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ON THE ∞-LAPLACIAN ON CARNOT GROUPS

FAUSTO FERRARI, NICOLÒ FORCILLO, AND JUAN J. MANFREDI

Dedicated to Professor Vladimir Maz'ya on the occasion of his 85^{th} *birthday.*

ABSTRACT. We prove Lipschitz estimates for viscosity solutions to Poisson problem for the infinity Laplacian in general Carnot groups.

CONTENTS

1. INTRODUCTION

The ∞-Laplacian operator has the property that viscosity sub- and super-solutions are Lipschitz continuous. This property holds in those settings where viscosity ∞-harmonic functions are absolute minimizers. These include the Euclidean space $\mathbb{R}^n$ ([ACJ04]), Riemannian vector fields ([Bie08]), the Heisenberg group ([Bie02]), Carnot groups ([Wan07], [Bie12]), and Grusǐn-type spaces ([Bie09]). We note that the case of general Carnot-Carathéodory spaces remains open. The converse direction, that absolute minimizers are ∞-harmonic functions was established in [BC05] for Carnot groups and in [DMV13] for general Carnot-Carathéodory spaces. Recently, this result was extended to more general Aronsson equations in [PVW22].

In the Euclidean case, solutions to the non-homogeneous equation

$$\Delta_\infty u = f, \tag{1.1}$$

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where f is bounded, continuous and does not change sign, are automatically Lipschitz continuous, since they are either sub- or super ∞-harmonic functions. In the Euclidean space $\mathbb{R}^n$, we can always add extra variables to reduce to this case. Suppose $u(x_1 \ldots, x_n)$ is a viscosity solution to (3.4) in a ball $B \subset \mathbb{R}^n$, then the function

$$v(x_1, \ldots, x_n, x_{n+1}) = u(x_1, \ldots, x_n) + C(x_{n+1})^{4/3},$$

where C is a constant, is a viscosity solution to

$$\Delta_\infty v = f + C^3 \frac{64}{81} \tag{1.2}$$

in the domain $B \times \mathbb{R}^+ \subset \mathbb{R}^{n+1}$. This is based on the identity

$$\Delta_\infty \left((x_{n+1})^{4/3} \right) = \frac{64}{81} \tag{1.3}$$

followed by an easy calculation assuming u is smooth, which requires justification in the viscosity sense (see [HF18]). We conclude that by choosing appropriate C, we can assume f is nonnegative. Then, the function v is a ∞-subhamonic, thus v and u are Lipschitz continuous. A variation of this argument works also in the case of Riemannian vector fields [FM21].

In this manuscript we extend the above argument to the case of Carnot groups.

2. Preliminaries and Statements of Results

A Carnot group $(\mathbb{G}, \cdot)$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = \bigoplus_{i=1}^{\nu} V^i \,, \tag{2.1}$$

where $\nu \in \mathbb{N}$, $\nu \geq 2$ and V^i is a vector subspace such that

$$\begin{aligned} &(i) \quad [V^1, V^i] = V^{i+1} \text{ if } \; i \leq \nu - 1, \\ &(ii) \quad [V^1, V^\nu] = \{0\} \,. \end{aligned} \tag{2.2}$$

Letting $n_i = \dim(V^i)$ and $n = n_1 + \cdots + n_\nu$, it is always possible to identify $(\mathbb{G}, \cdot)$ with a Carnot group whose underlying manifold is $\mathbb{R}^n$, and that satisfies the properties we describe next (see Chapter 2 of the book [BLU07]). A vector $h \in \mathfrak{g}$ can be written uniquely as

$$h = h_1 + h_2 + \ldots + h_\nu,$$

where $h_i \in V^i$ for $1 \leq i \leq \nu$.

We write points $x \in \mathbb{G}$ (identified with $\mathbb{R}^n$) as follows:

$$x = (x^{(1)}, \ldots, x^{(\nu)}) = (x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, \ldots, x_{\nu 1}, \ldots, x_{\nu n_\nu}),$$

where $x^{(i)}$ stands for the vector $(x_{i1}, \ldots, x_{in_i})$ for all $i = 1, \ldots, \nu$. The identity of the group is $0 \in \mathbb{R}^n$ and the inverse of $x \in \mathbb{R}^n$ is $x^{-1} = -x$.

The anisotropic dilations $\{\delta_\lambda\}_{\lambda > 0}$, defined as

$$\delta_\lambda(x) = (\lambda x^{(1)}, \ldots, \lambda^\nu x^{(\nu)}),$$

are group automorphisms. The number $Q = \sum_{i=1}^{\nu} i n_i$ is the homogeneous dimension of the group, and it agrees with the Hausdorff dimension of the metric space (G, d_{CC}), where d_{CC} is the Carnot-Carathéodory metric induced by the horizontal layer V^1, see Definition 2.1 below and [BLU07]. The Lebesgue measure of a measurable set $A \subset \mathbb{G}$ is denoted by $|A|$. It is left and right invariant, and δ_λ-homogeneous of degree Q; that is, we have $|\delta_\lambda(A)| = \lambda^Q |A|$ for all $\lambda > 0$.

The Jacobian basis of $\mathfrak{g}$ consists of left invariant vector fields

$$\{X_{11}, \dots, X_{1n_1}, \dots, X_{\nu 1}, \dots, X_{\nu n_\nu}\} = \{X_{ij}\}_{j=1,\dots,n_i}^{i=1,\dots,\nu}, \tag{2.3}$$

which coincide with $\{\partial_{x_{ij}}\}_{j=1,\dots,n_i}^{i=1,\dots,\nu}$ at the origin $x = 0$ and are adapted to the stratification; that is, for each $i = 1, \dots, \nu$ the collection $\{X_{i1}, \dots, X_{in_i}\}$ is a basis of the i-th layer V^i. As a consequence $X_{11}, \dots, X_{1n_1}$ are Lie generators of $\mathfrak{g}$, and will be referred to as horizontal vector fields. We shall relabel the horizontal layer as $X_1, \dots, X_{n_1}$.

The exponential map Exp: $\mathfrak{g} \longrightarrow \mathbb{G}$ written with respect to this basis is the identity, i.e.

$$\operatorname{Exp}\Big(\sum_{ij} h_{ij} X_{ij}\Big) = x \text{ and } x_{ij} = h_{ij}. \tag{2.4}$$

and $x_{ij} = h_{ij}$. In the above formula we used the convention that the sum $\sum_{ij}$ is extended to all indexes $i = 1, \dots, \nu$ and $j = 1, \dots, n_i$, which we will use throughout this exposition. By a slight abuse of notation we also denote by $X_{ij}(x)$ the vector in $\mathbb{R}^n$ whose components are the components of the the vector field X_{ij} with respect to the frame $\{\partial_{x_{kl}}\}_{l=1,\dots,n_k}^{k=1,\dots,\nu}$ at the point $x \in \mathbb{R}^n$. Denoting by $\star$ the group law in $\mathbb{G}$, the mapping $(x, y) \mapsto x \star y$ has polynomial entries when written in exponential coordinates.

Definition 2.1. *For $x, y \in \mathcal{G}$ and $r > 0$, let $AC_0(x, y, r)$ denote the set of all absolutely continuous functions $\varphi\colon [0,1] \mapsto \mathcal{G}$ such that $\varphi(0) = x$, $\varphi(1) = y$ and*

$$\varphi'(t) = \sum_{j=1}^{n_1} a_{1j}(t)\, X_{1j}(\varphi(t)) \quad \text{for a.e. } t \in [0,1] \tag{2.5}$$

for a vector of measurable functions $a = (a_{11}, \dots, a_{1n_1}) \in L^\infty([0,1], \mathbb{R}^{n_1})$ with

$$\|a\|_{L^\infty([0,1],\mathbb{R}^{n_1})} = \operatorname{ess\,sup}\left\{ |a(t)| = \left(\sum_{j=1}^{n_1} a_{1j}^2(t)\right)^{1/2} : t \in [0,1] \right\} < r. \tag{2.6}$$

Define the Carnot-Carathéodory distance as

$$d_{CC}(x, y) = \inf\{r > 0 \mid AC_0(x, y, r) \neq \emptyset\}.$$

Note that by the bracket generating property of $V^1 = \operatorname{span}\{X_1, \dots, X_{n_1}\}$ there is always an $r > 0$ such that $AC_0(x, y, r) \neq \emptyset$. The distance d_{CC} if left-invariant

$$d_{CC}(z \star x, z \star y) = d_{CC}(x, y)$$

and homogenous

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(z)) = \lambda\, d_{CC}(x, y).$$

We denote by $B(x_0, r)$ the open ball with respect to the metric d_{CC} centered at $x_0 \in \mathbb{G}$ with radius $r > 0$. Observe that

$$B(x_0, r) = x_0 \star B(0, r).$$

In this manuscript we consider the ∞-Laplacian determined by the horizontal distribution V^1 in a Carnot group $\mathbb{G}$. The horizontal gradient of a function u is denoted by

$$\mathfrak{X}u = \mathfrak{X}_1 u = (X_{11}u, \dots, X_{1n_1}u) \in V^1.$$

For the Taylor development in Carnot group we will need the second layer

$$\mathfrak{X}_2 u = (X_{21}u, \dots, X_{2n_2}u) \in V^2.$$

The second order horizontal derivative $\mathfrak{X}\mathfrak{X}u = \mathfrak{X}^2 u$ of u is the $n_1 \times n_1$ matrix with entries $X_i X_j u$. Notice that we use the super-index 2 for second derivatives in the first layer V^1 and the sub-index 2 for first derivatives in the second layer V^2.

For a smooth function $f\colon \mathbb{G} \mapsto \mathbb{R}$ and $x_0 \in \mathbb{G}$ the Taylor development of f at x_0 is

$$f(x_0 \star \delta_t(h)) = f(x_0) + t\,\langle \mathfrak{X}_1 f(x_0), h_1\rangle + t^2\,\langle \mathfrak{X}_2 f(x_0), h_2\rangle + t^2 \frac{1}{2}\langle \mathfrak{X}^2 f(x_0) h_1, h_1\rangle + o(t^2). \tag{2.7}$$

Note that if we set $x = x_0 \star \delta_t(h)$, we have $x_0^{-1} \star x = \delta_t(h) = (t\,h_1, t^2\,h_2, \dots, t^\nu h_\nu)$. Formula (2.7) follows from Chapter 20 in [BLU07].

The horizontal ∞-Laplacian is the left-invariant operator

$$\Delta_{\mathfrak{X},\infty} u = \langle \mathfrak{X}^2 u\, \mathfrak{X}u, \mathfrak{X}u\rangle = \sum_{i,j=1}^{n_1} X_i X_j u\, X_i u\, X_j u. \tag{2.8}$$

We consider the local regularity of viscosity solutions of the Poisson problem in a domain $\Omega \subset \mathbb{G}$

$$\Delta_{\mathfrak{X},\infty} u = f \text{ in } \Omega \tag{2.9}$$

where f is a continuous function.

Definition 2.2. *An upper semi-continuous function u is a viscosity subsolution of (2.9) in a domain Ω if whenever $\phi \in C^2(\Omega)$ touches u from above at a point $x_0 \in \Omega$ we have*

$$\Delta_{\mathfrak{X},\infty} \phi(x_0) \geq f(x_0).$$

A lower semi-continuous function v is a viscosity supersolution of (2.9) in a domain Ω if whenever $\phi \in C^2(\Omega)$ touches v from below at at a point $x_0 \in \Omega$ we have

$$\Delta_{\mathfrak{X},\infty} \phi(x_0) \leq f(x_0).$$

Recall that ϕ touches u from above at x_0 means $\phi(x) \leq u(x)$ for all x in a neighborhood of x_0 and $\phi(x_0) = u(x_0)$. To define ϕ touches u from below at x_0 just reverse the inequality. A viscosity solution is both a viscosity supersolution and a viscosity subsolution.

Theorem 2.3. *Let $\Omega \subset \mathbb{G}$ be a domain in a Carnot group $\mathbb{G}$ and $f\colon \Omega \mapsto \mathbb{R}$ be a continuous function. Let u be a continuous viscosity solution of the Poisson equation*

$$\Delta_{\mathfrak{X},\infty} u(x) = f(x) \text{ in } \Omega. \tag{2.10}$$

Then, the function u is locally Lipschitz continuous. More precisely, for all $x_0 \in \Omega$ such that $B(x_0, 2r) \subset \Omega$ we have

$$|u(x) - u(y)| \leq \|u\|_{L^\infty(B(x_0,2r))}\, d_{CC}(x,y), \tag{2.11}$$

for $x, y \in B(x_0, r)$.

Remark 2.4. *It would be expected that we can bound $\|u\|_{L^\infty(B(x_0,2r))}$ by some function of $\|f\|_{L^\infty(B(x_0,4r))}$. This is indeed the case in the Euclidean case ([LW08]), but the proof is based on knowing explicitly that $\Delta_\infty(|x|^{4/3})$ is a positive constant, a fact that is lacking in the case of general Carnot groups.*

Remark 2.5. *Note that this is an a-priori estimate. In general, uniqueness for the Poisson problem*

$$\begin{cases} \Delta_{\mathfrak{X},\infty} u(x) &= f(x) \text{ in } \Omega \\ u(x) &= g(x) \text{ on } \partial\Omega \end{cases} \tag{2.12}$$

fails in the Euclidean case when f changes sign, even when Ω is a ball and $g = 0$. See the counterexample in [LW08] and the discussion in [LW10].

When f does not change sign on Ω, the solutions of (3.1) are either viscosity subsolutions (when $f \geq 0$) of viscosity supersolutions (when $f \leq 0$) of the equation

$$\Delta_{\mathfrak{X},\infty} u(x) = 0 \text{ in } \Omega. \tag{2.13}$$

In this case Theorem 2.3 follows from the Lipschitz regularity of viscosity subsolutions of the equation (2.13). For the sake of completeness we will show below how this Lipschitz regularity follows from the comparison principle of Wang ([Wan07], and Bieske [Bie12]) and equivalence between absolute minimizer and comparison with cones of Champion-De Pascale [CDP07] .

To reduce the general case to the case when $f \geq 0$ we will lift the equation to the group $\mathbb{R} \oplus \mathbb{G}$. Elements in this group are pairs $(s, x) \in (\mathbb{R}, \mathbb{G})$ with following group law

$$(s, x) \cdot (t, y) = (s + t, x \star y),$$

and the homogeneous dilations are given by

$$\delta_\lambda(s, x) = (\lambda s, \delta_\lambda(x)).$$

The group $\mathbb{R} \oplus \mathbb{G}$ is a homogeneous stratified group of step r (Carnot group of step r) with horizontal layer $\{\frac{\partial}{\partial s}, X_1, \ldots, X_{n_1}\}$ and ambient manifold $\mathbb{R}^{n+1}$. See Section 4.1.5 in [BLU07] for a general discussion on the sum of Carnot groups.

For a function $U : \mathbb{R} \oplus \mathbb{G} \mapsto \mathbb{R}$ the horizontal gradient is the vector

$$\mathcal{Y}U = \left(\frac{\partial U}{\partial s}, X_1 U, \ldots, X_{n_1} U \right)$$

and we define the horizontal second derivative as the $(n_1 + 1) \times (n_1 + 1)$ matrix

$$\mathcal{Y}^2 U = \begin{pmatrix} \frac{\partial^2 U}{\partial s^2}, & \frac{\partial}{\partial s} X_1 U, & \ldots, & \frac{\partial}{\partial s} X_{n_1} U \\ X_1 \frac{\partial U}{\partial s}, & X_1 X_1 U, & \ldots, & X_1 X_{n_1} U \\ \ldots & \ldots & \ldots, & \ldots \\ X_{n_1} \frac{\partial U}{\partial s}, & X_{n_1} X_1 U, & \ldots, & X_{n_1} X_{n_1} U \end{pmatrix}.$$

Recall that this matrix, in general, $\mathcal{Y}^2 U$ is not symmetric, but that the quadratic form associated to is the same as that of its symmetrization $\frac{1}{2} \left(\mathcal{Y}^2 U + (\mathcal{Y}^2 U)^t \right)$ of $\mathcal{Y}^2 U$.

The horizontal infinity Laplacian of a smooth function of the form $U(s,x)=w(s)+u(x)$ in the group $\mathbb{R}\oplus\mathbb{G}$ is decoupled in a term on the s-variable plus a term on the x-variable

$$\begin{aligned}\langle \mathcal{Y}^2U\cdot\mathcal{Y}U,\mathcal{Y}U\rangle(s,x) &= \left\langle \begin{pmatrix} \frac{\partial^2 w}{\partial s^2}, & 0, & \dots, & 0\\ 0, & X_1X_1u, & \dots, & X_1X_{n_1u}\\ \dots & \dots & \dots, & \dots\\ 0, & X_{n_1}X_1u, & \dots, & X_{n_1}X_{n_1}u\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial w}{\partial s}\\ X_1u\\ \dots\\ X_{n_1}u\end{pmatrix},\begin{pmatrix}\frac{\partial w}{\partial s}\\ X_1u\\ \dots\\ X_{n_1}u\end{pmatrix}\right\rangle\\ &= \tfrac{\partial^2 w}{\partial s^2}(\tfrac{\partial w}{\partial s})^2+\langle\mathfrak{X}^2u\,\mathfrak{X}u,\mathfrak{X}u\rangle,\end{aligned}$$

which we write as

$$\Delta_{\mathcal{Y},\infty}U(s,x)=\Delta_\infty w(s)+\Delta_{\mathfrak{X},\infty}u(x).$$

In the viscosity sense, this decoupling is more subtle as we will see in the proof of Lemma 3.1 below.

3. Proofs

We begin with a generalization of the the result of Hong and Feng [HF18] that takes advantage of the decoupling above. It will allow us to reduce the proof of Theorem 2.3 to the case $f\ge 0$.

Lemma 3.1. *Let $\Omega\subset\mathbb{G}$ be a domain in a Carnot group $\mathbb{G}$ and $f\colon\Omega\mapsto\mathbb{R}$ be a continuous function. Let u be a continuous viscosity solution of the Poisson equation*

$$\Delta_{\mathfrak{X},\infty}u(x)=f(x)\ \text{in}\ \Omega. \tag{3.1}$$

Then, the function $U(s,x)=c\,s^{4/3}+u(x)$ is a viscosity solution of the equation

$$\Delta_{\mathcal{Y},\infty}U(s,x)=f(x)+c^3\frac{64}{81}\ \text{in}\ \{s\in\mathbb{R}\colon s>0\}\times\Omega. \tag{3.2}$$

Proof. Let us proof the subsolution case. Let $\phi\in C^2(\{s\in\mathbb{R}\colon s>0\}\times\Omega)$ touching U from above at a point $(s_0,x_0)\in\{s\in\mathbb{R}\colon s>0\}\times\Omega$. We have

$$\begin{cases} c\,s^{4/3}+u(x) \le \phi(s,x) \text{ for all } (s,x)\in\{s\in\mathbb{R}\colon s>0\}\times\Omega\\ c\,s_0^{4/3}+u(x_0) = \phi(s_0,x_0).\end{cases} \tag{3.3}$$

Then, the C^2 function $\psi(x)=\phi(s_0,x)-c\,s_0^{4/3}$ touches u from above at x_0. Therefore, we have

$$\Delta_{\mathfrak{X},\infty}\psi(x_0)\ge f(x_0). \tag{3.4}$$

Next, we compute $\Delta_{\mathcal{Y},\infty}\phi(s_0,x_0)$. Note that the function $s\mapsto\phi(s,x_0)-c\,s^{4/3}$ has a local minimum at s_0. Thus, we have

$$\frac{\partial}{\partial s}\phi(s_0,x_0)=\frac{4}{3}c\,s_0^{1/3} \tag{3.5}$$

and

$$\frac{\partial^2\phi}{\partial s^2}(s_0,x_0)-\frac{4}{9}c\,s_0^{-2/3}\ge 0. \tag{3.6}$$

Therefore, at the touching point (s_0, x_0), which we will often omit for the sake of notational brevity, we have

$$\begin{aligned}
\Delta_{\mathcal{Y},\infty}\phi(s_0,x_0) &= \langle \mathcal{Y}^2\phi\cdot\mathcal{Y}\phi, \mathcal{Y}\phi\rangle(s_0,x_0)\\
&= \left\langle \begin{pmatrix} \frac{\partial^2\phi}{\partial s^2}, & \frac{\partial}{\partial s}X_1\phi, & \dots, & \frac{\partial}{\partial s}X_{n_1}\phi \\ \frac{\partial}{\partial s}X_1\phi, & X_1X_1\psi, & \dots, & X_1X_{n_1}\psi \\ \dots & \dots & \dots, & \dots \\ \frac{\partial}{\partial s}X_{n_1}\phi, & X_{n_1}X_1\psi, & \dots, & X_{n_1}X_{n_1}\psi \end{pmatrix}\cdot\begin{pmatrix} \frac{4}{3}c\, s_0^{1/3} \\ X_1\psi \\ \dots \\ X_{n_1}\psi \end{pmatrix}, \begin{pmatrix} \frac{4}{3}c\, s_0^{1/3} \\ X_1\psi \\ \dots \\ X_{n_1}\psi \end{pmatrix}\right\rangle\\
&= \frac{\partial^2\phi}{\partial s^2}\left(\frac{4}{3}c\, s_0^{1/3}\right)^2 + 2\left(\frac{4}{3}c\, s_0^{1/3}\right)\left(\sum_{j=1}^{n_1} X_j\psi\frac{\partial}{\partial s}X_j\phi\right) + \Delta_{\mathfrak{X},\infty}\psi(x_0).
\end{aligned}$$

If $\mathfrak{X}\psi(x_0) = 0$, we obtain

$$\Delta_{\mathcal{Y},\infty}\phi(s_0,x_0) = \frac{\partial^2\phi}{\partial s^2}\left(\frac{4}{3}c\, s_0^{1/3}\right)^2 + \Delta_{\mathfrak{X},\infty}\psi(x_0) \geq \frac{64}{81}\, c^3 + f(x_0)$$

by (3.4) and (3.6). We assume from now on that $\mathfrak{X}\psi(x_0) \neq 0$. Let us write

$$Y(x_0) = \frac{1}{|\mathfrak{X}\psi(x_0)|}\left(\sum_{j=1}^{n_1} X_j\psi(x_0)\frac{\partial}{\partial s}X_j\phi(s_0,x_0)\right).$$

We get

$$\begin{aligned}
\Delta_{\mathcal{Y},\infty}\phi(s_0,x_0) &= \left\langle \begin{pmatrix} \frac{\partial^2\phi}{\partial s^2} & , & Y \\ Y & , & \frac{\Delta_{\mathfrak{X},\infty}\psi}{|\mathfrak{X}\psi|^2} \end{pmatrix}\cdot\begin{pmatrix} \frac{4}{3}c\, s_0^{1/3} \\ |\mathfrak{X}\psi| \end{pmatrix}, \begin{pmatrix} \frac{4}{3}c\, s_0^{1/3} \\ |\mathfrak{X}\psi| \end{pmatrix}\right\rangle\\
&= \left\langle \begin{pmatrix} \frac{\partial^2\phi}{\partial s^2} - \frac{4}{9}c\, s_0^{-2/3} & , & Y \\ Y & , & \frac{\Delta_{\mathfrak{X},\infty}\psi - f(x_0}{|\mathfrak{X}\psi|^2} \end{pmatrix}\cdot\begin{pmatrix} \frac{4}{3}c\, s_0^{1/3} \\ |\mathfrak{X}\psi| \end{pmatrix}, \begin{pmatrix} \frac{4}{3}c\, s_0^{1/3} \\ |\mathfrak{X}\psi| \end{pmatrix}\right\rangle\\
&+ \frac{4}{9}c\, s_0^{-2/3}\left(\frac{4}{3}c\, s_0^{1/3}\right)^2 + f(x_0)\\
&\geq \frac{64}{81}\, c^3 + f(x_0).
\end{aligned}$$

This will hold as long as we can show that the matrix

$$M = \begin{pmatrix} \frac{\partial^2\phi}{\partial s^2}(s_0,x_0) - \frac{4}{9}c\, s_0^{-2/3}, & Y(x_0) \\ Y(x_0), & \frac{\Delta_{\mathfrak{X},\infty}\psi(x_0) - f(x_0}{|\mathfrak{X}\psi(x_0)|^2} \end{pmatrix}$$

is positive definite.

For $h \in \overline{B(0,1)}$ and $s, t \in \mathbb{R}$ in a neighborhood of $(0,0)$ we have

$$c(s_0+s)^{4/3} + u(x_0 \star \delta_t(h)) \leq \phi(s_0+s, x_0 \star \delta_t(h))$$

with equality for $s = t = 0$. If follows that in a neighborhood of $(0,0)$ we have

$$u(x_0 \star \delta_t(h)) \leq \phi(s_0+s, x_0\star\delta_t(h)) - c\,(s_0+s)^{4/3}. \tag{3.7}$$

Note that the right hand side is smooth, so we can expand using the Taylor Theorem in Carnot groups. Let us write $h=(h_1,h_2,\dots,h_\nu)$

$$\begin{aligned}
u(x_0\star\delta_t(h)) &\le \phi(s_0+s, x_0\star\delta_t(h)) - c\,(s_0+s)^{4/3}\\
&= \phi(s_0,x_0) + s\tfrac{\partial\phi}{\partial s}(s_0,x_0) + t\langle \mathfrak{X}\phi(s_0,x_0),h_1\rangle + t^2\langle \mathfrak{X}_2\phi(s_0,x_0),h_2\rangle\\
&\quad +\tfrac12\left\langle \begin{pmatrix} \frac{\partial^2\phi}{\partial s^2}, & \frac{\partial}{\partial s}X_1\phi, & \dots, & \frac{\partial}{\partial s}X_{n_1}\phi\\ \frac{\partial}{\partial s}X_1\phi, & X_1X_1\phi, & \dots, & X_1X_{n_1}\phi\\ \dots & \dots & \dots, & \dots\\ \frac{\partial}{\partial s}X_{n_1}\phi, & X_{n_1}X_1\phi, & \dots, & X_{n_1}X_{n_1}\phi\end{pmatrix}\cdot\begin{pmatrix} s\\ t\,h_{11}\\ \dots\\ t\,h_{1n_1}\end{pmatrix}, \begin{pmatrix} s\\ t\,h_{11}\\ \dots\\ t\,h_{1n_1}\end{pmatrix}\right\rangle\\
&\quad -cs_0^{4/3} - c\tfrac43 s_0^{1/3}s - c\tfrac49 s_0^{-2/3}s^2 + o(s^2+t^2).
\end{aligned}$$

Using (3.3) and (3.5) we obtain

$$\begin{aligned}
u(x_0\star\delta_t(h)) - u(x_0) &\le t\,\langle \mathfrak{X}\phi(s_0,x_0),h_1\rangle + t^2\langle \mathfrak{X}_2\phi(s_0,x_0),h_2\rangle\\
&\quad +\tfrac12\left[\left(\tfrac{\partial^2\phi}{\partial s^2} - c\tfrac49 s_0^{-2/3}\right)s^2 + 2\,s\,t\sum_{j=1}^{n_1} h_i\frac{\partial}{\partial s}X_i\phi + t^2\left(\sum_{i,j=1}^{n_1} X_iX_j\phi\, h_{1i}h_{1j}\right)\right]\\
&\quad +o(s^2+t^2).
\end{aligned}$$

Setting $s=t$ we obtain

$$\begin{aligned}
u(x_0\star\delta_t(h)) - u(x_0) &\le t\,\langle \mathfrak{X}\phi(s_0,x_0),h_1\rangle + t^2\langle \mathfrak{X}_2\phi(s_0,x_0),h_2\rangle\\
&\quad +\tfrac12 t^2\left[\left(\tfrac{\partial^2\phi}{\partial s^2} - c\tfrac49 s_0^{-2/3}\right) + 2\sum_{j=1}^{n_1} h_{1i}\frac{\partial}{\partial s}X_i\phi + \left(\sum_{i,j=1}^{n_1} X_iX_j\phi\, h_{1i}h_{1j}\right)\right]\\
&\quad +o(t^2).
\end{aligned}$$

We conclude that

$$\begin{aligned}
a_h &= \limsup_{t\to 0}\frac{u(x_0\star\delta_t(h)) - u(x_0) - t\,\langle \mathfrak{X}\phi(s_0,x_0),h_1\rangle}{\frac12 t^2}\\
&\le 2\,\langle \mathfrak{X}_2\phi(s_0,x_0),h_2\rangle + \left(\tfrac{\partial^2\phi}{\partial s^2} - c\tfrac49 s_0^{-2/3}\right) + 2\sum_{j=1}^{n_1} h_{1i}\frac{\partial}{\partial s}X_i\phi + \left(\sum_{i,j=1}^{n_1} X_iX_j\phi\, h_{1i}\,h_{1j}\right) < \infty.
\end{aligned}$$

We can then find $t_h>0$ such that for for $0<t\le t_h$ we have

$$u(x\star\delta_t(h)) \le u(x_0) + t\,\langle \mathfrak{X}\phi(s_0,x_0),h_1\rangle + t^2\frac12(a_h+\epsilon)^2. \tag{3.8}$$

For $\delta>0$ set

$$\xi_0 = \frac{\mathfrak{X}\phi(s_0,x_0)}{|\mathfrak{X}\phi(s_0,x_0)|}\in V_1$$

and

$$C_\delta = \left\{\xi : |\xi_1| = 1, |\xi_i|\le 1,\ i=2,\dots\nu,\ \text{and } \langle \xi_1,\xi_0\rangle_{\mathbb{R}^{n_1}} \ge 1-\delta\right\}.$$

Recall that we have $n = n_1 + n_2 + \ldots + n_\nu$. From the continuity of u and the compactness of C_δ we conclude that $\inf\{t_h \colon h \in C_\delta\} = t_0 > 0$. We set

$$a = \limsup_{\xi \to \xi_0} a_\xi \tag{3.9}$$

For arbitrary $\epsilon > 0$ we can find $\delta > 0$ such that for $\xi \in C_\delta$ we have

$$a_\xi \le a + \epsilon.$$

For $0 < t < t_0$ and $\xi \in C_\delta$ we have

$$\begin{aligned} u(x_0 \star \delta_t(\xi)) &\le u(x_0) + t\,\langle \mathfrak{X}\phi(s_0,x_0), \xi_1\rangle + \tfrac{1}{2}(a_h + \epsilon)t^2 \\ &\le u(x_0) + t\,\langle \mathfrak{X}\phi(s_0,x_0), \xi_1\rangle + \tfrac{1}{2}(a + 2\epsilon)t^2. \end{aligned}$$

We can write ξ_1 uniquely as $\xi_1 = \langle \xi_1, \xi_0\rangle\, \xi_0 + \xi_1^\perp$, where $\xi_1^\perp$ is horizontal and perpendicular to ξ_0. Using $x = x_0 \star \delta_t(\xi)$ we write

$$t\,\xi_1^\perp = (x_0^{-1} \star x)_1 - \langle (x_0^{-1} \star x)_1, \xi_0\rangle\, \xi_0.$$

Consider the smooth function of x

$$\begin{aligned} g(x_0^{-1} \star x) = g(x_0 \star \delta_t(\xi)) &= u(x_0) + t\,\langle \mathfrak{X}\phi(s_0,x_0), \xi_1\rangle + \tfrac{1}{2}(a + 2\epsilon)t^2 + \tfrac{1}{2}t^2\, K|\xi_1^\perp|^2 \\ &= u(x_0) + \langle \mathfrak{X}\phi(s_0,x_0), (x_0^{-1} \star x)_1\rangle \\ &\quad + \tfrac{1}{2}(a + 2\epsilon)|(x_0^{-1} \star x)_1|^2 + K\,\left|(x_0^{-1} \star x)_1 - \langle (x_0^{-1} \star x)_1, \xi_0\rangle\, \xi_0\right|^2. \end{aligned}$$

It touches u from above at x_0 for K large enough by (3.8), since $|h_1^\perp| \ge \delta > 0$ whenever $h \notin C_\delta$ and we have a uniform bound for a_h. Therefore the pair $(\mathfrak{X}\phi(s_0,x_0), (a + 2\epsilon)I_{n_1} + D)$ is in the subelliptic superjet for u at the point x_0. Here D is an $n_1 \times n_1$ square matrix such that $D \cdot \mathfrak{X}\phi(s_0,x_0) = 0$. Since $\epsilon > 0$ is arbitrary, we conclude that $(\mathfrak{X}\phi(x_0), a\, I_{n_1} + D)$ is in the closure of the subelliptic superjet for u at x_0. Therefore, we have

$$a\,|\mathfrak{X}\phi(s_0,x_0)|^2 \ge f(x_0). \tag{3.10}$$

On the other hand, from (3.9) we deduce that given arbitrary $\epsilon > 0$ we can find $\delta > 0$ and a sequence $t_j \to 0$ and a vector $k \in C_\delta$ such that

$$u(x_0 \star \delta_{t_j}(k)) - u(x_0) - t_j\,\langle \phi(s_0,x_0), k_1\rangle > \frac{1}{2}t_j^2\,(a_k - \epsilon) > \frac{1}{2}t_j^2(a - 2\epsilon).$$

Therefore, we can write

$$\begin{aligned} \tfrac{1}{2}t_j^2(a - 2\epsilon) &\le 2\,t_j^2\langle \mathfrak{X}_2\phi(s_0,x_0), k_2\rangle \\ &\quad + \tfrac{1}{2}\left[\left(\tfrac{\partial^2\phi}{\partial s^2} - c\tfrac{4}{9}s_0^{-2/3}\right)s^2 + 2\,s\,t_j\sum_{j=1}^{n_1} k_i \frac{\partial}{\partial s}X_i\phi + t_j^2\left(\sum_{i,j=1}^{n_1} X_iX_j\phi\, k_ik_j\right)\right] \\ &\quad + o(s^2 + t_j^2). \end{aligned}$$

Next we set $s_j = \alpha\, t_j$, divide by $\frac{1}{2}t_j^2$ and let $t_j \to 0$ to get

$$(a_\delta - 2\epsilon) \le 2\langle \mathfrak{X}_2\phi(s_0,x_0), k_2\rangle + \left(\frac{\partial^2\phi}{\partial s^2} - c\frac{4}{9}s_0^{-2/3}\right)\alpha^2 + 2\alpha\sum_{j=1}^{n_1} k_{1i}\frac{\partial}{\partial s}X_i\phi + \left(\sum_{i,j=1}^{n_1} X_iX_j\phi\, k_{1i}\, k_{1j}\right).$$

We can now let $\delta \to 0$. The vector k converges to ξ_0 so that $k_2 \to 0$ and we get

$$(a - 2\epsilon) \le \left(\frac{\partial^2\phi}{\partial s^2} - c\frac{4}{9}s_0^{-2/3}\right)\alpha^2 + 2\alpha\sum_{j=1}^{n_1}((\xi_0)_{1i}\frac{\partial}{\partial s}X_i\phi + \left(\sum_{i,j=1}^{n_1}(X_iX_j\phi)\,(\xi_0)_{1i}\,(\xi_0)_{1j}\right).$$

Since this is true for every $\alpha \in \mathbb{R}$ the matrix

$$M_\epsilon = \begin{pmatrix} \frac{\partial^2\phi}{\partial s^2}(s_0,x_0) - \frac{4}{9}c\,s_0^{-2/3}, & \sum_{j=1}^{n_1}(\xi_0)_{1i}\frac{\partial}{\partial s}X_i\phi \\ \sum_{j=1}^{n_1}(\xi_0)_{1i}\frac{\partial}{\partial s}X_i\phi, & \left(\sum_{i,j=1}^{n_1} X_iX_j\phi\,(\xi_0)_{1i}\,(\xi_0)_j\right) - (a-2\epsilon) \end{pmatrix}$$

is positive semi-definite. Letting $\epsilon \to 0$ and using (3.10) we reach the desired conclusion. □

Next, we discuss the regularity of viscosity subsolutions. The Lipschitz regularity of viscosity sub- and super-solutions follows from the fact that ∞-harmonic functions are absolute minimizers with respect the d_{CC} metric (see the general discussion in [DMV13]). As mentioned in the Introduction, the fact that absolute minimizers are ∞-harmonic holds for general Carnot-Carathéodory spaces, but the reverse is only known, to the best of our knowledge, for Carnot groups and Riemannian vector fields. These are the cases when we have uniqueness of ∞-harmonic functions with given boundary values ([Wan07, Bie12]).

Denote by ω_{r,x_0} the ∞-harmonic function in the punctured ball $B(x_0,r)\setminus\{x_0\}$ with boundary values $\omega_{r,x_0}(x_0) = 0$ and $\omega_{r,x_0}(x) = r$ for $x \in \partial B(x_0,r)$. This is the metric cone with vertex x_0 and base $B(x_0,r)$. Let us recall the following estimate, whose proof we include for the sake of completeness.

Lemma 3.2. *For $x \in B(x_0,r)$ we have*

$$\omega_{r,x_0}(x) \le d_{CC}(x_0,x).$$

Proof. By a result of Monti and Serra Cassano [MSC01] we have $\|\mathfrak{X}(d_{CC}(x_0,x))\| = 1$ for a. e. $x \in B(x_0,r)$. Since ω_{r,x_0} is ∞-harmonic it is an absolute minimizer of the functional $\|\mathfrak{X}u\|_\infty$, see [Wan07, Bie12]. We conclude that $\|\mathfrak{X}(\omega_{r,x_0}(x))\|_\infty \le 1$, so that

$$\omega_{r,x_0}(x) = \omega_{r,x_0}(x) - \omega_{r,x_0}(x_0) \le \|\mathfrak{X}(\omega_{r,x_0}(x))\|_\infty\, d_{CC}(x_0,x) \le d_{CC}(x_0,x).$$

□

Lemma 3.3. *Let $\Omega \subset \mathbb{G}$ be a domain in a Carnot group $\mathbb{G}$. Let the function u satisfy*

$$\Delta_{\mathfrak{X},\infty}u(x) \ge 0 \text{ in } \Omega \tag{3.11}$$

in the viscosity sense. Then, if the function u is not identically $-\infty$, it is locally bounded and Lipschitz continuous with respect to the metric d_{CC}. Moreover, if $B(x_0,2r) \subset \Omega$ we have the estimate

$$|u(x) - u(y)| \le \|u\|_{L^\infty(B(x_0,2r))}\, d_{CC}(x,y),$$

for $x, y \in B(x_0,r)$.

Proof. Let us first recall the proof of the local boundedness of u. Since u is upper semicontinuous it is locally bounded above. We may further assume that $u \le 0$. Suppose $u(x_0) = -\infty$ for some $x_0 \in \Omega$ with $B(x_0, r) \subset \Omega$. For $k > 0$ we have

$$u(x) \le -k(r - \omega_{r,x_0}(x))$$

for all $x \in \partial \mathrm{B}(x_0, r) \cup \{x_0\}$.

By Lemma 3.2 we conclude that this inequality holds for all $x \in B(x_0, r)$. Letting $k \to \infty$ we get $u(x) = -\infty$ for all $x \in B(x_0, r)$, and thus in Ω by a connectivity argument, in contradiction with the local boundedness of u. Therefore, we conclude that u if finite everywhere.

Next, suppose that for $x_0 \in B(x_0, 2r) \subset \Omega$ there exists a sequence $x_k \in \Omega$ such that $x_k \to x_0$ and $\lim_{k\to\infty} u(x_k) = -\infty$. Select a subsequence, also denoted by x_k, such that $u(x_k) \le -k\, r$. By the comparison principle, we conclude that

$$u(x) \le -k(r - \omega_{r,x_k}(x))$$

for $x \in B(x_k, r)$. This inequality must also hold for x_0

$$u(x_0) \le -k(r - \omega_{r,x_k}(x_0)).$$

Using $\omega_{r,x_k}(x_0) \le d_{CC}(x_k, x_0)$ (see [CDP07]) we obtain for k large enough

$$u(x_0) \le -k\frac{r}{2}.$$

Letting $k \to \infty$ we get $u(x_0) = -\infty$, in contradiction with the fact that u is finite everywhere.

To prove the Lipschitz bound compare $u(x) - u(x_0)$ with the function $\|u\|_{L^\infty(B(x_0,r))}\, \omega_{r,x_0}(x)$ on $\partial \mathrm{B}(x_0, r) \cup \{x_0\}$ and conclude

$$u(x) - u(x_0) \le \|u\|_{L^\infty(B(x_0,r))}\, \omega_{r,x_0}(x) \le \|u\|_{L^\infty(B(x_0,r))}\, d_{CC}(x_0, x).$$

To prove the inequality for $u(x_0)-u(x)$ compare $u(y)-u(x)$ with the function $\|u\|_{L^\infty(B(x,r))}\, \omega_{r,x}(y)$. □

Proof of Theorem 2.3: By Lemma 3.1 the function $U(s, x) = c\, s^{4/3} + u(x)$ is a continuous viscosity solution of equation (3.2). Choosing

$$c = \frac{3^{4/3}}{4}\, \|f\|_{L^\infty(B(x_0,2r))}$$

we have

$$f(x) + c^3 \frac{64}{81} \ge 0, \text{ for } x \in B(x_0, 2r).$$

Therefore, the function $U(s, x)$ is ∞-subharmonic in $(\eta, 2\,\eta) \times B(x_0, 2r)$ for any $\eta > 0$. By Lemma 3.3 $U(s, x)$ is Lipschitz continuous with respect to the Carnot-Carathéodory metric in $\mathbb{R} \oplus \mathbb{G}$. Let us denote this metric by d'_{CC}. We then have

$$|U(s, x) - U(t, y)| \le \|U\|_{L^\infty((\eta,2\eta)\times B(x_0,2r))}\, d'_{CC}((s, x), (t, y)). \tag{3.12}$$

Next, we observe that for some constant $C > 0$ independent of η we have

$$\|U\|_{L^\infty((\eta,2\eta)\times B(x_0,2r))} \le C\eta^{4/3} + \|u\|_{L^\infty(B(x_0,2r))},$$

and that we have the estimate

$$d'_{CC}((s, x), (t, y)) \le |s - t| + d_{CC}(x, y),$$

since we can consider curves going from (s, x) to (t, x) and then from (t, x) to (t, y). Letting $s = t = 0$ in (3.12), we obtain

$$|u(x) - u(y)| \leq \left(C\, \eta^{4/3} + \|u\|_{L^\infty(B(x_0, 2r))} \right) d_{CC}(x, y).$$

Since $\eta > 0$ is arbitrary we have completed the proof. □

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