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DILATION THEORY IN FINITE DIMENSIONS AND MATRIX CONVEXITY

MICHAEL HARTZ AND MARTINO LUPINI

ABSTRACT. We establish a finite-dimensional version of the Arveson–Stinespring dilation theorem for unital completely positive maps on operator systems. This result can be seen as a general principle to deduce finite-dimensional dilation theorems from their classical infinite-dimensional counterparts. In addition to providing unified proofs of known finite-dimensional dilation theorems, we establish finite-dimensional versions of Agler’s theorem on rational dilation on an annulus, of Berger’s dilation theorem for operators of numerical radius at most 1, and of the Putinar–Sandberg numerical range dilation theorem. As a key tool, we prove versions of Carathéodory’s and of Minkowski’s theorem for matrix convex sets.

1. INTRODUCTION

1.1. **Background.** One of the cornerstones of the theory of operators on Hilbert space is Sz.-Nagy’s dilation theorem [45], which can be phrased as follows.

Theorem 1.1 (Sz.-Nagy). *Let T be a contraction on a Hilbert space H , i.e. a linear operator with $\|T\| \leq 1$. Then there exist a Hilbert space $K \supset H$ and a unitary operator U on K such that, for every polynomial p with complex coefficients,*

$$(1) \quad p(T) = P_H p(U)|_H.$$

The operator U in Sz.-Nagy’s theorem is called a *dilation* of T . This theorem frequently makes it possible to study contractions through their unitary dilations, the key advantage being that unitaries are well understood by virtue of the spectral theorem [46]. On the other hand, even if H is finite-dimensional, in which case T can be regarded as a matrix, then the unitary dilation U still typically acts on an infinite-dimensional space K . Indeed, one can show that unless T is itself unitary, K is necessarily infinite-dimensional. Thus, for contractive matrices T , it is not clear that the dilation U is always easier to understand.

This drawback was addressed by Egerváry [24], who established a finite-dimensional version of Sz.-Nagy’s dilation theorem.

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Theorem 1.2 (Egerváry). *Let T be a contraction on a finite-dimensional Hilbert space H and let $N \in \mathbb{N}$. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a unitary operator U on K such that, for every polynomial p with complex coefficients of degree at most N ,*

$$p(T) = P_H p(U)|_H.$$

In other words, by only requiring (1) to hold for a finite-dimensional space of polynomials, we can retain finite-dimensionality of the dilation space.

Egerváry's theorem was extended to pairs of commuting contractive matrices by M^cCarthy and Shalit [37]; their result is therefore a finite-dimensional version of Andô's dilation theorem [4]. More generally, M^cCarthy and Shalit proved a finite-dimensional dilation theorem for tuples of commuting matrices that admit a dilation to commuting unitaries. This last result was further generalized by Cohen [13] to d -tuples of commuting operators admitting a polynomial normal ∂X -dilation for compact subsets X of \mathbb{C}^d . A related finite-dimensional dilation result was proved by Davidson, Dor-On, Shalit and Solel [19, Theorem 7.1]. It is worth remarking that while Egerváry's proof explicitly constructs a unitary matrix on a larger space, the results of M^cCarthy–Shalit, Cohen and Davidson–Dor-On–Shalit–Solel all deduce the finite-dimensional dilation theorem from its infinite-dimensional counterpart. We also refer the reader to the survey article [36]; see also [35] for connections of finite-dimensional dilations with quantum information theory.

1.2. An abstract finite-dimensional dilation theorem. The goal of this article is to establish an abstract result that makes it possible to deduce finite-dimensional dilation theorems from their infinite-dimensional relatives under general assumptions. In particular, our result will imply all finite-dimensional dilation theorems mentioned in the preceding paragraph, as well as new ones.

To formulate such an abstract result, the framework of dilations of unital completely positive (u.c.p.) maps is very useful. Let A be a unital C^* -algebra. Recall that an *operator system* is a unital self-adjoint subspace $S \subset A$. A linear map $\varphi : S \rightarrow B(H)$ is said to be positive if it maps positive elements to positive elements, and completely positive if all amplifications $\varphi^{(n)} : M_n(S) \rightarrow M_n(B(H))$, defined by applying φ entrywise, are positive. Arveson's extension theorem shows that every u.c.p. map $\varphi : S \rightarrow B(H)$ extends to a u.c.p. map $\psi : A \rightarrow B(H)$. By Stinespring's dilation theorem, ψ dilates to a representation of A , that is, there exist a Hilbert space $K \supset H$ and a unital $*$ -homomorphism $\pi : A \rightarrow B(K)$ such that $\psi(a) = P_H \pi(a)|_H$ for all $a \in A$. In particular,

$$\varphi(s) = P_H \pi(s)|_H \quad (s \in S).$$

Conversely, every linear map $\varphi : S \rightarrow B(H)$ of this form is unital and completely positive. Seeking finite-dimensional dilations in this setting means asking whether we can achieve that $\dim(K) < \infty$.

In the sequel, we will say that a u.c.p. map $\varphi : S \rightarrow B(H)$ *dilates to a finite-dimensional representation of A* if there exist a finite-dimensional Hilbert space K

containing H and a unital $*$ -homomorphism $\pi : A \rightarrow B(K)$ such that $\varphi(s) = P_H \pi(s)|_H$ for all $s \in S$.

Question 1.3. Let A be a unital C^* -algebra, let $S \subset A$ be an operator system and let $\varphi : S \rightarrow B(H)$ be a u.c.p. map with $\dim(H) < \infty$. Does φ dilate to a finite-dimensional representation of A ?

As explained earlier, a dilation always exists on a possibly infinite-dimensional space by Arveson's extension theorem and Stinespring's dilation theorem.

Before stating our main result regarding Question 1.3, let us observe that the question can only have a positive answer if the C^* -algebra A has "enough" finite-dimensional representations. More precisely, a result of Courtney and Shulman [14] implies the following necessary condition.

Proposition 1.4. *Let A be a unital C^* -algebra with the property that for every operator system $S \subset A$ with $\dim(S) \leq 2$, each u.c.p. map $\varphi : S \rightarrow \mathbb{C}$ dilates to a finite-dimensional representation of A . Then every irreducible representation of A is finite-dimensional.*

Proof. Let $a \in A$ and consider the operator system $S = \text{span}\{1, a^*a\} \subset A$. By [9, II.6.3.3], there exists a state $\varphi : S \rightarrow \mathbb{C}$ with $\varphi(a^*a) = \|a^*a\|$. By assumption, φ dilates to a finite-dimensional representation π of A . Then

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| \geq \varphi(a^*a) = \|a^*a\| = \|a\|^2.$$

Since π is contractive, equality holds throughout. This shows that every element of A attains its norm on a finite-dimensional representation. By a result of Courtney and Shulman [14, Theorem 4.4], this is equivalent to saying that every irreducible representation of A is finite-dimensional. \square

C^* -algebras whose irreducible representations are all finite-dimensional are called FDI in [14]. We are exclusively concerned with unital C^* -algebras, in which case the class of FDI C^* -algebras coincides with the class of liminal (also called CCR) C^* -algebras; see [9, Section IV.1.3]. Examples of FDI C^* -algebras are commutative C^* -algebras, as every irreducible representation of a commutative C^* -algebra is one dimensional. A more general class of examples is given by r -subhomogeneous C^* -algebras. These are C^* -algebras whose irreducible representations all occur on a Hilbert space of dimension at most r ; see [9, Section IV.1.4]. In [14], examples of non-subhomogeneous FDI C^* -algebras are mentioned, such as full group C^* -algebras of certain Lie groups and algebras arising from mapping telescopes; these can be unitized if necessary without changing subhomogeneity or the FDI property. Clearly, every FDI C^* -algebra is residually finite-dimensional (RFD), meaning that finite-dimensional representations separate the elements of the C^* -algebra, but the converse is not true. An example of an RFD C^* -algebra that is not FDI is the full group C^* -algebra $C^*(\mathbb{F}_2)$ of the free group on two generators [12]. For more discussion about FDI C^* -algebras, the reader is referred to [14].

Our main result shows that if A is FDI and $\dim(S) < \infty$, then Question 1.3 has a positive answer. This result can be regarded as a finite-dimensional version of the Arveson–Stinespring dilation theorem.

Theorem 1.5. *Let A be a unital FDI (equivalently, unital liminal) C^* -algebra, let $S \subset A$ be an operator system with $\dim(S) < \infty$ and let $\varphi : S \rightarrow B(H)$ be a u.c.p. map with $\dim(H) < \infty$. Then φ dilates to a finite-dimensional representation of A .*

This result will be proved as Theorem 3.3 below. If A is commutative, or more generally subhomogeneous, then we obtain an explicit upper bound for the dimension of the dilation, see Proposition 3.4. An approximate version of Theorem 1.5, in which the C^* -algebra A is allowed to be RFD, is due to Alekseev, Netzer and Thom [3, Theorem 3.8].

To illustrate how Theorem 1.5 can be used to deduce concrete finite-dimensional dilation theorems from their infinite-dimensional relatives, let us explain how to prove Egevary's theorem from Sz.-Nagy's theorem and Theorem 1.5

Proof of Theorem 1.2 from Theorems 1.1 and 1.5. Let $T \in B(H)$ be a contraction with $\dim(H) < \infty$ and let $N \in \mathbb{N}$. By Sz.-Nagy's dilation theorem (Theorem 1.1), T admits a unitary dilation V on a (generally infinite-dimensional) Hilbert space $L \supset H$. The continuous functional calculus for V shows that V induces a representation $\sigma : C(\mathbb{T}) \rightarrow B(L)$ with $\sigma(p) = p(V)$ for all $p \in \mathbb{C}[z]$. Let

$$S = \text{span}\{1, z^k, \bar{z}^k : 1 \leq k \leq N\} \subset C(\mathbb{T}),$$

which is a finite-dimensional operator system. Then the map $\varphi : S \rightarrow B(H)$ defined by

$$\varphi(f) = P_H \sigma(f)|_H$$

is u.c.p. and satisfies $\varphi(p) = P_H p(V)|_H = p(T)$ for all $p \in \mathbb{C}[z]$ with $\deg(p) \leq N$. Applying Theorem 1.5 to the commutative C^* -algebra $C(\mathbb{T})$, we find a Hilbert space $K \supset H$ with $\dim(K) < \infty$ and a $*$ -representation $\pi : C(\mathbb{T}) \rightarrow B(K)$ with

$$\varphi(f) = P_H \pi(f)|_H \quad (f \in S).$$

Let $U = \pi(z)$. Then $U \in B(K)$ is unitary and

$$p(T) = \varphi(p) = P_H p(U)|_H$$

for all $p \in \mathbb{C}[z]$ with $\deg(p) \leq N$. □

The above proof shows that, roughly speaking, the operator system S encodes which relations should hold for the dilation. In particular, the necessity of the degree bound in Egevary's theorem shows that the assumption of finite-dimensionality of S in Theorem 1.5 is necessary.

Further applications of Theorem 3.3 will be given in Section 4. In particular, we establish a finite-dimensional dilation theorem for operators with numerical radius at most 1 and a finite-dimensional version of Agler's theorem of rational dilation on an annulus.

Here, we highlight one application regarding matrices with prescribed numerical range. Recall that the numerical range of an operator $T \in B(H)$ is defined to be

$$W(T) = \{\langle T\xi, \xi \rangle : \xi \in H, \|\xi\| = 1\}.$$

The Toeplitz–Hausdorff theorem shows that $W(T)$ is a convex set. Moreover, $\sigma(T) \subset \overline{W(T)}$ and $W(T)$ is compact if H is finite-dimensional. Currently, there is a large

amount of activity surrounding the numerical range in the context of Crouzeix's conjecture [16], which asserts that

$$\|p(T)\| \leq 2 \sup_{z \in W(T)} |p(z)|$$

should hold for all polynomials p and all $T \in B(H)$. Clearly, one may restrict to finite-dimensional Hilbert spaces H here. For recent work on this problem, see for instance [8, 15, 43] and the references therein. This conjecture still seems to be open, but it is known that it holds when the constant 2 is replaced with $1 + \sqrt{2}$, a result due to Crouzeix and Palencia [15], see also [43]. A theorem of Okubo and Ando [40] implies that Crouzeix's conjecture holds with constant 2 in the case when $W(T)$ is a disc. This result was proved using dilation theory and hence operator theory in infinite dimensions. On the other hand, some of the recent progress on Crouzeix's conjecture was obtained using special properties in finite dimensions, such as the existence of vectors on which the operator norm is attained; see for instance [8, 10].

In this context, we establish the following finite-dimensional dilation theorem, whose infinite-dimensional counterpart is due to Putinar and Sandberg [42]. If $\Omega \subset \mathbb{C}$ is a bounded open set with smooth boundary $\partial\Omega$, let $A(\Omega)$ be the algebra of all holomorphic functions on Ω that extend to be continuous on $\bar{\Omega}$. If $f \in A(\Omega)$, we let $C\bar{f}$ be the Cauchy transform of \bar{f} , which is defined by

$$(C\bar{f})(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\bar{f}(\zeta)}{\zeta - z} d\zeta \quad (z \in \Omega).$$

In particular, $C\bar{f}$ is holomorphic on Ω , so $(C\bar{f})(T)$ is defined whenever $T \in B(H)$ satisfies $\sigma(T) \subset \Omega$.

Theorem 1.6. *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with smooth boundary $\partial\Omega$. Let T be an operator on a finite-dimensional Hilbert space H with $W(T) \subset \Omega$ and let $\mathcal{A} \subset A(\Omega)$ be a finite-dimensional subspace. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a normal operator N on K with $\sigma(N) \subset \partial\Omega$ such that*

$$f(T) + (C\bar{f})(T)^* = 2P_H f(N)|_H$$

for all $f \in \mathcal{A}$.

This result will be proved in Corollary 4.10. There, we will also remark on the connection between the dilation result and some of the current approaches to Crouzeix's conjecture.

1.3. Matrix convex sets. To establish our main result, we will use tools from the theory of matrix convexity. Matrix convex sets were introduced by Wittstock [48] and by Effros and Winkler [23] and further studied by Webster and Winkler [47]. It is known that dilation theory is closely related with matrix convexity, see [19, 20, 29] for some recent work. In addition, matrix convexity has found applications in real algebraic geometry, see for instance [31, 32, 34].

We will state the precise definition of matrix convex sets in Section 2. For now, let us simply recall that a matrix convex set \mathbf{X} in a complex vector space V is of the form $\mathbf{X} = (X_n)_{n=1}^{\infty}$, where $X_n \subset M_n(V)$ for all $n \geq 1$. There are notions of matrix

convex combinations, matrix convex hull and of matrix extreme points. Moreover, Webster and Winkler [47] proved a version of the Krein–Milman theorem in this setting.

In the article [37] of M^cCarthy and Shalit and in subsequent works [13, 19] the authors crucially use a classical theorem of Carathéodory from convex analysis (see, for instance, [18, Theorem 16.1.8]) to obtain finite-dimensional dilations.

Theorem 1.7 (Carathéodory). *Let $X \subset \mathbb{R}^n$ be a set. If $x \in \mathbb{R}^n$ belongs to the convex hull of X , then x is a convex combination of at most $n + 1$ points in X .*

In the context of matrix convex sets, Davidson, Dor-On, Shalit and Solel proved a version of Carathéodory’s theorem for matrix ranges of normal tuples [19, Theorem 2.7]. Kriel established Carathéodory’s theorem for matrix convex sets consisting of tuples of self-adjoint matrices [34, Lemma 1.14]. For our purposes, the following Carathéodory theorem for general matrix convex sets will be useful. It will be proved in Theorem 2.3.

Theorem 1.8. *Let V be a finite-dimensional vector space and let $\mathbf{X} = (X_n)$ with $X_n \subset M_n(V)$ for $n \geq 1$. If $x \in M_n(V)$ belongs to the matrix convex hull of \mathbf{X} , then it is a matrix convex combination of points of \mathbf{X} of length at most $n^2(2 \dim(V) + 1)$.*

Carathéodory’s theorem is related with another classical result, due to Minkowski, which can be thought of as a strengthening of the Krein–Milman theorem in finite dimensions; see for example [18, Theorem 16.4.6]. The difference with the Krein–Milman theorem is that closure is not required.

Theorem 1.9 (Minkowski). *Let $K \subset \mathbb{R}^n$ be a compact convex set. Then K is the convex hull of its extreme points.*

Kriel obtained a version of Minkowski’s theorem in his setting of matrix convex sets; see Theorem 6.8 in [34]. In Theorem 2.9, we will prove the following version of Minkowski’s theorem for general matrix convex sets, which will be very useful in the proof of Theorem 1.5.

Theorem 1.10. *Let \mathbf{X} be a compact matrix convex set in a finite-dimensional locally convex vector space V . Then \mathbf{X} is the matrix convex hull of its matrix extreme points.*

For free spectrahedra, a particular class of matrix convex sets, a recent result of Evert and Helton [25] yields a stronger conclusion than Theorem 1.8 and Theorem 1.10 combined. In the result of Evert and Helton, it suffices to consider a more restrictive notion of extreme points, and they obtain a better bound on the length of the matrix convex combination. However, we will apply Theorem 1.8 and Theorem 1.10 to matrix convex sets that are typically not free spectrahedra. In the somewhat different setting of C^* -convexity, Carathéodory and Minkowski theorems were previously established by Farenick [27] and Morenz [38].

In light of the above mentioned results, it is not surprising that Carathéodory’s and Minkowski’s theorem hold for general matrix convex sets. Our contribution to matrix convexity in this article is the introduction of a device that makes it possible to relate questions about matrix convexity to questions about classical convexity.

Thus, we are able to deduce Theorem 1.8 and Theorem 1.10 from their classical counterparts. As a by-product, we also obtain another proof of the Krein–Milman theorem for matrix convex sets due to Webster and Winkler.

1.4. Outline. The remainder of this article is organized as follows. In Section 2, we establish Carathéodory’s and Minkowski’s theorem for matrix convex sets, i.e. Theorem 1.8 and Theorem 1.10. We also show how our methods yield another proof of the Krein–Milman theorem due to Webster and Winkler.

In Section 3, we establish our main result, Theorem 3.3, as well as the explicit dimension bound in the case of subhomogeneous C^* -algebras.

Section 4 consists of applications of the main result to various concrete dilation problems.

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2. CARATHÉODORY’S AND MINKOWSKI’S THEOREM FOR MATRIX CONVEX SETS

2.1. Matrix convexity. Let V be a complex vector space and let $\mathbf{X} = (X_n)_{n=1}^\infty$, where $X_n \subset M_n(V)$ for all $n \geq 1$. The identification $M_n(V) = M_n \otimes V$ makes it possible to multiply an element $x \in M_n(V)$ with a scalar $k \times n$ matrix on the left or with a scalar $n \times k$ matrix on the right. A *matrix convex combination* of elements $x_i \in M_{n_i}$, where $1 \leq i \leq s$, is an expression of the form

$$x = \sum_{i=1}^s \gamma_i^* x_i \gamma_i,$$

where $\gamma_i \in M_{n_i, n}$ and $\sum_{j=1}^s \gamma_j^* \gamma_j = I_n$. We refer to the integer s as the *length* of the matrix convex combination. (Notice that some of the elements x_i may be repeated without reducing the length of the matrix convex combination.) The matrix convex combination is called *proper* if each γ_i is surjective, and *trivial* if $n_i = n$ for all i and each x_i is unitarily equivalent to x . An element $x \in X_n$ is said to be a *matrix extreme point* of \mathbf{X} if whenever x is expressed as a proper matrix convex combination of elements of \mathbf{X} , the matrix convex combination is trivial. The matrix convex hull of \mathbf{X} is the smallest matrix convex set that contains \mathbf{X} , or equivalently, the set of all matrix convex combinations of elements of \mathbf{X} . If $\mathbf{X} = (X_n)_{n=1}^\infty$ is a matrix convex set in a topological vector space V , then we endow $M_n(V)$ with the product topology and say that \mathbf{X} is compact (respectively closed) if each X_n is compact (respectively closed). For more background on matrix convexity and matrix extreme points, see [47].

A *real structure* on V is a conjugate linear involution $*$ on V . If we set $V_{\mathbb{R}} = \{v \in V : v = v^*\}$, then $V_{\mathbb{R}}$ is a real vector space and $V = V_{\mathbb{R}} + iV_{\mathbb{R}}$, hence $\dim_{\mathbb{R}} V_{\mathbb{R}} = \dim_{\mathbb{C}} V$. A real structure on V induces a real structure on $M_n(V)$ for all $n \in \mathbb{N}$, via

$[v_{ij}]^* = [v_{ji}^*]$. An element $x \in M_n(V)$ is *self-adjoint* if $x = x^*$, and we write $M_n(V)_{sa}$ for the real vector space of all self-adjoint elements of $M_n(V)$.

Example 2.1. Let $V = \mathbb{C}^d$ and consider the involution given by coordinate-wise complex conjugation. Then $M_n(V)_{sa}$ can be naturally identified with the set of d -tuples of self-adjoint $n \times n$ matrices. This setting is frequently studied in free convexity; see for example [25, 31, 34].

2.2. Carathéodory's theorem. Our goal is to prove versions of Carathéodory's and Minkowski's theorems for matrix convex sets, that is, Theorem 1.8 and Theorem 1.10. To this end, we will reduce the matrix convex setting to the classical setting with the help of the following device. We let tr denote the normalized trace on M_n , so that $\text{tr}(I_n) = 1$. For $n \geq 1$, we define a subset of $M_n \oplus M_n(V)$ by

$$\Gamma_n(\mathbf{X}) = \{(\gamma^* \gamma, \gamma^* x \gamma) : \gamma \in M_{k,n}, \text{tr}(\gamma^* \gamma) = 1, k \in \mathbb{N}, x \in X_k\}.$$

This definition should be compared with the definition of Δ_n in [47], and with a device in the proof of Theorem 4.7 in [13]. A similar definition also occurs in the proof of Proposition 5.5 in [38] in the context of C^* -convexity.

The following simple lemma relates the matrix convex hull of \mathbf{X} to the convex hull of $\Gamma_n(\mathbf{X})$.

Lemma 2.2. *Let $\mathbf{X} = (X_n)$ with $X_n \subset M_n(V)$ for all $n \geq 1$. Let $x \in M_n(V)$ and let $r \in \mathbb{N}$. Then x is a matrix convex combination of elements of \mathbf{X} of length r if and only if (I_n, x) is a convex combination of r elements of $\Gamma_n(\mathbf{X})$.*

Proof. Let (I_n, x) be a convex combination of r elements of $\Gamma_n(\mathbf{X})$, say

$$(I_n, x) = \sum_{j=1}^r t_j (\gamma_j^* \gamma_j, \gamma_j^* x_j \gamma_j).$$

Let $\beta_j = t_j^{1/2} \gamma_j$. Then $\sum_{j=1}^r \beta_j^* \beta_j = I_n$ and $x = \sum_{j=1}^r \beta_j^* x_j \beta_j$, so x is a matrix convex combination of elements of \mathbf{X} of length r .

Conversely, suppose that $x = \sum_{j=1}^r \beta_j^* x_j \beta_j$ is a matrix convex combination of elements of \mathbf{X} of length r . We may without loss of generality assume that $\beta_j \neq 0$ for all j , so we may define $t_j = \text{tr}(\beta_j^* \beta_j) > 0$ and $\gamma_j = t_j^{-1/2} \beta_j$. Then $\text{tr}(\gamma_j^* \gamma_j) = 1$ for all j and

$$(I_n, x) = \sum_{j=1}^r t_j (\gamma_j^* \gamma_j, \gamma_j^* x_j \gamma_j)$$

is a convex combination of r elements of $\Gamma_n(\mathbf{X})$. □

We now obtain a more precise version of Theorem 1.8.

Theorem 2.3. *Let V be a finite-dimensional vector space and let $\mathbf{X} = (X_n)$ with $X_n \subset M_n(V)$ for $n \geq 1$.*

- (a) *If $x \in M_n(V)$ belongs to the matrix convex hull of \mathbf{X} , then it is a matrix convex combination of points of \mathbf{X} of length at most $n^2(2 \dim(V) + 1)$.*

- (b) Suppose that V has a real structure and that $X_n \subset M_n(V)_{sa}$ for $n \geq 1$. If $x \in M_n(V)$ belongs to the matrix convex hull of \mathbf{X} , then it is a matrix convex combination of points of \mathbf{X} of length at most $n^2(\dim(V) + 1)$.

Proof. (a) Since x belongs to the matrix convex hull of \mathbf{X} , Lemma 2.2 implies that (I_n, x) belongs to the convex hull of $\Gamma_n(\mathbf{X})$. By definition, $\Gamma_n(\mathbf{X})$ is contained in

$$\{(\alpha, v) : \alpha \in (M_n)_{sa}, \operatorname{tr}(\alpha) = 1, v \in M_n(V)\},$$

which is an affine subspace of real dimension $n^2 - 1 + 2n^2 \dim(V)$. The classical Carathéodory theorem shows that (I_n, x) is a convex combination of at most $n^2(2 \dim(V) + 1)$ points of $\Gamma_n(\mathbf{X})$. Applying Lemma 2.2 again, we find that x is a matrix convex combination of elements of \mathbf{X} of length at most $n^2(2 \dim(V) + 1)$.

- (b) In the setting of (b), the set $\Gamma_n(\mathbf{X})$ is contained in

$$\{(\alpha, v) : \alpha \in (M_n)_{sa}, \operatorname{tr}(\alpha) = 1, v \in M_n(V)_{sa}\},$$

which is an affine subspace of real dimension $n^2 - 1 + n^2 \dim(V)$, so the bound from the classical Carathéodory theorem is $n^2(\dim(V) + 1)$. \square

Remark 2.4. (1) No serious attempt was made to optimize the bounds in Theorem 2.3 and we do not know if the bounds are sharp. If $n = 1$, we recover the bounds in the classical Carathéodory theorem, which are known to be sharp in that case.

- (2) Kriel's setting in [34] corresponds to the self-adjoint case of Theorem 2.3; in Lemma 1.14 of [34], he obtains the slightly larger bound $2n^2 \dim(V) + 1$ in that case (with a different proof).

- (3) As mentioned in the introduction, Evert and Helton [25] obtain a better bound in the special case of (absolute) extreme points of compact free spectrahedra. In particular, they obtain a bound of the form $2n(\dim(V) + 1)$ in their setting.

As in classical convex analysis, the matrix convex version of Carathéodory's theorem has consequences for compactness of matrix convex hulls. This addresses a question raised in [26, Remark 3.2].

Corollary 2.5. *Let V be a finite-dimensional locally convex vector space and let $\mathbf{X} = (X_n)$ with $X_n \subset M_n(V)$ for $n \geq 1$. Suppose that each X_n is compact and that $X_n = \emptyset$ for all but finitely many $n \geq 1$. Then the matrix convex hull of \mathbf{X} is compact.*

Proof. Let $\mathbf{K} = (K_n)_{n=1}^{\infty}$ be the matrix convex hull of \mathbf{X} . Write

$$\{n \geq 1 : X_n \neq \emptyset\} = \{n_1, \dots, n_k\},$$

let $n \geq 1$ and let $r = n^2(2 \dim(V) + 1)$. Theorem 2.3 implies that for each $n \geq 1$,

$$K_n = \left\{ \sum_{j=1}^k \sum_{i=1}^r \gamma_{ij}^* x_{ij} \gamma_{ij} : x_{ij} \in X_{n_j}, \gamma_{ij} \in M_{n_j, n} \text{ with } \sum_{j=1}^k \sum_{i=1}^r \gamma_{ij}^* \gamma_{ij} = I_n \right\},$$

which is easily seen to be compact. \square

The following example shows that the assumption that $X_n = \emptyset$ for all but finitely many $n \geq 1$ in Corollary 2.5 cannot simply be omitted.

Example 2.6. For $n \geq 1$, let $X_n = \{(1 - 1/n)I_n\}$ and let $\mathbf{Y} = (Y_n)_{n=1}^\infty$ be the matrix convex hull of $\mathbf{X} = (X_n)_{n=1}^\infty$. It is not hard to check that $Y_1 = [0, 1]$, hence \mathbf{Y} is not compact.

2.3. Minkowski's theorem. To prove a version of Minkowski's theorem for matrix convex sets, we need the following lemma. In particular, part (b) shows that if \mathbf{X} is matrix convex, then in the definition of $\Gamma_n(\mathbf{X})$, we may assume that each γ is surjective and hence $k \leq n$.

Lemma 2.7. *Let $\mathbf{X} = (X_n)$ be a matrix convex set in V .*

- (a) *The set $\Gamma_n(\mathbf{X})$ is convex for all $n \in \mathbb{N}$.*
- (b) *The set $\Gamma_n(\mathbf{X})$ equals*

$$\{(\gamma^* \gamma, \gamma^* x \gamma) : \gamma \in M_{k,n} \text{ is surjective, } \text{tr}(\gamma^* \gamma) = 1, x \in X_k, k \leq n\}.$$

- (c) *If V is a topological vector space and if \mathbf{X} is a compact matrix convex set, then $\Gamma_n(\mathbf{X})$ is a compact convex set.*

Proof. The arguments are similar to the corresponding arguments in [47].

(a) Let $0 < t < 1$ and let $\gamma_i \in M_{k_i, n}$ and $x_i \in X_{k_i}$ for $i = 1, 2$ be as in the definition of $\Gamma_n(\mathbf{X})$. Let $k = k_1 + k_2$ and

$$\gamma = \begin{bmatrix} t^{1/2} \gamma_1 \\ (1-t)^{1/2} \gamma_2 \end{bmatrix} \in M_{k, n}.$$

Then $\gamma^* \gamma = t \gamma_1^* \gamma_1 + (1-t) \gamma_2^* \gamma_2$. In particular, $\text{tr}(\gamma^* \gamma) = 1$. Since \mathbf{X} is matrix convex, $x = x_1 \oplus x_2 \in X_k$, so

$$t(\gamma_1^* \gamma_1, \gamma_1^* x_1 \gamma_1) + (1-t)(\gamma_2^* \gamma_2, \gamma_2^* x_2 \gamma_2) = (\gamma^* \gamma, \gamma^* x \gamma) \in \Gamma_n(\mathbf{X}).$$

(b) Let $\gamma \in M_{k, n}$ and $x \in X_k$ be as in the definition of $\Gamma_n(\mathbf{X})$. Let r be the rank of γ , so that $1 \leq r \leq n$, and let $\delta \in M_{k, r}$ be an isometry onto the range of γ . Define $\beta = \delta^* \gamma \in M_{r, n}$. Then β is surjective, and

$$(\gamma^* \gamma, \gamma^* x \gamma) = (\gamma^* \delta \delta^* \gamma, \gamma^* \delta \delta^* x \delta \delta^* \gamma) = (\beta^* \beta, \beta^* (\delta^* x \delta) \beta).$$

Since \mathbf{X} is matrix convex, $\delta^* x \delta \in X_r$, so we have obtained the desired representation.

(c) We have seen in part (a) that $\Gamma_n(\mathbf{X})$ is convex. Part (b) implies that

$$\Gamma_n(\mathbf{X}) = \{(\gamma^* \gamma, \gamma^* x \gamma) : \gamma \in M_{k, n}, \text{tr}(\gamma^* \gamma) = 1, x \in X_k, k \leq n\},$$

which shows that $\Gamma_n(\mathbf{X})$ is compact since for each k , the set of all $\gamma \in M_{k, n}$ with $\text{tr}(\gamma^* \gamma) = 1$ is compact. \square

The following lemma shows that extreme points of $\Gamma_n(\mathbf{X})$ give rise to matrix extreme points of \mathbf{X} . In fact, we will see in Proposition 2.14 that every matrix extreme point arises in this way, but for the proof of Minkowski's theorem, the easier direction suffices.

Lemma 2.8. *Let $\mathbf{X} = (X_n)_{n=1}^\infty$ be a matrix convex set in a vector space V . Let $x \in X_k$ and let $\gamma \in M_{k,n}$ be surjective with $\text{tr}(\gamma^*\gamma) = 1$. If $(\gamma^*\gamma, \gamma^*x\gamma)$ is an extreme point of $\Gamma_n(\mathbf{X})$, then x is a matrix extreme point of \mathbf{X} .*

Proof. Let $x = \sum_{j=1}^s \gamma_j^* x_j \gamma_j$ be a proper matrix convex combination of x with $\gamma_j \in M_{k_j, k}$ and $x_j \in X_{k_j}$. Then

$$(\gamma^*\gamma, \gamma^*x\gamma) = \sum_{j=1}^s (\gamma^*\gamma_j^*\gamma_j\gamma, \gamma^*\gamma_j^*x_j\gamma_j\gamma).$$

Since γ and γ_j are surjective, we may define $t_j = \text{tr}(\gamma^*\gamma_j^*\gamma_j\gamma) > 0$ and $\beta_j = t_j^{-1/2}\gamma_j\gamma$. Then $\text{tr}(\beta_j^*\beta_j) = 1$ for all j and

$$(\gamma^*\gamma, \gamma^*x\gamma) = \sum_{j=1}^s t_j (\beta_j^*\beta_j, \beta_j^*x_j\beta_j).$$

Moreover, $\sum_{j=1}^s t_j = \text{tr}(\gamma^*\gamma) = 1$. Since $(\gamma^*\gamma, \gamma^*x\gamma)$ is an extreme point of $\Gamma_n(\mathbf{X})$, it follows that

$$(\gamma^*\gamma, \gamma^*x\gamma) = (\beta_j^*\beta_j, \beta_j^*x_j\beta_j)$$

for each j . Equality in the first component means that

$$\gamma^*\gamma = t_j^{-1}\gamma^*\gamma_j^*\gamma_j\gamma,$$

so surjectivity of γ implies that $\gamma_j^*\gamma_j = t_j$ for each j . Since each γ_j is also surjective, we find that $k_j = k$ for each j , and that $t_j^{-1/2}\gamma_j$ is unitary for each j . Equality in the second component means that

$$\gamma^*x\gamma = \gamma^*(t_j^{-1/2}\gamma_j)^*x_j(t_j^{-1/2}\gamma_j)\gamma,$$

so that $x = (t_j^{-1/2}\gamma_j)^*x_j(t_j^{-1/2}\gamma_j)$ by surjectivity of γ . Thus, the matrix convex combination was trivial, so that x is a matrix extreme point of \mathbf{X} . \square

We are now ready to prove Theorem 1.10 from the introduction.

Theorem 2.9. *Let \mathbf{X} be a compact matrix convex set in a finite-dimensional locally convex vector space V . Then \mathbf{X} is the matrix convex hull of its matrix extreme points.*

Proof. Let $n \geq 1$ and let $x \in X_n$. Then $\Gamma_n(\mathbf{X})$ is a compact convex set in a finite-dimensional space by part (c) of Lemma 2.7. Observe that $(I_n, x) \in \Gamma_n(\mathbf{X})$, hence by Minkowski's theorem, (I_n, x) is a finite convex combination of extreme points of $\Gamma_n(\mathbf{X})$, say

$$(I_n, x) = \sum_{j=1}^r t_j (\gamma_j^*\gamma_j, \gamma_j^*x_j\gamma_j).$$

By part (b) of Lemma 2.7, we may assume that each γ_j is surjective, so that $x_j \in X_{k_j}$ for some $k_j \leq n$. In this setting, Lemma 2.8 implies that each x_j is a matrix extreme point of \mathbf{X} . Lemma 2.2, applied to the collection x_1, \dots, x_r , shows that x is a matrix convex combination of matrix extreme points of \mathbf{X} . \square

Remark 2.10. The proof of Theorem 2.9 shows that each element of X_n is in fact a matrix convex combination of matrix extreme points in X_k for $k \leq n$.

As a by-product, our methods also yield a proof of the Krein–Milman theorem for matrix convex sets due to Webster and Winkler [47], which is arguably slightly simpler than the original proof.

Theorem 2.11 (Webster–Winkler). *Let \mathbf{X} be a compact matrix convex set in a locally convex vector space V . Then \mathbf{X} is the closed matrix convex hull of its matrix extreme points.*

Proof. Let $n \geq 1$ and let $x \in X_n$. By part (c) of Lemma 2.7, $\Gamma_n(\mathbf{X})$ is a compact convex set in the locally convex space $M_n \oplus M_n(V)$. Since $(I_n, x) \in \Gamma_n(\mathbf{X})$, the classical Krein–Milman theorem shows that (I_n, x) belongs to the closed convex hull of the set of extreme points of $\Gamma_n(\mathbf{X})$. Thus, given a neighborhood U of x in $M_n(V)$ and $0 < \varepsilon < 1$, there exist $y \in U$ and $\alpha \in M_n$ with $\|I_n - \alpha\| < \varepsilon$ such that (α, y) is a convex combination of extreme points of $\Gamma_n(\mathbf{X})$, say

$$(\alpha, y) = \sum_{j=1}^r t_j (\gamma_j^* \gamma_j, \gamma_j^* x_j \gamma_j).$$

By part (b) of Lemma 2.7, we may again assume that each γ_j is surjective, so that each x_j is a matrix extreme point of \mathbf{X} by Lemma 2.8. Note that α is positive and invertible. Let $\beta_j = t_j^{1/2} \gamma_j \alpha^{-1/2}$. Then

$$(I_n, \alpha^{-1/2} y \alpha^{-1/2}) = \sum_{j=1}^r (\beta_j^* \beta_j, \beta_j^* x_j \beta_j),$$

hence $\alpha^{-1/2} y \alpha^{-1/2}$ belongs to the matrix convex hull of the matrix extreme points x_1, \dots, x_r . This is true for every $0 < \varepsilon < 1$, so we can find a sequence (α_k) of positive invertible matrices tending to I_n so that $\alpha_k^{-1/2} y \alpha_k^{-1/2}$ belongs to the matrix convex hull of the matrix extreme points of \mathbf{X} for all k . Continuity of the continuous functional calculus (see, for instance, [9, II.2.3.2]) shows that $(\alpha_k^{-1/2})$ tends to I_n , hence $\alpha_k^{-1/2} y \alpha_k^{-1/2}$ tends to y in $M_n(V)$. Thus, $y \in U$ belongs to the closure of the matrix convex hull of the matrix extreme points of \mathbf{X} . This is true for every neighborhood U of x in $M_n(V)$, from which the result follows. \square

2.4. Matrix extreme points of \mathbf{X} vs. extreme points of $\Gamma_n(\mathbf{X})$. We will finish this section by establishing the converse of Lemma 2.8, thus showing that matrix extreme points of \mathbf{X} are in one-to-one correspondence with extreme points of $\Gamma_n(\mathbf{X})$. The first step is the following special case of Arveson’s boundary theorem, see for instance [28, p. 889].

Lemma 2.12. *Let $\alpha_1, \dots, \alpha_r \in M_n$ with $\sum_{i=1}^r \alpha_i^* \alpha_i = I_n$. If*

$$S = \left\{ \alpha \in M_n : \sum_{i=1}^r \alpha_i^* \alpha \alpha_i = \alpha \right\}$$

is an irreducible set of matrices, then each α_i is a scalar multiple of I_n .

Proof. Consider the u.c.p. map

$$\varphi : M_n \rightarrow M_n, \quad \alpha \mapsto \sum_{i=1}^r \alpha_i^* \alpha \alpha_i.$$

Arveson's boundary theorem [6, Theorem 2.1.1] implies that the identity representation on M_n is a boundary representation for S , meaning in particular that the identity map on S admits a unique extension to a u.c.p. map from M_n to M_n . Thus, φ is the identity map on M_n . The uniqueness part in Choi's theorem [11, Remark 4] then shows that each α_i is a scalar multiple of I_n . \square

The following lemma contains a different characterization of matrix extreme points. It implicitly appears (in a slightly different setting) in [28].

Lemma 2.13. *Let $\mathbf{X} = (X_n)$ be a matrix convex set in a vector space V and let $x \in X_n$. The following assertions are equivalent:*

- (i) x is a matrix extreme point of \mathbf{X} .
- (ii) Whenever $x = \sum_{i=1}^r \gamma_i^* x_i \gamma_i$ is a matrix convex combination of elements of \mathbf{X} , then there exist $t_i \geq 0$ with $\gamma_i^* \gamma_i = t_i I_n$ and $\gamma_i^* x_i \gamma_i = t_i x$ for $1 \leq i \leq r$.

Proof. (ii) \Rightarrow (i) Let $x = \sum_{i=1}^r \gamma_i^* x_i \gamma_i$ be a proper matrix convex combination of elements of \mathbf{X} . By assumption, there exist $t_i \geq 0$ with $\gamma_i^* \gamma_i = t_i I_n$ and $\gamma_i^* x_i \gamma_i = t_i x$ for $1 \leq i \leq n$. Since each γ_i is surjective, $t_i > 0$. Let $u_i = t_i^{-1/2} \gamma_i$. Then u_i is unitary and $u_i^* x_i u_i = x$ for $1 \leq i \leq n$, so the matrix convex combination is trivial.

(i) \Rightarrow (ii) Suppose that x is a matrix extreme point and let $x = \sum_{i=1}^r \gamma_i^* x_i \gamma_i$ be a matrix convex combination of elements of \mathbf{X} . We may assume that $\gamma_i \neq 0$ for $1 \leq i \leq r$. Moreover, as in the proof of part (b) of Lemma 2.7, there exist elements \tilde{x}_i of \mathbf{X} and surjective scalar matrices $\tilde{\gamma}_i$ of the appropriate size so that $\gamma_i^* \gamma_i = \tilde{\gamma}_i^* \tilde{\gamma}_i$ and $\gamma_i^* x_i \gamma_i = \tilde{\gamma}_i^* \tilde{x}_i \tilde{\gamma}_i$ for $1 \leq i \leq n$. Thus, we may without loss of generality assume that each γ_i is surjective, so that the matrix convex combination $x = \sum_{i=1}^r \gamma_i^* x_i \gamma_i$ is proper.

Since x is a matrix extreme point, there exist unitaries $u_i \in M_n$ so that $x_i = u_i^* x u_i$ for $1 \leq i \leq r$. Let $\alpha_i = u_i \gamma_i$, so that

$$(2) \quad x = \sum_{i=1}^r \alpha_i^* x \alpha_i.$$

We will show that $\alpha_i = \lambda_i I_n$ for some $\lambda_i \in \mathbb{C}$. Assuming this conclusion for the moment, it then follows that $\gamma_i^* \gamma_i = \alpha_i^* \alpha_i = |\lambda_i|^2 I_n$ and $\gamma_i^* x_i \gamma_i = \gamma_i^* u_i^* x u_i \gamma_i = \alpha_i^* x \alpha_i = |\lambda_i|^2 x$.

It remains to show that each α_i is a scalar multiple of I_n . Since $\sum_{i=1}^r \alpha_i^* \alpha_i = I_n$, it suffices by Lemma 2.12 to prove that the operator system

$$S = \left\{ \alpha \in M_n : \sum_{i=1}^r \alpha_i^* \alpha \alpha_i = \alpha \right\}$$

is irreducible. Let V^* denote the algebraic dual space of V and let

$$S_0 = \{(\text{id}_{M_n} \otimes v^*)(x) : v^* \in V^*\} \subset M_n.$$

From (2), we deduce that $S_0 \subset S$. We finish the proof by showing that S_0 is irreducible. Assume toward a contradiction that S_0 is reducible. Then there exist isometries $\beta \in M_{nk}$ and $\delta \in M_{nl}$ for some $1 \leq k, l < n$, so that $\beta\beta^* + \delta\delta^* = I_n$ and

$$\alpha = \beta\beta^*\alpha\beta\beta^* + \delta\delta^*\alpha\delta\delta^*$$

for all $\alpha \in S_0$. Since maps of the form $(\text{id}_{M_n} \otimes v^*)$ separate the points of $M_n(V)$, it follows that

$$(3) \quad x = \beta(\beta^*x\beta)\beta^* + \delta(\delta^*x\delta)\delta^*.$$

Matrix convexity of \mathbf{X} implies that $\beta^*x\beta \in X_k$ and $\delta^*x\delta \in X_l$, so (3) expresses x as a proper non-trivial matrix convex combination of elements of \mathbf{X} , contradicting the fact that x is a matrix extreme point of \mathbf{X} . \square

We are now ready to prove the converse of Lemma 2.8.

Proposition 2.14. *Let $\mathbf{X} = (X_n)_{n=1}^\infty$ be a matrix convex set in a vector space V . Let $x \in X_k$ and let $\gamma \in M_{k,n}$ be surjective with $\text{tr}(\gamma^*\gamma) = 1$. Then $(\gamma^*\gamma, \gamma^*x\gamma)$ is an extreme point of $\Gamma_n(\mathbf{X})$ if and only if x is a matrix extreme point of \mathbf{X} .*

Proof. The ‘‘only if’’ part is Lemma 2.8. Conversely, suppose that x is a matrix extreme point of \mathbf{X} and let

$$(\gamma^*\gamma, \gamma^*x\gamma) = \sum_{j=1}^s t_j(\beta_j^*\beta_j, \beta_j^*x_j\beta_j)$$

be a proper convex combination with $\beta_j \in M_{k_j,n}$ surjective, $\text{tr}(\beta_j^*\beta_j) = 1$ and $x_j \in X_{k_j}$ for each j (which we may assume by part (b) of Lemma 2.7). Since $\gamma \in M_{k,n}$ is surjective, there exists $\delta \in M_{n,k}$ with $\gamma\delta = I_k$, thus

$$(I_k, x) = \sum_{j=1}^s t_j((\beta_j\delta)^*(\beta_j\delta), (\beta_j\delta)^*x_j(\beta_j\delta)).$$

Let $\alpha_j = t_j^{1/2}\beta_j\delta \in M_{k_j,k}$, so that

$$\sum_{j=1}^s \alpha_j^*\alpha_j = I_k \quad \text{and} \quad x = \sum_{j=1}^s \alpha_j^*x_j\alpha_j.$$

Since x is a matrix extreme point, it follows from Lemma 2.13 that there exist scalars $\lambda_j \geq 0$ such that

$$(4) \quad \alpha_j^*\alpha_j = \lambda_j I_k \quad \text{and} \quad \alpha_j^*x_j\alpha_j = \lambda_j x.$$

We claim that

$$(5) \quad \beta_j\delta\gamma = \beta_j$$

for each j . Indeed, since $t_j\beta_j^*\beta_j \leq \gamma^*\gamma$, we find that $\ker(\gamma) \subset \ker(\beta_j)$. On the other hand, $\gamma(I_n - \delta\gamma) = 0$, hence also $\beta_j(I_n - \delta\gamma) = 0$, as asserted.

Using the definition of α_j and Equations (4) and (5), we conclude that $t_j\beta_j^*\beta_j = \lambda_j\gamma^*\gamma$ and $t_j\beta_j^*x_j\beta_j = \lambda_j\gamma^*x\gamma$ for each j . Taking traces in the first equation, we see that $\lambda_j = t_j$, so that our convex combination was trivial. \square

3. A FINITE-DIMENSIONAL ARVESON–STINESPRING THEOREM

The goal of this section is to prove Theorem 1.5. We begin with the following easy consequence of the usual proof of Stinespring’s dilation theorem.

Lemma 3.1. *Let A be a unital C^* -algebra, let $S \subset A$ be an operator system and let $\varphi : S \rightarrow B(H)$ be a u.c.p. map with $\dim(H) < \infty$. Then the following are equivalent:*

- (i) *The map φ dilates to a finite-dimensional representation of A .*
- (ii) *There exist a finite-dimensional unital C^* -algebra B , a unital $*$ -homomorphism $\sigma : A \rightarrow B$ and a u.c.p. map $\psi : B \rightarrow B(H)$ such that $\varphi(s) = (\psi \circ \sigma)(s)$ for all $s \in S$.*

Moreover, in the setting of (ii), we can achieve that $\dim(K) \leq \dim(B) \dim(H)$.

Proof. (i) \Rightarrow (ii) Let $\pi : A \rightarrow B(K)$ be a dilation of φ on a finite-dimensional Hilbert space K . Then we define $B = B(K)$, $\sigma = \pi$ and $\psi(b) = P_H b|_H$.

(ii) \Rightarrow (i) The usual proof of Stinespring’s dilation theorem (see, for example, [41, Theorem 4.1]) shows that in the setting of (ii), there exist a Hilbert space K with $\dim(K) \leq \dim(B) \dim(H)$ and a unital $*$ -homomorphism $\tau : B \rightarrow B(K)$ such that $\psi(b) = P_H \tau(b)|_H$ for all $b \in B$. Then $\tau \circ \sigma$ is a finite-dimensional representation of A that dilates φ . \square

Let S be an operator system. We will apply the results of the preceding section to the matrix state space of S , which is $\mathbf{X} = (X_n)_{n=1}^\infty$, where

$$X_n = \{\varphi : S \rightarrow M_n : \varphi \text{ is u.c.p.}\}.$$

Identifying the space of linear maps from S to M_n with $M_n(S^*)$, the matrix state space \mathbf{X} becomes a weak- $*$ compact matrix convex set in S^* . Elements of \mathbf{X} are also called *matrix states of S* . A matrix state $\varphi : S \rightarrow M_n$ is said to be *pure* if for every completely positive linear map $\psi : S \rightarrow M_n$ for which $\varphi - \psi$ is completely positive, there is a $\lambda \in [0, 1]$ with $\psi = \lambda\varphi$. It is a theorem of Farenick [28, Theorem B] that a matrix state of S is pure if and only if it is a matrix extreme point of the matrix state space of S . By a theorem of Arveson [5, Corollary 1.4.3], a matrix state of a unital C^* -algebra A is pure if and only if it dilates to an irreducible representation of A .

The following lemma connects Question 1.3 to matrix convexity.

Lemma 3.2. *Let A be a unital FDI C^* -algebra, let $S \subset A$ be an operator system and let $\varphi : S \rightarrow B(H)$ be a u.c.p. map with $\dim(H) < \infty$. Then the following are equivalent:*

- (i) *φ dilates to a finite-dimensional representation of A .*
- (ii) *φ is a matrix convex combination of restrictions of pure matrix states of A to S .*

Moreover, if A is r -subhomogeneous and the matrix convex combination in (ii) has length s , then φ dilates to a representation of A on a Hilbert space of dimension at most $sr^2 \dim(H)$.

Proof. (i) \Rightarrow (ii) Suppose that φ dilates to a finite-dimensional representation $\pi : A \rightarrow B(K)$ of A . Thus, there exists an isometry $\gamma : H \rightarrow K$ so that

$$\varphi(s) = \gamma^* \pi(s) \gamma \quad (s \in S).$$

Since $\dim(K) < \infty$, the representation π is a finite direct sum of irreducible representations $\pi_i : A \rightarrow B(K_i)$ of A for $1 \leq i \leq s$. Then we may regard the isometry γ as a column

$$\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_s \end{bmatrix},$$

where $\gamma_i \in B(H, K_i)$, so that

$$\varphi(s) = \sum_{i=1}^s \gamma_i^* \pi_i(s) \gamma_i \quad (s \in S).$$

Since irreducible representations of A are pure matrix states of A (for instance by [5, Corollary 1.4.3]), we see that φ is a matrix convex combination of restrictions of pure matrix states of A to S .

(ii) \Rightarrow (i) Suppose that φ is a matrix convex combination of restrictions of pure matrix states of A to S , say

$$\varphi(s) = \sum_{j=1}^s \gamma_j^* \varphi_j(s) \gamma_j \quad (s \in S),$$

where each $\varphi_j : A \rightarrow M_{k_j}$ is a pure matrix state. Then each φ_j dilates to an irreducible $*$ -representation of $\sigma_j : A \rightarrow B(K_j)$ by Corollary 1.4.3 of [5]. Thus, there are isometries $v_j : \mathbb{C}^{k_j} \rightarrow K_j$ such that

$$v_j^* \sigma_j(s) v_j = \varphi_j(s)$$

for all $s \in S$. Let $\sigma = \sigma_1 \oplus \dots \oplus \sigma_s$ and let $B = B(K_1) \oplus \dots \oplus B(K_s)$. Since A is FDI, $\dim(K_j) < \infty$ for all j , so that $\dim(B) < \infty$. Moreover, define

$$\psi : B \rightarrow B(H), \quad (b_1, \dots, b_s) \mapsto \sum_{j=1}^s \gamma_j^* v_j^* b_j v_j \gamma_j.$$

Then ψ is u.c.p. and $\varphi = \psi \circ \sigma$ on S . Thus, the implication (ii) \Rightarrow (i) of Lemma 3.1 shows that φ dilates to a finite-dimensional representation of A .

To prove the additional assertion, note that if A is r -subhomogeneous, then we can assume that $\dim(K_j) \leq r$ for all j , so that $\dim(B) \leq sr^2$, hence the dimension bound follows from the corresponding dimension bound in Lemma 3.1. \square

We are now ready to establish our main result, Theorem 1.5, which we restate for the reader's convenience.

Theorem 3.3. *Let A be a unital FDI C^* -algebra, let $S \subset A$ be a finite-dimensional operator system and let $\varphi : S \rightarrow B(H)$ be a u.c.p map with $\dim(H) < \infty$. Then φ dilates to a finite-dimensional representation of A , that is, there exist $K \supset H$ with $\dim(K) < \infty$ and a unital $*$ -representation $\pi : A \rightarrow B(K)$ such that $\varphi(s) = P_H \pi(s)|_H$ for all $s \in S$.*

Proof. We regard φ as an element of the matrix state space of S . Since $\dim(S) < \infty$, Minkowski's theorem for matrix convex sets (Theorem 2.9) implies that φ is a finite matrix convex combination of matrix states that are matrix extreme, say

$$\varphi = \sum_{j=1}^s \gamma_j^* \varphi_j \gamma_j,$$

where $\varphi_j : S \rightarrow M_{k_j}$. By Theorem B in [28], each φ_j extends to a pure matrix state on A . Thus, the implication (ii) \Rightarrow (i) of Lemma 3.2 shows that φ dilates to a finite-dimensional representation of A . \square

We also obtain the following quantitative bound in the preceding result in the case of subhomogeneous C^* -algebras.

Proposition 3.4. *In the setting of Theorem 3.3, suppose that A is also r -subhomogeneous. Then we may achieve that*

$$\dim(K) \leq r^2 \dim(H)^3 (\dim(S) + 1).$$

In particular, if A is commutative, we may achieve that

$$\dim(K) \leq \dim(H)^3 (\dim(S) + 1).$$

Proof. We bound the length s of the matrix convex combination in the proof of Theorem 3.3 using Carathéodory's theorem for matrix convex sets. To this end, recall that the matrix state space of S is a matrix convex set in S^* . Moreover, S^* has a real structure, given by the involution

$$\varphi^*(s) = \overline{\varphi(s^*)} \quad (\varphi \in S^*, s \in S).$$

The induced involution on $M_n(S^*) = \text{Hom}(S, M_n)$ is given by $\varphi^*(s) = \varphi(s^*)^*$, where $\varphi : S \rightarrow M_n$ and $s \in S$. Therefore, matrix states of S are self-adjoint with respect to the real structure, so part (b) of Theorem 2.3 applies and yields for the length s of the matrix convex combination the bound

$$s \leq \dim(H)^2 (\dim(S) + 1).$$

Thus, the dimension bound follows from the corresponding bound in Lemma 3.2. \square

4. APPLICATIONS

In this section, we will explore several consequences of Theorem 1.5 to concrete dilation problems in operator theory.

4.1. Known finite-dimensional dilation theorems. We already explained in the introduction how to obtain Egerváry's theorem from Theorem 1.5. In fact, the argument proves a more general result. Let $A(\mathbb{D})$ denote the disc algebra, that is, the algebra of all holomorphic functions on \mathbb{D} that extend continuously to $\overline{\mathbb{D}}$. Sz.-Nagy's dilation theorem (or von Neumann's inequality) shows that every contraction T has an $A(\mathbb{D})$ -functional calculus.

Corollary 4.1. *Let T be a contraction on a finite-dimensional Hilbert space H and let $\mathcal{A} \subset A(\mathbb{D})$ be a finite-dimensional subspace. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a unitary operator U on K such that*

$$f(T) = P_H f(U)|_H$$

for all $f \in \mathcal{A}$. We may achieve that $\dim(K) \leq 2 \dim(H)^3 (\dim(\mathcal{A}) + 1)$.

Proof. We argue exactly as in the introduction, but this time using the operator system

$$S = \text{span}\{1, f, \bar{f} : f \in \mathcal{A}\} \subset C(\mathbb{T}).$$

The dimension bound follows from Proposition 3.4 as $\dim(S) \leq 2 \dim(\mathcal{A}) + 1$. \square

A similar argument proves the theorem of McCarthy–Shalit [37, Theorem 1.2] about dilations of tuples of commuting contractive matrices.

Corollary 4.2 (McCarthy–Shalit). *Let $T = (T_1, \dots, T_d)$ be a tuple of commuting contractions on a finite-dimensional Hilbert space H that dilates to a tuple of commuting unitaries. Let $\mathcal{P} \subset \mathbb{C}[z_1, \dots, z_d]$ be a finite-dimensional subspace. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a tuple of commuting unitaries $U = (U_1, \dots, U_d)$ on K such that*

$$p(T) = P_H p(U)|_H$$

for all $p \in \mathcal{P}$. We may achieve that $\dim(K) \leq 2 \dim(H)^3 (\dim(\mathcal{P}) + 1)$.

Proof. We apply Theorem 1.5 to the C^* -algebra $A = C(\mathbb{T}^d)$ and the operator system

$$S = \text{span}\{1, f, \bar{f} : f \in \mathcal{P}\} \subset C(\mathbb{T}^d).$$

If $V = (V_1, \dots, V_d)$ is a tuple of commuting unitaries on $L \supset H$ that dilates T , then V induces a $*$ -representation

$$\sigma : C(\mathbb{T}^d) \rightarrow B(L) \quad \text{with } \sigma(p) = p(V)$$

for all $p \in \mathbb{C}[z_1, \dots, z_d]$, hence

$$\varphi : S \rightarrow B(H), \quad f \mapsto P_H \sigma(f)|_H,$$

is u.c.p. and satisfies $\varphi(p) = p(T)$ for all $p \in \mathcal{P}$. Theorem 1.5 yields a finite-dimensional Hilbert space $K \supset H$ and a dilation $\pi : C(\mathbb{T}^d) \rightarrow B(K)$ of φ . Defining $U_i = \pi(z_i)$ for $1 \leq i \leq d$, we obtain the desired dilation. The dimension bound once again follows from Proposition 3.4. \square

As mentioned in [37], the existence of a unitary dilation is automatic if $d = 2$ by Andô's dilation theorem.

M^cCarthy and Shalit also prove a theorem regarding regular dilations. This is a stronger notion of dilation to commuting unitaries. While there is no simple characterization of those tuples of commuting contractions that admit a unitary dilation, a clean characterization of those tuples that admit a regular dilation is known, see [46, Section I.9].

Let $T = (T_1, \dots, T_d)$ be a tuple of commuting contractions on H . If $n \in \mathbb{Z}^n$, let $n^+ = \max(n, 0)$ and $n^- = -\min(n, 0)$, where \max and \min are understood entry-wise. Thus, n^+ is the d -tuple of non-negative integers obtained from n by setting all negative entries equal to 0, and n^- is the d -tuple of non-negative integers obtained from $-n$ by setting all negative entries equal to 0. Define $T(n) = (T^*)^{n^-} T^{n^+}$ for $n \in \mathbb{Z}$. With this definition, a regular unitary dilation of T is a tuple of commuting unitaries U on a Hilbert space $K \supset H$ such that

$$T(n) = P_H U^n \Big|_H$$

for all $n \in \mathbb{Z}^d$. We can also obtain the result of M^cCarthy and Shalit regarding finite dimensional regular dilations [37, Theorem 1.7] from Theorem 1.5.

Corollary 4.3 (M^cCarthy–Shalit). *Let $T = (T_1, \dots, T_d)$ be a tuple of commuting contractions on a finite-dimensional Hilbert space H that admits a regular unitary dilation. Let $Z \subset \mathbb{Z}^d$ be a finite subset. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a tuple of commuting unitaries $U = (U_1, \dots, U_d)$ on K such that*

$$T(n) = P_H U^n \Big|_H$$

for all $n \in Z$. We may achieve that $\dim(K) \leq 2 \dim(H)^3(|Z| + 1)$.

Proof. We apply Theorem 1.5 to the C^* -algebra $A = C(\mathbb{T}^d)$, the operator system

$$S = \text{span}\{1, z^n, \bar{z}^n : n \in Z\} \subset C(\mathbb{T}^d)$$

and the unital map $\varphi : S \rightarrow B(H)$ defined by $\varphi(z^n) = T(n)$ for $n \in Z \cup -Z \cup \{0\}$, and extended linearly. The assumption that T admits a regular unitary dilation shows that φ dilates to a representation of $C(\mathbb{T}^d)$, and hence is completely positive. Theorem 1.5 yields a finite-dimensional Hilbert space $K \supset H$ and a dilation $\pi : A \rightarrow B(K)$ of φ , so defining $U_i = \pi(z_i)$ for $1 \leq i \leq d$ as before and appealing to Proposition 3.4 for the dimension bound finishes the proof. \square

Corollary 4.2 was extended by Cohen [13] in the following way. Let $X \subset \mathbb{C}^d$ be a compact set and let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on H . A polynomial normal ∂X -dilation of T is a d -tuple of commuting normal operators $N = (N_1, \dots, N_d)$ on a Hilbert space $K \supset H$ with $\sigma(N) \subset \partial X$ such that

$$p(T) = P_H p(N) \Big|_H$$

for all $p \in \mathbb{C}[z_1, \dots, z_d]$. (Here, the spectrum is computed in the unital commutative C^* -algebra generated by N_1, \dots, N_d .) Our abstract dilation result also implies Cohen's finite-dimensional dilation theorem.

Corollary 4.4 (Cohen). *Let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on a finite-dimensional Hilbert space H that admits a polynomial normal ∂X -dilation. Let $\mathcal{P} \subset \mathbb{C}[z_1, \dots, z_d]$ be a finite dimensional subspace. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a tuple $N = (N_1, \dots, N_d)$ of commuting normal operators on K with $\sigma(N) \subset \partial X$ such that*

$$p(T) = P_H p(N)|_H$$

for all $p \in \mathcal{P}$. We may achieve that $\dim(K) \leq 2 \dim(H)^3 (\dim(\mathcal{P}) + 1)$.

Proof. We apply Theorem 1.5 and Proposition 3.4 to the C^* -algebra $A = C(\partial X)$, the operator system

$$S = \text{span}\{1, p, \bar{p} : p \in \mathcal{P}\} \subset C(\partial X)$$

and the unique u.c.p. map $\varphi : S \rightarrow B(H)$ satisfying $\varphi(p) = p(T)$ for all $p \in \mathcal{P}$. \square

We also obtain the following result of Davidson, Dor-On, Shalit and Solel [19, Theorem 7.1] as a consequence.

Corollary 4.5 (Davidson–Dor-On–Shalit–Solel). *Let $X = (X_1, \dots, X_d)$ be a tuple of (not necessarily commuting) operators on a finite-dimensional Hilbert space H for which there exist a Hilbert space $L \supset H$ and a tuple $N = (N_1, \dots, N_d)$ of commuting normal operators on L such that $X_i = P_H N_i|_H$ for $1 \leq i \leq d$. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a tuple $Y = (Y_1, \dots, Y_d)$ of commuting normal operators on K with $\sigma(Y) \subset \sigma(N)$ such that $X_i = P_H Y_i|_H$ for $1 \leq i \leq d$. We may achieve that $\dim(K) \leq 2 \dim(H)^3 (d + 1)$.*

Proof. Let $X = \sigma(N)$. We apply Theorem 1.5 and Proposition 3.4 to the C^* -algebra $A = C(X)$, the operator system

$$S = \text{span}\{1, z_i, \bar{z}_i : 1 \leq i \leq d\} \subset C(X)$$

and the u.c.p. map $\varphi : S \rightarrow B(H)$ defined by $\varphi(1) = 1$, $\varphi(z_i) = X_i$ and $\varphi(\bar{z}_i) = X_i^*$, extended linearly. \square

4.2. Rational dilation. To illustrate how Theorem 3.3 can be used to prove new finite-dimensional dilation results, we establish a finite-dimensional version of Agler's theorem [1]. For $0 < r < 1$, let

$$A_r = \{z \in \mathbb{C} : r \leq |z| \leq 1\}$$

and let $\text{Rat}(A_r)$ denote the vector space of all rational functions with poles off A_r . If T is a bounded operator on H with $\sigma(T) \subset A_r$, we say that A_r is a *spectral set* for T if $\|f(T)\| \leq \sup_{z \in A_r} |f(z)|$ for all $f \in \text{Rat}(A_r)$.

Corollary 4.6. *Let T be an operator on a finite-dimensional Hilbert space H such that A_r is a spectral set for T . Let $\mathcal{R} \subset \text{Rat}(A_r)$ be a finite-dimensional subspace. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a normal operator N on K with $\sigma(N) \subset \partial A_r$ such that*

$$f(T) = P_H f(N)|_H$$

for all $f \in \mathcal{R}$. We may achieve that $\dim(K) \leq 2 \dim(H)^3 (\dim(\mathcal{R}) + 1)$.

Proof. We apply Theorem 1.5 with $A = C(\partial A_r)$ and

$$S = \text{span}\{1, f, \bar{f} : f \in \mathcal{R}\} \subset C(\partial A_r).$$

By Agler's theorem [1], there exist a Hilbert space $L \supset H$ and a normal operator B on L with $\sigma(B) \subset \partial A_r$ so that $f(T) = P_H f(B)|_H$ for all $f \in \text{Rat}(A_r)$. Since B induces a representation of $C(\partial A_r)$, there exists a u.c.p. map $\varphi : S \rightarrow B(H)$ with $\varphi(f) = f(T)$ for all $f \in \mathcal{R}$. By Theorem 1.5, φ dilates to a finite-dimensional representation π of $C(\partial A_r)$, so

$$f(T) = \varphi(f) = P_H \pi(f)|_H$$

for $f \in \mathcal{R}$. If we define $N = \pi(z)$, then $f(N) = \pi(f)$ for all $f \in \text{Rat}(A_r)$ since π is a homomorphism, so N has the required properties. The dimension bound is once again a consequence of Proposition 3.4. \square

We can in particular apply Corollary 4.6 for each $k \in \mathbb{N}$ to the space $\mathcal{R} = \text{span}\{z^n : -k \leq n \leq k\}$ to obtain a normal operator N on a finite-dimensional space with

$$T^n = P_H N^n|_H$$

for all $-k \leq n \leq k$.

For compact subsets $X \subset \mathbb{C}$ with more than one hole, it is in general no longer true that every operator for which X is a spectral set dilates to a normal operator with spectrum in ∂X ; see [2, 22]. In fact, there are typically finite-dimensional counterexamples; see [22, Section 7].

For general compact subsets X of \mathbb{C} or of \mathbb{C}^d , Theorem 1.5 implies a version of Corollary 4.4 for rational dilation. The authors are grateful to Michael Dritschel and to an anonymous referee for asking questions that led to the inclusion of this result.

Let $X \subset \mathbb{C}^d$ be compact and let

$$\text{Rat}(X) = \left\{ \frac{p}{q} : p, q \in \mathbb{C}[z_1, \dots, z_d] \text{ and } q(z) \neq 0 \text{ for all } z \in X \right\}.$$

Let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on H whose Taylor spectrum $\sigma_T(T)$ is contained in X (see [39, Chapter IV] for background on the Taylor spectrum; in finite dimensions, the Taylor spectrum agrees with various other notions of spectrum). A rational normal ∂X -dilation of T is a d -tuple of commuting normal operators $N = (N_1, \dots, N_d)$ on a Hilbert space $K \supset H$ with $\sigma(N) \subset \partial X$ such that

$$f(T) = P_H f(N)|_H$$

for all $f \in \text{Rat}(X)$. (Here, $f(T)$ can be defined by using that $q(T)$ is invertible if q is a polynomial that does not vanish on X , which follows from the spectral mapping property of the Taylor spectrum; see [39, Corollary 30.11].)

Corollary 4.7. *Let $X \subset \mathbb{C}^d$ be a compact set and let T be an operator on a finite-dimensional Hilbert space H with $\sigma_T(T) \subset X$ that admits a rational normal ∂X -dilation. Let $\mathcal{R} \subset \text{Rat}(X)$ be a finite-dimensional subspace. Then there exist a*

finite-dimensional Hilbert space $K \supset H$ and a tuple $N = (N_1, \dots, N_d)$ of commuting normal operators on K with $\sigma(N) \subset \partial X$ such that

$$f(T) = P_H f(N)|_H$$

for all $f \in \mathcal{R}$. We may achieve that $\dim(K) \leq 2 \dim(H)^3 (\dim(\mathcal{R}) + 1)$.

Proof. We apply Theorem 1.5 with $A = C(\partial X)$,

$$S = \text{span}\{1, f, \bar{f} : f \in \mathcal{R}\} \subset C(\partial X)$$

and the unique u.c.p. map $\varphi : S \rightarrow B(H)$ satisfying $\varphi(f) = f(T)$ for all $f \in \mathcal{R}$. Thus, we obtain a finite-dimensional representation π of $C(\partial A_r)$ with

$$f(T) = \varphi(f) = P_H \pi(f)|_H$$

for all $f \in \mathcal{R}$. Defining $N_i = \pi(z_i)$ for $1 \leq i \leq d$, we obtain a tuple N of commuting normal operators with $\sigma(N) \subset \partial X$. Since π is a homomorphism, $\pi(f) = f(N)$ for all $f \in \text{Rat}(X)$, so N has all desired properties. The dimension bound once again follows from Proposition 3.4. \square

4.3. Unitary ρ -dilations. Let $T \in B(H)$ and $\rho > 0$. A *unitary ρ -dilation* of T is a unitary operator U on a Hilbert space $K \supset H$ such that

$$T^n = \rho P_H U^n|_H \quad \text{for all } n \geq 1.$$

The class of operators C_ρ that admit a unitary ρ -dilation can be characterized intrinsically, see [46, Theorem 11.1]. In particular, C_1 consists of all contractions, and C_2 consists of all operators whose numerical radius is at most 1. We can also establish the existence of finite-dimensional ρ -dilations. The authors are grateful to John M^cCarthy for asking a question that led to this observation.

Corollary 4.8. *Let T be an operator on a finite-dimensional Hilbert space H and let $\rho > 0$. Suppose that T admits a unitary ρ -dilation and let $N \in \mathbb{N}$. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a unitary operator U on K such that*

$$T^n = \rho P_H U^n|_H \quad \text{for } 1 \leq n \leq N.$$

We may achieve that $\dim(K) \leq 2 \dim(H)^3 (N + 1)$.

Proof. As in the proof of Egeráry's theorem, we apply Theorem 1.5 to $A = C(\mathbb{T})$ and the operator system

$$S = \text{span}\{1, z^n, \bar{z}^n : 1 \leq n \leq N\} \subset C(\mathbb{T}),$$

but to a different u.c.p. map. Let V be a unitary ρ -dilation on a Hilbert space $K \supset H$, let $\sigma : C(\mathbb{T}) \rightarrow B(L)$ be the corresponding representation satisfying $\sigma(p) = p(V)$ for all $p \in \mathbb{C}[z]$ and let

$$\varphi : S \rightarrow B(H), \quad f \mapsto P_H \sigma(f)|_H.$$

Then φ is u.c.p. and satisfies $\varphi(z^n) = \rho^{-1} T^n$ for $1 \leq n \leq N$. By Theorem 1.5, there exist a finite-dimensional Hilbert space $K \supset H$ and a representation $\pi : C(\mathbb{T}) \rightarrow B(K)$ that dilates φ . Let $U = \pi(z)$. Then U is unitary and $T^n = \rho \varphi(z^n) = \rho P_H U^n|_H$ for all $1 \leq n \leq N$. The dimension bound follows from Proposition 3.4. \square

In particular, setting $\rho = 2$, we obtain the following finite-dimensional version of Berger's dilation theorem [7].

Corollary 4.9. *Let T be an operator on a finite-dimensional Hilbert space H with numerical radius at most 1. Let $N \in \mathbb{N}$. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a unitary operator U on K such that*

$$T^n = 2P_H U^n|_H \quad \text{for } 1 \leq n \leq N. \quad \square$$

4.4. Numerical range dilations. Next, we establish Theorem 1.6 regarding dilations of operators with prescribed numerical range. As mentioned in the introduction, this is a finite-dimensional version of a theorem of Putinar and Sandberg; see Theorem 2 and the discussion following it in [42]. It generalizes Corollary 4.9, which corresponds to the case where the set Ω below is the unit disc. For the reader's convenience, we restate the result.

Corollary 4.10. *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with smooth boundary $\partial\Omega$. Let T be an operator on a finite-dimensional Hilbert space H with $W(T) \subset \Omega$ and let $\mathcal{A} \subset A(\Omega)$ be a finite-dimensional subspace. Then there exist a finite-dimensional Hilbert space $K \supset H$ and a normal operator N on K with $\sigma(N) \subset \partial\Omega$ such that*

$$f(T) + (C\bar{f})(T)^* = 2P_H f(N)|_H$$

for all $f \in \mathcal{A}$. We may achieve that $\dim(K) \leq 2 \dim(H)^3 (\dim(\mathcal{A}) + 1)$.

Proof. We apply Theorem 1.5 with $A = C(\partial\Omega)$ and

$$S = \text{span}\{1, f, \bar{f} : f \in \mathcal{A}\} \subset C(\partial\Omega).$$

If $f \in A(\Omega)$, then by the Riesz–Dunford functional calculus,

$$\begin{aligned} f(T) + (C\bar{f})(T)^* &= \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta)(\zeta - T)^{-1} d\zeta + \left(\frac{1}{2\pi i} \int_{\partial\Omega} \overline{f(\zeta)}(\zeta - T)^{-1} d\zeta \right)^* \\ &= 2 \int_{\partial\Omega} f(\zeta) d\mu_T(\zeta), \end{aligned}$$

where μ_T is the operator-valued measure on $\partial\Omega$ given by

$$d\mu_T(\zeta) = \text{Re} \left(\frac{1}{2\pi i} (\zeta - T)^{-1} d\zeta \right).$$

The fact that $W(T) \subset \Omega$ implies that μ_T is a positive measure; see [42, Section 3] or [15, Section 2]. Thus, the map

$$\varphi : S \rightarrow B(H), \quad f \mapsto \int_{\partial\Omega} f(\zeta) d\mu_T(\zeta),$$

is u.c.p. (for instance by [41, Theorem 3.11]) and satisfies $2\varphi(f) = f(T) + (C\bar{f})(T)^*$ for all $f \in \mathcal{A}$. By Theorem 1.5, there exists a finite-dimensional Hilbert space $K \supset H$ and a representation $\pi : C(\partial\Omega) \rightarrow B(K)$ dilating φ . Let $N = \pi(z)$. Then N is a normal operator with $\sigma(N) \subset \partial\Omega$ and

$$f(T) + (C\bar{f})(T)^* = 2P_H \pi(f)|_H = 2P_H f(N)|_H$$

for all $f \in \mathcal{A}$. Proposition 3.4 yields the dimension bound. \square

It seems worth remarking that positivity of the operator-valued measure μ_T in the above proof plays a crucial role in most of the current approaches to Crouzeix's conjecture; see [43] for a very clear explanation. The arguments of Putinar and Sandberg [42, Section 3] and the above proof show that this is actually closely related to a dilation result by the Arveson–Stinespring dilation theorem.

4.5. q -commuting contractions. We finish this section with an application in which the C^* -algebra A in Theorem 1.5 is non-commutative. Let q be a complex number of modulus one. Two operators T_1, T_2 on H are said to be q -commuting if

$$T_2 T_1 = q T_1 T_2.$$

In particular, if $q = -1$, then T_1 and T_2 anti-commute. It was shown by Keshari and Mallick [33], extending previous work of Sebestyén [44], that any pair of q -commuting contractions dilates to a pair of q -commuting unitaries. We can also establish a finite-dimensional version of their dilation theorem.

Corollary 4.11. *Let $q = \exp(2\pi ia/b)$, where $a \in \mathbb{Z}$ and $b \in \mathbb{N} \setminus \{0\}$. Let T_1, T_2 be q -commuting contractions on a finite-dimensional Hilbert space H . Let $N \in \mathbb{N}$. Then there exist a finite-dimensional Hilbert space $K \supset H$ and q -commuting unitaries U_1, U_2 on K so that*

$$T_1^n T_2^m = P_H U_1^n U_2^m|_H \quad \text{for all } 0 \leq m, n \leq N.$$

We may achieve that $\dim(K) \leq 2b^2 \dim(H)^3 (N+1)^2$.

Proof. By [33, Theorem 2.3], there exist a Hilbert space $L \supset H$ and q -commuting unitaries V_1, V_2 on L so that

$$T_1^n T_2^m = P_H V_1^n V_2^m|_H \quad \text{for all } n, m \in \mathbb{N}.$$

Let $\mathcal{A}_{a/b}$ be the rational rotation algebra, that is, the universal C^* -algebra generated by two q -commuting unitaries u_1, u_2 . By [21, Proposition 1], $\mathcal{A}_{a/b}$ is b -subhomogeneous, and in particular FDI. The universal property of $\mathcal{A}_{a/b}$ yields a representation $\sigma : \mathcal{A}_{a/b} \rightarrow B(L)$ with $\sigma(u_1) = V_1$ and $\sigma(u_2) = V_2$. Let

$$S = \text{span}\{u_1^n u_2^m, u_2^{-m} u_1^{-n} : 0 \leq n, m \leq N\} \subset \mathcal{A}_{a/b}$$

and let

$$\varphi : S \rightarrow B(H), \quad a \mapsto P_H \sigma(a)|_H.$$

Then φ is u.c.p. and

$$\varphi(u_1^n u_2^m) = P_H V_1^n V_2^m|_H = T_1^n T_2^m \quad \text{for all } 0 \leq n, m \leq N.$$

By Theorem 1.5, the u.c.p. map φ dilates to a finite-dimensional representation $\pi : \mathcal{A}_{a/b} \rightarrow B(K)$. Let $U_1 = \pi(u_1)$ and $U_2 = \pi(u_2)$. Then U_1, U_2 are q -commuting unitaries on a finite-dimensional Hilbert space and

$$T_1^n T_2^m = \varphi(u_1^n u_2^m) = P_H \pi(u_1^n u_2^m)|_H = P_H U_1^n U_2^m|_H$$

for all $0 \leq n, m \leq N$. The dimension bound follows from Proposition 3.4, as $\dim(S) \leq 2(N+1)^2 - 1$. \square

Remark 4.12. The rationality assumption in Corollary 4.11 is essential. Indeed, if $q = \exp(2\pi i\theta)$ with θ irrational, then there are no q -commuting unitaries on a finite-dimensional Hilbert space, because the irrational rotation algebra \mathcal{A}_θ is simple and infinite-dimensional, see [17, Theorem VI.1.4]. On the other hand, it is easy to construct q -commuting contractions on a finite-dimensional Hilbert space, for instance

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, Corollary 4.11 fails without the rationality assumption.

In other words, the dilation theorem for q -commuting contractions has a finite dimensional version if and only if $q = \exp(2\pi i\theta)$ and θ is rational. This fact becomes very transparent on the level of C^* -algebras. Rational rotation algebras are subhomogeneous, whereas irrational rotation algebras are simple and infinite-dimensional and hence have no finite-dimensional representations.

A similar phenomenon occurs in [30, Theorem 6.1], where q -commuting unitaries are dilated to q' -commuting unitaries.

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