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On some variable coefficient Schrödinger operators on $\mathbb{R} \times \mathbb{R}^n$ and $\mathbb{R} \times \mathbb{T}^2$

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Abstract. We discuss some time-degenerate Schrödinger equations on $\mathbb{R} \times \mathbb{R}^n$ and on $\mathbb{R} \times \mathbb{T}^2$. We give weighted Strichartz estimates and local well-posedness results for the corresponding semilinear IVP (initial value problem). On $\mathbb{R}\times\mathbb{T}^2$ we also consider some nondegenerate space-variable coefficient Schrödinger equations and give a result about the local well-posedness of the cubic IVP.

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1 Introduction

In the present paper we will give an overview of some recent results concerning variable coefficient Schrödinger operators in two different settings, namely on $\mathbb{R}_t \times \mathbb{R}^n_x$ and on $\mathbb{R}_t \times \mathbb{T}^2_x$, where $\mathbb{T}^2$ stands for the two dimensional torus, which can be both rational and irrational. The operators we

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will be considering are of the following type:

$$\begin{aligned}
&\mathcal{L}_b := i\partial_t + b'(t)\Delta, && (t,x)\in\mathbb{R}\times\mathbb{R}^n,\\
&\mathcal{L}_g := i\partial_t + g'(t)\Delta, && (t,x)\in\mathbb{R}\times\mathbb{T}^2,\\
&\mathcal{L}_{a_1,a_2} := i\partial_t + a_1(x_1)\partial_{x_1}^2 + a_2(x_2)\partial_{x_2}^2, && (t,x)\in\mathbb{R}\times\mathbb{T}^2,
\end{aligned}$$

where $b\in C^1(\mathbb{R})$ and has a countable number of critical points, $g\in C^1(\mathbb{R})$ and is strictly monotone, while $a_1,a_2\in C^\infty(\mathbb{T})$ and are strictly positive. In addition, the functions b and g, as well as their first order derivatives b' and g', will be assumed to vanish at time $t=0$. This assumption is made in order to allow $\mathcal{L}_b$ and $\mathcal{L}_g$ to have a degeneracy at time $t=0$, which is the case of interest to us. Notice that, due to the properties of b, $\mathcal{L}_b$ can have finitely many degenerate points, while $\mathcal{L}_g$ is degenerate at time $t=0$ only. For each of these operators we will study the associated semilinear IVP (initial value problem) by means of the so called *Strichartz* estimates, estimates which are fundamental to characterize and prove well-posedness results for semilinear problems.

To the best of our knowledge the investigation of the class $\mathcal{L}_b$ with $b(t)=t^\alpha$ first appeared in [7], where the local well-posedness of the associated linear IVP was proved in Sobolev and Gevrey spaces. The same class, that is with $b(t)=t^\alpha$, was studied by the author and G. Staffilani in [10], where weighted smoothing estimates were proved and used to obtain local well-posedness results for NLIVPs (nonlinear IVPs) with polynomial and with derivative nonlinearities. As for the more general class $\mathcal{L}_b$, it was first treated by the author and M. Ruzhansky in [9], where weighted homogeneous smoothing estimates have been derived for time-degenerate Shrödinger operators of any order by means of comparison principles (see also [22] for comparison principles). In the same work, the authors proved weighted Strichartz estimates for the class $\mathcal{L}_b$. These fundamental estimates were used to prove a local well-posedness result for a suitable SLIVP (semilinear IVP).

It is important to mention that in the Euclidean setting operators like $\mathcal{L}_b$ are studied in the context of Bose-Einstein condensations and nonlinear optics, and that soliton solutions of different kind have been obtained depending on the time-dependent coefficients appearing in the equation (see [6, 18, 28, 29]). The classes $\mathcal{L}_g$ and $\mathcal{L}_{a_1,a_2}$, instead, have been considered by the author and G. Staffilani in [11]. From the analysis of these classes it came to light that for some variable coefficient Shrödinger operators on $\mathbb{R} \times \mathbb{T}^2$ *sharp* local-well-posedness results are still valid. Here *sharp* means that the local well-posedness holds true with the minimum regularity requirement on the initial datum, which, in accordance with Bourgain's sharp L^4-Strichartz estimate on $\mathbb{T}^2$ and related results (see [2]), amounts to ask to the initial datum to lie in the Sobolev space $H^\varepsilon(\mathbb{T}^2)$, with $\varepsilon > 0$. We will give a detailed presentation of these results below in the dedicated section. There particular attention will be given to the construction of suitable Bourgain spaces needed to carry out the study of $\mathcal{L}_g$.

As already mentioned above, the key role in the analysis of SLIVPs for dispersive equations, such as, in this case, the Shrödinger equation, is played by Strichartz estimates. These estimates were studied by several authors both in the Euclidean and in the compact manifold setting, and, more recently, also on stratified Lie groups. Their power lies in the information they give about the solution of the LIVP (linear IVP) and in the fact that they indicate the functional space where a contraction argument can be performed to prove well-posedness results for SLIVPs. We refer the interested reader to [12, 14, 15, 16, 17, 25, 27] and references therein for results in the Euclidean case for general dispersive equations with constant coefficients. In the variable coefficients case, specifically when the Laplacian is replaced by its compactly supported perturbation, Strichartz estimates were obtained in [24], while for asymptotically flat perturbations we refer to [19, 21]. In the general compact Riemannian manifold setting Strichartz estimates and related well-posedness results for the corresponding SLIVP were first studied in [4] (see also [13]). However, the sharp result in [2] on $\mathbb{T}^2$, that is the sharp L^4-Strichartz estimate on $\mathbb{T}^2$, is not covered by the general result in [4], which, instead, was proved to be sharp on spheres. For other results on manifolds we refer the interested reader to [13, 19, 20, 23], while for results on two-step stratified Lie groups see [1].

In this paper we focus on the classes presented above. We shall show suitable Strichartz estimates for $\mathcal{L}_b$ and $\mathcal{L}_g$ in the proper settings. These inequalities will be employed to get local well-posedness results for some SLIVPs. For $\mathcal{L}_{a_1,a_2}$, instead, we apply a strategy allowing to use the results in [2, 3] to get sharp local well-posedness results for space-variable coefficient operators. The interest for these classes of operators is motivated by the presence of the time degeneracy in the first two classes $\mathcal{L}_b$ and $\mathcal{L}_g$, and by the nondegenerate space-variable coefficients in the third class $\mathcal{L}_{a_1,a_2}$.

We conclude this introduction by giving the plan of the paper.

In Section 2 we discuss the class $\mathcal{L}_b$ in the Euclidean setting. We will state the weighted Strichartz estimates holding in this case and give a local well-posedness result for a SLIVP.

In Section 3 we change the setting and consider the class $\mathcal{L}_g$ on $\mathbb{R} \times \mathbb{T}^d, d \geq 1$. Here we will describe some functional spaces of Bourgain type on which our analysis is based. Then, we will present weighted Strichartz estimates on $\mathbb{R} \times \mathbb{T}^d$, multilinear estimates on $\mathbb{R} \times \mathbb{T}^2$, and local well-posedness results for a cubic SLIVP on $\mathbb{R} \times \mathbb{T}^2$.

In Section 4 we will deal with the class $\mathcal{L}_{a_1,a_2}$ and give a local well-posedness result for the associated cubic SLIVP.

2 Time-degenerate Schrödinger operators on $\mathbb{R} \times \mathbb{R}^n$

This section is devoted to the analysis of the calss of time-degenerate Schrödinger operators of the form $\mathcal{L}_b$, that is,

$$\mathcal{L}_b := i\partial_t + b'(t)\Delta, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.1}$$

where $b \in C^1(\mathbb{R})$, $b(0) = b'(0) = 0$, and b is such that it can have finitely many critical points. For this class of operators we are interested in the analysis of the local well-posedness of the SLIVP

$$\begin{cases} \partial_t u + ib'(t)\Delta u = \mu |b'(t)||u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \tag{2.2}$$

with $p > 1$ suitable, and $0 \neq \mu \in \mathbb{R}$. Since we will be working in a finite time interval, we will simply say that b satisfies the following condition:

(H) $b \in C^1(\mathbb{R})$, $b(0) = b'(0) = 0$, and, for any $\tilde{T} < \infty$, $\sharp\{t \in [0, \tilde{T}], b'(t) = 0\} = k < \infty$, with $k \geq 1$.

Let us remark that our results below are true for nondegenerate time-variable coefficient operators too, which means that if $b(0) = 0$ but b has no critical points, our results can still be applied. This fact is particularly important, since it allows to recover classical results for constant coefficient operators as a particular case of those for time dependent nondegenerate operators.

In order to solve (2.2), it is useful to go back to the LIVP

$$\begin{cases} \partial_t u + ib'(t)\Delta u = f(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{2.3}$$

and observe that the solution of (2.3) can be computed explicitely. Indeed, for $f = 0$, one applies the Fourier transform in space and solves the resulting ode in time. Then, by Duhamel's principle, one gets that the solution of (2.3) is given by

$$u(t,x) = e^{ib(t)\Delta} u_0(x) + \int_0^t e^{i(b(t)-b(s))\Delta} f(s,x) ds,$$

where $e^{i(b(t)-b(s))\Delta}$ is the *solution operator* giving the solution at time t starting at time s of the HIVP (homogeneous IVP), that is

$$e^{i(b(t)-b(s))\Delta} v(s,x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi + i(b(t)-b(s))|\xi|^2} \widehat{v}(s,\xi) d\xi,$$

where here $v(s,x)$ is the datum at time s. Recall that $b(0) = 0$, so $e^{i(b(t)-b(0))\Delta} = e^{ib(t)\Delta}$. Observe also that, when $b(t) = t$, that is when $b'(t) = 1$, we obtain the well known formulas for $\mathcal{L}_t = \partial_t + i\Delta$.

The explicit knowledge of the solution of the LIVP allowed us to prove Strichartz estimates suitable to our time-degenerate setting, which, in turn, correspond to the weighted version of the classical estimates. To state the result we will make use of the so called *admissible pairs*.

Let $n \geq 1$, then a pair of exponents (q,p) is called n*-admissible* if $2 \leq q,p \leq \infty$, and

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \quad \text{with} \quad (q,p,n) \neq (2,\infty,2).$$

Theorem 2.1 (Local weighted Strichartz estimates)**.** *Let $b \in C^1([0,T])$ be such that it satisfies condition (H). Then, on denoting by $L^q_t L^p_x := L^q([0,T]; L^p(\mathbb{R}^n))$, we have that for any (q,p) n-admissible pair, with $2 < q,p < \infty$, the following estimates hold*

$$\| |b'(t)|^{1/q} e^{ib(t)\Delta} \varphi \|_{L^q_t L^p_x} \leq C \|\varphi\|_{L^2_x(\mathbb{R}^n)}, \tag{2.4}$$

$$\| e^{ib(t)\Delta} \varphi \|_{L^\infty_t L^2_x} \leq \|\varphi\|_{L^2_x(\mathbb{R}^n)}, \tag{2.5}$$

$$\left\| |b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \right\|_{L^q_t L^p_x} \leq C \| |b'|^{1/q'} g \|_{L^{q'}_t L^{p'}_x}, \tag{2.6}$$

and

$$\left\| \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \right\|_{L^\infty_t L^2_x} \leq C \| |b'|^{1/q'} g \|_{L^{q'}_t L^{p'}_x}, \tag{2.7}$$

with $C = C(k,n,q,p)$.

We will not give the proof of Theorem 2.1 here, which is based on classical tools, and we refer the interested reader to [9].

Remark 2.2. The estimates above involve only one admissible pair (p,q) instead of two admissible pairs $(p,q), (\tilde{p},\tilde{q})$ as in the classical case when $b(t) = t$. However, when $b(t) = t$ we get back Strichartz estimates for $\mathcal{L}_t = \partial_t + i\Delta$. Note aslo that the weight appearing in the estimates depends on the time-dependent coefficient in $\mathcal{L}_b$. This fact dictates which SLIVP can be solved by using Theorem 2.1, that is the one where b' appears in the nonlinear term. Finally, observe that the previous inequalities describe some integrability properties of the solution depending on the integrability properties of the initial datum u_0 and of that of the inhomogeneous term f.

Remark 2.3. In [9] global weighted Strichartz estimates for $\mathcal{L}_b$ have been proved too. The form of the global version does not allow the application of a contraction argument to solve (2.2), for that we have to use the local version above.

We now state the local well-posedness result for (2.2).

Theorem 2.4. *Let $1 < p < \frac{4}{n}+1$ and $b \in C^1([0,+\infty))$ satisfying condition (H). Then, for all $u_0 \in L^2(\mathbb{R}^n)$, there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (2.2) in the time interval $[0,T]$ with*

$$u \in C([0,T]; L^2(\mathbb{R}^n)) \bigcap L_t^q([0,T]; L_x^{p+1}(\mathbb{R}^n))$$

and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot,t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0,T); L^2(\mathbb{R}^n))$, is continuous.

The proof of this theorem is based on the standard contraction argument by means of the estimates in Theorem 2.1. For details see [9].

We now conclude this section with some examples of operators of the form $\mathcal{L}_b$ to which Theorem 2.4 applies.

Example 2.5. The first example is the one also treated in [7, 10], that is

$$\mathcal{L}_b = \mathcal{L}_{\frac{t^{\alpha+1}}{\alpha+1}} = \partial_t + it^\alpha \Delta, \quad \alpha \geq 1.$$

The operator is degenerate only at $t = 0$, and the local well-posedness of (2.2) is guaranteed by Theorem 2.4.

Example 2.6. An other example is given by

$$\mathcal{L}_b = \mathcal{L}_{e^t - t - 1} = \partial_t + i(e^t - 1)\Delta.$$

This operator is, once more, degenerate at $t = 0$, and the associated cubic IVP is locally well-posed by Theorem 2.4.

Example 2.7. The last example we give is the following

$$\mathcal{L}_b = \mathcal{L}_{\cos(t)} := \partial_t u - i\sin(t)\Delta,$$

which represents an operator having more than one degenerate point depending on the time interval of existence of the solution. In this case, if the time interval is big enough, we cross more than one degenerate point.

3 A class of time-degenerate Schrödinger operators on $\mathbb{R} \times \mathbb{T}^2$

We now change the setting and consider $\mathbb{R} \times \mathbb{T}^d$ as the ambient space. We shall consider the class of operators $\mathcal{L}_g$ of the form

$$\mathcal{L}_g := i\partial_t + g'(t)\Delta, \tag{3.1}$$

where $g \in C^1(\mathbb{R})$, $g(0) = g'(0) = 0$, and g is striclty monotone. The monotonicity of the function g is needed in order to define some functional spaces that we introduce below.

As in the case of $\mathcal{L}_b$, we can, once more, write explicity the solution of the LIVP for $\mathcal{L}_g$ on $\mathbb{R} \times \mathbb{T}^d$, $d \geq 1$, that is,

$$u(t,x) := e^{i(g(t)-g(s))\Delta}u_s(x) + \int_s^t e^{i(g(t)-g(\tau))\Delta} f(\tau,x)d\tau, \tag{3.2}$$

where u_s is the initial datum given at time s, f is the inhomogeneous term of the equation, and $e^{i(g(t)-g(s))\Delta}$ is the solution operator now defined as

$$e^{i(g(t)-g(s))\Delta}v(x) := \sum_{k\in\mathbb{Z}^d} e^{ix\cdot k + i(g(t)-g(s))|k|^2}\widehat{v}(k), \tag{3.3}$$

where $\widehat{v}$ is the Fourier transform on $\mathbb{T}^d$. Note finally that when $g(t) = t$ we get the formulas corresponding to the case $\mathcal{L}_t = i\partial_t + \Delta$. These formulas hold on $\mathbb{R}\times\mathbb{T}^d$, $d\geq 1$, but, later on, we shall restrict ourselves to the case $d=2$ for the reasons explained below.

Before going into the details of the results available in this time-degenerate case, let us recall the *sharp L^4-Strichartz estimate* for the homogeneous solutions of $\mathcal{L}_t = i\partial_t + \Delta$ on $\mathbb{R}\times\mathbb{T}^2$, with $\mathbb{T}^2$ being either rational or irrational:

$$\|e^{it\Delta}v_0\|_{L^4_{t,x}(\mathbb{T}\times\mathbb{T}^2)} \lesssim \|v_0\|_{H^s(\mathbb{T}^2)}, \quad s>0. \tag{3.4}$$

Such estimate was proved on $\mathbb{T}^d$, $d\geq 1$, and with L^p, $p\geq 4$, in place of L^4, for suitable values of s. However, our interest in the L^4-estimate has two motivations: one is the sharpness in the case $d=2$, where here the sharpness is measured by s which dictates the regularity to assign to the initial datum; the other one is that the L^4-estimate allows to close the contraction argument to prove the local well-posedness of the SLIVP. Inequality (3.4) was proved by Bourgain in [2] for the flat torus and by Bourgain and Demeter in [3] for the irrational one.

We remark, once more, that Strichartz estimates were proved on general compact Riemaniann manifolds in [4], and that, however, the sharp estimate (3.4) is not covered by the result in [4].

So we take (3.4) as our prototype estimate, and we show that a suitable version of that is true in the time-degenerate setting as well. This allows to obtain information on the solution of the HIVP and to prove sharp results (in terms of the regularity of the initial datum) for a SLIVP associated with $\mathcal{L}_g$.

Bourgain spaces

A key tool to prove local well-posedness results for SLIVPs in the manifold setting is given by the following spaces, also called Bourgain spaces.

Definition 3.1. Let X be the space of functions on $\mathbb{R}_t\times\mathbb{T}^d_x$ such that

- $u:\mathbb{R}_t\times\mathbb{T}^d_x\to\mathbb{C}$,

- $t \to u(t,x)$ is in $\mathcal{S}(\mathbb{R}), \forall x \in \mathbb{T}^d$;
- $x \to u(\cdot,x)$ is $C^\infty(\mathbb{T}^d)$.

Then, for $s, b \in \mathbb{R}$, the space $X^{s,b}(\mathbb{R}\times\mathbb{T}^d)$ is defined as the completion of X with respect to the norm

$$\|u\|_{X^{s,b}} := \left(\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{2s}\int_{\mathbb{R}}(1+|\tau+|k|^2|)^{2b}|\widehat{u}(\tau,k)|^2 d\tau\right)^{1/2}.$$

Remark 3.2. Definition 3.1 is not the general definition of Bourgain spaces, but is the one suitable to the study of $\mathcal{L}_t$. In fact the norm $\|\cdot\|_{X^{s,b}}$ depends on the symbol of the operator $\mathcal{L}_t$, which is exactly $\tau+|k|^2$. For other dispersive equations, such as, for instance, the KdV equation, one can define the appropriate $X^{s,b}$ spaces by using the corresponding pseudodifferential symbol. These spaces are used in the analysis of dispersive equations since solutions to these equations belong, locally in time, to such spaces. For more details see [26].

By Remark 3.2 it is obvious that the space in Definition 3.1 is not suitable to study a general $\mathcal{L}_g$ with $g(t)\neq t$. To define the right space we need to introduce a Fourier transform in time which depends on the function g appearing in $\mathcal{L}_g$.

Definition 3.3. Let $g \in C^\infty(\mathbb{R})$ be such that $g(0)=0$ and g is strictly monotone. Then we define the FT (Fourier transform) and the IFT (inverse Fourier transform) subordinate to g as

$$(\tilde{\mathcal{F}}_g u)(\tau) := \int_{\mathbb{R}} e^{-ig(t)\tau}u(t)dt$$

and

$$(\tilde{\mathcal{F}}_g^{-1} v)(t) := g'(t)\int_{\mathbb{R}} e^{ig(t)\tau}v(\tau)d\tau,$$

where $u, v \in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$. We shall denote by $\tilde{u}(\tau) := (\tilde{\mathcal{F}}_g u)(\tau)$.

Some properties of $\tilde{\mathcal{F}}_g$ are the following:

- $u(t) = (\tilde{\mathcal{F}}_g^{-1}\tilde{u})(t)$;
- $\tilde{\mathcal{F}}_g(\partial_t u)(\tau) = (-i\tau)\tilde{\mathcal{F}}_g(g'u)(\tau)$;
- $\|\tilde{u}\|_{L^2(\mathbb{R}_\tau)} = \|\frac{1}{\sqrt{|g'|}}u\|_{L^2(\mathbb{R}_t)}$;
- $\|\widetilde{g'u}\|_{L^2(\mathbb{R}_\tau)} = \|\sqrt{|g'|}\,u\|_{L^2(\mathbb{R}_t)}$;
- $\|\tilde{\mathcal{F}}_g(\sqrt{|g'|}u)\|_{L^2(\mathbb{R}_\tau)} = \|u\|_{L^2(\mathbb{R}_t)}$;

Below we shall use the notation $\tilde{u}(\tau, k)$ for the space-time transform

$$\tilde{u}(\tau, k) := \int_{\mathbb{R}\times\mathbb{T}^d} e^{-i(g(t)\tau + k\cdot x)} u(t,x)\,dt dx,$$

being the modified Fourier transform in time and the standard Fourier transform in space of a function u on $\mathbb{R}\times\mathbb{T}^d$.

We are now ready to introduce the appropriate Bourgain spaces to study $\mathcal{L}_g$.

Definition 3.4. Given a strictly monotone function $g \in C^\infty(\mathbb{R}_t)$, with $g(0) = 0$, we define the space $X_g^{s,b}(\mathbb{R}\times\mathbb{T}^d)$ as the completion of the space X as in Definition 3.1 with respect to the norm

$$\|u\|_{X_g^{s,b}} := \left(\sum_{k\in\mathbb{Z}^d}(1+|k|)^{2s}\int_{\mathbb{R}}(1+|\tau+|k|^2|)^{2b}|\tilde{u}(\tau,k)|^2 d\tau\right)^{1/2}.$$

Moreover, we define the spaces $\tilde{X}_g^{s,b}(\mathbb{R}\times\mathbb{T}^d)$ as

$$\tilde{X}_g^{s,b} := \{u \in X_g^{s,b}; g'u \in X_g^{s,b}\},$$

where

$$\|u\|_{\tilde{X}_g^{s,b}} := \left(\sum_{k\in\mathbb{Z}^d}(1+|k|)^{2s}\int_{\mathbb{R}}(1+|\tau+|k|^2|)^{2b}|\widetilde{g'u}(\tau,k)|^2 d\tau\right)^{1/2}.$$

Remark 3.5. The spaces in Definition 3.4 enjoy the same properties as the ones in Definition 3.1. In fact, the application of the modified FT subordinate to g in time and the standard FT in space to the homogeneous equation gives

$$\mathcal{L}_g u(t,x) = i\partial_t u + g'(t)\Delta u = 0 \underset{\tilde{\mathcal{F}}_g \mathcal{F}_{x\to k}}{\Longrightarrow} (\tau + |k|^2)\widetilde{g'u}(\tau,k) = 0,$$

meaning that if u solves $\mathcal{L}_g u = 0$ then $\widetilde{g'u}$ is supported in $\{(\tau,k) \in \mathbb{R} \times \mathbb{Z}^d; \tau + |k|^2 = 0\}$. Additionally, for $u \in H^s(\mathbb{T}^d)$ and η being a smooth cutoff function in time, one has that $\eta e^{ig(t)\Delta} u \in \tilde{X}_g^{s,b}(\mathbb{R}\times\mathbb{T}^d)$. These are exactly the same properties holding for $\mathcal{L}_t$, properties that can also be recovered from those above by taking $g(t) = t$. This suggests that the spaces $\tilde{X}_g^{s,b}$ are the right ones to carry out the analysis of $\mathcal{L}_g$.

We shall present here some other spaces we will need in the analysis of $\mathcal{L}_g$ and in the next section to study $\mathcal{L}_{a_1,a_2}$.

Definition 3.6 ($X_\Phi^{s,b}$, $X_{g,\Phi}^{s,b}$, $X_{\Phi,\tilde{\alpha}}^{s,b}$, $X_{g,\Phi,\tilde{\alpha}}^{s,b}$, and $\tilde{X}_{g,\Phi,\tilde{\alpha}}^{s,b}$ spaces)**.** Let $\Phi \in C^\infty(\mathbb{T}^d)$ and let g be as in Definition 3.4. Let also $\tilde{\alpha} : \mathbb{R}\times\mathbb{T}^d \to \mathbb{R}\times\mathbb{T}^d$ be such that $\tilde{\alpha}(t,x) := (t,\alpha(x))$, where $\alpha : \mathbb{T}^d \to \mathbb{T}^d$ is a diffeomorphism. Then we define the spaces $X_\Phi^{s,b}$, $X_{g,\Phi}^{s,b}$, $X_{\Phi,\tilde{\alpha}}^{s,b}$, $X_{g,\Phi,\tilde{\alpha}}^{s,b}$, and $\tilde{X}_{g,\Phi,\tilde{\alpha}}^{s,b}$ as

$$\begin{aligned}
X_\Phi^{s,b}(\mathbb{R}\times\mathbb{T}^d) &:= \{f : \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}; e^\Phi f \in X^{s,b}(\mathbb{R}\times\mathbb{T}^d)\},\\
X_{g,\Phi}^{s,b}(\mathbb{R}\times\mathbb{T}^d) &:= \{f : \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}; e^\Phi f \in X_g^{s,b}(\mathbb{R}\times\mathbb{T}^d)\},\\
\tilde{X}_{g,\Phi}^{s,b}(\mathbb{R}\times\mathbb{T}^d) &:= \{f : \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}; e^\Phi f \in \tilde{X}_g^{s,b}(\mathbb{R}\times\mathbb{T}^d)\},\\
X_{\Phi,\tilde{\alpha}}^{s,b}(\mathbb{R}\times\mathbb{T}^d) &:= \{f : \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}; (e^\Phi f)\circ\tilde{\alpha} \in X^{s,b}(\mathbb{R}\times\mathbb{T}^d)\},\\
X_{g,\Phi,\tilde{\alpha}}^{s,b}(\mathbb{R}\times\mathbb{T}^d) &:= \{f : \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}; (e^\Phi f)\circ\tilde{\alpha} \in X_g^{s,b}(\mathbb{R}\times\mathbb{T}^d)\},\\
\tilde{X}_{g,\Phi,\tilde{\alpha}}^{s,b}(\mathbb{R}\times\mathbb{T}^d) &:= \{f : \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}; (e^\Phi f)\circ\tilde{\alpha} \in \tilde{X}_g^{s,b}(\mathbb{R}\times\mathbb{T}^d)\}.
\end{aligned}$$

Definition 3.7 ($H^{p,b}$ and $H_g^{p,b}$ spaces)**.** Let $p \in [1,\infty)$ and $b \in \mathbb{R}$, then we define the spaces $H^{p,b}(\mathbb{R})$ and $H_g^{p,b}(\mathbb{R})$ as

$$H^{p,b}(\mathbb{R}) := \{f \in L^p(\mathbb{R}); \widehat{f}, \widehat{D^b f} \in L^p(\mathbb{R})\}$$

equipped with the norm

$$\|f\|^p_{H^{p,b}} := \int_{\mathbb{R}} \langle\tau\rangle^{pb} |\widehat{f}(\tau)|^p d\tau,$$

with $\langle\tau\rangle := (1+|\tau|^2)^{1/2}$, and

$$H^{p,b}_g(\mathbb{R}) := \{f \in L^p(\mathbb{R}); \|f\|_{H^{p,b}_g} < \infty\},$$

where $\|f\|^p_{H^{p,b}_g} := \int_{\mathbb{R}} \langle\tau\rangle^{pb} |\tilde{f}(\tau)|^p d\tau$.

Strichartz, multilinear estimates and local well-posedness

We now give the general statement of the Strichartz estimates proved in [11] on $\mathbb{R}\times\mathbb{T}^d$, with $d \geq 1$, but later on we shall restrict ourselves to the case $d = 2$.

Below we shall assume, without loss of generality, that g is strictly increasing, and we shall denote by $S(t) := e^{ig(t)\Delta}$ the solution operator.

Theorem 3.8 (Weighted Strichartz estimates)**.** *Let I be a finite interval of time and ϕ a function on $\mathbb{T}^d$. Then, for $p \geq 2$,*

$$\begin{aligned}
&\|g'(t)^{1/p} S(t)\phi\|_{L^p(I\times\mathbb{T}^d)} \lesssim \|\phi\|_{L^2(\mathbb{T}^2)}, \quad p < \tfrac{2(d+2)}{d};\\
&\|g'(t)^{1/p} S(t)\phi\|_{L^p(I\times\mathbb{T}^d)} \lesssim \|\phi\|_{H^s(\mathbb{T}^2)}, \quad s > 0, \quad p = \tfrac{2(d+2)}{d};\\
&\|g'(t)^{1/p} S(t)\phi\|_{L^p(I\times\mathbb{T}^d)} \lesssim \|\phi\|_{H^s(\mathbb{T}^2)}, \quad s > \tfrac{d}{2} - \tfrac{d+2}{p}, \quad p > \tfrac{2(d+2)}{d}.
\end{aligned} \tag{3.5}$$

Observe that when $g(t) = t$ the estimates in Theorem 3.8 give back the result for the homogeneous solutions of $\mathcal{L}_t$, and that, for $g(t) \neq t$, $p = 2, d = 2$, we have a weighted version of (3.4).

We now focus on the case $p = 4$ and $d = 2$. By using Theorem 3.8 we can get estimates in $X^{s,b}_g$-spaces which are fundamental to prove the local well-posedness results stated at the end of this section.

Proposition 3.9. *Assume that $|I'| := |g(I)| = \delta$, then*

$$\|\chi_I(t) g'(t) S(t) u_0\|_{X^{s,b}_g} \lesssim \delta^{1/2-b} \|u_0\|_{H^s}, \quad \forall u_0 \in H^s(\mathbb{T}^2), \tag{3.6}$$

$$\left\| g'(t) \int_0^t g'(s) S(t,s) w(s) ds \right\|_{X_g^{s,b}} \lesssim \|g'(t) w\|_{X_g^{s,b-1}}, \tag{3.7}$$

$$\begin{aligned} \|\chi_I(t) g'(t) |u|^2 u\|_{X_g^{s,b-1}} &\lesssim \|\chi_I g'(t) u\|^2_{X_g^{s,b'}} \|\chi_I g'(t) u\|_{X_g^{s,b}}, \\ &\textit{for } b > 1/2,\ 1/4 < b' < b,\ s > 0; \end{aligned} \tag{3.8}$$

$$\|\chi_I g'(t) u\|_{X_g^{s,b'}} \lesssim \delta^{\frac{b-b'}{8}} \|g'(t) u\|_{X_g^{s,b}}. \tag{3.9}$$

For our purposes, that is to solve SLIVPs, the following multilinear estimates proved in [11] are crucial.

Proposition 3.10. *Let $s > 0$, $b \in (1/2, 1)$, $b' < b$, and $H_g^{p,b}(\mathbb{R})$ as in Definition 3.7. Then, for $h \in H^1(\mathbb{T}^2)$ and $\beta \in H^{s+2b}(\mathbb{T}^2)$, we have*

$$\|g'(t) f(t) u\|_{X_g^{s,b}} \lesssim \|g' f\|_{H_g^{1,b}} \|g' u\|_{X_g^{s,b}}, \tag{3.10}$$

$$\|g'(t) \chi_I \beta\|_{X_g^{s,b}} \lesssim \|g' \chi_I\|_{H_g^{2,b}} \|\beta\|_{H_x^{s+2b}}, \tag{3.11}$$

$$\|g'(t)\, \chi_I u_1\, \chi_I u_2\|_{X_g^{s,b-1}} \lesssim \|g'(t) \chi_I u_1\|_{X_g^{s,b}} \|g'(t) \chi_I u_2\|_{X_g^{s,b'}}, \tag{3.12}$$

and, for $p_1 > 1/2, s_1 > 1$,

$$\begin{aligned} \|g'(t)\, \chi_I h\, \chi_I u_1\, \chi_I u_2\, \chi_I u_3\|_{X_g^{s,b-1}} &\lesssim \|g'(t) \chi_I h\|_{H_g^{2,p_1} H_x^{s_1}} \|g'(t) \chi_I u_1\|_{X_g^{s,b}} \\ &\times \|g'(t) \chi_I u_2\|_{X_g^{s,b}} \|g'(t) \chi_I u_2\|_{X_g^{s,b'}}. \end{aligned} \tag{3.13}$$

Below we state our result concerning the local well-posedness of a cubic time-degenerate IVP on $\mathbb{R} \times \mathbb{T}^2$. Recall that the function g is assumed to be as described at the beginning of the section.

Theorem 3.11. *Let $s > 0$ and $b \in (1/2, 1)$. Then, for every $u_0 \in H^s(\mathbb{T}^2)$, there exists a unique solution of the IVP*

$$\begin{cases} i\partial_t u + g'(t) \Delta_x u = g'(t) |u|^2 u, \\ u(0,x) = u_0(x), \end{cases} \tag{3.14}$$

in the time interval $[-T,T]$ *for a suitable time* $T = T(\|u_0\|_{H^s})$. *Moreover the solution* u *satisfies*

$$u \in C([-T,T]; H^s),$$

and, for I *closed neighborhood of* $[-T,T]$, *and* χ_I *a smooth cutoff function such that* $\chi_I \equiv 1$ *on* $[-T,T]$, *we have that there exists* $b \in (1/2,1)$ *such that*

$$\chi_I u \in \tilde{X}_g^{s,b}(\mathbb{R} \times \mathbb{T}^2).$$

A more general version of Theorem 3.11 is the one that follows, where, in particular, the nonlinear term depends on a time-dependent function non necessarily being the time-degenerate coefficient g'.

Theorem 3.12. *Let* $s > 0$, $b \in (1/2,1)$, *and* $f \in H_g^{1,b}(\mathbb{R})$. *Then, for every* $u_0 \in H^s(\mathbb{T}^2)$, *there exists a unique solution of the IVP*

$$\begin{cases} i\partial_t u + g'(t)\Delta_x u = f(t)|u|^2 u, \\ u(0,x) = u_0(x), \end{cases} \tag{3.15}$$

in the time interval $[-T,T]$ *for a suitable time* $T = T(\|u_0\|_{H^s})$. *Moreover the solution* u *satisfies*

$$u \in C([-T,T]; H^s)$$

and, for a closed neighborhood I *of* $[-T,T]$, *we have that there exists* $b \in (1/2,1)$ *such that*

$$\chi_I u \in \tilde{X}_g^{s,b}(\mathbb{R} \times \mathbb{T}^2)$$

with χ_I *being a smooth cutoff function such that* $\chi_I \equiv 1$ *on* $[-T,T]$.

The proof of the previous theorems is based on the standard contraction argument in the Bourgain spaces $\tilde{X}_g^{s,b}(\mathbb{R}\times\mathbb{T}^2)$ and on the use of the results in Proposition 3.9. For the proofs see [11].

Remark 3.13. In [11] the corresponding versions of Theorem 3.11 and Theorem 3.12 for the quintic NLS (with $\mathcal{L}_g$) on $\mathbb{R} \times \mathbb{T}$ are given. We will not state such results here, but we refer the interested reader to [11].

4 A class of nondegenerate space-variable coefficient Schrödinger operators on $\mathbb{R} \times \mathbb{T}^2$

We now discuss the class of Schrödinger operators

$$\mathcal{L}_{a_1,a_2} := i\partial_t u + a_1(x_1)\partial_{x_1}^2 u + a_2(x_2)\partial_{x_2}^2, \tag{4.1}$$

where $a_1, a_2 \in C^\infty(\mathbb{T})$ are real valued and strictly positive.

We will give a sharp local well-posedness result for the cubic IPV

$$\begin{cases} i\partial_t u + a_1(x_1)\partial_{x_1}^2 u + a_2(x_2)\partial_{x_2}^2 u = u|u|^2, \\ u(0,x) = u_0(x), \end{cases} \tag{4.2}$$

where, once again, here sharp means that it suffices to require that the intial datum belongs to $H^\varepsilon(\mathbb{T}^2)$, with $\varepsilon > 0$, to have local well-posedness.

The strategy employed here consists in combinig a change of variables and a suitable gauge transform. This, in turn, reduces our problem to a suitable one for the constant coefficient Schrödinger operator $\mathcal{L}_{1,1} = i\partial_t + \Delta$.

In what follows we will explain the main steps to reduce problem (4.2) to a suitable one for $\mathcal{L}_{1,1}$ and give the statement of our local well-posedness result for (4.2).

The first step is to apply in (4.2) the change of variables

$$(x_1, x_2) = (\alpha_1(y_1), \alpha_2(y_2)) := \alpha(y),$$

with $\alpha_1(y_1), \alpha_2(y_2)$ such that $\partial_{y_1}\alpha_1(y_1) = \sqrt{a_1(\alpha_1(y_1))}$ and $\partial_{y_2}\alpha_2(y_2) = \sqrt{a_2(\alpha_2(y_2))}$. Then, on denoting by $v(t,y) := u(t,\alpha(y))$ and by $v_0(y) := u_0(\alpha(y))$, assuming that u solves (4.2), and using that $(\partial_{x_j}u)(t,\alpha(y)) = (\partial_{y_j}v(t,y))\partial_{y_j}\alpha_j(y_j)$, we get that v solves the IVP

$$\begin{cases} i\partial_t v(t,y) + \Delta_y v(t,y) - (\partial_{y_1}v(t,y))\frac{\partial_{y_1}^2\alpha_1(y_1)}{\partial_{y_1}\alpha_1(y_1)} - (\partial_{y_2}v(t,y))\frac{\partial_{y_2}^2\alpha_2(y_2)}{\partial_{y_2}\alpha_2(y_2)} = v|v|^2, \\ v(0,y) = v_0(y). \end{cases} \tag{4.3}$$

At this stage we apply a gauge transform on the left and right hand side of (4.3), that is we apply the transformation

$$Tf(t,y):=e^{\Phi(y)}f(t,y)=\exp\Big\{-\frac{1}{2}\int_0^{y_1}\frac{\alpha_1''(s_1)}{\alpha_1'(s_1)}ds_1-\frac{1}{2}\int_0^{y_2}\frac{\alpha_2''(s_2)}{\alpha_2'(s_2)}ds_2\Big\}f(t,y).$$

Now, defining w as $w(t,y):=e^{\Phi(y)}v(t,y)$, we obtain that w solves

$$\begin{cases} i\partial_t w+\Delta_y w=e^{-2\Phi}w|w|^2-\beta w, \\ w(0,y)=w_0(y), \end{cases} \tag{4.4}$$

with $\beta=\beta(y)=\partial_{y_1}^2\Phi+\partial_{y_2}^2\Phi+(\partial_{y_1}\Phi)^2+(\partial_{y_2}\Phi)^2$ and $w_0(y)=e^{\Phi(y)}u_0(\alpha(y))$. Summarizing, after the change of variables and the application of the gauge transform, we have reduced the study of (4.2) to that of (4.4). We will not state the result for (4.4) here in order to mantain the exposition selfcontained. We refer to [11] for details about the result for (4.4). Therefore, solving (4.4), applying the inverse gauge transform to the solution of (4.4) (which will give the solution of (4.3)) and changing variables, we get the following result for (4.2).

Theorem 4.1. *Let $s>0$ and $b\in(1/2,1)$. Then, for every $u_0\in H^s(\mathbb{T}^2)$, there exists a unique solution of the IVP*

$$\begin{cases} i\partial_t u+a_1(x_1)\partial_{x_1}^2u+a_2(x_2)\partial_{x_2}^2u=u|u|^2, \\ u(0,x)=u_0(x), \end{cases} \tag{4.5}$$

in the time interval $[-T,T]$ for a suitable time $T=T(\|u_0\|_{H^s})$. Moreover the solution u satisfies

$$u\in C([-T,T];H^s),$$

and, for a closed neighborhood I of $[-T,T]$, we have that there exists $b\in(1/2,1)$ such that

$$\chi_I u\in X_\Phi^{s,b}(\mathbb{R}\times\mathbb{T}^2),$$

with χ_I being a smooth cutoff function such that $\chi_I\equiv 1$ on $[-T,T]$, and where $X_{\Phi,\tilde{\alpha}}^{s,b}(\mathbb{R}\times\mathbb{T}^2)$ is the Banach space in Definition 3.6.

Remark 4.2. In [11], the quintic NLS on $\mathbb{R} \times \mathbb{T}$ for $\mathcal{L}_a = i\partial_t u + a(x)\partial_x^2$, and the cubic IVP

$$\begin{cases} i\partial_t u + g'(t)\Big(a_1(x_1)\partial_{x_1}^2 u + a_2(x_2)\partial_{x_2}^2 u\Big) = f(t)u|u|^2, \\ u(0,x) = u_0(x), \end{cases} \tag{4.6}$$

for operators with space-time variable coefficients on $\mathbb{R} \times \mathbb{T}^2$, are also discussed. For the quintic NLIVP for $\mathcal{L}_a$, a result similar to the one holding for the cubic NLS for $\mathcal{L}_{a_1,a_2}$ on $\mathbb{R}\times\mathbb{T}^2$ is still valid, so we refer the interested reader to [11] for more details about this problem. As for (4.6) on $\mathbb{R} \times \mathbb{T}^2$, it can be solved by combining the strategies used for the operators $\mathcal{L}_{a_1,a_2}$ and $\mathcal{L}_g$ (see Remark 4.3 in [11]).

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