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# GROUP AMENABILITY AND ACTIONS ON $\mathcal{Z}$ -STABLE C\*-ALGEBRAS

EUSEBIO GARDELLA AND MARTINO LUPINI

**ABSTRACT.** We study strongly outer actions of discrete groups on C\*-algebras in relation to (non)amenability. In contrast to related results for amenable groups, where uniqueness of strongly outer actions on the Jiang-Su algebra is expected, we show that uniqueness fails for all nonamenable groups, and that the failure is drastic. Our main result implies that if  $G$  contains a copy of  $\mathbb{F}_2$ , then there exist uncountably many, non-cocycle conjugate strongly outer actions of  $G$  on any tracial, unital, separable C\*-algebra that absorbs tensorially the Jiang-Su algebra. Similar conclusions hold for outer actions on McDuff  $\text{II}_1$  factors. We moreover show that  $G$  is amenable if and only if the Bernoulli shift on any finite strongly self-absorbing C\*-algebra absorbs the trivial action on the Jiang-Su algebra. Our methods consist in a careful study of weak containment for the Koopman representations of certain generalized Bernoulli actions.

## INTRODUCTION

Amenability for discrete groups was first introduced by von Neumann in the context of the Banach-Tarski paradox. One of the main early results in the theory, proved by Tarski, asserts that a group is amenable if and only if it admits no paradoxical decompositions. The fact that the Banach-Tarski paradox only makes use of free groups led Day to conjecture that a discrete group is nonamenable if and only if it contains the free group  $\mathbb{F}_2$  as a subgroup. This conjecture, known as *the von Neumann problem*, was open for about 40 years, until it was disproved by Ol'shanskii.

Amenability admits a surprisingly large number of equivalent formulations. Here, we are concerned with those characterizations that are phrased in terms of actions of the group. These usually come in the form of a dichotomy: roughly speaking, they assert that there is an object in the relevant category, on which every amenable group acts in an essentially unique way, while every nonamenable group admits a continuum of non-equivalent actions. The following is an illustrative example:

**Theorem.** Let  $G$  be a discrete group, and let  $(X, \mu)$  be a standard atomless probability space.

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- (1) If  $G$  is amenable, then all free, measure preserving, ergodic actions of  $G$  on  $(X, \mu)$  are orbit equivalent.
- (2) If  $G$  is not amenable, then there exist uncountably many non-orbit equivalent free, measure preserving, ergodic actions of  $G$  on  $(X, \mu)$ .

Part (1) is a combination of classical results of Dye and Ornstein-Weiss. In reference to (2), the first result in this direction is a theorem of Connes-Weiss, asserting that every nonamenable group without property (T) admits two such actions. Much more recently, Ioana proved part (2) for groups containing a copy of  $\mathbb{F}_2$  ([19]), using the corresponding result for  $\mathbb{F}_2$  due to Gaboriau-Popa ([10]), and finally Epstein extended the result to all nonamenable groups ([7]). In recent work ([14, 15]), the authors strengthened the conclusion in part (2) above: the relation of orbit equivalence of actions of nonamenable groups is not Borel.

In the context of von Neumann algebras, and specifically for the hyperfinite  $\text{II}_1$  factor  $\mathcal{R}$ , amenability can also be characterized in terms of actions:

**Theorem.** Let  $G$  be a discrete group, and let  $\mathcal{R}$  be the hyperfinite  $\text{II}_1$  factor.

- (1) If  $G$  is amenable, then all outer actions of  $G$  on  $\mathcal{R}$  are cocycle conjugate.
- (2) If  $G$  is not amenable, then there exist uncountably many non-cocycle conjugate outer actions of  $G$  on  $\mathcal{R}$ .

Part (1) is due to Ocneanu ([29]), although particular cases were proved by Connes for cyclic groups ([5]), and by Jones for finite groups ([20]). Part (2) is a recent result due to Brothier-Vaes (Theorem B in [4]), which also shows that the relation of cocycle conjugacy of outer actions of  $G$  on  $\mathcal{R}$  is not Borel when  $G$  is not amenable. This result generalizes previous results of Popa ([32]) and Jones ([21]).

In both theorems recalled above, the amenable case was resolved relatively early. On the other hand, the nonamenable case took much longer, and it required the invention of new and powerful tools such as Popa's celebrated deformation/rigidity theory. Indeed, it was realized that certain nonamenable groups (or certain nonamenable  $\text{II}_1$  factors) exhibit striking rigidity phenomena, which are best seen in the presence of property (T). The richness of the nonamenable world drove researchers in both Ergodic Theory and in von Neumann algebras to study actions of nonamenable groups on the standard atomless probability space as well as on  $\mathcal{R}$ , with particular focus on the complexity of their classification.

This work revolves around analogs of the above results in the context of  $C^*$ -algebras, the central theme being the case of nonamenable groups. Strongly self-absorbing  $C^*$ -algebras can be seen as the  $C^*$ -analog of the hyperfinite  $\text{II}_1$  factor. (Recall that a unital, separable  $C^*$ -algebra  $\mathcal{D}$  is said to be *strongly self-absorbing* if  $\mathcal{D} \neq \mathbb{C}$  and there is an isomorphism  $\varphi: \mathcal{D} \rightarrow \mathcal{D} \otimes_{\min} \mathcal{D}$  which is approximately unitarily equivalent to the first tensor factor embedding; see [37].) Examples of such algebras are the UHF-algebras of infinite type, the Jiang-Su algebra  $\mathcal{Z}$ , the Cuntz algebras  $\mathcal{O}_2$  and  $\mathcal{O}_\infty$ , and their tensor products. Moreover, it is conjectured that these are the only strongly self-absorbing  $C^*$ -algebras. By a result of Winter ([38]), any strongly self-absorbing  $C^*$ -algebra absorbs  $\mathcal{Z}$  tensorially.

$C^*$ -analogs of part (1) in the theorem above were explored in the early 1990s by Bratteli, Evans and Kishimoto ([3]), who studied a concrete family of outer actions of  $\mathbb{Z}$  on a specific UHF-algebra. Their results show that outerness in (finite)  $C^*$ -algebras is too weak a condition for an analog of Ocneanu's result to hold. They also provided evidence for the fact that a uniqueness result may hold if one assumes outerness not only for the action, but also for its extension to the weak closure in the GNS representation. This notion is now called *strong outerness*.

Motivated by a recent breakthrough of Szabó [36], the following conjecture has been proposed in [16], the first part of which had already been suggested in [36].

We refer the reader to the introductions of [36] and [16] for motivation and relevant references (in particular, for the reason why groups with torsion must be excluded).

**Conjecture A.** Let  $G$  be a torsion-free countable group, and let  $\mathcal{D}$  be a strongly self-absorbing  $C^*$ -algebra.

- (1) If  $G$  is amenable, then any two strongly outer actions of  $G$  on  $\mathcal{D}$  are cocycle conjugate.
- (2) If  $G$  is not amenable, then there exist uncountably many non-cocycle conjugate strongly outer actions of  $G$  on  $\mathcal{D}$ .

The main result of [36] asserts that part (1) holds when  $\mathcal{D}$  is either a UHF-algebra or the Jiang-Su algebra and  $G$  is abelian, while [16] asserts that part (2) holds when  $\mathcal{D}$  is a UHF-algebra and for groups containing a subgroup with relative property (T).

In this work, we continue the study of strongly outer actions on  $C^*$ -algebras, and particularly on strongly self-absorbing  $C^*$ -algebras. We are interested in constructing many non-cocycle conjugate actions for a given nonamenable group. To this end, we focus on a specific and very rich class of actions, which we call *generalized (noncommutative) Bernoulli shifts*. These are constructed as follows: given a strongly self-absorbing  $C^*$ -algebra  $\mathcal{D}$  and an action  $G \curvearrowright^\sigma X$  of a discrete group  $G$  on a countable set  $X$ , we consider the action of  $G$  on  $\bigotimes_{x \in X} \mathcal{D} \cong \mathcal{D}$  given by permuting the tensor factors according to  $\sigma$ .

For an arbitrary group  $G$ , it seems difficult to produce actions of this form other than the usual Bernoulli shift  $\beta_{\mathcal{D}} : G \curvearrowright \bigotimes_{g \in G} \mathcal{D}$  and the trivial action of  $G$  on  $\mathcal{D}$ . However, considering these actions leads to a new characterization of amenability. The implication (1) $\Rightarrow$ (2) in Theorem B appeared in [13], and can also be deduced from [34].

**Theorem B.** Let  $G$  be a countable discrete group, and let  $\mathcal{D}$  be a finite strongly self-absorbing  $C^*$ -algebra. Then the following are equivalent:

- (1)  $G$  is amenable;
- (2) the Bernoulli shift  $\beta_{\mathcal{D}}$  absorbs tensorially (up to cocycle conjugacy) the trivial action  $\text{id}_{\mathcal{Z}}$  on  $\mathcal{Z}$ .

This result implies, in particular, a weak form of part (2) of Conjecture A: every nonamenable group admits two strongly outer actions on  $\mathcal{D}$  which are not cocycle conjugate, namely, the Bernoulli shift and its stabilization with  $\text{id}_{\mathcal{Z}}$ .

We obtain stronger results for groups having sufficiently many finite subquotients. A particular instance of our main result (Theorem 4.5) confirms part (2) of Conjecture A for groups containing  $\mathbb{F}_2$ :

**Theorem C.** Let  $G$  be a discrete group containing a nonabelian free group, and let  $A$  be a tracial, separable, unital  $\mathcal{Z}$ -absorbing  $C^*$ -algebra. (For example, a finite strongly self-absorbing  $C^*$ -algebra.) Then there exist uncountably many non-cocycle conjugate, strongly outer actions of  $G$  on  $A$ , acting via asymptotically inner automorphisms of  $A$ . When  $A$  is strongly self-absorbing, these actions can also be chosen to be weak mixing.

The fact that the actions we construct are pointwise asymptotically inner implies that these actions are not distinguishable by any kind of  $K$ - or  $KK$ -theoretical invariant, nor are they classified using the Cuntz semigroup. The way in which we distinguish them is via the weak equivalence class of the associated Koopman representation.

Theorem C is the  $C^*$ -version of Ioana's result on non-orbit equivalent actions from [19]. Epstein later combined Ioana's result with Gaboriau-Lyon's measurable

solution to the von Neumann problem [9] to generalize Ioana's work to *all* nonamenable groups. A very interesting and promising problem is to find an analog of the main result in [9] in the context of strongly outer actions on  $C^*$ -algebras, or at least for outer actions on  $\mathcal{R}$ . A satisfactory solution would allow one to extend Theorem C (and Theorem D below) to include all nonamenable groups.

Finally, our methods allow one to replace  $\mathcal{Z}$  with the hyperfinite  $\text{II}_1$  factor  $\mathcal{R}$ , thus obtaining the following result.

**Theorem D.** Let  $G$  be a discrete group containing a copy of  $\mathbb{F}_2$ , and let  $M$  be a McDuff  $\text{II}_1$  factor. Then there exist uncountably many non-cocycle conjugate, outer actions of  $G$  on  $M$ , acting via asymptotically inner automorphisms of  $M$ . When  $M = \mathcal{R}$ , these actions can also be chosen to be weak mixing.

It was shown by Brothier and Vaes [4, Theorem B], using Popa's deformation/rigidity theory from [33, 32], that an arbitrary nonamenable group  $G$  admits uncountably many pairwise non-cocycle conjugate outer actions on  $\mathcal{R}$ . The arguments of this paper produce actions of  $G$  on  $\mathcal{R}$  that are furthermore weak mixing, under the additional assumption that  $G$  contains  $\mathbb{F}_2$ . It would be interesting to know whether there exist uncountably many pairwise non-cocycle conjugate *weak mixing* outer actions of  $G$  on  $\mathcal{R}$  for an arbitrary nonamenable group  $G$ .

The rest of the paper is organized as follows. In Section 1, we establish a number of basic facts about subgroups with finite index that will be important in the later sections. In Section 2, we study the generalized Bernoulli shift associated with an action  $G \curvearrowright^\sigma X$  on a discrete set  $X$ , and relate its Koopman representation to the canonical unitary representation of  $G$  on  $\ell^2(X)$ . In Section 3, we specialize to a particular family of generalized Bernoulli shifts, obtained from finite subquotients of  $G$ . Finally, Section 4 contains the proofs of our main results (Theorem 4.4 and Theorem 4.5), from which Theorems B, C and D follow.

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## 1. QUASIREGULAR REPRESENTATIONS

We begin by recalling the notion of quasiregular representation.

**Definition 1.1.** Let  $G$  be a discrete group, and let  $H$  be subgroup. We denote by  $\lambda_{G/H}: G \rightarrow \mathcal{U}(\ell^2(G/H))$  the unitary representation induced by the canonical left translation action of  $G$  on  $G/H$ . We call  $\lambda_{G/H}$  the *quasiregular representation* associated with  $H$ .

When  $H$  is a normal subgroup of  $G$ , the quasiregular representation  $\lambda_{G/H}$  is precisely the left regular representation of the quotient group  $G/H$ .

Let  $G$  be a discrete group and let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$  and  $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$  be unitary representations. Recall that  $\mu$  is said to be (*unitarily*) *contained* in  $\nu$ , written  $\mu \subseteq \nu$ , if there exists an isometry  $\varphi: \mathcal{H}_\mu \rightarrow \mathcal{H}_\nu$  satisfying  $\varphi \circ \mu_g = \nu_g \circ \varphi$  for all  $g \in G$ . When  $\varphi$  can be chosen to be surjective, we say that  $\mu$  and  $\nu$  are *unitarily equivalent*, and write  $\mu \cong \nu$ .

**Lemma 1.2.** Let  $G$  be a discrete group, and let  $H_1, \dots, H_n$  be subgroups of  $G$  whose indices in  $G$  are finite. Set  $H = H_1 \cap \dots \cap H_n$ . Then:

- (1)  $H$  has finite index in  $G$ , and  $[G : H] \leq [G : H_1] \cdots [G : H_n]$ . If the indices of  $H_1, \dots, H_n$  in  $G$  are pairwise coprime, then equality holds.
- (2) We have  $\lambda_{G/H} \subseteq \lambda_{G/H_1} \oplus \dots \oplus \lambda_{G/H_n}$ , and this containment is an equivalence whenever the indices of  $H_1, \dots, H_n$  in  $G$  are pairwise coprime.

*Proof.* It is enough to prove both parts for  $n = 2$ . We begin with some notation. Consider the diagonal action of  $G$  on  $\ell^2(G/H_1) \otimes \ell^2(G/H_2) \cong \ell^2(G/H_1 \times G/H_2)$  via  $\lambda_{G/H_1} \otimes \lambda_{G/H_2}$ . Set  $x = (H_1, H_2) \in G/H_1 \times G/H_2$ . Then the stabilizer of  $x$  is precisely  $H_1 \cap H_2$ . Define a map  $\psi: G \rightarrow G/H_1 \times G/H_2$  by  $\psi(g) = g \cdot x$  for all  $g \in G$ .

(1). Since  $\psi$  is the orbit map associated with  $x$ , we have

$$[G : H_1 \cap H_2] = |\psi(x)| \leq [G : H_1][G : H_2].$$

When  $[G : H_1]$  and  $[G : H_2]$  are coprime, one checks that  $\psi$  is surjective, so we get equality.

(2). Consider the induced map  $\hat{\psi}: G/(H_1 \cap H_2) \rightarrow G/H_1 \times G/H_2$ , which is given by  $\hat{\psi}(g(H_1 \cap H_2)) = (gH_1, gH_2)$  for all  $g \in G$ . Define

$$\varphi: \ell^2(G/(H_1 \cap H_2)) \rightarrow \ell^2(G/H_1 \times G/H_2)$$

on the canonical orthonormal basis by setting  $\varphi(\delta_{g(H_1 \cap H_2)}) = \delta_{\hat{\psi}(g)}$  for all  $g \in G$ . It is clear that  $\varphi$  is an isometry, and that it is a unitary if  $[G : H_1]$  and  $[G : H_2]$  are coprime. It remains to show that  $\varphi$  intertwines  $\lambda_{G/(H_1 \cap H_2)}$  and  $\lambda_{G/H_1} \oplus \lambda_{G/H_2}$ . Given  $g, k \in G$ , we have

$$\begin{aligned} (\lambda_{G/H_1} \oplus \lambda_{G/H_2})_k (\varphi(\delta_{g(H_1 \cap H_2)})) &= (\lambda_{G/H_1} \oplus \lambda_{G/H_2})_k (\delta_{(gH_1, gH_2)}) \\ &= \delta_{kgH_1, kgH_2} \\ &= \varphi(\delta_{kg(H_1 \cap H_2)}) \\ &= \varphi((\lambda_{G/(H_1 \cap H_2)})_k (\delta_{g(H_1 \cap H_2)})), \end{aligned}$$

as desired. This finishes the proof.  $\square$

Our final lemma is well-known, so we only sketch the proof; see [22, Lemma 3.3].

**Lemma 1.3.** Let  $G$  be a discrete group, let  $S$  be a subgroup of  $G$ , and let  $H$  be a subgroup of  $S$  with  $[S : H] < \infty$ . Then  $\lambda_{G/S} \subseteq \lambda_{G/H}$ . In particular, if  $S$  is a *finite* subgroup of  $G$ , then  $\lambda_{G/S} \subseteq \lambda_G$ .

*Proof.* Let  $\pi: G/H \rightarrow G/S$  be the canonical quotient map. Then  $\varphi: \ell^2(G/S) \rightarrow \ell^2(G/H)$  given by  $\varphi(\xi) = \frac{1}{\sqrt{[S:H]}} \xi \circ \pi$  for  $\xi \in \ell^2(G/S)$ , is an equivariant isometry.  $\square$

## 2. GENERALIZED BERNOULLI SHIFTS

In this section, we study a class of group actions on  $C^*$ -algebras which are obtained from permutation actions of  $G$  on (discrete) sets. First, we need to discuss how the GNS construction behaves with respect to infinite tensor products.

**2.1. Infinite tensor products and the GNS construction.** We briefly review the GNS construction.

**Definition 2.1.** Let  $D$  be a unital  $C^*$ -algebra, and let  $\phi: D \rightarrow \mathbb{C}$  be a state on  $D$ . Define  $\langle \cdot, \cdot \rangle_\phi: D \times D \rightarrow \mathbb{C}$  by  $\langle a, b \rangle_\phi = \phi(a^*b)$  for all  $a, b \in D$ . Let  $\|\cdot\|_{2,\phi}$  be the corresponding seminorm on  $D$ , given by  $\|a\|_{2,\phi} = \phi(a^*a)^{1/2}$ . Let  $\mathcal{H}_\phi^D$  denote the Hilbert space obtained as the Hausdorff completion of  $D$  with respect to the seminorm  $\|\cdot\|_{2,\phi}$ . We denote by  $\iota_\phi^D: D \rightarrow \mathcal{H}_\phi^D$  the canonical map with dense image. When  $D$  is clear from the context, we will simply write  $\mathcal{H}_\phi$  and  $\iota_\phi$ .

We turn to infinite tensor products of Hilbert spaces.

**Definition 2.2.** Let  $\mathcal{H}$  be a Hilbert space, let  $\eta \in \mathcal{H}$  be a unit vector, and let  $X$  be a discrete set. We define the tensor product  $\bigotimes_{x \in X} \mathcal{H}$  of  $\mathcal{H}$  over  $X$  (along  $\eta$ ) to be the completion of

$$\text{span} \left\{ \bigotimes_{x \in X} \xi_x : \xi_x \in \mathcal{H}, \text{ and } \xi_x = \eta \text{ for all but finitely many } x \in X \right\},$$

in the norm induced by the pre-inner product given by

$$\left\langle \bigotimes_{x \in X} \xi_x, \bigotimes_{x \in X} \zeta_x \right\rangle = \prod_{x \in X} \langle \xi_x, \zeta_x \rangle.$$

(Observe that all but finitely many of the multiplicative factors above are equal to 1, so that the product is indeed well-defined.)

It will be convenient to have a description of an orthonormal basis of an infinite tensor product of Hilbert spaces.

**Lemma 2.3.** Let  $\mathcal{H}$  be a Hilbert space, let  $\eta \in \mathcal{H}$  be a unit vector, and let  $X$  be a discrete set. Denote by  $\kappa$  the dimension of  $\mathcal{H}$ . Let  $\{\eta_n : n \in \kappa\}$  be an orthonormal basis for  $\mathcal{H}$  with  $\eta_0 = \eta$ . Set

$$\mathcal{F} = \{f : X \rightarrow \kappa \text{ such that } \{x \in X : f(x) \neq 0\} \text{ is finite}\}.$$

In particular,  $\mathcal{F}$  contains the function  $f_0 : X \rightarrow \kappa$  that is constantly equal to 0. Then an element  $f \in \mathcal{F}$  can be canonically identified with the element  $\bigotimes_{x \in X} \eta_{f(x)}$  of  $\bigotimes_{x \in X} \mathcal{H}$ . In turn, this allows one to identify  $\mathcal{F}$  with an orthonormal basis for  $\bigotimes_{x \in X} \mathcal{H}$ .

We will need infinite (minimal) tensor products of unital C\*-algebras. Let  $D$  be a unital C\*-algebra, and let  $X$  be a countable set. Write  $\mathbb{P}_f(X)$  for the set of all finite subsets of  $X$ , ordered by inclusion. We define the tensor product  $\bigotimes_{x \in X} D$  to be the direct limit of the minimal tensor products  $\bigotimes_{x \in S} D$ , for  $S \in \mathbb{P}_f(X)$ , with the canonical connecting maps  $\iota_{S,T} : \bigotimes_{x \in S} D \rightarrow \bigotimes_{x \in T} D$  given by  $\iota_{S,T}(d) = d \otimes 1_{T \setminus S}$  for  $d \in \bigotimes_{x \in S} D$ , whenever  $S, T \in \mathbb{P}_f(X)$  satisfy  $S \subseteq T$ . If  $\phi$  is a state on  $D$ , then the direct limit of the states  $\bigotimes_{x \in S} \phi$ , for  $S \in \mathbb{P}_f(X)$ , defines a state on  $\bigotimes_{x \in X} D$ , which we denote by  $\bigotimes_{x \in X} \phi$ .

Next, we show that GNS constructions commute with infinite tensor products. The result is folklore and well-known, and we include a proof for the convenience of the reader.

**Theorem 2.4.** Let  $D$  be a unital C\*-algebra, let  $X$  be a discrete set, and let  $\phi : D \rightarrow \mathbb{C}$  be a state. Set  $\tilde{D} = \bigotimes_{x \in X} D$  and  $\tilde{\phi} = \bigotimes_{x \in X} \phi$ . Then there is a canonical unitary

$$u : \bigotimes_{x \in X} \mathcal{H}_\phi^D \rightarrow \mathcal{H}_{\tilde{\phi}}^{\tilde{D}}$$

determined on a dense subset by

$$u \left( \bigotimes_{x \in X} \iota_\phi(a_x) \right) = \iota_{\tilde{\phi}} \left( \bigotimes_{x \in X} a_x \right),$$

where  $a_x \in D$  for all  $x \in X$ , and  $a_x = 1_D$  for all but finitely many  $x \in X$ . (The tensor product  $\bigotimes_{x \in X} \mathcal{H}_\phi^D$  is taken along  $\eta = \iota_\phi(1_D) \in \mathcal{H}_\phi^D$ .)

*Proof.* Let  $x \in X$  and write  $\psi_x : D \rightarrow \bigotimes_{x \in X} D = \tilde{D}$  for the  $x$ -th tensor factor embedding. Since  $\tilde{\phi} = \tilde{\phi} \circ \psi_x$ , it follows that  $\psi_x$  induces a Hilbert space isometry  $u_x : \mathcal{H}_\phi \rightarrow \mathcal{H}_{\tilde{\phi}}$  satisfying  $u_x \circ \iota_\phi = \iota_{\tilde{\phi}} \circ \psi_x$ .



Denote by  $s_x: \mathcal{H}_\phi^D \rightarrow \bigotimes_{x \in X} \mathcal{H}_\phi^{\bar{D}}$  the canonical isometry as the  $x$ -th tensor factor. By the universal property of the tensor product, there exists a bounded linear map  $u: \bigotimes_{x \in X} \mathcal{H}_\phi^D \rightarrow \mathcal{H}_\phi^{\bar{D}}$  satisfying  $u_x = u \circ s_x$  for all  $x \in X$ . It is then easy to check that  $u$  is a unitary, and that it satisfies the identity in the statement. We omit the details.  $\square$

**2.2. Generalized Bernoulli shifts.** In classical dynamical systems, the Bernoulli shift is the transformation on the space  $\{0, 1\}^{\mathbb{Z}}$  of bi-infinite binary sequences given by sending  $(a_n)_{n \in \mathbb{Z}}$  to the shifted sequence  $(a_{n+1})_{n \in \mathbb{Z}}$ ; see [30, 27]. More generally, one can replace  $\{0, 1\}$  with an arbitrary finite alphabet or a compact space  $T$  (which is called the *base* or *state space*), in which case one considers the space  $T^{\mathbb{Z}}$  endowed with the product topology. Even more generally, one can replace  $\mathbb{Z}$  with an arbitrary discrete group  $G$ , which then naturally acts on  $T^G$  by setting  $g \cdot (t_h)_{h \in G} = (t_{gh})_{h \in G}$ .

A generalized Bernoulli shift is defined by replacing the left translation action  $G \curvearrowright G$  with an arbitrary action of  $G$  on a set  $X$ , that is, a homomorphism  $\sigma: G \rightarrow \text{Perm}(X)$  into the group  $\text{Perm}(X)$  of permutations of  $X$ . We usually abbreviate this to  $G \curvearrowright^\sigma X$ , and when the action  $\sigma$  is clear from the context, we write  $g \cdot x$  instead of  $\sigma_g(x)$  for  $g \in G$  and  $x \in X$ . These notions admit natural generalizations to noncommutative  $C^*$ -algebras, which we proceed to define.

For a  $C^*$ -algebra  $D$  and a state  $\phi$  on it, we say that an action  $\alpha: G \rightarrow \text{Aut}(D)$  is  $\phi$ -preserving, or that  $\phi$  is  $\alpha$ -invariant, if  $\phi \circ \alpha_g = \phi$  for all  $g \in G$ .

**Definition 2.5.** Let  $G$  be a countable group, let  $X$  be a countable set, and let  $G \curvearrowright^\sigma X$  be an action. Endow  $X$  with the counting measure, and let  $D$  be a unital  $C^*$ -algebra.

- (1) The *unitary representation associated with  $\sigma$*  is the unitary representation  $u_\sigma: G \rightarrow \mathcal{U}(\ell^2(X))$  given by  $(u_\sigma)_g(\delta_x) = \delta_{g \cdot x}$  for all  $g \in G$  and all  $x \in X$ .
- (2) The *generalized Bernoulli shift associated with  $\sigma$*  is the action  $\beta_{\sigma, D}: G \rightarrow \text{Aut}(\bigotimes_{x \in X} D)$  given by permuting the tensor factors according to  $G \curvearrowright^\sigma X$ .

**Notation 2.6.** Let  $G$  be a discrete group. We will denote by  $\text{Lt}_G$  the action of left translation  $G \curvearrowright G$ , so that  $u_{\text{Lt}_G}$  is the left regular representation  $\lambda_G: G \rightarrow \mathcal{U}(\ell^2(G))$ . Similarly, if  $H$  is a subgroup of  $G$ , we will denote by  $\text{Lt}_{G/H}$  the canonical action  $G \curvearrowright G/H$  by left translation of left cosets, so that  $u_{\text{Lt}_{G/H}}$  is the quasiregular representation  $\lambda_{G/H}: G \rightarrow \mathcal{U}(\ell^2(G/H))$  from Definition 1.1.

We will also need the Koopman construction, which is a way of obtaining unitary representations from group actions. In measurable dynamics, an invertible measure-preserving transformation  $T$  of the standard probability space  $(X, \mu)$  gives rise to a unitary operator  $U_T$  on  $L^2(X, \mu)$ , called *Koopman operator*, defined by  $U_T(f) = f \circ T^{-1}$  for  $f \in L^2(X, \mu)$ ; see [23]. Thus, a measure-preserving action  $\alpha: G \rightarrow \text{Aut}(X, \mu)$  induces a unitary representation  $\kappa(\alpha): G \rightarrow \mathcal{U}(L^2(X, \mu))$ , called the *Koopman representation* associated to  $\alpha$ , which is given by  $\kappa(\alpha)_g = U_{\alpha_g}$  for all  $g \in G$ . In the definition below, we recall its natural noncommutative analogue.

**Definition 2.7.** Let  $G$  be a countable group, let  $(D, \phi)$  be a unital  $C^*$ -algebra with a state  $\phi$ , and let  $\alpha$  be a  $\phi$ -preserving action of  $G$  on  $D$ .

- The *Koopman representation of  $\alpha$  (with respect to  $\phi$ )* is the unitary representation  $\kappa_\phi(\alpha): G \rightarrow \mathcal{U}(\mathcal{H}_\phi^D)$  determined by  $\kappa_\phi(\alpha)_g(\iota_\phi(a)) = \iota_\phi(\alpha_g(a))$  for all  $g \in G$  and all  $a \in D$ .
- The *reduced Koopman representation of  $\alpha$  (with respect to  $\phi$ )*, denoted by  $\kappa_\phi^{(0)}(\alpha)$ , is the restriction of  $\kappa_\phi(\alpha)$  to the orthogonal complement of  $\iota_\phi(1_D)$ .

**Remark 2.8.** In the notation of the definition above, and with  $1_G$  denoting the trivial representation, there is a unitary equivalence  $\kappa_\phi(\alpha) \cong \kappa_\phi^{(0)}(\alpha) \oplus 1_G$ .

We will need the following easy observation, whose proof is straightforward.

**Lemma 2.9.** Let  $G$  be a countable group, let  $(A, \phi_A)$  and  $(D, \phi_D)$  be unital  $C^*$ -algebras with states, and let  $\alpha: G \rightarrow \text{Aut}(A)$  and  $\beta: G \rightarrow \text{Aut}(D)$  be actions preserving  $\phi_A$  and  $\phi_D$ , respectively. Then the canonical unitary

$$u: \mathcal{H}_{\phi_A}^A \otimes \mathcal{H}_{\phi_D}^D \rightarrow \mathcal{H}_{\phi_A \otimes \phi_D}^{A \otimes D}$$

determined by  $u(\iota_{\phi_A}^A(a) \otimes \iota_{\phi_D}^D(d)) = \iota_{\phi_A \otimes \phi_D}^{A \otimes D}(a \otimes d)$  for all  $a \in A$  and  $d \in D$ , implements a unitary equivalence between  $\kappa_{\phi_A}(\alpha) \otimes \kappa_{\phi_D}(\beta)$  and  $\kappa_{\phi_A \otimes \phi_D}(\alpha \otimes \beta)$ .

If  $G \curvearrowright^\sigma X$  and  $G \curvearrowright^\rho Y$  are actions on countable sets, we let  $\sigma \times \rho$  be the action  $G \curvearrowright X \times Y$  defined by

$$(\sigma \times \rho)_g(x, y) = (\sigma_g(x), \rho_g(y))$$

for  $g \in G$ , for  $x \in X$ , and for  $y \in Y$ . We also let  $\sigma \sqcup \rho$  be the unique action of  $G$  on the disjoint union  $X \sqcup Y$  which extends the actions  $\sigma$  and  $\rho$ . The disjoint union of an  $n$ -tuple of actions, or even an infinite sequence of actions, is defined similarly.

We proceed to collect some elementary lemmas that will be needed later.

**Notation 2.10.** Let  $G$  be a countable group, let  $G \curvearrowright^\sigma X$  be an action of  $G$  on a countable set  $X$ , and let  $\mathcal{H}$  be a separable Hilbert space with a distinguished unit vector  $\eta$ . We denote by  $\kappa_\sigma^\mathcal{H}: G \rightarrow \mathcal{U}(\bigotimes_{x \in X} \mathcal{H})$  the unitary representation given by permuting the tensor factors according to  $G \curvearrowright^\sigma X$ . When  $\mathcal{H}$  and  $\eta$  are clear from the context, we simply write  $\kappa_\sigma$ . We define  $\kappa_\sigma^{(0)}$  to be the restriction of  $\kappa_\sigma$  to the orthogonal complement of  $\eta$  in  $\bigotimes_{x \in X} \mathcal{H}$ .

**Remark 2.11.** Let  $G$  be a countable group, let  $(D, \phi)$  be a unital  $C^*$ -algebra endowed with a state  $\phi$ , and let  $G \curvearrowright^\sigma X$  be an action of  $G$  on a countable set  $X$ . Write  $\kappa_\sigma$  for  $\kappa_\sigma^{\mathcal{H}_\phi^D}$ . Then:

- (1) The Koopman representation of  $\beta_{\sigma, D}$  is unitarily equivalent to  $\kappa_\sigma$ .
- (2) The reduced Koopman representation of  $\beta_{\sigma, D}$  is unitarily equivalent to  $\kappa_\sigma^{(0)}$ .

**Lemma 2.12.** For  $n \in \mathbb{N}$ , let  $G \curvearrowright^{\sigma_n} X_n$  be actions, and let  $G \curvearrowright^\sigma X$  denote their disjoint union. Let  $\mathcal{H}$  be a Hilbert space with a distinguished unit vector  $\eta$ .

- (1) The representations  $u_\sigma$  and  $\bigoplus_{n \in \mathbb{N}} u_{\sigma_n}$  are unitarily equivalent.
- (2) The representations  $\kappa_\sigma^\mathcal{H}$  and  $\bigotimes_{n \in \mathbb{N}} \kappa_{\sigma_n}^\mathcal{H}$  are unitarily equivalent.

*Proof.* Set  $X = \bigsqcup_{n \in \mathbb{N}} X_n$ . Then  $\ell^2(X)$  is canonically isometrically isomorphic to  $\bigoplus_{n \in \mathbb{N}} \ell^2(X_n)$ , as witnessed by a unitary intertwining the representations  $u_\sigma$  and  $u_{\bigoplus_{n \in \mathbb{N}} \sigma_n}$ , which shows (1). Similarly,  $\bigotimes_{n \in \mathbb{N}} \bigotimes_{x \in X_n} \mathcal{H}$  is canonically isometrically isomorphic to  $\bigotimes_{x \in X} \mathcal{H}$ , as witnessed by a unitary that intertwines  $\kappa_\sigma$  and  $\bigotimes_{n \in \mathbb{N}} \kappa_{\sigma_n}$ .  $\square$

**Lemma 2.13.** For  $n \in \mathbb{N}$ , let  $D_n$  be a unital  $C^*$ -algebra with a state  $\phi_n$ , and  $\beta_n$  be a  $\phi_n$ -preserving action of  $G$  on  $D_n$ . Set  $D = \bigotimes_{n \in \mathbb{N}} D_n$  and  $\phi = \bigotimes_{n \in \mathbb{N}} \phi_n$ , and let  $\beta: G \rightarrow \text{Aut}(D)$  be the infinite tensor product of the actions  $\beta_n$ , for  $n \in \mathbb{N}$ . Then  $\phi$  is  $\beta$ -invariant and there is a unitary equivalence

$$\kappa_\phi(\beta) \cong \bigotimes_{n \in \mathbb{N}} \kappa_{\phi_n}(\beta_n).$$

*Proof.* It is immediate that  $\phi$  is  $\beta$ -invariant. It suffices to observe that  $\mathcal{H}_\phi^D$  is canonically isometrically isomorphic to  $\bigotimes_{n \in \mathbb{N}} \mathcal{H}_{\phi_n}^{D_n}$ , with the tensor product taken

along the unit vectors  $\iota_{\phi_n}^{D_n}(1_{D_n}) \in \mathcal{H}_{\phi_n}^{D_n}$ , as witnessed by a unitary that intertwines the representations  $\kappa_\phi(\beta)$  and  $\bigotimes_{n \in \mathbb{N}} \kappa_{\phi_n}(\beta_n)$ .  $\square$

**2.3. Weak containment of representations.** We recall the definition of weak containment for representations in the sense of Zimmer.

**Definition 2.14.** Let  $G$  be a discrete group, and let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$  and  $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$  be unitary representations. We say that  $\mu$  is *weakly contained in  $\nu$  in the sense of Zimmer*, in symbols  $\mu \prec_Z \nu$ , if for any  $\varepsilon > 0$ , for any  $\xi_1, \dots, \xi_n \in \mathcal{H}_\mu$ , for any finite subset  $F \subseteq G$ , and for any  $\varepsilon > 0$ , there exist  $\eta_1, \dots, \eta_n \in \mathcal{H}_\nu$  satisfying

$$|\langle \mu_g(\xi_j), \xi_k \rangle - \langle \nu_g(\eta_j), \eta_k \rangle| < \varepsilon$$

for all  $g \in F$  and for all  $j, k = 1, \dots, n$ .

We say that  $\mu$  and  $\nu$  are *weakly equivalent in the sense of Zimmer*, written  $\mu \sim_Z \nu$ , if  $\mu \prec_Z \nu$  and  $\nu \prec_Z \mu$ .

We will not be using the standard notion of weak containment, which is weaker. It is obvious that, when  $G$  is countable,  $\mu \prec_Z \nu$  if and only if  $\mu' \prec_Z \nu$  for every separable subrepresentation  $\mu'$  of  $\mu$ .

Below, we present a characterization of weak containment in the sense of Zimmer that will be convenient for our purposes. We need a short discussion on ultrapowers of unitary representations first. Let  $\mathcal{U}$  be a nonprincipal ultrafilter over  $\mathbb{N}$  and let  $\mathcal{H}$  be a Hilbert space. Set

$$\mathcal{H}^{\mathcal{U}} = \ell^\infty(\mathbb{N}, \mathcal{H}) / \{(\xi_j)_{j \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathcal{H}) : \lim_{j \rightarrow \mathcal{U}} \|\xi_j\| = 0\}.$$

The class in  $\mathcal{H}^{\mathcal{U}}$  of a sequence  $\xi \in \ell^\infty(\mathbb{N}, \mathcal{H})$  is denoted by  $[\xi]$ . Then  $\mathcal{H}^{\mathcal{U}}$  is a Hilbert space with respect to the inner product given by

$$\langle [\xi], [\eta] \rangle = \lim_{j \rightarrow \mathcal{U}} \langle \xi_j, \eta_j \rangle$$

for all  $\xi, \eta \in \ell^\infty(\mathbb{N}, \mathcal{H})$ . If  $\nu: G \rightarrow \mathcal{U}(\mathcal{H})$  is a unitary representation of a discrete group  $G$  on  $\mathcal{H}$ , then there is an induced representation  $\nu^{\mathcal{U}}: G \rightarrow \mathcal{U}(\mathcal{H}^{\mathcal{U}})$  given by  $\nu_g^{\mathcal{U}}([\xi]) = [(\nu_g(\xi_j))_{j \in \mathbb{N}}]$  for all  $g \in G$  and all  $\xi \in \ell^\infty(\mathbb{N}, \mathcal{H})$ .

**Remark 2.15.** Adopt the notation from the discussion above. If  $\nu_1$  and  $\nu_2$  are unitary representations, then it is easy to verify that  $(\nu_1 \oplus \nu_2)^{\mathcal{U}}$  is unitarily equivalent to  $\nu_1^{\mathcal{U}} \oplus \nu_2^{\mathcal{U}}$ .

For the convenience of the reader, we now recall several well-known properties of weak containment of representations.

**Proposition 2.16.** Let  $G$  be a countable discrete group, and let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$  and  $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$  be unitary representations, with  $\mathcal{H}_\mu$  separable. Let  $\mathcal{U}$  be a nonprincipal ultrafilter on  $\mathbb{N}$ . Then the following assertions are equivalent:

- (1)  $\mu \prec_Z \nu$ ;
- (2)  $\mu \subseteq \nu^{\mathcal{U}}$ .

*Proof.* Both directions follow from general results in model theory for metric structures [2]. (In this case, the structures are unitary representations of  $G$ .) Precisely, (2)  $\Rightarrow$  (1) follows from Los' theorem, while (1)  $\Rightarrow$  (2) follows from countable saturation of ultrapowers; see for example, Sections 2.3 and 4.3 in [8]. We include a proof for the sake of completeness.

(1)  $\Rightarrow$  (2): Fix an increasing sequence  $(F_n)_{n \in \mathbb{N}}$  of finite subsets of  $G$  such that  $F_1$  contains the unit of  $G$  and  $\bigcup_{n \in \mathbb{N}} F_n = G$ . Fix an increasing sequence  $(Q_n)_{n \in \mathbb{N}}$  of finite subsets of  $\mathbb{Q}$  such that  $\bigcup_{n \in \mathbb{N}} Q_n = \mathbb{Q}$ . Fix also an increasing sequence  $H_n$  of finite subsets of  $\mathcal{H}_\mu$  with dense union such that  $\sum_{j=1}^n a_j \mu_{g_j}(\xi_j)$  belongs to  $H_{n+1}$  for every  $n \in \mathbb{N}$ , for every  $g_1, \dots, g_n \in F_n$ , for every  $a_1, \dots, a_n \in Q_n$ , and for every

$\xi_1, \dots, \xi_n \in H_n$ . Set  $H = \bigcup_{n \in \mathbb{N}} H_n$ . By assumption, for every  $n \in \mathbb{N}$  there exists a function  $s_n: H_n \rightarrow \mathcal{H}_\nu$  such that

$$|\langle \mu_g(\xi), \eta \rangle - \langle \nu_g(s_n(\xi)), s_n(\eta) \rangle| < 2^{-n}$$

for every  $\xi, \eta \in H_n$  and for every  $g \in F_n$ . Set  $s_n(\xi) = 0$  for  $n \in \mathbb{N}$  and  $\xi \in H \setminus H_n$ . Let  $s: H \rightarrow \mathcal{H}_\nu^\mathcal{U}$  be determined by letting  $s(\xi)$  be the class of  $(s_n(\xi))_{n \in \mathbb{N}}$ , for  $\xi \in H_n$ . One checks that  $s$  is  $\mathbb{Q}$ -linear, isometric, and  $G$ -equivariant, so it extends to a linear isometry  $s: \mathcal{H}_\mu \rightarrow \mathcal{H}_\nu^\mathcal{U}$  satisfying  $s(\mu_g(\xi)) = \nu_g^\mathcal{U}(s(\xi))$  for every  $g \in G$  and every  $\xi \in \mathcal{H}_\mu$ .

(2)  $\Rightarrow$  (1): Suppose that  $s: \mathcal{H}_\mu \rightarrow \mathcal{H}_\nu^\mathcal{U}$  is a linear isometry satisfying  $s(\mu_g(\xi)) = \nu_g^\mathcal{U}(s(\xi))$  for every  $g \in G$  and  $\xi \in \mathcal{H}_\mu$ . Fix finite subsets  $F \subseteq G$  and  $H \subseteq \mathcal{H}_\mu$ , and fix  $\varepsilon > 0$ . We need to find elements  $f(\xi) \in \mathcal{H}_\nu$ , for  $\xi \in H$ , satisfying

$$|\langle \mu_g(\xi), \eta \rangle - \langle \nu_g(f(\xi)), f(\eta) \rangle| < \varepsilon$$

for every  $\xi, \eta \in H$  and  $g \in F$ . For every  $\xi \in \mathcal{H}_\mu$ , fix a representative sequence  $(s_n(\xi))_{n \in \mathbb{N}}$  of  $s(\xi)$ . By definition of the representation  $\nu^\mathcal{U}$  on  $\mathcal{H}_\nu^\mathcal{U}$ , there exists  $n \in \mathbb{N}$  such that

$$|\langle \nu_g^\mathcal{U}(s(\xi)), s(\eta) \rangle - \langle \nu_g(s_n(\xi)), s_n(\eta) \rangle| < \varepsilon$$

for every  $\xi, \eta \in H$  and  $g \in F$ . Since  $s$  is an equivariant isometry, it follows that

$$\langle \nu_g^\mathcal{U}(s(\xi)), s(\eta) \rangle = \langle s(\mu_g(\xi)), s(\eta) \rangle = \langle \mu_g(\xi), \eta \rangle$$

and hence  $|\langle \mu_g(\xi), \eta \rangle - \langle \nu_g(s_n(\xi)), s_n(\eta) \rangle| < \varepsilon$ . The proof is concluded by setting  $f = s_n$ .  $\square$

**Proposition 2.17.** Let  $G$  be a countable discrete group, and let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$  and  $\nu_j: G \rightarrow \mathcal{U}(\mathcal{H}_j)$ , for  $j = 1, \dots, n$ , be unitary representations. Assume that  $\mu$  is irreducible and finite-dimensional, and that  $\mu \prec_Z \nu_1 \oplus \dots \oplus \nu_n$ . Then there exists  $k \in \{1, \dots, n\}$  such that  $\mu \prec_Z \nu_k$ .

*Proof.* Let  $\mathcal{U}$  be any nonprincipal ultrafilter on  $\mathbb{N}$ . Use Proposition 2.16 to choose an equivariant isometry  $s: \mathcal{H}_\mu \rightarrow (\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n)^\mathcal{U}$  witnessing the fact that  $\mu \prec_Z \nu_1 \oplus \dots \oplus \nu_n$ . We identify  $(\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n)^\mathcal{U}$  equivariantly with  $\mathcal{H}_1^\mathcal{U} \oplus \dots \oplus \mathcal{H}_n^\mathcal{U}$  in a canonical way via Remark 2.15. For  $j = 1, \dots, n$ , we denote by  $s_j: \mathcal{H}_\mu \rightarrow \mathcal{H}_j^\mathcal{U}$  the composition of  $s$  with the canonical projection onto  $\mathcal{H}_j^\mathcal{U}$ . Since  $s$  is nonzero, there exists  $k \in \{1, \dots, n\}$  such that  $s_k$  is nonzero. Since  $\mu$  is irreducible and finite-dimensional, by Schur's lemma  $s_k$  is a scalar multiple of an isometry. This concludes the proof.  $\square$

**Lemma 2.18.** Let  $G$  be a discrete group, and let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$  and  $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$  be unitary representations. Let  $\pi_\mu: C^*(G) \rightarrow \mathcal{B}(\mathcal{H}_\mu)$  and  $\pi_\nu: C^*(G) \rightarrow \mathcal{B}(\mathcal{H}_\nu)$  denote the canonical unital homomorphisms induced by  $\mu$  and  $\nu$ , respectively. If  $\mu \prec_Z \nu$ , then  $\ker(\pi_\nu) \subseteq \ker(\pi_\mu)$ .

*Proof.* Let  $x \in \ker(\pi_\nu)$  be a positive contraction. We need to show that  $\pi_\mu(x) = 0$ . Fix  $\varepsilon > 0$  and a unit vector  $\xi$  of  $\mathcal{H}_\mu$ . For  $g \in G$ , we denote by  $u_g \in C^*(G)$  the canonical unitary associated to  $g$ . Find a finite subset  $F \subseteq G$  and scalars  $a_g \in \mathbb{C}$ , for  $g \in F$ , such that  $y = \sum_{g \in F} a_g u_g$  is a contraction with  $\|x - y\| < \varepsilon$ .

Fix  $\delta > 0$  such that  $\sum_{g \in F} |a_g| \delta < \varepsilon$ . Since  $\mu \prec_Z \nu$ , we can find a unit vector  $\eta \in \mathcal{H}_\nu$  such that  $|\langle \mu_g(\xi), \xi \rangle - \langle \nu_g(\eta), \eta \rangle| < \delta$  for  $g \in F$ . Thus,

$$\begin{aligned} \langle \pi_\mu(x)\xi, \xi \rangle &\leq \left| \sum_{g \in F} a_g \langle \mu_g(\xi), \xi \rangle \right| + \varepsilon \\ &\leq \left| \sum_{g \in F} a_g \langle \nu_g(\eta), \eta \rangle \right| + \sum_{g \in F} |a_g| \delta + \varepsilon \\ &= |\langle \pi_\nu(y)\eta, \eta \rangle| + 2\varepsilon \leq \|\pi_\nu(y)\| + 2\varepsilon < 3\varepsilon. \end{aligned}$$

This concludes the proof.  $\square$

**Lemma 2.19.** Let  $G$  be a discrete group admitting a finite-dimensional representation  $\mu$  such that  $\mu \prec_Z \bigoplus_{n \in \mathbb{N}} \lambda_G$ . Then  $G$  is amenable.

*Proof.* Denote by  $\bar{\mu}$  and  $\bar{\lambda}_G$  the conjugate representations of  $\mu$  and  $\lambda_G$ , respectively. Then the assumption implies that  $\mu \otimes \bar{\mu} \prec_Z \bigoplus_{n \in \mathbb{N}} \lambda_G \otimes \bar{\lambda}_G$ . Since  $\mu$  is finite-dimensional, the trivial representation  $1_G$  of  $G$  is contained in  $\mu \otimes \bar{\mu}$ . Therefore  $1_G \prec_Z \bigoplus_{n \in \mathbb{N}} \lambda_G \otimes \bar{\lambda}_G$ . Hence  $G$  is amenable by [1, Theorem 5.1].  $\square$

**2.4. Koopman representations of generalized Bernoulli shifts.** Let  $D$  be a unital, separable  $C^*$ -algebra and let  $\mathcal{U}$  be a nonprincipal ultrafilter over  $\mathbb{N}$  which is fixed throughout. We denote by  $D^\mathcal{U}$  the  $C^*$ -algebra ultrapower of  $D$ , and we identify  $D$  with its image inside  $D^\mathcal{U}$  under the diagonal embedding. With a slight abuse of notation, we denote by  $[a_n]$  the element of  $D^\mathcal{U}$  with representative sequence  $(a_n)_{n \in \mathbb{N}}$ . We let  $D_\mathcal{U} = D' \cap D^\mathcal{U}$  be the relative commutant of  $D$  inside the ultrapower (also called the  $\mathcal{U}$ -central sequence algebra). A state  $\phi$  on  $D$  extends to a state  $\phi^\mathcal{U}$  on  $D^\mathcal{U}$ , and we let  $\phi_\mathcal{U}$  denote its restriction to  $D_\mathcal{U}$ . If  $G$  is a discrete group, and  $\alpha$  is an action of  $G$  on  $D$ , then  $\alpha$  induces actions  $\alpha^\mathcal{U}: G \rightarrow \text{Aut}(D^\mathcal{U})$  and  $\alpha_\mathcal{U}: G \rightarrow \text{Aut}(D_\mathcal{U})$ .

**Remark 2.20.** Adopt the notation from the discussion above. Then there is a canonical linear isometry  $s: \mathcal{H}_{\phi_\mathcal{U}}^{D_\mathcal{U}} \rightarrow (\mathcal{H}_\phi^D)^\mathcal{U}$ , determined by  $s(\iota_{\phi_\mathcal{U}}^{D_\mathcal{U}}([a_n])) = [\iota_\phi^D(a_n)]$ . Moreover, if  $\phi$  is  $\alpha$ -invariant, then  $s$  intertwines  $\kappa_{\phi_\mathcal{U}}^{(0)}(\alpha_\mathcal{U})$  and  $(\kappa_\phi^{(0)}(\alpha))^\mathcal{U}$ . If  $\mathcal{H}_\phi^D$  is separable, then it follows from Proposition 2.16 that  $\kappa_{\phi_\mathcal{U}}^{(0)}(\alpha_\mathcal{U}) \prec_Z \kappa_\phi^{(0)}(\alpha)$ .

Many properties of  $C^*$ -algebras are defined or characterized in terms of realizing certain configurations in the central sequence algebra. Examples of such properties include tensorial absorption of a given strongly self-absorbing  $C^*$ -algebra ([37, Theorem 2.2]) or  $G$ - $C^*$ -algebra ([35, Theorem 3.7]), as well as the Rokhlin property for compact group actions ([18, Definition 3.2], [12, Definition 2.3]), and some of its variations. The property we consider in the following definition is defined in terms of (weakly) realizing a certain configuration *in the GNS representation* of the central sequence algebra with respect to a given invariant state.

**Definition 2.21.** Let  $G$  be a countable group, let  $D$  be a unital, separable  $C^*$ -algebra, let  $\phi$  be a state on  $D$ , and let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$  be a unitary representation. An action  $\alpha: G \rightarrow \text{Aut}(D)$  is said to *commutant weakly contain  $\mu$  with respect to  $\phi$* , if  $\phi_\mathcal{U}$  is  $\alpha_\mathcal{U}$ -invariant and  $\mu \prec_Z \kappa_{\phi_\mathcal{U}}^{(0)}(\alpha_\mathcal{U})$ .

The terminology in Definition 2.21 is chosen for consistency with the notion of *commutant positive weak containment* for actions on  $C^*$ -algebras, which was defined in Section 3.2 of [17]. We briefly recall this notion, and show in Proposition 2.22 how it relates to Definition 2.21. Recall that  $\mathcal{U}$  denotes a fixed nonprincipal ultrafilter on  $\mathbb{N}$ . For a discrete group  $G$  and actions  $\alpha: G \rightarrow \text{Aut}(A)$  and  $\beta: G \rightarrow \text{Aut}(B)$  on

unital  $C^*$ -algebras  $A$  and  $B$ , we say that  $\beta$  is *commutant positively weakly contained in  $\alpha$* , if there is a unital, equivariant, injective homomorphism  $(B, \beta) \rightarrow (A_{\mathcal{U}}, \alpha_{\mathcal{U}})$

**Proposition 2.22.** Let  $G$  be a discrete group, let  $A$  and  $D$  be unital, separable  $C^*$ -algebras, and suppose that  $D$  has a unique trace  $\tau_D$ . Let  $\tau$  be a trace on  $A$  such that  $\tau_{\mathcal{U}}$  is  $\alpha_{\mathcal{U}}$ -invariant, and suppose that  $\beta$  is commutant positively weakly contained in  $\alpha$ . Then  $\alpha$  comutant weakly contains  $\kappa_{\tau_D}^{(0)}(\beta)$  with respect to  $\tau$ .

*Proof.* Let  $\varphi: (D, \beta) \rightarrow (A_{\mathcal{U}}, \alpha_{\mathcal{U}})$  be a unital, equivariant homomorphism. Since  $\tau_D$  is the unique trace on  $D$ , we have  $\tau_D = \tau_{\mathcal{U}} \circ \varphi$ . Therefore,  $\varphi$  induces a linear isometry  $s: \mathcal{H}_{\tau_D}^D \rightarrow \mathcal{H}_{\tau_{\mathcal{U}}}^{A_{\mathcal{U}}}$  which is determined by  $s(\iota_{\tau_D}^D(d)) = \iota_{\tau_{\mathcal{U}}}^{A_{\mathcal{U}}}(\varphi(d))$  for all  $d \in D$ . Since  $\varphi$  is equivariant, it follows that  $s$  intertwines  $\kappa_{\tau_D}(\beta)$  and  $\kappa_{\tau_{\mathcal{U}}}(\alpha_{\mathcal{U}})$ . Since  $s(\iota_{\tau_D}^D(1_D)) = \iota_{\tau_{\mathcal{U}}}^{A_{\mathcal{U}}}(1_A)$ , we deduce that  $\kappa_{\tau_D}^{(0)}(\beta) \subseteq \kappa_{\tau_{\mathcal{U}}}^{(0)}(\alpha_{\mathcal{U}})$ . We conclude that  $\kappa_{\tau_D}^{(0)}(\beta)$  is commutant weakly contained in  $\alpha$  with respect to  $\tau$ .  $\square$

Next, we use generalized Bernoulli shifts to construct examples of actions that commutant weakly contain the representation  $u_{\sigma}$  associated to an action  $G \curvearrowright^{\sigma} X$  as in Definition 2.5.

**Proposition 2.23.** Let  $G$  be a countable group, let  $D$  be a unital, separable  $C^*$ -algebra, and let  $\phi$  be a state on  $D$  that is not a character. Let  $G \curvearrowright^{\sigma} X$  be an action on a countable set  $X$  satisfying  $\sigma \cong \text{id}_{\mathbb{N}} \times \sigma$ . Set  $\tilde{D} = \bigotimes_{n \in \mathbb{N}} D$  and  $\tilde{\phi} = \bigotimes_{n \in \mathbb{N}} \phi$ .

- (1) The action  $\beta_{\sigma, D}: G \rightarrow \text{Aut}(\tilde{D})$  commutant weakly contains  $u_{\sigma}$  with respect to the invariant state  $\tilde{\phi}$ .
- (2) Let  $\alpha: G \rightarrow \text{Aut}(A)$  be any action on a separable, unital  $C^*$ -algebra  $A$ , and let  $\psi$  be an  $\alpha$ -invariant state on  $A$ . Then  $\alpha \otimes \beta_{\sigma, D}$  commutant weakly contains  $u_{\sigma}$  with respect to  $\psi \otimes \tilde{\phi}$ .

*Proof.* (1) Using that  $\sigma \cong \text{id}_{\mathbb{N}} \times \sigma$ , we identify  $\tilde{D}$  with  $\bigotimes_{\mathbb{N} \times X} D$ , and  $\tilde{\phi}$  with  $\bigotimes_{\mathbb{N} \times X} \phi$ , and will show that the action  $\beta_{\text{id}_{\mathbb{N}} \times \sigma, D}$  of  $G$  on  $\tilde{D}$  commutant weakly contains  $u_{\sigma}$  with respect to  $\tilde{\phi}$ . Note that  $\tilde{\phi}$  is  $\beta_{\text{id}_{\mathbb{N}} \times \sigma, D}$ -invariant, and hence  $\tilde{\phi}_{\mathcal{U}}$  is  $(\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}}$ -invariant. By Choi's multiplicative domain theorem (see, for example, Theorem 3.18 in [31]), there exists a positive contraction  $d_0 \in D$  such that  $\phi(d_0^2) > \phi(d_0)^2$ . Set

$$d = \frac{1}{\sqrt{\phi(d_0^2) - \phi(d_0)^2}}(d_0 - \phi(d_0)1_D) \in D,$$

and observe that

$$(2.1) \quad \phi(d) = 0 \quad \text{and} \quad \phi(d^*d) = 1.$$

For  $n \in \mathbb{N}$  and  $x \in X$ , denote by  $j_x^{(n)}: D \rightarrow \tilde{D}$  the canonical embedding into the  $(n, x)$ -th tensor factor, and set  $d_x^{(n)} = j_x^{(n)}(d) \in \tilde{D}$ . Given  $x \in X$ , consider the sequence  $(d_x^{(n)})_{n \in \mathbb{N}}$  in  $\tilde{D}$ , and let  $d_x \in \tilde{D}^{\mathcal{U}}$  denote the induced equivalence class in the sequence algebra.

Fix  $x \in X$ . We claim that  $d_x$  belongs to the relative commutant  $\tilde{D}_{\mathcal{U}} = \tilde{D}' \cap \tilde{D}^{\mathcal{U}}$ . To this end, fix  $y \in X$ ,  $m \in \mathbb{N}$  and  $c \in D$ , and set  $c_y^{(m)} = j_y^{(m)}(c) \in \tilde{D}$ . Then

$$\|c_y^{(m)}d_x - d_x c_y^{(m)}\| = \lim_{n \rightarrow \mathcal{U}} \|c_y^{(m)}d_x^{(n)} - d_x^{(n)}c_y^{(m)}\|.$$

Note that  $d_x^{(n)} = j_x^{(n)}(d)$  and  $c_y^{(m)} = j_y^{(m)}(c)$  are commuting elements of  $\tilde{D}$  whenever  $n > m$ . Since  $\mathcal{U}$  is nonprincipal, we have

$$\left\{ n \in \mathbb{N}: \|c_y^{(m)}d_x^{(n)} - d_x^{(n)}c_y^{(m)}\| = 0 \right\} \in \mathcal{U}$$

and hence

$$\|c_y^{(m)}d_x - d_x c_y^{(m)}\| = \lim_{n \rightarrow \mathcal{U}} \|c_y^{(m)}d_x^{(n)} - d_x^{(n)}c_y^{(m)}\| = 0.$$

It follows that  $d_x$  commutes with  $c_y^{(m)}$ . Since elements of the form  $c_y^{(m)}$ , for  $c \in D$ ,  $y \in X$  and  $m \in \mathbb{N}$ , generate  $\tilde{D}$  as a  $C^*$ -algebra, the claim follows.

Using the notation introduced in Definition 2.1, define a bounded linear map  $s: \ell^2(X) \rightarrow \mathcal{H}_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}$  by setting  $s(\delta_x) = \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(d_x)$  for all  $x \in X$ . We will show that  $s$  implements a containment of  $u_\sigma$  into the reduced Koopman representation of the action on  $\tilde{D}_{\mathcal{U}} = \tilde{D}' \cap \tilde{D}^{\mathcal{U}}$  induced by  $\beta_{\text{id}_{\mathbb{N}} \times \sigma, D}$ , with respect to  $\tilde{\phi}_{\mathcal{U}}$ .

First, we show that  $s$  is an isometry. Given  $x \in X$ , we have

$$\begin{aligned} \langle s(\delta_x), s(\delta_x) \rangle &= \left\langle \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(d_x), \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(d_x) \right\rangle \\ &= \tilde{\phi}_{\mathcal{U}}(d_x^* d_x) \\ &= \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}((d_x^{(n)})^* d_x^{(n)}) \\ &= \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}(j_x^{(n)}(d)^* j_x^{(n)}(d)) \\ &= \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}(j_x^{(n)}(d^* d)) \\ &= \phi(d^* d) \stackrel{(2.1)}{=} 1. \end{aligned}$$

On the other hand, for  $x, y \in X$  with  $x \neq y$ , we have

$$\begin{aligned} \langle s(\delta_x), s(\delta_y) \rangle &= \left\langle \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(d_x), \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(d_y) \right\rangle \\ &= \tilde{\phi}_{\mathcal{U}}(d_x^* d_y) \\ &= \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}((d_x^{(n)})^* d_y^{(n)}) \\ &= \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}(j_x^{(n)}(d)^* j_y^{(n)}(d)) \\ &= \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}(j_x^{(n)}(d^*)) \tilde{\phi}(j_y^{(n)}(d)) \\ &= \phi(d^*) \phi(d) \stackrel{(2.1)}{=} 0. \end{aligned}$$

In particular,  $s$  is an isometry. Moreover, for  $x \in X$  we have

$$\begin{aligned} \left\langle \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(1_{\tilde{D}_{\mathcal{U}}}), s(\delta_x) \right\rangle &= \left\langle \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(1_{\tilde{D}_{\mathcal{U}}}), \iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(d_x) \right\rangle \\ &= \tilde{\phi}_{\mathcal{U}}(d_x) = \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}(d_x^{(n)}) \\ &= \lim_{n \rightarrow \mathcal{U}} \tilde{\phi}(j_x^{(n)}(d)) = \phi(d) \stackrel{(2.1)}{=} 0. \end{aligned}$$

Thus,  $s$  maps  $\ell^2(X)$  to the orthogonal complement of  $\iota_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}(1_{\tilde{D}_{\mathcal{U}}})$  inside  $\mathcal{H}_{\tilde{\phi}_{\mathcal{U}}}^{\tilde{D}_{\mathcal{U}}}$ . We now check equivariance of  $s$ . Fix  $g \in G$  and  $x \in X$ . Using that  $(\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_g \circ j_x^{(n)} = j_{g \cdot x}^{(n)}$ , one readily checks that

$$(2.2) \quad ((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_g)_*(d_x) = d_{g \cdot x}.$$

Using the definition of the Koopman representation  $\kappa_{\tilde{\phi}_{\mathcal{U}}}^{(0)}((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_g)$  from Definition 2.7 at the second step; using the definition of  $s$  at the fourth step; and using

Definition 2.5 at the last step, we get

$$\begin{aligned} \kappa_{\tilde{\phi}\mathcal{U}}^{(0)}((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})_g(s(\delta_x)) &= \kappa_{\tilde{\phi}\mathcal{U}}((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})_g(\iota_{\tilde{\phi}\mathcal{U}}^{\tilde{D}\mathcal{U}}(d_x)) \\ &= \iota_{\tilde{\phi}\mathcal{U}}^{\tilde{D}\mathcal{U}}(((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})_g(d_x)) \\ &\stackrel{(2.2)}{=} \iota_{\tilde{\phi}\mathcal{U}}^{\tilde{D}\mathcal{U}}(d_{g \cdot x}) = s(\delta_{g \cdot x}) = s(u_{\sigma}(\delta_x)). \end{aligned}$$

It follows that  $s$  intertwines  $u_{\sigma}$  and  $\kappa_{\tilde{\phi}\mathcal{U}}^{(0)}((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})$ . We have shown that  $u_{\sigma} \subseteq \kappa_{\tilde{\phi}\mathcal{U}}^{(0)}((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})$ , and in particular  $u_{\sigma} \prec_{\mathbb{Z}} \kappa_{\tilde{\phi}\mathcal{U}}^{(0)}((\beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})$ , as desired.

(2) Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action on a separable, unital C\*-algebra  $A$ , and let  $\psi$  be an  $\alpha$ -invariant state on  $A$ . Since  $\psi \otimes \tilde{\phi}$  is  $\alpha \otimes \beta_{\text{id}_{\mathbb{N}} \times \sigma, D}$ -invariant, the state  $(\psi \otimes \tilde{\phi})_{\mathcal{U}}$  is  $(\alpha \otimes \beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}}$ -invariant.

We keep the notation for the elements  $d_x^{(n)} \in \tilde{D}$ , for  $x \in X$  and  $n \in \mathbb{N}$ , from part (1). For  $x \in X$  and  $n \in \mathbb{N}$ , we set  $a_x^{(n)} = 1_A \otimes d_x^{(n)} \in A \otimes \tilde{D}$ , and let  $a_x \in (A \otimes \tilde{D})^{\mathcal{U}}$  denote the equivalence class determined by the sequence  $(a_x^{(n)})_{n \in \mathbb{N}}$ . The same proof as above shows that  $a_x$  belongs to the commutant  $(A \otimes \tilde{D})_{\mathcal{U}}$  of  $A \otimes \tilde{D}$  inside  $(A \otimes \tilde{D})^{\mathcal{U}}$ .

As above, one defines a bounded linear operator  $t: \ell^2(X) \rightarrow \mathcal{H}_{(\psi \otimes \tilde{\phi})_{\mathcal{U}}}^{(A \otimes \tilde{D})_{\mathcal{U}}}$  by setting  $t(\delta_x) = \iota_{(\psi \otimes \tilde{\phi})_{\mathcal{U}}}^{(A \otimes \tilde{D})_{\mathcal{U}}}(a_x)$  for all  $x \in X$ . Then  $t$  is a linear isometry whose range is contained in the orthogonal complement of  $\iota_{(\psi \otimes \tilde{\phi})_{\mathcal{U}}}^{(A \otimes \tilde{D})_{\mathcal{U}}}(1_{(A \otimes \tilde{D})_{\mathcal{U}}})$ . Furthermore, the same computation as above shows that  $t$  intertwines  $u_{\sigma}$  and the reduced Koopman representation  $\kappa_{(\psi \otimes \tilde{\phi})_{\mathcal{U}}}^{(0)}((\alpha \otimes \beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})$ . This shows that there is a containment  $u_{\sigma} \subseteq \kappa_{(\psi \otimes \tilde{\phi})_{\mathcal{U}}}^{(0)}((\alpha \otimes \beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}})$  and, in particular,

$$u_{\sigma} \prec_{\mathbb{Z}} \kappa_{(\psi \otimes \tilde{\phi})_{\mathcal{U}}}^{(0)}((\alpha \otimes \beta_{\text{id}_{\mathbb{N}} \times \sigma, D})_{\mathcal{U}}).$$

We conclude that  $u_{\sigma}$  is commutant weakly contained in  $\alpha \otimes \beta_{\text{id}_{\mathbb{N}} \times \sigma, D}$  with respect to  $\psi \otimes \tilde{\phi}$ , as desired.  $\square$

The following definition is standard.

**Definition 2.24.** Let  $G$  be a discrete group, and let  $\alpha: G \rightarrow \text{Aut}(A)$  and  $\beta: G \rightarrow \text{Aut}(B)$  be actions of  $G$  on unital C\*-algebras  $A$  and  $B$ . We say that  $\alpha$  and  $\beta$  are *cocycle conjugate* if there exist an isomorphism  $\theta: B \rightarrow A$  and a function  $u: G \rightarrow \mathcal{U}(A)$  satisfying

$$u_{gh} = u_g \alpha_g(u_h) \quad \text{and} \quad \beta_g = \theta^{-1} \circ (\text{Ad}(u_g) \circ \alpha_g) \circ \theta$$

for all  $g, h \in G$ . The function  $u$  is called an  $\alpha$ -cocycle.

Let  $\alpha, \beta: G \rightarrow \text{Aut}(D)$  be actions of a discrete group  $G$  on a unital C\*-algebra  $D$ . If  $\alpha$  and  $\beta$  are cocycle conjugate, there is in general no relationship between  $\kappa(\alpha)$  and  $\kappa(\beta)$ , even if they both preserve the same tracial state. This can be seen, for example, by letting  $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(M_2)$  be the trivial action, and  $\beta: \mathbb{Z}_2 \rightarrow \text{Aut}(M_2)$  be the inner action determined by the order two unitary  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . (In this case, with respect to the unique tracial state,  $\kappa(\alpha)$  is conjugate to  $\bigoplus_{j=1}^4 1_{\mathbb{Z}_2}$ , while  $\kappa(\beta)$  is conjugate to  $\bigoplus_{j=1}^2 \lambda_{\mathbb{Z}_2}$ .) In particular, if  $\kappa(\alpha)$  contains a given unitary representation, we cannot conclude that so does  $\kappa(\beta)$ .



**Lemma 2.25.** Let  $G$  be a discrete group, and let  $\alpha: G \rightarrow \text{Aut}(A)$  and  $\beta: G \rightarrow \text{Aut}(B)$  be actions of  $G$  on unital  $C^*$ -algebras  $A$  and  $B$ . Assume that  $\alpha$  and  $\beta$  are cocycle conjugate, and let  $\theta: B \rightarrow A$  and  $u: G \rightarrow \mathcal{U}(A)$  be as in Definition 2.24. Let  $\phi$  be a state on  $A$ , and set  $\psi = \phi \circ \theta$ .

- (1)  $\alpha_{\mathcal{U}}$  is conjugate to  $\beta_{\mathcal{U}}$  via  $\theta_{\mathcal{U}}$ .
- (2) If  $\phi_{\mathcal{U}}$  is  $\alpha_{\mathcal{U}}$ -invariant, then  $\psi_{\mathcal{U}}$  is  $\beta_{\mathcal{U}}$ -invariant and  $\kappa_{\phi_{\mathcal{U}}}^{(0)}(\alpha_{\mathcal{U}}) \cong \kappa_{\psi_{\mathcal{U}}}^{(0)}(\beta_{\mathcal{U}})$ .
- (3) Let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H})$  be a unitary representation. If  $\alpha$  commutant weakly contains  $\mu$  with respect to  $\phi$ , then  $\beta$  commutant weakly contains  $\mu$  with respect to  $\psi$ .
- (4) If  $\phi$  is tracial and  $\alpha$ -invariant, then  $\psi$  is tracial and  $\beta$ -invariant.

*Proof.* (1) Let  $g \in G$ . Note that  $\text{Ad}(u_g)$  induces the trivial automorphism on  $A_{\mathcal{U}}$ . Then

$$\begin{aligned} \theta_{\mathcal{U}} \circ \beta_{\mathcal{U},g} &= \theta_{\mathcal{U}} \circ (\theta^{-1} \circ \text{Ad}(u_g) \circ \alpha_g \circ \theta)_{\mathcal{U}} \\ &= \theta_{\mathcal{U}} \circ \theta_{\mathcal{U}}^{-1} \circ (\text{Ad}(u_g))_{\mathcal{U}} \circ \alpha_{\mathcal{U},g} \circ \theta_{\mathcal{U}} \\ &= \theta_{\mathcal{U}} \circ \theta_{\mathcal{U}}^{-1} \circ \alpha_{\mathcal{U},g} \circ \theta_{\mathcal{U}} \\ &= \alpha_{\mathcal{U},g} \circ \theta_{\mathcal{U}}, \end{aligned}$$

as desired.

- (2) Since  $\phi_{\mathcal{U}}$  is  $\alpha_{\mathcal{U}}$ -invariant, we have

$$\psi_{\mathcal{U}} \circ \beta_{\mathcal{U},g} = \phi_{\mathcal{U}} \circ \theta_{\mathcal{U}} \circ \beta_{\mathcal{U},g} = \phi_{\mathcal{U}} \circ \alpha_{\mathcal{U},g} \circ \theta_{\mathcal{U}} = \phi_{\mathcal{U}} \circ \theta_{\mathcal{U}} = \psi_{\mathcal{U}}$$

for all  $g \in G$ . Thus,  $\psi_{\mathcal{U}}$  is  $\beta$ -invariant. Moreover, since  $\alpha_{\mathcal{U}}$  and  $\beta_{\mathcal{U}}$  are conjugate via  $\theta_{\mathcal{U}}$  and  $\psi_{\mathcal{U}} \circ \theta_{\mathcal{U}} = \phi_{\mathcal{U}}$ , it follows that  $\theta_{\mathcal{U}}$  induces a unitary operator  $\mathcal{H}_{\psi_{\mathcal{U}}}^{B_{\mathcal{U}}} \rightarrow \mathcal{H}_{\phi_{\mathcal{U}}}^{A_{\mathcal{U}}}$  that intertwines  $\kappa_{\psi_{\mathcal{U}}}^{(0)}(\beta_{\mathcal{U}})$  and  $\kappa_{\phi_{\mathcal{U}}}^{(0)}(\alpha_{\mathcal{U}})$ .

- (3) This follows directly from (2) in view of Definition 2.21.

- (4) If  $\phi$  is tracial and  $\alpha$ -invariant, then clearly  $\psi$  is tracial as well. Furthermore,

$$\psi \circ \beta_g = \phi \circ \text{Ad}(u_g) \circ \alpha_g \circ \theta = \phi \circ \alpha_g \circ \theta = \phi \circ \theta = \psi$$

for all  $g \in G$ . Thus,  $\psi$  is  $\beta$ -invariant.  $\square$

### 3. ACTIONS INDUCED BY FINITE SUBQUOTIENTS

In this section, we specialize the discussion to Bernoulli shifts associated with a particular class of actions on countable sets which are constructed from quasi-regular representations (see Definition 1.1). If  $H$  is a subgroup of a discrete group  $F$ , then the quasiregular representation  $\lambda_{F/H}$  induces a unital homomorphism  $\pi_H: C^*(F) \rightarrow \mathcal{B}(\ell^2(F/H))$ , by universality of  $C^*(F)$ . When  $H$  is normal in  $F$ , then the image of  $\pi_H$  is  $C_r^*(F/H)$ .

**Definition 3.1.** Let  $F$  be a discrete group, let  $\mathcal{P} \subseteq \mathbb{N}$  be a set of pairwise coprime numbers. A family  $\{H_p\}_{p \in \mathcal{P}}$  of subgroups of  $F$  is said to be *separated* if it satisfies the following properties:

- (S.1)  $H_p$  has index  $p$  in  $F$  for every  $p \in \mathcal{P}$ , and
- (S.2) for  $p \in \mathcal{P}$  and  $Q \subseteq \mathcal{P}$ , if  $\bigcap_{q \in Q} \ker(\pi_{H_q}) \subseteq \ker(\pi_{H_p})$ , then  $p \in Q$ .

We say that  $F$  is *separated over*  $\mathcal{P}$  if it contains a separated family of subgroups indexed over  $\mathcal{P}$ . Finally, we say that  $F$  is *infinitely separated* if it is separated over an infinite set  $\mathcal{P}$  of pairwise coprime numbers.

We do not require the subgroups  $H_p$  in the definition above to be normal. Our standard example of an infinitely separated group is  $\mathbb{F}_{\infty}$ , as we show next.

**Lemma 3.2.** The free group  $\mathbb{F}_{\infty}$  is infinitely separated.

*Proof.* Let  $\{x_n : n \in \mathbb{N}\}$  be free generators of  $\mathbb{F}_\infty$ , and let  $\mathcal{P} \subseteq \mathbb{N}$  denote the set of prime numbers. For  $p \in \mathcal{P}$ , let  $H_p$  be the normal subgroup generated by

$$\{x_1, \dots, x_{p-1}, x_p^p, x_{p+1}, \dots\} \subseteq \mathbb{F}_\infty.$$

It is clear that  $\mathbb{F}_\infty/H_p \cong \mathbb{Z}/p\mathbb{Z}$ , so property (S.1) is satisfied.

We proceed to check property (S.2). For  $p \in \mathcal{P}$ , the quotient map

$$\pi_{H_p} : C^*(\mathbb{F}_\infty) \rightarrow C^*(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{C}^p$$

can be described as follows. We identify  $\mathbb{F}_\infty$  with its image inside the unitary group of  $C^*(\mathbb{F}_\infty)$ . We also identify  $C^*(\mathbb{F}_\infty)$  with the full free product  $*_{n=1}^\infty C(S^1)$  amalgamated over  $\mathbb{C}$ , by regarding  $x_n \in \mathbb{F}_\infty$  as the canonical generator of the  $n$ -th free factor  $C(S^1)$ . Given  $f \in C(S^1)$  and  $n \in \mathbb{N}$ , set  $f^{(n)} = 1 * \dots * f * 1 \dots \in *_{n=1}^\infty C(S^1)$ , where the nontrivial entry is in the  $n$ -th position. Then

$$\pi_{H_p}(f^{(n)}) = \begin{cases} (f(1), \dots, f(1)), & \text{if } n \neq p \\ (f(1), f(e^{2\pi i/p}), \dots, f(e^{2\pi i(p-1)/p})) & \text{if } n = p. \end{cases}$$

Now fix  $p \in \mathcal{P}$  and  $Q \subseteq \mathcal{P}$  and suppose that  $p \notin Q$ . Let  $f \in C(S^1)$  be any function satisfying  $f(1) = 0$  and  $f(e^{2\pi i/p}) \neq 0$ . Then  $f^{(p)}$  belongs to  $\ker(\pi_{H_q})$  for all  $q \in Q$ , but not to  $\ker(\pi_{H_p})$ . Thus (S.2) is satisfied as well.  $\square$

**Notation 3.3.** Let  $G$  be a discrete group, let  $F$  be a subgroup, let  $\mathcal{P}$  be a set of pairwise coprime natural numbers, and suppose that  $F$  is separated over  $\mathcal{P}$ , as witnessed by a family  $\{H_p\}_{p \in \mathcal{P}}$  as in Definition 3.1.

Given  $p \in \mathcal{P}$ , we establish the following notations:

- we write  $G_p = G/H_p$  and  $F_p = F/H_p$  for the left coset spaces;
- we write  $G \curvearrowright^{\text{Lt}_{G_p}} G_p$  and  $F \curvearrowright^{\text{Lt}_{F_p}} F_p$  for the canonical left translation actions; see Notation 2.6.

**Definition 3.4.** Let  $G$  be a discrete group, let  $F$  be a subgroup, let  $\mathcal{P}$  be a set of pairwise coprime natural numbers, and suppose that  $F$  is separated over  $\mathcal{P}$ , as witnessed by a family  $\{H_p\}_{p \in \mathcal{P}}$  as in Definition 3.1.

For a (possibly empty) subset  $P \subseteq \mathcal{P}$ , set

$$X_G^P = \bigsqcup_{n \in \mathbb{N}} G \sqcup \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{p \in P} G_p \quad \text{and} \quad X_F^P = \bigsqcup_{n \in \mathbb{N}} F \sqcup \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{p \in P} F_p,$$

and define actions  $G \curvearrowright^{\sigma_G^P} X_G^P$  and  $F \curvearrowright^{\sigma_F^P} X_F^P$  as follows:

- $\sigma_G^P$  acts on each copy of  $G$  via  $\text{Lt}_G$ , and on each copy of  $G_p$  via  $\text{Lt}_{G_p}$ ;
- $\sigma_F^P$  acts on each copy of  $F$  via  $\text{Lt}_F$ , and on each copy of  $F_p$  via  $\text{Lt}_{F_p}$ .

We isolate one example in the definition above, which will be needed in the proof of Theorem 4.4.

**Example 3.5.** Adopt the assumptions and notation from Definition 3.4, and assume that  $G$  is at most countable. Let  $P = \emptyset$  be the empty set. Then  $X_G^\emptyset = \mathbb{N} \times G$  and  $\sigma_G^\emptyset = \text{id}_{\mathbb{N}} \times \text{Lt}_G$ . For a unital, separable  $C^*$ -algebra  $D$ , consider the generalized Bernoulli shifts

$$\beta_{\sigma_G^\emptyset, D} : G \rightarrow \text{Aut}(\otimes_{(n,g) \in \mathbb{N} \times G} D) \quad \text{and} \quad \beta_{\text{Lt}_G, D} : G \rightarrow \text{Aut}(\otimes_{g \in G} D).$$

Then  $\beta_{\sigma_G^\emptyset, D}$  can be naturally identified with  $\otimes_{n \in \mathbb{N}} \beta_{\text{Lt}_G, D}$ . Set  $\tilde{D} = \otimes_{n \in \mathbb{N}} D$ . Then a rearranging of the tensor factors shows that  $\otimes_{n \in \mathbb{N}} \beta_{\text{Lt}_G, D}$  is conjugate to  $\beta_{\text{Lt}_G, \tilde{D}}$ , and thus

$$(3.1) \quad \beta_{\sigma_G^\emptyset, D} \cong \beta_{\text{Lt}_G, \tilde{D}}.$$

For later use, we identify the restriction of  $\sigma_G^P$  to  $F$  with an amplification of  $\sigma_F^P$ .

**Lemma 3.6.** Let the notation be as in Definition 3.4. Then the restriction  $\mathbf{Lt}_G|_F$  of  $\mathbf{Lt}_G$  to  $F$  is conjugate to  $\mathrm{id}_{G/F} \times \mathbf{Lt}_F$ . Similarly, for  $p \in \mathcal{P}$ , the restriction  $\mathbf{Lt}_{G_p}|_F$  is conjugate to  $\mathrm{id}_{G/F} \times \mathbf{Lt}_{F_p}$ .

*Proof.* Choose a section  $t: G/F \rightarrow G$  for the canonical quotient map, that is, a function satisfying  $t_{gF} \in gF$  for all  $g \in G$ . Let  $f: G \rightarrow G/F \times F$  be given by  $f(g) = (g^{-1}F, gt_{g^{-1}F})$  for all  $g \in G$ . Then  $f$  is a bijection.

We claim that  $f$  implements a conjugacy between  $\mathbf{Lt}_G|_F$  and  $\mathrm{id}_{G/F} \times \mathbf{Lt}_F$ . Given  $k \in F$  and  $g \in G$ , we have

$$\begin{aligned} f((\mathbf{Lt}_G)_k(g)) &= f(kg) \\ &= (g^{-1}k^{-1}F, kgt_{g^{-1}k^{-1}F}) \\ &= (g^{-1}F, kgt_{g^{-1}F}) \\ &= (\mathrm{id}_{G/F} \times \mathbf{Lt}_F)_k(f(g)). \end{aligned}$$

This concludes the proof that  $\mathbf{Lt}_G|_F$  is conjugate to  $\mathrm{id}_{G/F} \times \mathbf{Lt}_F$ .

The proof that  $\mathbf{Lt}_{G_p}|_F$  is conjugate to  $\mathrm{id}_{G/F} \times \mathbf{Lt}_{F_p}$  is completely analogous, and is left to the reader.  $\square$

Recall the notation  $u_\sigma$  from Definition 2.5.

**Corollary 3.7.** Let  $G$  be a discrete group, let  $F$  be a subgroup, let  $\mathcal{P}$  be a set of pairwise coprime natural numbers, and suppose that  $F$  is separated over  $\mathcal{P}$ , as witnessed by a family  $\{H_p\}_{p \in \mathcal{P}}$  as in Definition 3.1. Let  $P \subseteq \mathcal{P}$  be a subset, and use the notation introduced in Definition 3.4. Then

$$u_{\sigma_G^P}|_F \cong \bigoplus_{n \in \mathbb{N}} \lambda_F \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{p \in P} \lambda_{F_p}.$$

In particular, if  $P$  is nonempty, then  $1_G \subseteq u_{\sigma_G^P}|_F$ .

*Proof.* The first assertion follows immediately from Lemma 3.6 and part (1) of Lemma 2.12. If  $P$  is nonempty and  $p \in P$ , then  $1_G \subseteq \lambda_{F_p}$  by Lemma 1.3, and hence  $1_G \subseteq u_{\sigma_G^P}|_F$  by the first part.  $\square$

Recall the notation  $\kappa_\sigma^{\mathcal{H}}$  from Notation 2.10.

**Lemma 3.8.** Let  $G$  be a discrete group, let  $F$  be a subgroup, let  $\mathcal{P}$  be a set of pairwise coprime natural numbers, and suppose that  $F$  is separated over  $\mathcal{P}$ , as witnessed by a family  $\{H_p\}_{p \in \mathcal{P}}$  as in Definition 3.1.

Let  $(\mathcal{H}, \eta)$  be a separable Hilbert space with a distinguished unit vector, let  $P \subseteq \mathcal{P}$  be a (possibly empty) subset, and let  $\sigma_G^P$  be as in Definition 3.4. Following Notation 2.10, we abbreviate  $\kappa_{\sigma_G^P}^{\mathcal{H}}$  to simply  $\kappa_{\sigma_G^P}$ . Then

$$\kappa_{\sigma_G^P}^{(0)}|_F \subseteq u_{\sigma_G^P}|_F.$$

*Proof.* We begin by providing an alternative description of the restriction  $\kappa_{\sigma_G^P}^{(0)}|_F$  of  $\kappa_{\sigma_G^P}^{(0)}$  to  $F$ . Find an orthonormal basis  $\{\eta_n : n \in \mathbb{N}\}$  of  $\mathcal{H}$  with  $\eta_0 = \eta$ , and set

$$\mathcal{F} = \{\xi : X_G^P \rightarrow \mathbb{N} : \{x \in X_G^P : \xi(x) \neq 0\} \text{ is finite}\}.$$

In particular,  $\mathcal{F}$  contains the function  $\xi_0$  which is constantly equal to 0. By Lemma 2.3, we can identify  $\mathcal{F}$  with an orthonormal basis for  $\bigotimes_{x \in X_G^P} \mathcal{H}$ . Moreover, such an orthonormal basis of  $\bigotimes_{x \in X_G^P} \mathcal{H}$  is invariant under the unitary representation  $\kappa_{\sigma_G^P}|_F$ . Thus, the unitary representation  $\kappa_{\sigma_G^P}|_F$  induces a (set-theoretic) action of  $F$  on the countable set  $\mathcal{F}$ , which is easily seen to be given by

$$(\kappa_{\sigma_G^P})_g(\xi) = \xi \circ (\sigma_G^P)_{g^{-1}}$$

for all  $g \in F$  and all  $\xi \in \mathcal{F}$ . Observe that  $\xi_0$  is fixed by this action.

We set  $\mathcal{F}_0 = \mathcal{F} \setminus \{\xi_0\}$  and  $\mathcal{H}^{(0)} = \overline{\text{span}} \mathcal{F}_0$ . Then  $\mathcal{H}^{(0)}$  is the orthogonal complement of  $\xi_0$  in  $\bigotimes_{x \in X_G^P} \mathcal{H}$ , and  $\mathcal{F}_0$  is an orthonormal basis for  $\mathcal{H}^{(0)}$ . Moreover, the set  $\mathcal{F}_0$  (and hence also the subspace  $\mathcal{H}^{(0)}$ ) is invariant under  $\kappa_{\sigma_G^P}|_F$ , and the restriction of  $\kappa_{\sigma_G^P}|_F$  to  $\mathcal{H}^{(0)}$  is  $\kappa_{\sigma_G^P}^{(0)}|_F$ .

For  $\xi \in \mathcal{F}_0$ , we let  $[\xi]$  denote its  $F$ -orbit

$$[\xi] = \{(\kappa_{\sigma_G^P})_g(\xi) : g \in F\}.$$

We denote by  $\mathcal{G}_0$  the  $F$ -orbit space  $\{[\xi] : \xi \in \mathcal{F}_0\}$  of the action  $\kappa_{\sigma_G^P}|_F$  of  $F$  on  $\mathcal{F}_0$ . For  $\xi \in \mathcal{F}_0$ , we write  $\text{Stab}_F(\xi)$  for the stabilizer subgroup  $\{g \in F : (\kappa_{\sigma_G^P})_g(\xi) = \xi\}$  of  $F$ . Since each  $F$ -orbit  $[\xi]$  is  $F$ -invariant and  $F$  acts transitively on  $[\xi]$ , it is easy to see that

$$(3.2) \quad \kappa_{\sigma_G^P}^{(0)}|_F \cong \bigoplus_{[\xi] \in \mathcal{G}_0} \lambda_{F/\text{Stab}_F(\xi)}.$$

Fix  $\xi \in \mathcal{F}_0$ . Then  $\text{supp}(\xi) = \{x \in X_G^P : \xi(x) \neq 0\}$  is finite and nonempty.

**Claim:** *If  $\text{supp}(\xi)$  meets one of the copies of  $G$  in  $X_G^P$ , then  $\lambda_{F/\text{Stab}_F(\xi)}$  is unitarily contained in  $\lambda_F$ .*

*Proof of the claim.* Let  $h_1, \dots, h_n$  be the elements of such a copy of  $G$  that belong to  $\text{supp}(\xi)$ . Note that if  $g \in \text{Stab}_F(\xi)$ , then  $\text{supp}(\xi)$  is invariant under  $(\sigma_G^P)_g$ . In particular, this implies that  $gh_1 \in \{h_1, \dots, h_n\}$  and hence there exists  $j \in \{1, \dots, n\}$  with  $g = h_j h_1^{-1}$ . This shows that  $\text{Stab}_F(\xi)$  is finite. (In fact, the same argument shows that  $\text{Stab}_G(\xi)$  is finite.) Thus  $\lambda_{F/\text{Stab}_F(\xi)} \subseteq \lambda_F$  by Lemma 1.3, as desired.

**Claim:** *If  $\text{supp}(\xi)$  does not meet any of the copies of  $G$  in  $X_G^P$ , then  $\lambda_{F/\text{Stab}_F(\xi)}$  is unitarily contained in  $\bigoplus_{p \in \mathcal{P}} \lambda_{F_p}$ .*

*Proof of the claim.* Since  $\text{supp}(\xi)$  is finite, the set

$$\mathcal{P}_\xi = \{p \in \mathcal{P} : \text{supp}(\xi) \text{ meets some copy of } G_p \text{ inside } X_G^P\}$$

is also finite. Let  $p_1, \dots, p_n$  be an enumeration of  $\mathcal{P}_\xi$ . Set  $H = H_{p_1} \cap \dots \cap H_{p_n}$ . Since  $H_p$  is a subgroup of  $F$  for every  $p \in \mathcal{P}$ , we deduce that  $H \subseteq F$ . We proceed to show that  $H$  is contained in  $\text{Stab}_F(\xi)$ . For this, let  $g \in H$ . We need to show that  $(\kappa_{\sigma_G^P})_g(\xi) = \xi$  or, equivalently, that  $\xi \circ (\sigma_G^P)_{g^{-1}} = \xi$ . Fix  $x \in X_G^P$ .

If  $x$  belongs to some copy of  $G$ , then the same is true for  $(\sigma_G^P)_{g^{-1}}(x)$ . In this case,  $\xi((\sigma_G^P)_{g^{-1}}(x)) = \xi(x) = 0$ , since  $\text{supp}(\xi)$  does not meet any copy of  $G$  in  $X_G^P$ .

If  $x$  belongs to some copy of  $G_q$ , for  $q \in \mathcal{P}$ , then the same is true for  $(\sigma_G^P)_{g^{-1}}(x)$ . If  $G_q$  does not meet  $\text{supp}(\xi)$ , then  $\xi((\sigma_G^P)_{g^{-1}}(x)) = \xi(x) = 0$ . Finally, suppose that  $G_q$  does meet  $\text{supp}(\xi)$ . Then  $q$  belongs to  $\mathcal{P}_\xi$ , and hence  $H \subseteq H_q$ . Therefore  $(\sigma_G^P)_{g^{-1}}(x) = x$  since  $x \in G_q = G/H_q$ ,  $g \in H_q$ , and  $\sigma_G^P$  restricted to  $G/H_q$  is the canonical left translation action. Since  $(\sigma_G^P)_{g^{-1}}(x) = x$ , we also have  $\xi((\sigma_G^P)_{g^{-1}}(x)) = \xi(x)$ . This concludes the proof that  $\xi \circ (\sigma_G^P)_{g^{-1}} = \xi$ , and hence  $(\kappa_{\sigma_G^P})_g(\xi) = \xi$ . We deduce that  $H \subseteq \text{Stab}_F(\xi)$ .

Since  $H_{p_1}, \dots, H_{p_n}$  have finite index in  $G$ , the same is true for  $H$  by Lemma 1.2. Applying Lemma 1.3 with  $S = \text{Stab}_F(\xi)$  at the first step; and applying part (2) of Lemma 1.2 at the second step together with property (S.1) in Definition 3.1, we get

$$\lambda_{F/\text{Stab}_F(\xi)} \subseteq \lambda_{F/H} \cong \lambda_{F_{p_1}} \oplus \dots \oplus \lambda_{F_{p_n}} \subseteq \bigoplus_{p \in \mathcal{P}} \lambda_{F_p}.$$

This concludes the proof of the claim.

As a consequence of the previous two claims, we have

$$(3.3) \quad \bigoplus_{[\xi] \in \mathcal{G}_0} \lambda_{F/\text{Stab}_F(\xi)} \subseteq \bigoplus_{n \in \mathbb{N}} \lambda_F \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{p \in P} \lambda_{F_p}.$$

Finally, using Corollary 3.7 at the third step, we conclude that

$$\kappa_{\sigma_G^P}^{(0)}|_F \stackrel{(3.2)}{\cong} \bigoplus_{[\xi] \in \mathcal{G}_0} \lambda_{F/\text{Stab}_F(\xi)} \stackrel{(3.3)}{\subseteq} \bigoplus_{n \in \mathbb{N}} \lambda_F \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{p \in P} \lambda_{F_p} \cong u_{\sigma_G^P}|_F. \quad \square$$

Note that only condition (S.1) from Definition 3.1 was used in the preceding proof. On the other hand, condition (S.2) will be crucial in the proof of Lemma 3.10.

We define the following order on representations, in terms of weak containment.

**Definition 3.9.** Let  $G$  be a discrete group, and let  $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$  and  $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$  be unitary representations. We set  $\mu \prec_{\text{fin}} \nu$  if for every finite-dimensional irreducible subrepresentation  $\pi$  of  $\mu$ , one has that  $\pi \prec_Z \nu$ . Moreover, we set  $\mu \sim_{\text{fin}} \nu$  if  $\mu \prec_{\text{fin}} \nu$  and  $\nu \prec_{\text{fin}} \mu$ .

**Lemma 3.10.** Let  $G$  be a discrete group, let  $F$  be a nonamenable subgroup, let  $\mathcal{P}$  be a set of pairwise coprime natural numbers, and suppose that  $F$  is separated over  $\mathcal{P}$ , as witnessed by a family  $\{H_p\}_{p \in \mathcal{P}}$  as in Definition 3.1. Let  $P, Q \subseteq \mathcal{P}$  satisfy  $u_{\sigma_G^P}|_F \prec_{\text{fin}} u_{\sigma_G^Q}|_F$ . Then  $P \subseteq Q$ .

*Proof.* By Corollary 3.7, there are unitary equivalences

$$u_{\sigma_G^P}|_F \cong \bigoplus_{n \in \mathbb{N}} \lambda_F \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{p \in P} \lambda_{F_p} \quad \text{and} \quad u_{\sigma_G^Q}|_F \cong \bigoplus_{n \in \mathbb{N}} \lambda_F \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \lambda_{F_q}.$$

Fix  $p \in P$ . We will show that  $p \in Q$ . If  $\mu \subseteq \lambda_{F_p}$  is irreducible and (automatically) finite-dimensional, then  $\mu \prec_Z u_{\sigma_G^Q}|_F$ . Since  $F$  is nonamenable and  $\mu$  is finite-dimensional, by Lemma 2.19 we cannot have  $\mu \prec_Z \bigoplus_{n \in \mathbb{N}} \lambda_F$ . Therefore  $\mu \prec_Z \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \lambda_{F_q}$  by Proposition 2.17. Since this applies to every irreducible subrepresentation of  $\lambda_{F_p}$ , and  $\lambda_{F_p}$  (being finite-dimensional) is equivalent to the direct sum of its irreducible subrepresentations, it follows that

$$(3.4) \quad \lambda_{F_p} \prec_Z \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \lambda_{F_q}.$$

As in Definition 3.1, we let  $\pi_{H_p}: C^*(F) \rightarrow \mathcal{B}(\ell^2(F_p))$  be the unital homomorphism induced by the representation  $\lambda_{F_p}$  of  $F$ . Similarly, we let  $\pi_{H_q}: C^*(F) \rightarrow \mathcal{B}(\ell^2(F_q))$ , for  $q \in Q$ , be the unital homomorphism induced by the representation  $\lambda_{F_q}$  of  $F$ . Combining Lemma 2.18 with (3.4) at the second step, we get

$$\bigcap_{q \in Q} \ker(\pi_{H_q}) = \ker \left( \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \pi_{H_q} \right) \subseteq \ker(\pi_{H_p}).$$

By property (S.2) in Definition 3.1, we conclude that  $p \in Q$ . Since  $p \in P$  is arbitrary, this shows that  $P \subseteq Q$ .  $\square$

**Corollary 3.11.** Let the assumptions be as in the preceding lemma. For subsets  $P, Q \subseteq \mathcal{P}$ , the following assertions are equivalent:

- (1)  $P = Q$ ;
- (2)  $u_{\sigma_G^P} \cong u_{\sigma_G^Q}$ ;
- (3)  $u_{\sigma_G^P}|_F \sim_Z u_{\sigma_G^Q}|_F$ ;
- (4)  $u_{\sigma_G^P}|_F \sim_{\text{fin}} u_{\sigma_G^Q}|_F$ .

*Proof.* The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) are immediate, while the implication (4)  $\Rightarrow$  (1) is a consequence of Lemma 3.10.  $\square$

The following lemma is well known; see for example [28, Lemma 1.1]. Recall that we denote by  $\otimes$  the minimal tensor product of  $C^*$ -algebras.

**Lemma 3.12.** Let  $A$  and  $D$  be unital  $C^*$ -algebras, and let  $\tau$  be a tracial state on  $A \otimes D$ . Let  $\tau_D$  be the tracial state on  $D$  given by  $\tau_D(d) = \tau(1 \otimes d)$  for  $d \in D$ , and define  $\tau_A$  similarly. If  $\tau_D$  is an extreme tracial state of  $D$ , then  $\tau = \tau_A \otimes \tau_D$ .

*Proof.* Fix a positive contraction  $a \in A$ . It suffices to show that  $\tau(a \otimes d) = \tau_A(a)\tau_D(d)$  for all  $d \in D$ .

If  $\tau(a \otimes 1) = 0$  then  $\tau(a \otimes d) = 0$  for every  $d \in D$ , since  $a \otimes d \leq a \otimes (\|d\|1)$  and  $\tau$  is order-preserving. We deduce that

$$\tau(a \otimes d) = 0 = \tau(a \otimes 1)\tau(1 \otimes d) = \tau_A(a)\tau_D(d)$$

for every  $d \in D$ , as desired.

Assume now that  $\tau(a \otimes 1) > 0$ . Define  $f_0, f_1: D \rightarrow \mathbb{C}$  by

$$f_0(d) = \tau(a \otimes d) \quad \text{and} \quad f_1(d) = \tau((1 - a) \otimes d).$$

for all  $d \in D$ . Then  $f_0$  and  $f_1$  are positive tracial linear functionals on  $D$  satisfying  $\tau_D = f_0 + f_1$ . Since  $\tau_D$  is an extreme tracial state on  $D$  and  $f_0$  is not zero, we must have  $\tau_D = f_0/f_0(1)$ . Therefore

$$\tau_A(a)\tau_D(d) = f_0(1)\tau_D(d) = f_0(d) = \tau(a \otimes d),$$

for all  $d \in D$ , as desired. This finishes the proof  $\square$

In the following lemma, we will use the conventions introduced in Notation 2.10. We denote by  $1_G^\infty$  the trivial representation of  $G$  on  $\ell^2(\mathbb{N})$ .

**Lemma 3.13.** Let  $G$  be a countable discrete group, let  $A$  and  $D$  be tracial, separable, unital  $C^*$ -algebras, and assume that  $D$  has a unique tracial state  $\tau_D$  which is not a character. Let  $G \curvearrowright^\sigma X$  and  $G \curvearrowright^\rho Y$  be actions of  $G$  on countable discrete spaces  $X$  and  $Y$ . Suppose that the actions  $\text{id}_A \otimes \beta_{\sigma,D}$  and  $\text{id}_A \otimes \beta_{\rho,D}$  are cocycle conjugate. Then  $u_\sigma \prec_Z 1_G^\infty \otimes \kappa_\rho^{\mathcal{H}_D^D}$ .

*Proof.* Set  $\tilde{D} = \bigotimes_{n \in \mathbb{N}} D$ . Let  $\theta: A \otimes \tilde{D} \rightarrow A \otimes \tilde{D}$  be an isomorphism and let  $u: G \rightarrow \mathcal{U}(A \otimes \tilde{D})$  be a cocycle for  $\text{id}_A \otimes \beta_{\sigma,D}$  satisfying

$$\text{Ad}(u_g) \circ (\text{id}_A \otimes \beta_{\sigma,D}) = \theta \circ (\text{id}_A \otimes \beta_{\rho,D}) \circ \theta^{-1}$$

for all  $g \in G$ . Let  $\tau$  be a tracial state on  $A$ , and set  $\tilde{\tau}_D = \bigotimes_{n \in \mathbb{N}} \tau_D$ . Use Lemma 3.12 to find a tracial state  $\tau'$  on  $A$  such that  $(\tau \otimes \tilde{\tau}_D) \circ \theta = \tau' \otimes \tilde{\tau}_D$ . Note that

$$(3.5) \quad \kappa_{\tau'}(\text{id}_A) \subseteq 1_G^\infty \quad \text{and} \quad \kappa_{\tilde{\tau}_D}(\beta_{\rho,D}) \cong \kappa_\rho^{\mathcal{H}_D^D},$$

where the second of these follows from part (1) of Remark 2.11.

By Proposition 2.23, the action  $\text{id}_A \otimes \beta_{\sigma,D}$  commutant weakly contains  $u_\sigma$  with respect to  $\tilde{\tau}_D \otimes \tau$ . Thus Lemma 2.25 implies that  $\text{id}_A \otimes \beta_{\rho,D}$  commutant weakly contains  $u_\sigma$  with respect to  $\tau' \otimes \tilde{\tau}_D$ , that is,

$$(3.6) \quad u_\sigma \prec_Z \kappa_{(\tau' \otimes \tilde{\tau}_D)_u}^{(0)}((\text{id}_A \otimes \beta_{\rho,D})u).$$

In the following, we use Remark 2.20 for  $\text{id}_A \otimes \beta_{\rho,D}$  at the second step; and Lemma 2.9 at the fourth step, to get the desired weak containment:

$$\begin{aligned}
u_\sigma &\stackrel{(3.6)}{\prec_Z} \kappa_{(\tau' \otimes \tilde{\tau}_D)\mathcal{U}}^{(0)}((\text{id}_A \otimes \beta_{\rho,D})\mathcal{U}) \\
&\prec_Z \kappa_{\tau' \otimes \tilde{\tau}_D}^{(0)}(\text{id}_A \otimes \beta_{\rho,D}) \\
&\subseteq \kappa_{\tau' \otimes \tilde{\tau}_D}(\text{id}_A \otimes \beta_{\rho,D}) \\
&\cong \kappa_{\tau'}(\text{id}_A) \otimes \kappa_{\tilde{\tau}_D}(\beta_{\rho,D}) \\
&\stackrel{(3.5)}{\subseteq} 1_G^\infty \otimes \kappa_\rho^{\mathcal{H}_{\tau_D}^D}.
\end{aligned}$$

□

**Theorem 3.14.** Let  $A$  and  $D$  be tracial, separable, unital  $C^*$ -algebras, and assume that  $D$  has a unique tracial state which is not a character. Let  $G$  be a discrete group containing a nonamenable subgroup  $F$  which is separated over some set  $\mathcal{P} \subseteq \mathbb{N}$ , as witnessed by a family  $\{H_p : p \in \mathcal{P}\}$ . For non-empty  $P, Q \subseteq \mathcal{P}$ , the following are equivalent:

- (1)  $P = Q$ .
- (2)  $\text{id}_A \otimes \beta_{\sigma_G^P, D}$  is conjugate to  $\text{id}_A \otimes \beta_{\sigma_G^Q, D}$ .
- (3)  $\text{id}_A \otimes \beta_{\sigma_G^P, D}$  is cocycle conjugate to  $\text{id}_A \otimes \beta_{\sigma_G^Q, D}$ .

*Proof.* The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) are trivial. We now prove (3)  $\Rightarrow$  (1).

Assume that  $\beta_{\sigma_G^P, D} \otimes \text{id}_A$  is cocycle conjugate to  $\beta_{\sigma_G^Q, D} \otimes \text{id}_A$ . We abbreviate  $\kappa_{\sigma_G^Q}^{\mathcal{H}_{\tau_D}^D}$  to  $\kappa_{\sigma_G^Q, D}$ , and similarly for its reduced version. Then  $u_{\sigma_G^P} \prec_Z 1_G^\infty \otimes \kappa_{\sigma_G^Q, D}$  by Lemma 3.13. Using this at the first step, we get

$$(3.7) \quad u_{\sigma_G^P}|_F \prec_Z (1_G^\infty \otimes \kappa_{\sigma_G^Q, D})|_F \cong 1_F^\infty \otimes \kappa_{\sigma_G^Q, D}|_F.$$

On the other hand, using Remark 2.8 at the first step, and using Lemma 3.8 at the second step, we have

$$(3.8) \quad \kappa_{\sigma_G^Q, D}|_F \cong \kappa_{\sigma_G^Q, D}^{(0)}|_F \oplus 1_F \subseteq u_{\sigma_G^Q}|_F \oplus 1_F.$$

Since  $\sigma_G^Q \cong \text{id}_\mathbb{N} \times \sigma_G^Q$ , we have  $u_{\sigma_G^Q} \cong 1_G^\infty \otimes u_{\sigma_G^Q}$  by Lemma 2.12. In particular,

$$(3.9) \quad 1_F^\infty \otimes u_{\sigma_G^Q}|_F \cong u_{\sigma_G^Q}|_F.$$

Combining these facts, we get

$$(3.10) \quad u_{\sigma_G^P}|_F \stackrel{(3.7)}{\prec_Z} 1_F^\infty \otimes \kappa_{\sigma_G^Q, D}|_F \stackrel{(3.8)}{\subseteq} 1_F^\infty \otimes (u_{\sigma_G^Q}|_F \oplus 1_F) \stackrel{(3.9)}{\cong} u_{\sigma_G^Q}|_F \oplus 1_F^\infty.$$

We claim that  $u_{\sigma_G^P}|_F \prec_{\text{fin}} u_{\sigma_G^Q}|_F$ ; see Definition 3.9. Let  $\mu$  be an irreducible finite-dimensional subrepresentation of  $u_{\sigma_G^P}|_F$ ; we need to show that  $\mu \prec_Z u_{\sigma_G^Q}|_F$ . Suppose first that  $\mu = 1_F$ . Using that  $Q$  is not empty, choose some  $q \in Q$ . Using Corollary 3.7 at the last step, we get

$$\mu = 1_F \subseteq \lambda_{F/H_q} \subseteq u_{\sigma_G^Q}|_F,$$

as desired. Suppose now that  $\mu$  is not the trivial representation of  $F$ . Then

$$\mu \subseteq u_{\sigma_G^P}|_F \stackrel{(3.10)}{\prec_Z} u_{\sigma_G^Q}|_F \oplus 1_F^\infty.$$

Since  $\mu$  is irreducible, by Proposition 2.17 we have  $\mu \prec_Z u_{\sigma_G^Q}|_F$  or  $\mu \prec_Z 1_F^\infty$ . Since  $\mu$  is not trivial, we must have  $\mu \prec_Z u_{\sigma_G^Q}|_F$ . Since this holds for every irreducible finite-dimensional subrepresentation of  $u_{\sigma_G^P}|_F$ , we conclude that  $u_{\sigma_G^P}|_F \prec_{\text{fin}} u_{\sigma_G^Q}|_F$ .

Reversing the roles of  $P$  and  $Q$ , one shows that  $u_{\sigma_G^Q}|_F \prec_{\text{fin}} u_{\sigma_G^P}|_F$ , and hence  $u_{\sigma_G^Q}|_F \sim_{\text{fin}} u_{\sigma_G^P}|_F$ . By Corollary 3.11 this implies that  $P = Q$ . □

## 4. MAIN RESULTS

In this section, we prove Theorem B and Theorem C from the introduction, which make significant contributions to part (2) of Conjecture A.

The following definition, which is standard by now, originates from the work of Kishimoto [25], and in this form can be found, among others, in the work of Matui-Sato [26].

**Definition 4.1.** Let  $A$  be a tracial  $C^*$ -algebra, and let  $\theta \in \text{Aut}(A)$ . We say that  $\theta$  is *strongly outer* if for every tracial state  $\tau$  on  $A$  satisfying  $\tau \circ \theta = \tau$ , the weak extension  $\bar{\theta}^\tau \in \text{Aut}(\bar{A}^\tau)$  is outer. An action  $\alpha: G \rightarrow \text{Aut}(A)$  of a discrete group  $G$  is said to be *strongly outer* if  $\alpha_g \in \text{Aut}(A)$  is strongly outer for every  $g \in G \setminus \{1\}$ .

We also recall the following definitions from [24].

**Definition 4.2.** Let  $G$  be an infinite, countable group, let  $A$  be a unital  $C^*$ -algebra with a unique tracial state  $\tau$ , and let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action. Then  $\alpha$  is said to be *weak mixing* if for every finite subset  $F \subseteq A$  and  $\varepsilon > 0$  there exists  $g \in G$  such that  $|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon$  for all  $a, b \in F$ .

The following fact is well-known and easy to see.

**Lemma 4.3.** Let  $G$  be an infinite, countable group, let  $D$  be a unital  $C^*$ -algebra, and let  $G \curvearrowright^\sigma X$  be an action on a countable set  $X$  with a infinite orbits. Denote by  $\beta_{\sigma,D}$  the corresponding generalized Bernoulli shift. Then:

- (a)  $\beta_{\sigma,D}$  is strongly outer. More generally, if  $A$  is any  $C^*$ -algebra and  $\alpha: G \rightarrow \text{Aut}(A)$  is any action, then  $\alpha \otimes \beta_{\sigma,D}$  is strongly outer.
- (b) If  $D$  has a unique tracial state, then  $\beta_{\sigma,D}$  is weak mixing.

Recall from [37, 38] that any strongly self-absorbing  $C^*$ -algebra  $\mathcal{D}$  is simple, nuclear,  $\mathcal{Z}$ -stable, and satisfies  $\mathcal{D} \cong \bigotimes_{n \in \mathbb{N}} \mathcal{D}$ . Moreover,  $\mathcal{D}$  is finite if and only if it has a unique tracial state. Moreover, for a countable group  $G$ , the Bernoulli shift  $\beta_{\text{Lt}_G, \mathcal{D}}$  (see Definition 2.5) is commonly abbreviated to  $\beta_{\mathcal{D}}$ .

We are now ready to prove Theorem B.

**Theorem 4.4.** Let  $G$  be a countable group, and let  $\mathcal{D}$  be a finite strongly self-absorbing  $C^*$ -algebra. Then the following are equivalent:

- (1)  $G$  is amenable;
- (2) The Bernoulli shift  $\beta_{\mathcal{D}}: G \curvearrowright \bigotimes_{g \in G} \mathcal{D}$  is cocycle conjugate to  $\beta_{\mathcal{D}} \otimes \text{id}_{\mathcal{Z}}$ .

*Proof.* That (1) implies (2) is a consequence of Corollary 4.8 in [13] (and it can also be deduced from the proof of Theorem 1.1 in [34]). We prove the converse. For clarity, we write  $\beta_{\text{Lt}_G, \mathcal{D}}$  in place of  $\beta_{\mathcal{D}}$ . Let  $\tau_{\mathcal{D}}$  and  $\tau_{\mathcal{Z}}$  be the unique tracial states of  $\mathcal{D}$  and  $\mathcal{Z}$ , respectively. Being unique, they are invariant by any group action on the respective algebras. Assume that  $\beta_{\text{Lt}_G, \mathcal{D}} \otimes \text{id}_{\mathcal{Z}}$  and  $\beta_{\text{Lt}_G, \mathcal{D}}$  are cocycle conjugate; we will prove that  $G$  is amenable.

Let  $G \curvearrowright^{\text{id}_{\mathbb{N}}} \mathbb{N}$  be the trivial action. We identify  $\mathcal{Z}$  with  $\bigotimes_{n \in \mathbb{N}} \mathcal{Z}$  throughout, so that  $\tau_{\mathcal{D}}$  is identified with  $\bigotimes_{n \in \mathbb{N}} \tau_{\mathcal{D}}$  and  $\text{id}_{\mathcal{Z}}$  is identified with  $\beta_{\text{id}_{\mathbb{N}}, \mathcal{Z}} = \text{id}_{\bigotimes_{n \in \mathbb{N}} \mathcal{Z}}$ .

Observe that  $u_{\text{id}_{\mathbb{N}}}$  is the trivial representation of  $G$  on  $\ell^2(\mathbb{N})$ , so in particular  $u_{\text{id}_{\mathbb{N}}}$  contains the 1-dimensional trivial representation  $1_G$ . Applying part (2) of Proposition 2.23 with  $D = \mathcal{Z}$  and  $A = \mathcal{D}$ , we deduce that  $\beta_{\text{Lt}_G, \mathcal{D}} \otimes \text{id}_{\mathcal{Z}} \cong \beta_{\text{Lt}_G, \mathcal{D}} \otimes \beta_{\text{id}_{\mathbb{N}}, \mathcal{Z}}$  commutant weakly contains  $u_{\text{id}_{\mathbb{N}}}$  with respect to  $\tau_{\mathcal{D}} \otimes \tau_{\mathcal{Z}}$ . Thus

$$1_G \subseteq u_{\text{id}_{\mathbb{N}}} \prec_{\mathcal{Z}}^{(0)} \kappa_{(\tau_{\mathcal{D}} \otimes \tau_{\mathcal{Z}})u}((\beta_{\text{Lt}_G, \mathcal{D}} \otimes \text{id}_{\mathcal{Z}})u).$$

Since  $\mathcal{D}$  has a unique tracial state, and since  $\beta_{\text{Lt}_G, \mathcal{D}}$  is assumed to be cocycle conjugate to  $\beta_{\text{Lt}_G, \mathcal{D}} \otimes \text{id}_{\mathcal{Z}}$ , it follows from part (3) of Lemma 2.25 (applied to



$\mu = 1_G$ ) that  $\beta_{\text{Lt}_G, \mathcal{D}}$  commutant weakly contains  $1_G$  with respect to  $\tau_{\mathcal{D}}$ . Using this at the first step, and using Remark 2.20 at the second step, we deduce that

$$(4.1) \quad 1_G \prec_Z \kappa_{(\tau_{\mathcal{D}})_U}^{(0)}((\beta_{\text{Lt}_G, \mathcal{D}})_U) \prec_Z \kappa_{\tau_{\mathcal{D}}}^{(0)}(\beta_{\text{Lt}_G, \mathcal{D}}).$$

Using Example 3.5, and particularly (3.1), there is a conjugacy of actions

$$(4.2) \quad \beta_{\text{Lt}_G, \mathcal{D}} \cong \beta_{\sigma_G^\emptyset, \mathcal{D}}.$$

In the notation of Lemma 3.8, we take  $G = F$  and  $\mathcal{P} = P = \emptyset$ , so that  $\sigma_G^P = \sigma_G^\emptyset$  is just  $\text{Lt}_G \times \text{id}_{\mathbb{N}}$ . Using part (1) of Lemma 2.12 at the second step, and the comments in Notation 2.6 at the third step, we get

$$(4.3) \quad u_{\sigma_G^\emptyset} = u_{\text{Lt}_G \times \text{id}_{\mathbb{N}}} \cong \bigoplus_{n \in \mathbb{N}} u_{\text{Lt}_G} \cong \bigoplus_{n \in \mathbb{N}} \lambda_G.$$

Using part (2) of Remark 2.11 at the second step, and using Lemma 3.8 at the third step, we get

$$(4.4) \quad \kappa_{\tau_{\mathcal{D}}}^{(0)}(\beta_{\text{Lt}_G, \mathcal{D}}) \stackrel{(4.2)}{\cong} \kappa_{\tau_{\mathcal{D}}}^{(0)}(\beta_{\sigma_G^\emptyset, \mathcal{D}}) \cong \kappa_{\sigma_G^\emptyset}^{(0)} \subseteq u_{\sigma_G^\emptyset} \stackrel{(4.3)}{\cong} \bigoplus_{n \in \mathbb{N}} \lambda_G.$$

Combining (4.1) and (4.4), we conclude that  $1_G \prec_Z \bigoplus_{n \in \mathbb{N}} \lambda_G$ . This implies that  $G$  is amenable by Lemma 2.19, as desired.  $\square$

The theorem above complements the results in [34] and [13]: while every strongly outer action of an amenable group on a finite strongly self-absorbing  $C^*$ -algebra absorbs the identity on  $\mathcal{Z}$  tensorially, this result fails for every nonamenable group. In particular, we deduce a weak version of part (2) of Conjecture A: any non-amenable group admits at least two strongly outer actions on  $\mathcal{D}$  which are not cocycle conjugate, namely, the Bernoulli shift  $\beta_{\mathcal{D}}$  and  $\beta_{\mathcal{D}} \otimes \text{id}_{\mathcal{Z}}$ .

Our strongest result concerns groups that contain a non-amenable infinitely separated subgroup.

**Theorem 4.5.** Let  $G$  be a countable group containing a non-amenable infinitely separated subgroup  $F$ , let  $\mathcal{D}$  be a finite strongly self-absorbing  $C^*$ -algebra, and let  $A$  be a separable, unital  $C^*$ -algebra with a trace. Then there exist uncountably many pairwise non-cocycle conjugate, strongly outer actions of  $G$  on  $A \otimes \mathcal{D}$ , which are moreover pointwise asymptotically inner. When  $A = \mathbb{C}$ , these actions are also weak mixing.

*Proof.* By assumption, there exist an infinite set  $\mathcal{P} \subseteq \mathbb{N}$  of pairwise coprime numbers, and a separated family  $\{H_p\}_{p \in \mathcal{P}}$  of subgroups of  $F$  as in Definition 3.1. We use the notation introduced in Definition 3.4. By Theorem 3.14, the family  $\{\text{id}_A \otimes \beta_{\sigma_{G, \mathcal{D}}^P} : P \subseteq \mathcal{P} \text{ is non-empty}\}$  consists of pairwise non-cocycle conjugate actions of  $G$  on  $A \otimes \mathcal{D}$ . Since  $\mathcal{P}$  is infinite, this family is uncountable. These actions are strongly outer by Lemma 4.3, and pointwise asymptotically inner by [6, Theorem 2.2]. When  $A = \mathbb{C}$ , these actions are weak mixing by Lemma 4.3.  $\square$

Since  $\mathbb{F}_\infty$  is infinitely separated by Lemma 3.2, the above result implies Theorem C, which is the noncommutative analog of Ioana's celebrated result [19].

The methods of this paper apply to the case of  $\text{II}_1$  factors as well. In this case, one defines generalized Bernoulli shifts as in the case of  $C^*$ -algebras (see Definition 2.5), by replacing minimal  $C^*$ -algebra tensor products with von Neumann algebra tensor products (with respect to the unique normal tracial state). One obtains the analogue of Definition 2.21 for an action of  $G$  on a  $\text{II}_1$  factor  $M$  by replacing states with the unique normal tracial state on  $M$ , and by replacing the  $C^*$ -algebra ultrapower with the von Neumann algebra ultrapower. The rest of the proofs apply without change in this setting. We thus obtain the following.

**Theorem 4.6.** Let  $G$  be a countable group containing a nonabelian free group, and let  $M$  be a McDuff  $\text{II}_1$  factor with separable predual. Then there exist uncountably many pairwise non-cocycle conjugate, outer actions of  $G$  on  $M$ . When  $M = \mathcal{R}$ , these actions are also weak mixing.

It was shown by Brothier and Vaes [4, Theorem B] that an arbitrary nonamenable group admits uncountably many pairwise non-cocycle conjugate outer actions on  $\mathcal{R}$ . While our conclusions only hold for groups that contain  $\mathbb{F}_2$ , the actions on  $\mathcal{R}$  that we construct are ergodic (in fact, weak mixing), unlike the actions produced in the proof of [4, Theorem B]. This naturally raises the following:

**Problem 4.7.** For a nonamenable group  $G$ , construct uncountably many pairwise non-cocycle conjugate *weak mixing* outer actions of  $G$  on  $\mathcal{R}$ .

In the measurable setting, Epstein [7] combined Ioana’s result from [19] with Gaboriau-Lyons’ solution [9] to the von Neumann problem, to show that *any* non-amenable group admits a continuum of non-orbit equivalent free, ergodic actions. In order to prove part (2) of Conjecture A for all nonamenable groups, one could attempt a similar approach of inducing actions from  $\mathbb{F}_2$  to any amenable group. For this approach to work, however, one would need a noncommutative analog of the result of Gaboriau-Lyons. This suggests the following interesting problem:

**Problem 4.8.** Is there an analog of Gaboriau-Lyons’ measurable solution to the von Neumann problem in the context of strongly outer actions on strongly self-absorbing  $C^*$ -algebras? And for outer actions on the hyperfinite  $\text{II}_1$  factor?

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