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Probability Solutions of the Sincov's Functional Equation on the Set of Non-negative Integers

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Abstract. In this note we establish when the bivariate discrete Schur-constant models possess the Sibuya-type aging property. It happens that the corresponding class is large, solving the counterpart of classical Sincov's functional equation on the set of non-negative integers.

1 Introduction

Let us recall the functional equation

$$f(x + y) = a^{xy} f(x) f(y) \quad \text{for all} \quad x, y \in \mathbb{R} = (-\infty, \infty) \tag{1}$$

of Sincov (1903) where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and a is a positive constant. Defining the continuous function $g(x) = a^{-x^2/2} f(x)$, equation (1) transforms into the multiplicative Cauchy equation $g(x + y) = g(x)g(y)$ with solutions $g(x) = 0$ or $g(x) = b^x$ where b is a positive constant. Therefore, the solutions of (1) are $f(x) = 0$ and $f(x) = a^{x^2/2+cx}$, where $c = \frac{\ln b}{\ln a}$. If we are interested in all functions $f(\cdot)$ that satisfy the above conditions and are not identically zero, we must have $f(0) = 1$. Moreover, if it happens that the function $f(\cdot)$ is not continuous with $f(0) = 1$ and $f(1) > 0$, then the solutions of (1) for positive rational x are again of the form $f(x) = a^{x^2/2+cx}$, but with $c = -0.5 + \frac{\ln f(1)}{\ln a}$, see Aczel (1966), pages 64–65.

In this note we will further restrict the domain of Sincov's functional equation (1) on the set of non-negative integers, i.e, assuming that $x, y \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Surprisingly, the corresponding solutions determine a subclass of discrete Schur-constant models exhibiting a particular aging property, called Sibuya-type and introduced by Kolev (2020). To proceed, we describe briefly these two notions.

1.1 Discrete Schur-constancy

Let us denote by $S_{X,Y}(x, y)$ the joint survival function of the non-negative integer-valued random vector (X, Y), i.e., $S_{X,Y}(x, y) = \mathbb{P}(X \geq x, Y \geq y)$ for $x, y \in \mathbb{N}_0$. Then, the *bivariate discrete Schur-constant model* is specified by the relation

$$S_{X,Y}(x, y) = G(x + y), \quad x, y \in \mathbb{N}_0 \tag{2}$$

for a 2-monotone on $\mathbb{N}_0$ univariate survival function $G(x) = \mathbb{P}(X \geq x)$, see Proposition 2.2 in Castaner et al. (2015). We remind that a function $G : \mathbb{N}_0 \to [0, 1]$ is 2-monotone if $G(x) \geq 0$, $G(x + 1) \leq G(x)$ and $G(x + 2) - 2G(x + 1) + G(x) \geq 0$ for all $x \in \mathbb{N}_0$.

Nowadays, the properties of multivariate Schur-constant discrete distributions and their applications have been investigated extensively by several authors, e.g., see Lefevre at al. (2018); Castaner and Claramunt (2019); Kolev and Mulinacci (2021).

1.2 Discrete Sibuya-type aging property

The notion of the discrete Sibuya-type aging property is recently introduced by Kolev (2020) as follows.

Definition 1.1. Let $(X_t, Y_t) = [(X - t, Y - t) \mid X > t, Y > t]$ for $t \in \mathbb{N}_0$ be the residual lifetime vector with joint survival function $S_{X_t, Y_t}(x, y)$, corresponding to the non-negative integer-valued random vector (X, Y) having joint survival function $S_{X,Y}(x, y)$. The vector (X, Y) with marginal survival functions $S_X(x)$ and $S_Y(y)$ possesses the "discrete Sibuya-type bivariate aging property" (to be denoted DS–BAP), if and only if

$$\frac{S_{X_t, Y_t}(x, y)}{S_{X_t}(x) S_{Y_t}(y)} = \frac{S_{X,Y}(x, y)}{S_X(x) S_Y(y)} = B(x, y) \tag{3}$$

and the marginal survival functions of (X_t, Y_t) are given by

$$S_{X_t}(x) = S_X(x) \exp\{-a_1 x t\} \quad \text{and} \quad S_{Y_t}(y) = S_Y(y) \exp\{-a_2 y t\}, \tag{4}$$

for all $x, y, t \in \mathbb{N}_0$ and non-negative parameters a_1 and a_2.

Definition 1.1 indicates that the random vector (X, Y) and its residual lifetime vector (X_t, Y_t) should share, for all $t \in \mathbb{N}_0$, the same Sibuya's dependence function $B(x, y)$, i.e., it has to be tail invariant under the marginal restrictions (4). In other words, the dependence structure of the residual lifetime vector (X_t, Y_t) might change in the time, but it has no impact on the Sibuya's dependence function.

Characterizations of DS–BAP (consult Theorems 5.1, 5.2 and 5.3 in Kolev (2020)), are summarized in the following statement.

Lemma 1.2. *The discrete random vector* (X, Y) *possesses DS–BAP specified by (3) and (4) if and only if*

1. its joint survival function $S_{X,Y}(x, y)$ *is given by*

$$S_{X,Y}(x, y) = \begin{cases} S_X(x - y) \exp\left\{-\frac{2a_0 - a_1 - a_2}{2} y - a_1 x y - \frac{a_2 - a_1}{2} y^2\right\}, & \text{if } x > y \geq 0; \\ \exp\left\{-\frac{2a_0 - a_1 - a_2}{2} x - \frac{a_1 + a_2}{2} x^2\right\}, & \text{if } x = y \geq 0; \\ S_Y(y - x) \exp\left\{-\frac{2a_0 - a_1 - a_2}{2} x - a_2 x y - \frac{a_1 - a_2}{2} x^2\right\}, & \text{if } y > x \geq 0. \end{cases} \tag{5}$$

The non-negative parameters a_0, a_1 *and* a_2 *are such that* $2a_0 - a_1 - a_2 \geq 0$ *and can be computed as*

$$a_0 = -\ln[S_{X,Y}(1, 1)], \ a_1 = -\ln\left[\frac{S_{X,Y}(2, 1)}{S_X(1) S_{X,Y}(1, 1)}\right] \quad \text{and} \quad a_2 = -\ln\left[\frac{S_{X,Y}(1, 2)}{S_Y(1) S_{X,Y}(1, 1)}\right]. \tag{6}$$

2. Moreover, $S_{X,Y}(x, y)$ *specified by (5) solves the functional equation*

$$S_{X,Y}(x + t, y + t) = S_{X,Y}(x, y) S_{X,Y}(t, t) \exp\{-(a_1 x + a_2 y)t\}, \quad x, y, t \in \mathbb{N}_0. \tag{7}$$

In Section 2 we will find a subclass of bivariate discrete Schur-constant models possessing the Sibuya-type bivariate aging property. We will establish that survival function $G(\cdot)$ defined in (2) of these models solves the Sincov's functional equation (1) for all $x, y \in \mathbb{N}_0$. We obtain two- and three-parameter solutions leading to well known and new bivariate discrete probability distributions. We conclude with a discussion.

2 Solutions of the Sincov's functional equation on $\mathbb{N}_0$

Remind that the bivariate Schur-constant models are not only exchangeable, but must satisfy the functional equation (2) as well. So, we will consider further exchangeable bivariate models exhibiting DS–BAP, sharing the same marginal survival function. In this case, one should substitute $a_1 = a_2$ in relations (5) and (7) given in Lemma 1.2. Setting $p_i = \exp\{-a_i\}$ for $i = 0, 1$ in (6) we have $p_0 = G(2)$ and $p_1 = \frac{G(3)}{G(1)G(2)}$. Thus, the functional equation (7) takes the form

$$G(x + y + 2t) = G(x + y)G(2t)p_1^{(x+y)t} \quad \text{for all} \quad x, y, t \in \mathbb{N}_0.$$

Substitute $x_1 = x + y$, $y_1 = 2t$ and $f(\cdot) = G(\cdot)$ in last equation to get

$$f(x_1 + y_1) = a^{x_1 y_1} f(x_1) f(y_1) \quad \text{for all} \quad x_1 \in \mathbb{N}_0, y_1 = 0, 2, 4, \ldots \tag{8}$$

where $a = p_1^{1/2} \in (0, 1]$ is a real constant. Notice that (8) is a discrete version of Sincov's functional equation (1) on the set of non-negative integers.

Set $x_1 = 2$ and $y_1 = 2(t - 1)$ for $t = 1, 2, \ldots$ in (8) to get $f(2t) = a^{4(t-1)} f(2) f(2t - 2)$. Applying iteratively backward this relation, one obtains

$$f(2t) = a^{2t(t-1)} f^t(2) \quad \text{for} \quad t = 1, 2, \ldots, \tag{9}$$

that is,

$$f(x) = a^{x(x-2)/2} f(2)^{x/2} \quad \text{for even } x. \tag{10}$$

By analogy, setting $x_1 = 1$ and $y_1 = 2t$ for $t \in \mathbb{N}_0$ in (8) yields $f(1 + 2t) = a^{2t} f(1) f(2t)$ and using (9), we obtain $f(1 + 2t) = a^{2t^2} f(1) f^t(2)$, implying

$$f(x) = \frac{f(1)\sqrt{a}}{\sqrt{f(2)}} a^{x(x-2)/2} f(2)^{x/2} \quad \text{for odd } x. \tag{11}$$

Therefore, the general solution of (8) can be obtained, if $f(1)$, $f(2)$ and $a = p_1^{1/2}$ are given (pre-specified). Comparing the solutions (10) and (11) for even and odd arguments, one will find an extra term $\frac{f(1)\sqrt{a}}{\sqrt{f(2)}} = d > 0$ in (11). So, we will analyze two cases: $d = 1$ and $d \neq 1$, leading to a two- and three-parameter solutions of functional equation (8), correspondingly. We will assume further that $f(1) = p \in (0, 1)$ and $f(2) = p^\alpha$ for some $\alpha > 1$.

2.1 Two-parameter solutions

To unify the solutions of (8) for all $x \in \mathbb{N}_0$ we will first assume that $\frac{f(1)\sqrt{a}}{\sqrt{f(2)}} = d = 1$ in (11) being equivalent to $a = p^{\alpha-2}$, which leads to the following two-parameter solution

$$f(x; p, \alpha) = p^{(\alpha-2)x^2/2 + (4-\alpha)x/2} \quad \text{for} \quad x \in \mathbb{N}_0, \ p \in (0, 1). \tag{12}$$

Since $a \in (0, 1]$ and $p \in (0, 1)$, then, a standard analysis implies that $f(x; p, \alpha)$ is a proper survival function for all $p \in (0, 1)$ and $\alpha \in [2, +\infty)$. In this case we have $f(3; p, \alpha) = [f(2; p, \alpha)/f(1; p, \alpha)]^3$.

It remains to find the admissible values of the parameter p for $\alpha \in [2, +\infty)$, such that $f(x; p, \alpha)$ given by (12) is a 2-monotone function, i.e., satisfying the inequality

$$f(x + 2; p, \alpha) - 2f(x + 1; p, \alpha) + f(x; p, \alpha) \geq 0 \quad \text{for} \quad x \in \mathbb{N}_0. \tag{13}$$

After some algebra, it can be shown that (13) is satisfied if and only if $p^\alpha - 2p + 1 \geq 0$ and there exist $p^*(\alpha) \in (0, 1]$ such that the last inequality is fulfilled when $p \in (0, p^*(\alpha)] \cap (0, 1)$.

In fact, we established the following characterization.

Theorem 2.1. *The discrete bivariate Schur-constant models defined by (2) with support on* $\mathbb{N}_0 \times \mathbb{N}_0$ *possess the DS–BAP property, defined by (3) and (4), if and only if the corresponding survival function* $f(x; p, \alpha) = G(x; p, \alpha)$ *is specified by (12) for* $\alpha \in [2, +\infty)$ *and* $p \in (0, p^*(\alpha)] \cap (0, 1)$ *with* $p^*(\alpha) \in (0, 1]$ *satisfying equation* $p^\alpha - 2p + 1 = 0$.

Therefore, $f(x; p, \alpha)$ given by (12) is a solution of Sincov's functional equation (8) on the set of non-negative integers. Note that this solution can be rewritten as $f(x; a, c) = a^{x^2/2 + cx}$, i.e., in the same form as the solution of the classical Sincov's equation (1), when $f(.)$ is continuous function on $\mathbb{R}$. The difference is in the restricted parameter space, of course.

We list below several particular cases, recursion relations and a closure property related to the DS–BAP of the Schur-constant models.

- If $\alpha = 2$ in (12), one gets the *Geometric law* with survival function $G(x; p, 2) = p^x$. It satisfies (13) for all $p \in (0, 1)$;
- If $\alpha = 3$ in (12), then the *modified Geometric law* with survival function $G(x; p, 3) = p^{x(x+1)/2}$ holds and from (13) one concludes that $p \in \left(0, \frac{\sqrt{5}-1}{2}\right]$;
- If $\alpha = 4$ in (12), the *discrete Rayleigh law* with $G(x; p, 4) = p^{x^2}$ results. From (13) we obtain that $p \in \left(0, \frac{\sqrt[3]{(17+3\sqrt{33})^2} - \sqrt[3]{17+3\sqrt{33}} - 2}{3\sqrt[3]{17+3\sqrt{33}}}\right]$;
- For admissible values of parameters p and α, the following recursions hold as well

$$G(x + 1; p, \alpha) = G(x; p, \alpha) p^{x(\alpha-2)+1} \quad \text{for} \quad x \in \mathbb{N}_0$$

and

$$G(x; p, \alpha) = G(x; p, \alpha - 1) p^{x(x-1)/2} \quad \text{for} \quad x \in \mathbb{N}_0;$$

- *Closure property*: If $G(x; p_1, \alpha_1)$ and $G(x; p_2, \alpha_2)$ are two solutions of (8), then their product $G(x; p_1, \alpha_1) G(x; p_2, \alpha_2)$ also solves (8). This property might be used to generate new Schur-constant models having DS–BAP property based on existing ones.

2.2 Three-parameter solutions

We will now consider the general case in which we allow for $\frac{f(1)\sqrt{a}}{\sqrt{f(2)}} = d \neq 1$ in (11). Equivalently, set $a = p^{2\beta}$ for some $\beta \in [0, +\infty)$. Hence, using (10) and (11) we obtain the three-parameter solution of (8), written as

$$f(x; p, \alpha, \beta) = d(x) p^{\beta(x^2 - 2x) + \alpha x/2}, \quad \text{with} \quad d(x) = \begin{cases} 1, & \text{for } x \text{ even;} \\ p^{\beta+1-\alpha/2}, & \text{for } x \text{ odd.} \end{cases} \tag{14}$$

The following general characterization holds true.

Theorem 2.2. *The discrete bivariate Schur-constant model defined by (2) with support on* $\mathbb{N}_0 \times \mathbb{N}_0$ *possesses the DS–BAP property, defined by (3) and (4), if and only if the corresponding survival function* $f(x; p, \alpha, \beta) = G(x; p, \alpha, \beta)$ *is specified by (14) for* $\alpha \in (1, +\infty)$, $\beta \in [0, +\infty)$ *and* $p \in (0, p^*(\alpha, \beta)] \cap (0, 1)$, *where* $p^*(\alpha, \beta) \in (0, 1]$ *is the smallest among the solutions of equations* $p^\alpha - 2p + 1 = 0$ *and* $p^{\alpha+4\beta} - 2p^{\alpha-1} + 1 = 0$.

Proof. It can be easily checked that $f(x; p, \alpha, \beta)$ in (14) is a proper survival function for all $p \in (0, 1)$ and $\alpha \in (1, +\infty)$. Furthermore, $f(x; p, \alpha, \beta)$ is a 2-monotone function, i.e.,

$$f(x + 2; p, \alpha, \beta) - 2f(x + 1; p, \alpha, \beta) + f(x; p, \alpha, \beta) \geq 0 \quad \text{for} \quad x \in \mathbb{N}_0, \tag{15}$$

if and only if the parameters $p \in (0, 1)$, $\alpha \in (1, +\infty)$ and $\beta \in [0, +\infty)$ satisfy inequalities

$$p^\alpha - 2p + 1 \geq 0 \quad \text{and} \quad p^{\alpha+4\beta} - 2p^{\alpha-1} + 1 \geq 0. \tag{16}$$

In fact, (15) is satisfied if and only if

$$d(x+2)p^{4\beta x+\alpha} - 2d(x+1)p^{2\beta x-\beta+\alpha/2} + d(x) \geq 0 \tag{17}$$

where $d(x)$ is defined via (14) depending on x being even or odd number. If x is even, then setting $z = p^{2\beta x} \in (0,1]$ in (17) we get

$$p^{\alpha}z^2 - 2pz + 1 \geq 0. \tag{18}$$

Since the left-hand side of (18) is a decreasing function of $z \in (0,1]$, then (17) is fulfilled for x even if and only if (18) is satisfied at $z = 1$, that is if and only if $p^{\alpha} - 2p + 1 \geq 0$.

Considering the function $h(p) = p^{\alpha} - 2p + 1$, we have that $h(0) = 1$ and $h(1) = 0$. Moreover, $h(p)$ is a decreasing function of $p \in (0,1]$ if and only if $\alpha \leq 2$, while it admits a minimum in $(2/\alpha)^{\frac{1}{\alpha-1}}$ when $\alpha > 2$. Thus, for all $\alpha > 1$ there exists $p^{**}(\alpha) \in (0,1]$ such $h(p) \geq 0$ if and only if $p \in (0, p^{**}(\alpha)]$.

In the case when x is odd, put $y = p^{2\beta x} \in (0,1]$ in (17) to obtain

$$p^{\alpha}y^2 - 2p^{\alpha-2\beta-1}y + 1 \geq 0. \tag{19}$$

Similarly, the left-hand side of (19) is a decreasing function of $y \in (0,1]$. Hence (17) is satisfied for x odd if and only if (19) holds at $y = p^{2\beta}$, i.e., if and only if $p^{\alpha+4\beta} - 2p^{\alpha-1} + 1 \geq 0$.

Define now the function $g(p) = p^{\alpha+4\beta} - 2p^{\alpha-1} + 1$, with $g(0) = 1$ and $g(1) = 0$. Observe that $g(p)$ is a decreasing function of $p \in (0,1]$ if and only if $\beta \leq \frac{\alpha-2}{4}$, while it admits a minimum at $\left(\frac{2(\alpha-1)}{\alpha+4\beta}\right)^{\frac{1}{4\beta+1}}$ when $\beta > \frac{\alpha-2}{4}$. Hence, for all $\alpha > 1$ and $\beta \geq 0$ there exists $p^{**}(\alpha,\beta) \in (0,1]$ such $g(p) \geq 0$ if and only if $p \in (0, p^{**}(\alpha,\beta)]$. Choose $p^*(\alpha,\beta) = \min\{p^{**}(\alpha), p^{**}(\alpha,\beta)\}$ to conclude that for all $\alpha \in (1,+\infty)$ and $\beta \in [0,+\infty)$ there exists $p^*(\alpha,\beta) \in (0,1]$ such that inequalities (16) are satisfied, if and only if $p \in (0, p^*(\alpha,\beta)]$.   □

Remark 2.3. Setting $\beta = \frac{\alpha-2}{2}$ in (14), we recover the two-parameter model considered in Subsection 2.1. Notice that, in this case, (19) reduces to (18) and therefore, the only condition required is $p^{\alpha} - 2p + 1 \geq 0$.

Naturally, because of the additional parameter β, the class of DS–BAP distributions with survival function specified by (14) is more general. We just mention two examples that introduce generalizations of the classical geometric and discrete Rayleigh laws considered above.

- *Extended Gometric law*: If $\beta = 0$ in (14), one gets the survival function $G(x;p,\alpha,0) = d(x)p^{\alpha x/2}$ with

$$d(x) = \begin{cases} 1, & \text{for } x \text{ even;} \\ p^{1-\alpha/2}, & \text{for } x \text{ odd.} \end{cases}$$

  It can be checked that $G(x;p,\alpha,0)$ is a 2-monotone, if and only if $p^{\alpha} - 2p + 1 \geq 0$ when $\alpha \geq 2$ and $p^{\alpha} - 2p^{\alpha-1} + 1 \geq 0$ when $\alpha \in (1,2)$.
  In Figure 1 we compare the survival functions when $p = \frac{1}{3}$, i.e., $G(x;1/3,\alpha,0)$ for various values of α (1.5, 2 and 4). One can observe that the differences of the shapes for $x \geq 2$.

- *Extended discrete Rayleigh law*: If $\beta > \frac{1}{4}$ and $\alpha = 4\beta$ in (14), one gets the survival function $G(x;p,4\beta,\beta) = d(x)p^{\beta x^2}$ where

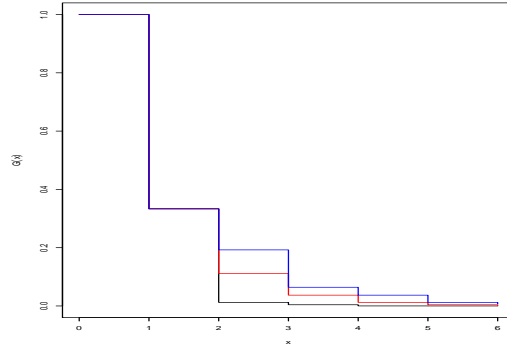$$d(x) = \begin{cases} 1, & \text{for } x \text{ even;} \\ p^{1-\beta}, & \text{for } x \text{ odd.} \end{cases}$$

Figure 1 *Extended Geometric law survival function $G(x; 1/3, \alpha, 0)$. Blue line: $\alpha = 1.5$, red line: $\alpha = 2$ (the standard geometric law) and black line: $\alpha = 4$*
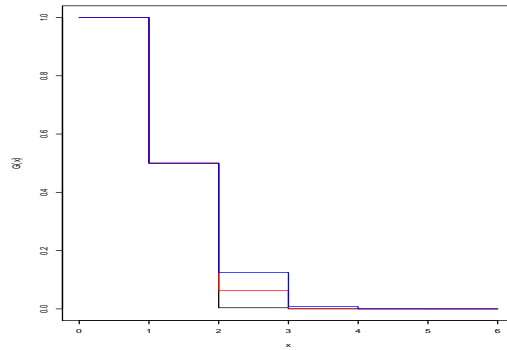

Figure 2 *Extended Rayleigh law survival function $G(x; 0.5, 4\beta, \beta)$: Blue line: $\beta = \frac{3}{4}$, red line: $\beta = 1$ (the standard Rayleigh law) and black line: $\beta = 2$.*

Now, $G(x; p, 4\beta, \beta)$ is 2-monotone if and only if $p^{4\beta} - 2p + 1 \geq 0$ and $p^{8\beta} - 2p^{4\beta-1} + 1 \geq 0$.

In Figure 2 we display the survival functions $G(x; 0.5, 4\beta, \beta)$ for $\beta = 0.75, 1$ and 2. The differences of the shapes occur when $x \geq 2$.

To finalize, note that the three-parameter solutions (14) satisfy the closure property as well: if $G(x; p_1, \alpha_1, \beta_1)$ and $G(x; p_2, \alpha_2, \beta_2)$ are two solutions of (8), then their product also solves (8).

3 Conclusions

In this note we suggest a further contribution to the theory of the bivariate Schur-constant discrete models. We establish that the subclass of Schur-constant models possessing discrete Sibuya-type bivariate aging property is rich. A connection is drawn between this subclass and Sincov's functional equation on the set of non-negative integers. In fact, we got its two- and three-parameter solutions (12) and (14), see the characterization Theorems 2.1 and 2.2. Moreover, the closer property allows generation of plenty new models. In particular, the discrete Rayleigh law can be included in the list of distributions in Table 8.1 given by Nair et al. (2018).

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