, z))^*) \in \overline{J}^{2,+}(u, x_L)$$

and rewriting the inequalities (6.4) as

$$(6.8) \quad -(\mu + \|M_L\|) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \leq \begin{pmatrix} X_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} \leq M_L + \frac{1}{\mu} (M_L^2).$$

Using the fact that  $u$  is viscosity subsolution of (4.1) and (6.7) to get

$$(6.9) \quad \begin{aligned} & f(x_L, u(x_L), \xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z)) \\ & \leq \langle [X_\mu + A (D_{\mathfrak{X}}^{2,x} d^2(x_L, z))^*] \cdot (\xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z)), (\xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z)) \rangle_g. \end{aligned}$$

Using the fact that  $u$  is viscosity supersolution of (4.1) and the fourth statement in (6.5) we obtain

$$(6.10) \quad f(y_L, u(y_L), \eta_L) \geq \langle Y_\mu \cdot \eta_L, \eta_L \rangle_g.$$

Adding these estimates we get

$$(6.11) \quad \begin{aligned} & f(x_L, u(x_L), \xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z)) - f(y_L, u(y_L), \eta_L) \\ & \leq \langle [X_\mu + A (D_{\mathfrak{X}}^{2,x} d^2(x_L, z))^*] \cdot (\xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z)), (\xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z)) \rangle_g \\ & \quad - \langle Y_\mu \cdot \eta_L, \eta_L \rangle_g. \end{aligned}$$

Expanding the right hand side of (6.11) we obtain

$$(6.12) \quad \begin{aligned} & \langle X_\mu \cdot \xi_L, \xi_L \rangle_g - \langle Y_\mu \cdot \eta_L, \eta_L \rangle_g \\ & + 2A \langle X_\mu \cdot \xi_L, D_{\mathfrak{X}}^x d^2(x_L, z) \rangle_g \\ & + A^2 \langle X_\mu \cdot D_{\mathfrak{X}}^x d^2(x_L, z), D_{\mathfrak{X}}^x d^2(x_L, z) \rangle_g \\ & + A \langle (D_{\mathfrak{X}}^{2,x} d^2(x_L, z))^* \cdot \xi_L, \xi_L \rangle_g \\ & + 2A^2 \langle (D_{\mathfrak{X}}^{2,x} d^2(x_L, z))^* \cdot \xi_L, D_{\mathfrak{X}}^x d^2(x_L, z) \rangle_g \\ & + A^3 \langle (D_{\mathfrak{X}}^{2,x} d^2(x_L, z))^* \cdot D_{\mathfrak{X}}^x d^2(x_L, z), D_{\mathfrak{X}}^x d^2(x_L, z) \rangle_g \\ & = T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \end{aligned}$$

**Claim 6.3.** *Estimate of  $T_1$ :*

$$\begin{aligned} T_1 & \leq 4\alpha^3(\alpha-1)L^3d^{3\alpha-4} + \frac{L^4}{\mu} (c_0\alpha^4d^{4\alpha-4} + 8\alpha^4(\alpha-1)^2d^{2\alpha-6}) \\ & = 4(\alpha-1)\alpha^3L^3d^{3\alpha-4} \left(1 + \frac{2L\alpha(\alpha-1)d^{\alpha-2}}{\mu} + \frac{c_0\alpha Ld^\alpha}{\mu^4(\alpha-1)}\right) \end{aligned}$$

*Proof.* From the upper bound in (6.8) we get

$$\langle X_\mu \cdot \xi_L, \xi_L \rangle_g - \langle Y_\mu \cdot \eta_L, \eta_L \rangle_g \leq \langle \left(M_L + \frac{1}{\mu}(M_L^2)\right) \cdot \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix}, \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix} \rangle_g$$

Recall that  $M_L = L(D_3^2 d^\alpha(x_L, y_L))^*$ . We need to estimate

$$\langle M_L \cdot \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix}, \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix} \rangle_g \text{ and } \langle M_L^2 \cdot \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix}, \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix} \rangle_g.$$

Using Lemma 3.2, we get

$$\begin{aligned} \langle M_L \cdot \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix}, \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix} \rangle_g & = L^3 \langle (D_3^2 d^\alpha(x_L, y_L))^* \cdot \begin{pmatrix} D_{\mathfrak{X}}^x d^\alpha(x_L, y_L) \\ D_{\mathfrak{X}}^y d^\alpha(x_L, y_L) \end{pmatrix}, \begin{pmatrix} D_{\mathfrak{X}}^x d^\alpha(x_L, y_L) \\ D_{\mathfrak{X}}^y d^\alpha(x_L, y_L) \end{pmatrix} \rangle_g \\ & = L^3 4\alpha^3(\alpha-1)d(x_L, y_L)^{3\alpha-4}, \end{aligned}$$

and invoking Lemma 3.6 we get

$$\langle M_L^2 \cdot \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix}, \begin{pmatrix} \xi_L \\ \eta_L \end{pmatrix} \rangle_g \leq L^4 (c_0\alpha^4d^{4\alpha-4} + 8\alpha^4(\alpha-1)^2d^{2\alpha-6}).$$

□

For a fixed  $\beta \in \mathbb{R}$  to be determined below set  $\mu = \beta (2L\alpha(\alpha-1)d^{\alpha-2})$ , so that we have

$$\begin{aligned} 1 + \frac{2\alpha(\alpha-1)Ld^{\alpha-2}}{\mu} + \frac{c_0\alpha Ld^\alpha}{\mu 4(\alpha-1)} &= 1 + \frac{1}{\beta} + \frac{c_0\alpha d^\alpha}{\beta(2\alpha(\alpha-1)d^{\alpha-2})4(\alpha-1)} \\ &= 1 + \frac{1}{\beta} + \frac{c_0 d^2}{8\beta(\alpha-1)^2}. \end{aligned}$$

Since  $\beta < 0$  we have

$$1 + \frac{1}{\beta} + \frac{c_0 d^2}{8\beta(\alpha-1)^2} \geq 1 + \frac{1}{\beta} + \frac{c_0}{8\beta(\alpha-1)^2}.$$

We can now choose  $\beta$  depending only on  $c_0$  and  $\alpha$  so that

$$(6.13) \quad 1 + \frac{2L\alpha(\alpha-1)d^{\alpha-2}}{\mu} + \frac{Lc_0\alpha d^2}{\mu 4(\alpha-1)} \geq \frac{1}{2}.$$

Our next task is to estimate the norm  $\|X_\mu\|$  using Proposition 2.9 and (6.8). Using the pair of vectors  $(v, 0) \in \mathbb{R}^n \times \mathbb{R}^n$  in (6.8) we get

$$-(\mu + \|M_L\|)|v|_g^2 \leq \langle X_\mu \cdot v, v \rangle_g \leq \left( \|M_L\| + \frac{\|M_L\|^2}{\mu} \right) |v|_g^2.$$

We estimate the norm of  $M_L$  by using Proposition 2.9

$$\begin{aligned} \|M_L\| &\leq L\|D_3^2 d^\alpha(x_L, y_L)^*\| \\ &\leq L\|\alpha d^{\alpha-1}(D_3^2 d)^* + \alpha(\alpha-1)d^{\alpha-2}(D_3 d \otimes D_3 d)\| \\ &\leq L\alpha d^{\alpha-1}\|(D_3^2 d)^*\| + L\sqrt{2}\alpha|\alpha-1|d^{\alpha-2} \\ &\leq L\alpha d^{\alpha-1}\frac{c_1}{d} + L\sqrt{2}\alpha|\alpha-1|d^{\alpha-2} \\ &\leq c_2 L\alpha d^{\alpha-2} \end{aligned}$$

for some constant  $c_2 \geq 1$ . Note that we can choose  $\beta$  sufficiently negative so that  $c_2 \leq \beta(\alpha-1)$  we can guarantee that  $\|M_L\| \leq \mu/2$ . For the upper bound we compute

$$\begin{aligned} \langle X_\mu \cdot v, v \rangle_g &\leq \left( c_2 L\alpha d^{\alpha-2} + \frac{(c_2 L\alpha d^{\alpha-2})^2}{\mu} \right) |v|_g^2 \\ &\leq \left( \|M_L\| + \frac{\|M_L\|^2}{\mu} \right) |v|_g^2 \\ &\leq \frac{3}{4}\mu |v|_g^2. \end{aligned}$$

For the lower bound

$$\begin{aligned} \langle X_\mu \cdot v, v \rangle_g &\geq -(\mu + \|M_L\|) |v|_g^2 \\ &\geq -\frac{3}{2}\mu |v|_g^2. \end{aligned}$$

Combining both estimates we get

$$(6.14) \quad \|X_\mu\| \leq \frac{3}{4}\mu \leq \frac{3}{4}\beta(2L\alpha(\alpha-1)d^{\alpha-2}) \leq c_4 L\alpha d^{\alpha-2}.$$

**Claim 6.4.** *Estimate of  $T_2$ :*

$$T_2 \leq c_5 \alpha^2 L^2 d^{2\alpha-3}.$$

**Claim 6.5.** *Estimate of  $T_3$ :*

$$T_3 \leq c_6 \alpha^2 L d^{\alpha-2}.$$

**Claim 6.6.** *Estimate of  $T_4$ :*

$$T_4 \leq c_7 \alpha^2 L^2 d^{2\alpha-2}.$$

**Claim 6.7.** *Estimate of  $T_5$ :*

$$T_5 \leq c_8 \alpha L d^{\alpha-1}.$$

**Claim 6.8.** *Estimate of  $T_6$ :*

$$T_6 \leq c_9.$$

Let us now estimate the left-hand side of (6.11) using condition (1.9). We have as  $L \rightarrow \infty$

$$\begin{aligned} |f(x_L, u(x_L), \xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z))| &\leq C_2 |\xi_L + A D_{\mathfrak{X}}^x d^2(x_L, z)|_g^\beta + C_3 \\ &= C_2 |L D_{\mathfrak{X}}^x(x_L, y_L) d^\alpha + A D_{\mathfrak{X}}^x d^2(x_L, z)|_g^\beta + C_3 \\ &\leq C_4 |L \alpha d^{\alpha-1} D_{\mathfrak{X}}^x d(x_L, y_L)|^\beta + C_5 |d(x_L, z) D_{\mathfrak{X}}^x d(x_L, z)|^\beta + C_3 \\ &\leq C_6 L^\beta d^{\beta(\alpha-1)} + C_7 \\ &\leq C_8 L^\beta d^{\beta(\alpha-1)}, \end{aligned}$$

and similarly for the term  $|\langle Y_\mu \cdot \eta_L, \eta_L \rangle|$ . Combining these estimates we get

$$\begin{aligned} -C_9 L^\beta d^{\beta(\alpha-1)} &\leq 4(\alpha-1) \alpha^3 L^3 d^{3\alpha-4} \left( 1 + \frac{2L \alpha(\alpha-1) d^{\alpha-2}}{\mu} + \frac{c_0 \alpha L d^2}{\mu 4(\alpha-1)} \right) \\ &\quad + c_5 \alpha^2 L^2 d^{2\alpha-3} \\ &\quad + c_6 \alpha^2 L d^{\alpha-2} \\ &\quad + c_7 \alpha^2 L^2 d^{2\alpha-2} \\ &\quad + c_8 \alpha L d^{\alpha-1} \\ &\quad + c_9. \end{aligned} \tag{6.15}$$

Using (6.13) and  $\alpha - 1 < 0$  we rewrite it as

$$\begin{aligned} -C_9 L^\beta d^{\beta(\alpha-1)} &\leq L^3 d^{3\alpha-4} \left[ 2(\alpha-1) \alpha^3 \right. \\ &\quad + c_5 \alpha^2 L^{-1} d^{-\alpha+1} \\ &\quad + c_6 \alpha^2 L^{-2} d^{-2\alpha+2} \\ &\quad + c_7 \alpha^2 L^{-1} d^{-\alpha+2} \\ &\quad + c_8 \alpha L^{-2} d^{-2\alpha+3} \\ &\quad \left. + c_9 L^{-3} d^{-3\alpha+4} \right]. \end{aligned} \tag{6.16}$$

We now let  $L \rightarrow \infty$  and use the fact that  $L d^\alpha$  is bounded (6.2) to get

$$L^3 d^{3\alpha-4} \leq C_{10} L^\beta d^{\beta\alpha} d^{-\beta} \leq (L d^\alpha)^\beta d^{-\beta} \leq C_{11} d^{-\beta},$$

which implies the boundedness of  $L^3 d^{3\alpha-4+\beta}$  as  $L \rightarrow \infty$ .

Since  $d \rightarrow 0$  by (6.3) we obtain the desired contradiction whenever  $3\alpha - 4 - \beta < 0$ .

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ESTIMATES FOR THE  $\infty$ -LAPLACIAN

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