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# Geometry and thermodynamics of coherent quantum black holes

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## Abstract

We present a quantum description of black holes given by coherent states of gravitons sourced by a matter core. The expected behaviour in the weak-field region outside the horizon is recovered, with arbitrarily good approximation, but the classical central singularity is not resolved because the coherent states may not contain modes of arbitrarily short wavelength and the matter core must therefore have finite size. Ensuing quantum corrections both in the interior and exterior are also estimated by assuming the mean-field approximation holds everywhere. These deviations from the classical black hole geometry can be viewed as quantum hair and lead to a quantum corrected horizon radius and thermodynamics.

## 1 Introduction and motivation

The gravitational collapse of compact objects generates geodesically incomplete spacetimes in general relativity if a trapping surface appears [1] and eternal point-like sources are not mathematically compatible with Einstein's field equations [2]. We expect the quantum theory will fix this inconsistent classical picture of the gravitational interaction, in the same way that quantum mechanics explains the stability of atoms by not admitting quantum states corresponding to the classical ultraviolet catastrophe. Although the fundamental dynamics remains unaffected, the condition for the existence of a quantum state can greatly modify the effective description of any systems. At the same time, the quantum state of a macroscopic black hole must be such that the effective description reproduces the phenomenology of spacetime we detect experimentally [3, 4].

There are many quantum models of black holes in the literature (for a very partial list, see Refs. [5–7]). In particular, the corpuscular picture [5] belongs to the class of approaches for which geometry should only emerge at suitable (macroscopic) scales from the underlying (microscopic) quantum field theory of gravitons [8, 9]. The key idea is that the constituents of black holes are soft gravitons marginally bound in their own potential and forming a condensate [5] with characteristic

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Compton-de Broglie wavelength  $\lambda_G \sim R_H$ , where the gravitational (or Schwarzschild) radius of the black hole of Arnowitt-Deser-Misner (ADM) mass [11]  $M$  is given by <sup>1</sup>

$$R_H = 2 G_N M . \quad (1.1)$$

The energy scale of the gravitons is correspondingly given by  $\epsilon_G \sim \hbar/\lambda_G$  and, if one assumes that the total mass of the black hole  $M \simeq N_G \epsilon_G$ , there immediately follows the scaling relation for the total number of gravitons

$$N_G \sim \frac{M^2}{m_p^2} \sim \frac{R_H^2}{\ell_p^2} , \quad (1.2)$$

which reproduces Bekenstein's conjecture for the horizon area quantisation [12].

The nonlinearity of the gravitational interaction plays a crucial role in this picture. This can be seen by considering that the (negative) gravitational energy of a source of mass  $M$  localised inside a sphere of radius  $R_s$  is given by  $U_N \sim M V_N(R_s)$ , where

$$V_N = -\frac{G_N M}{r} \quad (1.3)$$

here is the Newtonian potential. This potential can be obtained as the expectation value of a scalar field on a coherent state, whose normalisation then yields the graviton number (1.2) for any values of  $R_s \gtrsim R_H$  [13–16]. In addition to that, assuming most gravitons have the same wavelength  $\lambda_G$ , the binding energy of each graviton is given by

$$\epsilon_G \sim \frac{U_N}{N_G} \sim -\frac{\ell_p m_p}{R_s} , \quad (1.4)$$

which yields the typical Compton-de Broglie length  $\lambda_G \sim R_s$ . The graviton self-interaction energy hence reproduces the (positive) post-Newtonian energy,

$$U_{GG} \sim N_G \epsilon_G V_N(R_s) \sim \frac{G_N^2 M^3}{R_s^2} , \quad (1.5)$$

and the fact that gravitons in a black hole are marginally bound [5], that is  $U_N + U_{GG} \simeq 0$ , finally yields the scaling  $\lambda_G \sim R_s \simeq R_H$  [13, 16].

A key feature of the above scenario is that, like for all bound states in quantum physics, it does not contain modes of arbitrarily small wavelengths and the classical central singularity is therefore not realised. <sup>2</sup> However, viewing a black hole as a quantum state made of only gravitons with one typical wavelength  $\lambda_G$  cannot reproduce the gravitational field in the accessible outer spacetime, even in the simple Newtonian approximation. Of course, general relativity is a metric theory and a complete description of quantum gravity remains beyond our scope, but this (conceptually and phenomenologically) important issue can be addressed in a simplified form for static and spherically

<sup>1</sup>Units with  $c = 1$  are used throughout and the Newton constant  $G_N = \ell_p/m_p$ , where  $\ell_p$  is the Planck length and  $m_p$  the Planck mass, so that  $\hbar = \ell_p m_p$ .

<sup>2</sup>This assumption can be viewed as a manifestation of the classicalization of gravity [17]. Independent arguments in support of this condition, based on the quantum nature of the source, were also given in Refs. [18, 19]. For other similar considerations, see Refs. [21].

symmetric systems [13, 22–24]. In particular, we limit the quantum description to the potential function (1.3) appearing in the Schwarzschild metric

$$ds^2 = -(1 + 2V_N) dt^2 + \frac{dr^2}{1 + 2V_N} + r^2 d\Omega^2 . \quad (1.6)$$

In fact, the function  $V_N$  plays the role of a potential in the geodesic equation of radial motion,

$$\frac{d^2 r}{d\tau^2} \equiv \ddot{r} = -\frac{G_N M}{r^2} = -V'_N , \quad (1.7)$$

where the geodesic  $r = r(\tau)$  is parameterised by the proper time  $\tau$ ,<sup>3</sup> and this equation is the starting point to derive a discrete spectrum for a collapsing dust core [19].

The first important step in a field theoretic description of gravity is to identify the quantum vacuum  $|0\rangle$ , which we assume is the realisation of a universe where no modes (of matter or gravity) are excited. We could then associate the Minkowski metric  $\eta_{\mu\nu}$  to such an absolute vacuum, since this is the metric used to describe linearised gravity and to define both matter and gravitational excitations in this regime. The linearised theory should provide a reliable description for small matter sources (say with total energy  $M \ll m_p$ ), for which it allows one to recover the Newtonian potential from simple tree-level graviton exchanges in the non-relativistic limit. In this context, the potential  $V_N$  is not a fundamental scalar, but it emerges from the (non-propagating) longitudinal polarisation of virtual gravitons [8]. For large sources (that is, with  $M \gg m_p$ ), one can in principle reconstruct the complete classical dynamics [9],<sup>4</sup> but to obtain the proper quantum state from the excitations of the linearised theory appears rather hopeless in this highly non-linear regime. In order to circumvent this issue and, at the same time, to compare with experimental data, one usually assumes that there exists a classical background geometry to replace  $\eta_{\mu\nu}$  with a suitable solution  $g_{\mu\nu}$  of the corresponding classical Einstein's equations. However, in our perspective, the Einstein equations should emerge as effectively describing the macroscopic dynamics and a mean-field metric should correspond to a suitable quantum state. Since we are interested in static and spherically symmetric configurations representing a black hole, we just require that the relevant quantum state of gravity effectively reproduces (as closely as possible) the expected Schwarzschild geometry (1.6) like it was done in Ref. [22]. This approach is thus akin to quantising the longitudinal mode of gravity [8, 26] and can be obtained by employing coherent states in a suitable Fock space built upon the Minkowski vacuum. The use of coherent states is generically motivated by their property of minimising the quantum uncertainty, and is further supported by studies of electrodynamics [14, 27], linearised gravity [15] and the de Sitter spacetime [28].

In the original corpuscular picture, as briefly reviewed above, baryonic matter sourcing the gravitational field and triggering the gravitational collapse is argued to become essentially irrelevant after the black hole forms [5]. In Refs. [19, 22], we instead found that a finite size  $R_s$  of the matter source is both a consequence of the quantum nature of collapsing matter and a necessary condition for the existence of a proper quantum state which (approximately) reproduces the outer Schwarzschild geometry.<sup>5</sup> This supports the point of view of Refs. [13, 16, 18, 29], according to which

<sup>3</sup>One should also notice that the general relativistic function  $V_N$  depends on the areal radius  $r$ , whereas the post-Newtonian viewpoint in Eqs. (1.3)–(1.5) makes use of the harmonic radius [19, 20, 25].

<sup>4</sup>See Ref. [10] for a critical discussion of this conjecture.

<sup>5</sup>In a fully quantum picture, one expects that  $R_s$  be at best the expectation value of some emergent operator in the relevant state of matter fields. For more considerations on the scale  $R_s$  for quantum black holes, see the concluding remarks and Refs. [19, 30].

matter inside the black hole still plays a very significant role in defining the structure of the interior of astrophysical black holes (and possibly microscopic ones) and leads to the existence of quantum hair [31]. In fact, the quantum version of  $V_N$  can be employed in order to reconstruct a quantum corrected complete metric [22]. In this work, we analyse in details the quantum corrected black hole geometry and thermodynamics, thus complementing the study of the entropy of the matter core of Ref. [32].

In the next section, we will briefly review how coherent states of a massless scalar field on a reference flat spacetime can be used to reproduce a classical static field configuration, in general; a consistent quantum state for the metric function (1.3) is then introduced in Section 3, where the corresponding geometry and thermodynamics will be analysed in details; final remarks, outlook and connections with other works will be given in Section 4.

## 2 Quantum coherent states for classical static configurations

We will first review how to describe a generic static  $V = V(r)$  as the mean field of the coherent state of a free massless scalar field (for more details, see, *e.g.* Ref. [14, 16]). It is important to remark again that this description is not fundamental and the use of a scalar field to represent the (non-perturbative) behaviour of the true degrees of freedom of Einstein gravity in the (rather unphysical) static limit is a simplification supported by more refined analyses [15, 28].

We first rescale the dimensionless  $V$  so as to obtain a canonically normalised real scalar field  $\Phi = \sqrt{m_p/\ell_p} V$ , and then quantise  $\Phi$  as a massless field satisfying the free wave equation

$$\left[ -\frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \right] \Phi(t, r) \equiv (-\partial_t^2 + \Delta) \Phi = 0 . \quad (2.1)$$

Solutions to Eq. (2.1) can be conveniently written as

$$u_k(t, r) = e^{-i k t} j_0(k r) , \quad (2.2)$$

where  $k > 0$  and  $j_0 = \sin(k r)/k r$  are spherical Bessel functions satisfying the orthogonality relation

$$4 \pi \int_0^\infty r^2 dr j_0(k r) j_0(p r) = \frac{2 \pi^2}{k^2} \delta(k - p) . \quad (2.3)$$

The quantum field operator and its conjugate momentum read

$$\hat{\Phi}(t, r) = \int_0^\infty \frac{k^2 dk}{2 \pi^2} \sqrt{\frac{\hbar}{2 k}} \left[ \hat{a}_k u_k(t, r) + \hat{a}_k^\dagger u_k^*(t, r) \right] \quad (2.4)$$

$$\hat{\Pi}(t, r) = i \int_0^\infty \frac{k^2 dk}{2 \pi^2} \sqrt{\frac{\hbar k}{2}} \left[ \hat{a}_k u_k(t, r) - \hat{a}_k^\dagger u_k^*(t, r) \right] , \quad (2.5)$$

which satisfy the equal time commutation relations,

$$\left[ \hat{\Phi}(t, r), \hat{\Pi}(t, s) \right] = \frac{i \hbar}{4 \pi r^2} \delta(r - s) , \quad (2.6)$$

provided the creation and annihilation operators obey the commutation rules

$$\left[ \hat{a}_k, \hat{a}_p^\dagger \right] = \frac{2 \pi^2}{k^2} \delta(k - p) . \quad (2.7)$$

The Fock space of quantum states is then built from the vacuum defined by  $\hat{a}_k |0\rangle = 0$  for all  $k > 0$ . The choice of the flat Minkowski metric in Eq. (2.1) follows from the expectation that this vacuum  $|0\rangle$  is meant to describe a completely empty spacetime devoid of any matter source and without excitations of the gravitational field, as we argued in the introductory Section 1.

Classical configurations of the scalar field that can be realised in the quantum theory must correspond to suitable states in this Fock space, and a natural choice is given by coherent states  $|g\rangle$  such that

$$\hat{a}_k |g\rangle = g_k e^{i\gamma_k(t)} |g\rangle . \quad (2.8)$$

In particular, we are interested in those  $|g\rangle$  for which the expectation value of the quantum field  $\hat{\Phi}$  reproduces the classical potential, namely

$$\sqrt{\frac{\ell_p}{m_p}} \langle g | \hat{\Phi}(t, r) | g \rangle = V(r) . \quad (2.9)$$

From the expansion (2.4), we obtain

$$\langle g | \hat{\Phi}(t, r) | g \rangle = \int_0^\infty \frac{k^2 dk}{2\pi^2} \sqrt{\frac{2\ell_p m_p}{k}} g_k \cos[\gamma_k(t) - kt] j_0(kr) . \quad (2.10)$$

If we now write

$$V = \int_0^\infty \frac{k^2 dk}{2\pi^2} \tilde{V}(k) j_0(kr) , \quad (2.11)$$

we immediately obtain <sup>6</sup>

$$\gamma_k = kt \quad (2.12)$$

and

$$g_k = \sqrt{\frac{k}{2}} \frac{\tilde{V}(k)}{\ell_p} . \quad (2.13)$$

The coherent state finally reads

$$|g\rangle = e^{-N_G/2} \exp \left\{ \int_0^\infty \frac{k^2 dk}{2\pi^2} g_k \hat{a}_k^\dagger \right\} |0\rangle , \quad (2.14)$$

where

$$N_G = \int_0^\infty \frac{k^2 dk}{2\pi^2} g_k^2 \quad (2.15)$$

is the total occupation number. We note in particular that the value of  $N_G$  measures the “distance” in the Fock Space of  $|g\rangle$  from the vacuum  $|0\rangle$  corresponding to  $N_G = 0$ . Another quantity of interest is given by

$$\langle k \rangle = \int_0^\infty \frac{k^2 dk}{2\pi^2} k g_k^2 , \quad (2.16)$$

from which one obtains the “average” wavelength  $\lambda_G = N_G / \langle k \rangle$ .

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<sup>6</sup>This formal choice of phases eliminates the time dependence from the normal modes (2.2) and allows for reproducing static configurations. From a physical point of view, one can consider such a limiting approximation holds for time intervals shorter than  $\Delta t \sim k^{-1}$ .

### 3 Quantum Schwarzschild black holes

We will now apply results from the previous section to the Schwarzschild metric (1.6). This geometry contains only the function  $V_N = \sqrt{G_N} \Phi$ , and all of the relevant expressions introduced can be explicitly computed from the coefficients  $g_k$  representing the occupation numbers of the modes  $u_k$ .

In particular, by inverting Eq. (2.11), we find

$$\tilde{V}_N = -4\pi G_N \frac{M}{k^2} \quad (3.1)$$

and the coefficients

$$g_k = -\frac{4\pi M}{\sqrt{2} k^3 m_p} . \quad (3.2)$$

For such a coherent state, we obtain

$$N_G = \frac{4 M^2}{m_p^2} \int_0^\infty \frac{dk}{k} \quad (3.3)$$

and

$$\langle k \rangle = \frac{4 M^2}{m_p^2} \int_0^\infty dk . \quad (3.4)$$

The number of quanta  $N_G$  contains a logarithmic divergence both in the infrared (IR) and the ultraviolet (UV), whereas  $\langle k \rangle$  only diverges (linearly) in the UV.

The meaning of such divergences was already explored in details in previous works [16, 18, 22]. In particular, we recall that the UV divergences are due to the vanishing size of the source, hence they would not be present if the density were regular. Instead of smoothing out the source, the UV divergences can be formally regularised by introducing a cut-off  $k_{UV} \sim 1/R_s$ , where  $R_s$  can be interpreted as the finite radius of a would-be-regular matter source. Such a cut-off is just a mathematically simple way of describing the fact that the very existence of a proper quantum state  $|g\rangle$  requires the coefficients  $g_k$  to depart from their purely classical expression (3.2) for  $k \rightarrow 0$  and  $k \rightarrow \infty$ . Of course, for a black hole spacetime, we must have  $R_s < R_H$ . Likewise, we introduce a IR cut-off  $k_{IR} = 1/R_\infty$  to account for the necessarily finite life-time  $\tau \sim R_\infty$  of a realistic source [16]<sup>7</sup> and rewrite

$$N_G = \frac{4 M^2}{m_p^2} \int_{k_{IR}}^{k_{UV}} \frac{dk}{k} = 4 \frac{M^2}{m_p^2} \ln \left( \frac{R_\infty}{R_s} \right) \quad (3.5)$$

and

$$\langle k \rangle = \frac{4 M^2}{m_p^2} \int_{k_{IR}}^{k_{UV}} dk = 4 \frac{M^2}{m_p^2} \left( \frac{1}{R_s} - \frac{1}{R_\infty} \right) . \quad (3.6)$$

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<sup>7</sup>Any disturbance in the source will propagate (at most) at the speed of light. Alternatively, we could take for  $R_\infty$  the size of the observable Universe as an upper bound [24]. In any case, modes with wavelength  $k^{-1}$  many orders of magnitude larger than  $R_H$  do not contribute significantly to the determination of  $V = V(r)$  in Eq. (2.9) (for the details, see Ref. [18, 33]) and we shall assume  $k_{IR} \rightarrow 0$  whenever possible.



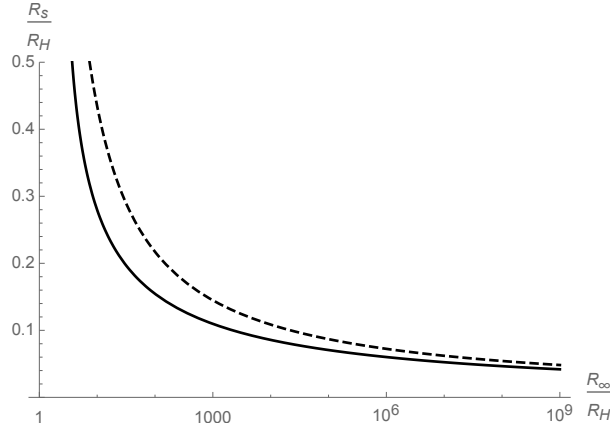


Figure 1: Exact solution to Eq. (3.8) (solid line) compared to its approximation (3.9) (dashed line).

The corpuscular scaling (1.2) for the number  $N_G$  with the square of the energy  $M$  of the system already appears at this stage, whereas the second crucial result

$$\lambda_G = \frac{N_G}{\langle k \rangle} \sim \ell_p \frac{M}{m_p} , \quad (3.7)$$

is obtained from Eqs. (3.5) and (3.6) only provided the cut-offs satisfy

$$\ln \left( \frac{R_\infty}{R_s} \right) \simeq \frac{R_H}{R_s} . \quad (3.8)$$

Assuming  $R_s \lesssim R_H \ll R_\infty$ , the above yields (see Fig. 1 for a graphical comparison with the exact solution)

$$R_s \simeq \frac{R_H}{\ln(R_\infty/R_H)} , \quad (3.9)$$

so that the size of the inner source and the radius of the outer region containing a gravitational field appear connected in the quantum description. We will further comment about possible consequences of this result in the concluding Section 4.

### 3.1 Consistent black hole states

We started from the condition in Eq. (2.9), which demands that the coherent state  $|g\rangle$  reproduces the classical potential everywhere. We then found that acceptable occupation numbers  $g_k$  do not exist which satisfy this requirement for  $k \rightarrow 0$  and  $k \rightarrow \infty$ . We next observe that, for a black hole, the mean field needs only reproduce the classical  $V_N$  with sufficient accuracy to comply with experimental bounds at most in the region of outer communication outside the horizon. This means that the coherent state  $|g_{\text{BH}}\rangle$  representing a black hole must give

$$\sqrt{\frac{\ell_p}{m_p}} \langle g_{\text{BH}} | \hat{\Phi}(t, r) | g_{\text{BH}} \rangle \simeq V_N(r) \quad \text{for } r \gtrsim R_H , \quad (3.10)$$

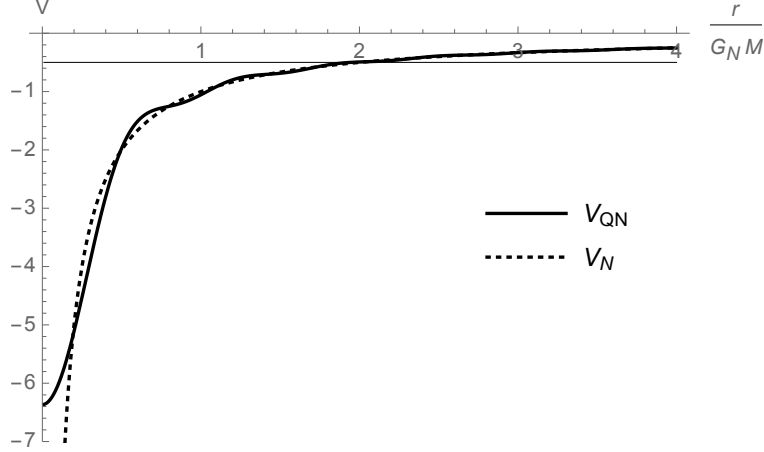


Figure 2: Quantum metric function  $V_{\text{QN}}$  in Eq. (3.13) (solid line) compared to  $V_N$  (dashed line) for  $R_s = R_H/20$ . The horizontal thin line marks the location of the horizon for  $V = -1/2$ .

where we recall that  $V_N(R_H) = -1/2$  and the approximate equality is subject to experimental precision. In practice, this weaker condition means that  $|g_{\text{BH}}\rangle$  does not need to contain the modes of infinitely short wavelength that are necessary to resolve the classical singularity at  $r = 0$ .

In fact, Eq. (3.10) can be satisfied by building the coherent state  $|g_{\text{BH}}\rangle$  according to Eq. (2.14) with modes of wavelength  $k^{-1}$  larger than some fraction of the size of the gravitational radius  $R_H$  of the source, which we can further identify with the UV cut-off  $R_s$ . By momentarily considering also the IR scale  $k_{\text{IR}}$ , we thus have that only the modes  $k$  satisfying

$$R_\infty^{-1} \sim k_{\text{IR}} \lesssim k \lesssim k_{\text{UV}} \sim R_s^{-1} \quad (3.11)$$

are significantly populated in the quantum state  $|g_{\text{BH}}\rangle$ . This yields an effective quantum potential

$$\begin{aligned} V_{\text{QN}} &\simeq \int_{k_{\text{IR}}}^{k_{\text{UV}}} \frac{k^2 dk}{2\pi^2} \tilde{V}_N(k) j_0(kr) \\ &\simeq -\frac{2\ell_p M}{\pi m_p r} \int_0^{r/R_s} dz \frac{\sin z}{z}, \end{aligned} \quad (3.12)$$

where we defined  $z = kr$  and let  $k_{\text{IR}} = 1/R_\infty \rightarrow 0$  as mentioned above. We thus find

$$\begin{aligned} V_{\text{QN}} &\simeq -\frac{2G_N M}{\pi r} \text{Si}\left(\frac{r}{R_s}\right) \\ &\simeq V_N \left\{ 1 - \left[ 1 - \frac{2}{\pi} \text{Si}\left(\frac{r}{R_s}\right) \right] \right\}, \end{aligned} \quad (3.13)$$

where Si denotes the sine integral function (see Fig. 2 for an example).

It is important to remark that different shapes for the deviation from the classical potential  $V_N$  would be obtained if one employed a different UV cut-off in the integral (3.12). In particular, one could consider a smooth window function rather than the hard cut-off  $k < R_s^{-1}$ . Assuming such a smooth function is physically related to the distribution of matter in the black hole interior, one could in principle detect different matter profiles from analysing (test particle motion in) the

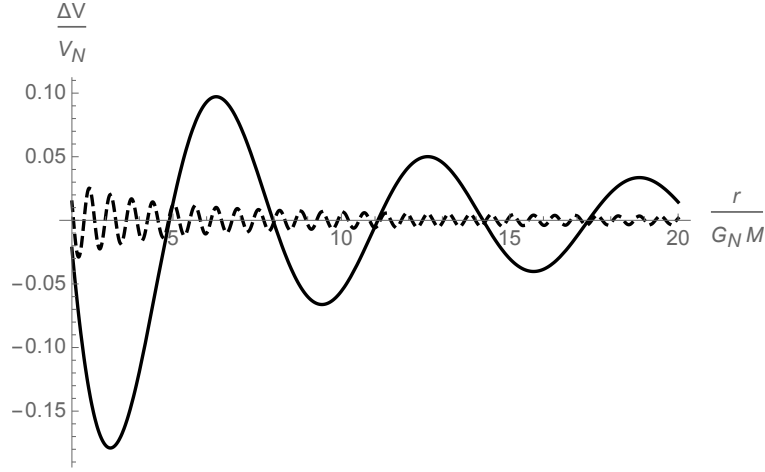


Figure 3: Oscillations of the quantum potential  $V_{\text{QN}}$  in Eq. (3.13) around the Schwarzschild expression  $V_N$  for  $R_s = G_N M = R_H/2$  (solid line) and  $R_s = R_H/20$  (dashed line) in the region outside the horizon  $R_H = 2 G_N M$ .

black hole exterior, whose geometry is going to be described in details next. However, we shall not endeavour in the survey of other possibilities in the present work, as very little can be accomplished analytically.

### 3.2 Geometry

The potential (3.13) can be used to reconstruct the full spacetime metric straightforwardly as [22]

$$ds^2 = -(1 + 2 V_{\text{QN}}) dt^2 + \frac{dr^2}{1 + 2 V_{\text{QN}}} + r^2 d\Omega^2, \quad (3.14)$$

where the dependence of  $V_{\text{QN}}$  on  $R_s$  therefore results in a quantum violation of the no-hair theorem [31]. Near the origin, this quantum corrected metric function reads

$$V_{\text{QN}} \simeq -\frac{2 G_N M}{\pi R_s} \left[ 1 - \frac{\pi r^2}{18 R_s^2} \right], \quad (3.15)$$

so that it is bounded and its derivative vanishes for  $r = 0$  (see Fig. 2). This suggests that gravitational forces remain finite, as we shall show in more details next.

In the classical Schwarzschild spacetime (1.6), the Kretschmann scalar  $R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \sim R^2 \sim r^{-6}$  for  $r \rightarrow 0$ , whereas for the above quantum corrected metric we have

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \simeq R^2 \simeq \frac{64 G_N^2 M^2}{\pi R_s^2 r^4}. \quad (3.16)$$

This ensures that tidal forces remain finite all the way to the centre, as can be seen more explicitly from the relative acceleration of radial geodesics approaching  $r = 0$ , to wit

$$\frac{\ddot{r}}{\delta r} = -R^1_{010} \simeq \frac{8 G_N^2 M^2}{9 \pi^2 R_s^4} \left( 1 - \frac{\pi R_s}{4 G_N M} \right), \quad (3.17)$$

where  $\delta r$  is the separation between two nearby radial geodesics and a dot denotes again the derivative with respect to the proper time, like in Eq. (1.7). We recall that, in the Schwarzschild spacetime,  $\ddot{\delta r}/\delta r \sim r^{-4}$ , which causes the so-called “spaghettification” of matter approaching the central singularity. The point  $r = 0$  in the quantum corrected geometry can instead be seen as an integrable singularity [34], where some geometric invariants still diverge but no harmful effects occur to matter.

One can further compute the effective energy-momentum tensor  $T_{\mu\nu}$  from the Einstein tensor  $G_{\mu\nu}$  of the metric (3.13) and find the effective energy density

$$\rho = -\frac{G^0_0}{8\pi G_N} = \frac{M}{2\pi^2 r^3} \sin\left(\frac{r}{R_s}\right), \quad (3.18)$$

the effective radial pressure

$$p_r = \frac{G^1_1}{8\pi G_N} = -\rho \quad (3.19)$$

and the effective tension

$$p_t = \frac{G^2_2}{8\pi G_N} = \frac{M}{4\pi^2 r^3} \left[ \sin\left(\frac{r}{R_s}\right) - \frac{r}{R_s} \cos\left(\frac{r}{R_s}\right) \right]. \quad (3.20)$$

The integrals of these quantities over space are finite and, in particular, one obtains

$$4\pi \int_0^\infty r^2 \rho(r) dr = -4\pi \int_0^\infty r^2 p_r(r) dr = M \quad (3.21)$$

and

$$4\pi \int_0^\infty r^2 p_t(r) dr = \frac{M}{2}. \quad (3.22)$$

Another quality of this quantum corrected geometry is that it does not contain a (inner) Cauchy horizon (whenever there exists the outer event horizon), a property which generalises to electrically charged black holes [23]. This excludes potentially serious casual issues which are often present in regular black hole candidates (see, *e.g.* Refs. [35] and references therein).

In light of the above results, the quantum geometry given by coherent states is consistent with a inner matter source that does not collapse to a singularity [19], similarly to what is found in the bootstrapped Newtonian approach [36,37]. The latter could therefore provide a compatible effective description of the quantum black hole interior including (some of the) nonlinearities. Moreover, the term inside square brackets in Eq. (3.13) induces oscillations around the classical  $V_N$  whose effects on test bodies could be observed at  $r > R_H$ . Such oscillations are determined by the (quantum) size  $R_s$  of the matter source and another important observation is that the amplitude of these fluctuations around  $V_N$  in the outer region (for  $r > R_H$ ) decreases for decreasing values of  $R_s/R_H$  (see Fig. 3). Therefore, one can always choose (finite) values of  $R_s/R_H$  so that the oscillations are too small to be measured by a distant observer. On the other hand, one could interpret this effect as a damping of transients as  $R_s$  shrinks and the collapse proceeds inside the horizon.

As can be seen from Fig. 2, the quantum corrected metric still contains a horizon when the cut-off scale  $R_s \lesssim R_H$ . The horizon radius  $r = r_H$  is given by the solution of  $2V_{QN} = -1$ , which can be computed numerically for different values of  $R_s$  (see Fig. 4). In particular, one can see that  $r_H = R_H$  for very specific values of  $R_s$ , the largest one being  $R_s \simeq 0.53 R_H$ , which is very close to the

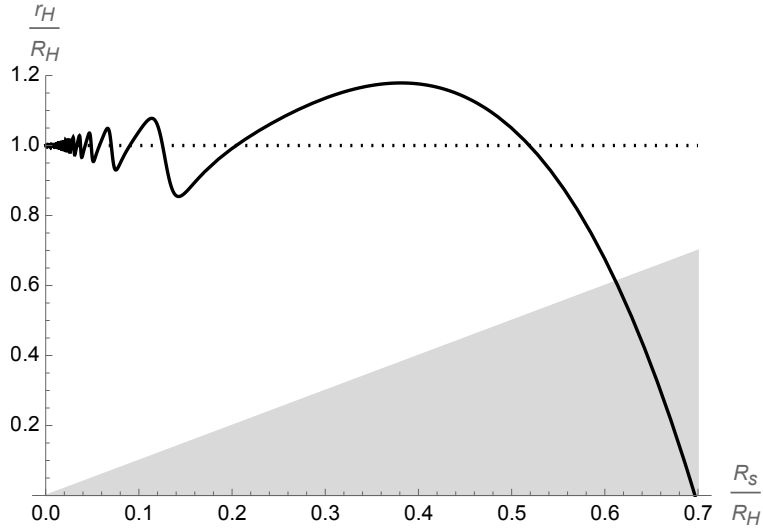


Figure 4: Radius  $r_H$  of the quantum corrected horizon (solid line) compared to the classical horizon  $R_H = 2G_N M$  (dotted line) for different values of  $R_s$ . Shaded regions cover points with  $r_H < R_s$  and do not correspond to black holes.

preferred case obtained in Ref. [19]. A remarkable result is that the horizon radius  $r_H$  shrinks for  $R_s$  approaching  $R_H$ . In fact,  $r_H = R_s$  for  $R_s \simeq 0.6 R_H$  and  $r_H$  would further vanish for  $R_s \simeq 0.7 R_H$ . This means that the material core cannot be too close in size to the classical gravitational radius  $R_H$  in order for it to lie inside the actual horizon. The shaded regions in Fig. 4 correspond to values of  $r_H < R_s$ , which are therefore not black holes.

### 3.3 Thermodynamics

The above departure of  $r_H$  from the Schwarzschild radius  $R_H = 2G_N M$  will also give rise to corrections for the horizon area  $\mathcal{A}_H$  and Bekenstein-Hawking entropy [12],

$$S_{\text{QBH}} = \frac{\mathcal{A}_H}{4\ell_p^2} = \frac{\pi r_H^2}{\ell_p^2} \quad (3.23)$$

and for the black hole temperature [38]

$$T_Q = \frac{\hbar \kappa}{2\pi} = \frac{\hbar}{2\pi} \left. \frac{\partial V_Q}{\partial r} \right|_{r=r_H}, \quad (3.24)$$

where  $\kappa$  is the surface gravity at the horizon. In particular, Fig. 4 shows that the quantum corrected horizon radius  $r_H \simeq R_H$  for  $R_s \lesssim R_H/2 = G_N M$ . Therefore, the quantum corrected gravitational entropy (3.23) remains very close to its classical Schwarzschild value  $S_{\text{BH}} = 4\pi M^2/\ell_p^2$  as long as the matter core is sufficiently smaller than  $R_H$ .

The quantum corrected Hawking temperature can also be computed only numerically and is displayed in Fig. 5. One can then see that quantum corrected black holes are colder than their classical counterpart, that is  $T_Q < T_H$ , if  $R_s \gtrsim R_H/2$ . In fact,  $T_Q$  would vanish for  $R_s \simeq 0.7 R_H$ , which precisely correspond to  $r_H = 0$  but falls outside the range  $r_H > R_s$  representing proper black

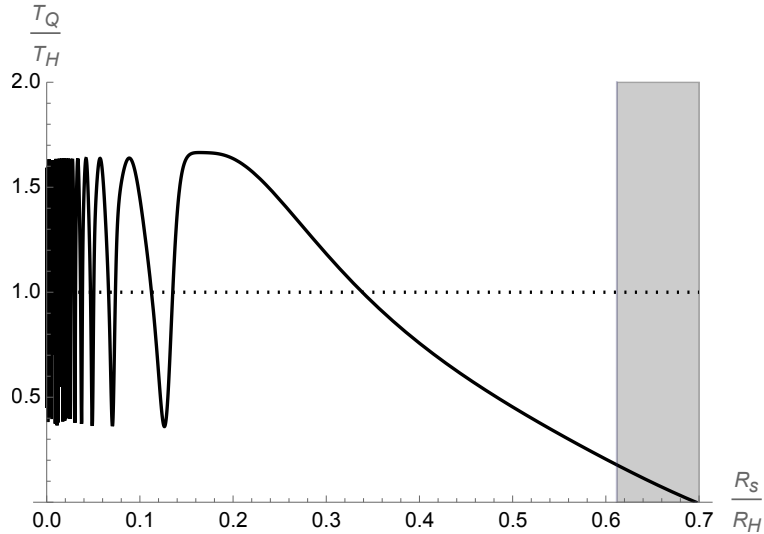


Figure 5: Temperature  $T_Q$  of the quantum corrected black hole (solid line) compared to the Hawking result  $T_H = \hbar/4\pi R_H$  (dotted line) for different values of  $R_s$ . Shaded regions cover points with  $r_H < R_s$  and do not correspond to black holes.

holes. One could argue that the shaded regions in Fig. 4 still represent some intermediate stage in the gravitational collapse or will perhaps become relevant at the end of the Hawking evaporation.<sup>8</sup>

## 4 Conclusions and outlook

In this work we started from the description of the static and spherically symmetric Schwarzschild geometry in terms of the coherent state of a massless scalar field on a reference flat spacetime, as previously employed, *e.g.* in Refs. [13, 14, 16, 18]. We recall that, in order to grant the coherent state a proper normalisation, we must relax the condition that its mean field reproduces the classical behaviour everywhere inside the horizon of a black hole [22]. In fact, it remains questionable whether the mean field approximation should make sense at all in the interior of a black hole as a (macroscopic) quantum object [6, 7, 19, 30], but here we took the less drastic viewpoint that a mean field exists, although it cannot reproduce the expected classical solution.

The weaker requirement in Eq. (3.10) that the coherent state yields the classical behaviour in the exterior can be achieved by removing modes of wavelength shorter than a UV cut-off  $R_s \lesssim R_H$ , where  $R_s$  can be physically interpreted as the size of the (quantum) matter source [19]. The expectation value of the gravitational potential then displays oscillations around the classical  $V_N$ , whose amplitudes decrease for shrinking  $R_s$ . We already commented that the specific shape of such deviations will be affected by the form of the UV cut-off, but the qualitative dependence on the scale  $R_s$  should be fairly well captured by the simplest choice of a sharp cut-off in Eq. (3.12). One could then interpret this effect as a decay in time due to the relative collapse of the inner material core of size  $R_s$  with respect to the outer boundary  $R_\infty$  of the Newtonian region. Eq. (3.8) then tells us that a core that shrinks down to  $R_s \simeq R_H/65$  roughly corresponds to an increase by 26 orders of magnitude in the outer Newtonian region of size  $R_\infty$ . Assuming the last signals

<sup>8</sup>For the late stage of the Hawking evaporation in the corpuscular picture, see *e.g.* [29] and references therein.

generated by the black hole formation (when  $R_s$  crosses  $R_H$ ) are approximately located at the edge of the outer Newtonian potential, and  $R_\infty$  expands at the speed of light, for a solar mass black hole with  $R_H \sim 1$  km, this means a time of around  $10^{10}$  years, the present age of the Universe. Such a consideration could be relevant for scenarios which predict a semiclassical bounce [39]. It is also tempting to push the model towards smaller and smaller black hole masses and conjecture that the oscillations shown in Fig. 3 grow significantly and totally disrupt the classical picture for  $R_s \sim R_H \sim \ell_p$ . This would exclude the existence of stable remnants of Planckian size at the end of the Hawking evaporation [38], in agreement with other approaches among those in Refs. [7].

The simple analysis we presented in this work neglects the precise quantum nature of the matter source and a better understanding of the interior might be obtained from the bootstrapped Newtonian picture [16, 20, 36, 37] or other modifications of the gravity-matter coupling [40]. For a quantum source, the size  $R_s$  needs not be sharply defined and one should include the effects of its uncertainty onto the gravitational state, like it was done by considering the bootstrapped Newtonian potential generated by a source of finite size in Ref. [18], or like in Ref. [19], where a bound on the compactness was obtained from the quantisation of the geodesic motion of the surface of a sphere of dust. The latter result in particular provides a clear motivation for including the UV scale  $R_s \sim G_N M$  in the definition (3.10) of the coherent state of gravitons, although the use of a sharp cut-off in Eq. (3.12) is justified mostly by the simplicity of the calculation. Moreover, we note that the natural time evolution of quantum states would imply that the oscillations around the expected classical potential could make the region around the horizon rather fuzzy [7, 41], with possibly relevant implications for the causal structure of astrophysical black holes and their interaction with infalling matter.

We conclude by remarking that the modified thermodynamics described in Section 3.3 complements the results from Ref. [32], in which different forms of information entropy were computed for the matter core of Ref. [19]. In particular, it was found that matter entropy grows with the ADM mass  $M$  like  $\ln(M^\alpha)$ , where the power  $0.4 < \alpha < 0.9$ . Since the quantum corrected gravitational entropy (3.23) is still proportional to the horizon area and grows like  $M^2$  for sufficiently small matter cores, the entropy of quantum black holes is expected to be dominated by the gravitational contribution, like in their classical counterparts. It will be interesting to investigate whether this conclusion is also supported by a direct calculation of the information entropy for the coherent states of the geometry [42].

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