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The Unreasonable Effectiveness of Physics in Mathematics

Daniele Molinini

Abstract

The philosophical problem that stems from the successful application of mathematics in the empirical sciences has recently attracted growing interest within philosophers of mathematics and philosophers of science. Nevertheless, little attention has been devoted to the converse applicability issue of how physical considerations find successful application in mathematics. In this article, focusing on some case studies, I address the latter issue and argue that some successful applications of physics to mathematics essentially depend on the use of conservation principles. I conclude with a short discussion of how the successful interplay between conservation principles and mathematics can be accounted for.

1 Introduction

2 Applications of Physics to Mathematics

3 Why Successful?

4 The Role of Physical Principles in Converse Applications

5 The Reasonable Effectiveness of Physical Principles in Mathematics

6 Conclusion

1 Introduction

How can abstract mathematical objects like functions and numbers, which lack spatiotemporal location and causal efficacy, be successful in science and enlighten us about the behaviour and the properties of concrete entities like apples and electrons? The philosophical issue is known today as 'the problem of applicability of mathematics' and, not surprisingly, the interest in it goes back to the Greeks and to the Pythagorean investigation of harmonic intervals in terms of arithmetic.[1]

The problem of the applicability of mathematics has been at the centre of attention in many recent discussions in philosophy of mathematics and philosophy of science. The motivation for this consideration comes not only from the fact that successful interactions between mathematics and the empirical (that is, the natural and social) sciences have grown massively over the last decades. It also results from the many connections that the applicability issue has with other debates that have recently received a renewed attention, as for instance that concerning the role of idealizations and mathematical models in science or that over the (purported) explanatory function of mathematics in science. Take, for instance, the debate on mathematical explanation. Articulating a plausible account of the applicability of mathematics in the empirical sciences is particularly important for those philosophers who maintain that mathematics can play an explanatory role in science. Indeed, successful applicability (of mathematics) is a necessary, although not sufficient, condition for mathematical explanation of empirical facts. As observed by Shapiro ([1983], p. 525), 'strictly speaking, a mathematical description, model, structure, theory, or whatever, cannot serve as an explanation of a non-mathematical event without an account of the relationship between mathematics *per se* and scientific reality *per se*'.[2]

The pressing issue has therefore become that of accounting for the success, or effectiveness, of mathematics in science. Call this the 'direct applicability problem', and 'direct applications' the successful applications of mathematics in science.[3] The direct applicability problem is surely central to the more general problem—which we may call the 'general problem of applicability'—of accounting for the several ways in which the empirical sciences and mathematics successfully interact. Nevertheless, it is just a component of it. In fact, there is another

[1] Many philosophical studies have addressed the problem of accounting for the success of mathematics in the natural and social sciences; for instance, accounts of applicability have been offered in (Steiner [1998], [2005]; Pincock [2004]; Bueno and Colyvan [2011]; Rizza [2013]; Bueno and French [2018]; McCullough-Benner [2020]). Some of these studies also serve as surveys on the applicability issue. None of these, however, offers an historical reconstruction of how the topic has emerged and developed in the history of philosophy. An historically oriented overview of the applicability problem is given in (Panza and Molinini [forthcoming]).

[2] Yet, surprisingly enough, philosophers interested in mathematical explanation in science typically tend to focus on explanation alone, without addressing the applicability issue (see, for example, Lyon and Colyvan [2008]; Baker [2009]; Batterman [2010]; Pincock [2015]; Lange [2017]; some exceptions are Bueno and Colyvan [2011]; Bueno and French [2011]; Bangu [2012]; Baron [2019]).

[3] Additionally, there are applications of mathematics that are fully internal to mathematics, as for instance the application of a mathematical theory to another mathematical theory (see Steiner [1998], [2005]). In my analysis I will concentrate on the interplay between mathematics and physics, and therefore I shall not consider such 'internal' applications.

aspect of the general problem of applicability that has not been addressed yet. This aspect has to do with the successful application of (methods and ideas that are proper to) science in mathematics. In this article I shall focus on such 'converse' applications and on the 'converse applicability problem' that stems from them, namely, the philosophical problem of accounting for the effectiveness of science in mathematics. More precisely, I will concentrate on the successful application of physics to mathematics.[4]

Although the general problem of applicability is usually conflated with the direct applicability problem, we have many examples of converse applications from scientific practice. These examples clearly show how the converse applicability problem is another crucial (though less known and less studied) component of the philosophical analysis of the successful interplay between science and mathematics.

In Section 2, I shall present some examples of converse applications. I shall report four cases in which a mathematical result is obtained using physical, and therefore non-mathematical, considerations. Next, in Section 3, I shall show why these cases are successful applications. In particular, I shall propose two different criteria to evaluate the success of direct and converse applications. In Section 4, I shall maintain that the examples explored in Section 2 are instances of a specific kind of converse application. I shall show that the success of some converse applications essentially depends on the application of a conservation principle, which operates as a bridge between the concrete entities of physics and the abstract entities of mathematics. Thus my suggestion will be that there is a cluster of cases of converse applications for which the converse applicability problem can be restated in terms of the successful application of some physical principles, such as conservation of energy and conservation of mass, in mathematics. Given this view, it remains to explore the issue of how physical principles and mathematics successfully interact, namely, to account for the effectiveness of physical principles in mathematics. In Section 5, I shall sketch some proposals that may be used to address this issue. Finally, in the concluding section, I shall resume the results of my analysis and point to the import that the present article, which provides a step towards the investigation of the unreasonable effect-

[4] By limiting my analysis to the successful interplay between physics and mathematics, I will not directly address the broader problem of how other empirical sciences—besides physics—and mathematics successfully interact. Nevertheless, the approach suggested here may have interesting repercussions on such broader discussion. I will point to some of these outcomes in my concluding remarks.

iveness of a natural science—physics—in mathematics, may have for the direct applicability problem.

Before moving to Section 2, in which I illustrate four cases of successful application of physics to mathematics, let me add here a brief word about the content of the next section. In my presentation of the examples, I have decided not to skip many mathematical details that are often omitted when a case study coming from scientific practice is presented in a philosophical article. By proceeding in this way, I am aware that the risk of taxing the patience of the reader is high. On the other hand, my motivation for this choice is twofold. First, by reporting a more comprehensive mathematical treatment I want to show how these cases entirely belong to mathematical practice and they are not just toy cases that have been set up and manipulated for philosophical discussion. Second, I believe that a more detailed rendering of what is going on in these cases not only provides the reader with a better understanding of the cases themselves but it also helps to highlight how physics is applied to mathematics. And since the interplay between the two disciplines is central to the philosophical topic addressed here, the choice of presenting the case studies in a more detailed way is also instrumental in developing the analysis contained in the remaining sections.[5]

2 Applications of Physics to Mathematics

In recent decades, many textbooks have been devoted to illustrating how physics—mainly mechanics—finds successful applications in mathematics (see, for example, Uspenskii [1961]; Kogan [1974]; Levi [2009]). Nevertheless, the interest in applications of physics to mathematics is not novel and it goes back at least to Archimedes. In his work *The Method of Mechanical Theorems, for Eratosthenes* (henceforth *Method*), which is a private communication to Eratosthenes, Archimedes describes how the application of some mechanical laws in mathematics has led him to the discovery of many important results. The first result reported by Archimedes appears in Proposition 1 and concerns the area of a parabolic segment: any

[5] Concerning Archimedes' example, namely, the first case presented in Section 2, there is a further point that could be mentioned: as observed by many historians of science and mathematics (see, for example, Dijksterhuis [1987], p. 50), for a modern reader it may be very difficult to appreciate how Archimedes uses mechanical considerations in his mathematical treatment without a presentation of how he develops the mathematical arguments. Hence the need to report Archimedes' line of reasoning in a more detailed way.

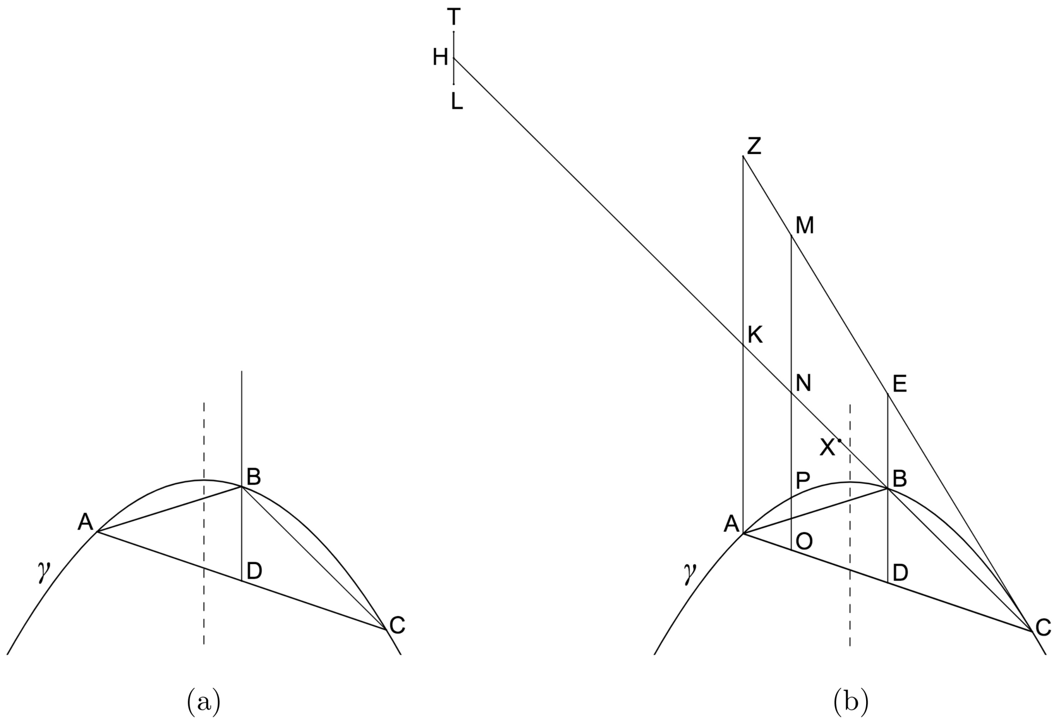

Figure 1. Geometric diagrams for Proposition 1 of the *Method*.

segment of a parabola is four-thirds of the triangle that has the same base and equal height.[6] Archimedes offers a geometric proof of this proposition in his treatise *Quadrature of the Parabola*. Nevertheless, in the *Method* he puts forwards an argument that shows how, combining geometric results with the law of the lever, he was led to the discovery.[7]

How did Archimedes use physics to discover Proposition 1, namely, the proposition that 'any segment of a parabola is four-thirds of the triangle that has the same base and equal height'? Let's have a look at his argument.

Consider a parabolic segment ABC (henceforth ABC_{ps}), bounded by the straight line AC and the parabola γ (see Figure 1a). Let AC be bisected in D and DB be drawn parallel to the axis of symmetry, or diameter, of the parabola (the dashed line in Figure 1a). Now join AB and BC. Archimedes shows that the area of the parabolic segment ABC_{ps} is equal to four-thirds of the area of the triangle $\triangle ABC$ inscribed in the parabola.

From A draw the straight line AKZ parallel to DB (see Figure 1b). Line AKZ meets the

[6] A parabolic segment is the region bounded by a parabola and a line.

[7] The first edition of the *Method* was made by Johan Ludvig Heiberg ([1909]), who discovered Archimedes' *Palimpsest* (which includes the *Method*) in Constantinople in 1906. In this article I am adopting Dijksterhuis' ([1987]) exposition, which adds some clarity to the mathematical content. Thanks to the recent (re)discovery of the *Palimpsest*, a new translation and commentary of the works of Archimedes is in progress (see <www.archimedespalimpsest.org>).

tangent to the curve at C in Z and the straight line CB in K. Point E is the point of intersection of lines CZ and DB. Extend now line CK to H such that $CK = KH$. Next, consider an arbitrary point O on AC and draw a straight line parallel to DB that meets the parabola γ in P, CK in N and CZ in M.

Since ABC_{ps} is a parabolic segment, CE its tangent and CD an ordinate, then $DB = BE$.[8] For this reason, and because AZ and OM are both parallel to DE, we also have that $ON = NM$ and $AK = KZ$.[9] Now, since $CA : AO = MO : OP$ (proved in *Quadrature of the Parabola*, Proposition 5), $CA : AO = CK : KN$ (application of Euclid's *Elements*, Proposition 2 from Book VI and Proposition 18 from Book V) and $CK = KH$, we have that $KH : KN = MO : OP$.

After this purely geometrical treatment, Archimedes begins to introduce physical considerations. In particular, he makes use of some lemmas on centres of gravity (proved in his treatise *On the Equilibrium of Planes*) and then he applies the law of the lever (proved in his *On the Equilibrium of Planes*, Propositions 6 and 7 of Book I).

He observes that point N is the centre of gravity of the straight line OM (the centre of gravity of a straight line is its middle point). Therefore, if we take a segment LT equal to OP with H as its centre of gravity (so that $LH = HT$), LT will be in equilibrium with OM. Indeed, using the law of the lever we can observe that HN is divided into two segments (HK and KN) that are inversely proportional to the 'weights' LT and MO, namely, in such a way that $HK : KN = MO : LT$, so that K is the centre of gravity of the combined weight of the two.[10]

In the same way, all the straight lines parallel to DE that can be drawn in the triangle $\triangle AZC$ will be in equilibrium with their portions 'cut off' by the parabola, when transferred to H (as we did for OP). And K will be the centre of gravity of the combined weight of each straight line and its portion. But since the triangle $\triangle AZC$ is made up of all the parallel lines inside it, and since the parabolic segment ABC_{ps} consists of all the parallel lines drawn inside it in the

[8] This result, concerning the property of the subtangent, is proved by Archimedes in the *Quadrature of the Parabola*, Proposition 2.

[9] Archimedes is implicitly applying three results obtained by Euclid in the *Elements*, namely, Proposition 4 of Book VI and Propositions 11 and 9 of Book V.

[10] The law of the lever, found by Archimedes in the treatise *On the Equilibrium of Planes*, states that bodies placed on opposite sides of the fulcrum are in equilibrium at distances reciprocally proportional to their weights. For instance, if two bodies of masses m_1 and m_2 placed on the arms of a straight lever of fulcrum K, and if d_1 and d_2 are the distances of the bodies' centres of mass from the fulcrum, then the two bodies will balance just in case $m_1/d_2 = m_2/d_1$.

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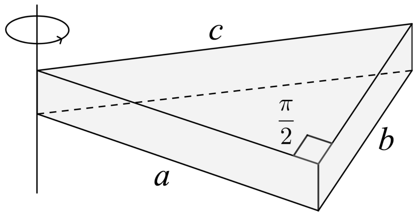

Figure 2. The water-filled prism mounted on a spindle.

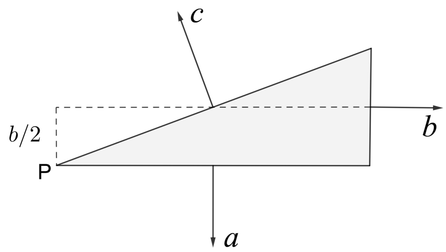

Figure 3. Water pushing on the side faces of the prism.

manner of OP, the triangle $\triangle AZC$ will balance about the point K the parabolic segment placed about H as centre of gravity, so that K is their common centre of gravity.

After having established the equilibrium (with centre of gravity K) between the triangle $\triangle AZC$ and the parabolic segment ABC_{ps} transferred to H, Archimedes finds the centre of gravity of triangle $\triangle AZC$. If we take point X on CK such that $CK = 3XK$, then X will be the centre of gravity of $\triangle AZC$.[11] Now, since there is equilibrium between the triangle $\triangle AZC$ and the parabolic segment ABC_{ps} about H, and their centre of gravity is K, we have the following result: the triangle $\triangle AZC$ is to the parabolic segment ABC_{ps} transferred to H as its centre of gravity as HK is to KX (that is, $\triangle AZC : ABC_{ps} = HK : KX$). But $HK = KC = 3KX$, and therefore the triangle $\triangle AZC$ is three times the segment of parabola ABC_{ps}. Moreover, $\triangle AZC$ is four times the triangle $\triangle ABC$, since $ZK = KA$ and $AD = DC$.[12] Hence, if $\triangle AZC = 3ABC_{ps}$ and $\triangle AZC = 4\triangle ABC$, we can conclude that $3ABC_{ps} = 4\triangle ABC$.

We have thus shown that the parabolic segment is four-thirds the triangle it encloses.

As second example of successful application of physics to mathematics, let's consider a mechanical proof of the Pythagorean theorem given by Levi ([2009]).[13]

Take a right triangular prism. The prism is filled with water and it is mounted on a vertical spindle at one of the acute angles of the base triangle (see Figure 2). The legs of the triangle are a, b, and the hypotenuse c. The prism is initially at rest, with no disturbances (we have a closed system at equilibrium). And even if it can rotate freely around the spindle, it will stay at rest. The reason for this is that the rotation is forbidden by the law of conservation of energy.

[11] Archimedes uses here a result that appears without proof in Proposition 15, Book I, of *On the Equilibrium of Planes*. Although unproven, this result is equivalent to what Archimedes demonstrates in Proposition 14, Book I, of the same treatise.

[12] The result follows from an application of Euclid's *Elements*, Proposition 1 of Book VI.

[13] In the same book, Levi offers several physical proofs of the Pythagorean theorem, not just one.

Curiously enough, we can use these physical considerations to get the Pythagorean theorem. How? Let's consider the forces that act on the prism. The vertical gravitational pull is balanced by the spindle's resistance. Moreover, the water pushes from inside on each of the side faces of the prism, in three competing directions (see Figure 3). Such push is proportional to the area of the side and, since all the sides are rectangles of the same height, the pressure force is also proportional to the lengths a, b, and c. Each pressure force tries to rotate the tank around P. Indeed, the pressure force at each side generates a torque as if it was concentrated at the centre of gravity of the side, which coincides with the centre of the side. Now, since the torque of each force is the force's magnitude times the distance from the line of force to P, and since the torques balance (because of energy conservation), we can write the zero-torque condition by considering the three forces a, b, c and the corresponding levers $a/2$, $b/2$ and $c/2$:

$$a \cdot \frac{a}{2} + b \cdot \frac{b}{2} - c \cdot \frac{c}{2} = 0. \tag{1}$$

Equation (1) gives the Pythagorean theorem $a^2 + b^2 = c^2$.

The two examples above show how physical considerations can be applied in mathematics, and more specifically in Euclidean geometry. In Archimedes' case, the application is given by the law of the lever, which states that bodies placed on opposite sides of the fulcrum are in equilibrium at distances reciprocally proportional to their weights. In the prism-example what is applied is the law of conservation of energy, which guarantees the vanishing of the combined torque around point P. In fact, since the prism is initially at rest, and it is considered as a closed system at equilibrium, it will not begin to rotate because such rotation would violate the law of conservation of energy (as a consequence, the net torque on the prism must be zero).[14]

At this point, it may be thought that only geometry, and not other areas of mathematics, admits of applications of physics. Nevertheless, it is possible to find other examples that show how geometry is not the only area of mathematics in which we find the application of physics

[14] Incidentally, Skow ([2015]) has considered Archimedes' Proposition 1 and the mechanical proof of the Pythagorean example. Skow recognizes the importance that physical considerations have, in these examples, when they are used to reach mathematical conclusions. Nevertheless, he does not address the question of how the application of physics in mathematics is possible. Since Skow's article is specifically devoted to show that there exist physical explanations of mathematical truths, I take it as representative of the attitude described in Footnote 2.

(although it is undoubtedly the area where such applications have received more attention). The third and fourth examples presented below provide an illustration of how physical considerations find application also in number theory.

The third example is taken from Vladimir Uspenskii's book *Some Applications of Mechanics to Mathematics* ([1961]) and it concerns the use of mechanics to establish a purely arithmetical result. Consider a row of positive numbers $P_1, P_2, \ldots, P_n$, with $n > 2$. In order to reduce the length of the row, we subtract from each of its end-terms P_1 and P_n a number P that is equal to the smaller of these end-terms and we add P to the middle terms (if there is only one middle term, then we add the number $2P$ to this one term). What we get is a derived row whose length is less by one or two terms than the length of the preceding one. One or both of the end-terms have become zero and the sum of all terms has remained unaltered. Now, if we iterate the process of passing from a row to its derived row, we will finally arrive at a row whose length is one or two terms. Call this last row the 'characteristic' of the row $P_1, P_2, \ldots, P_n$ with which we started. For instance, take $1, 7, 2, 3$ as our starting row. The first derived row will be $8, 3, 2$. The last derived row, which is the characteristic of $1, 7, 2, 3$, will be $6, 7$.

If we consider as row an initial segment of the succession of natural numbers $(1, 2, 3, \ldots, n)$, we can notice the following regularity: the characteristic of the row consists of one term if $n = 3k + 1$, otherwise it consists of two terms. This regularity is expressed by the following theorem:

> The characteristic of the row $1, 2, 3, \ldots, n$ consists of one term, when $n = 3k + 1$. It consists of two terms when $n = 3k$ or $n = 3k + 2$.

The arithmetical result contained in this theorem, which concerns an initial segment of the succession of natural numbers $(1, 2, 3, \ldots, n)$, can be obtained through a purely mechanical interpretation of the arithmetic involved. The key insight is that we can represent a row of numbers $P_1, P_2, \ldots, P_n$ by a rod r (see Figure 4). The rod is loaded at points $A_1, A_2, \ldots, A_n$ by means of weights $P_1, P_2, \ldots, P_n$ such that the distance between two consecutive points is always the same $(d(A_1, A_2) = d(A_2, A_3) = \ldots = d(A_{n-1}, A_n))$.

We now perform the same operation we saw in the arithmetical case and we move from the original system of loads to a derived system of loads. The new set of loads will correspond to

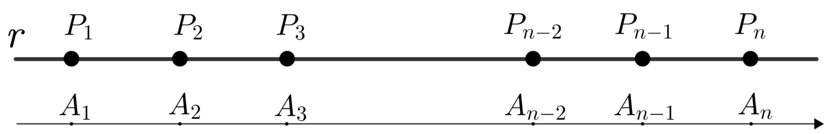

Figure 4. Rod r representing a row of numbers $P_1, P_2, \ldots, P_n$.

our (mathematical) derived row. As we iterate the process, the centre of gravity of the mechanical system does not change and each (derived) system of loads has the same centre of gravity of the original system of loads. This is also true for the mechanical system corresponding to the characteristic of the row: the characteristic of our initial set of loads will be given by one or two loads having the same centre of gravity as the initial system of loads. Furthermore, the positions of the loads in the characteristic will coincide with the positions of some of the loads of the initial system (that is, some of the points $A_1, A_2, \ldots, A_n$).

The abscissa of the centre of gravity of the initial system, which coincides with the centre of gravity of the characteristic of such a system, is given by the following formula:

$$\frac{P_1 a_1 + P_2 a_2 + \cdots + P_n a_n}{P_1 + P_2 + \cdots + P_n}, \tag{2}$$

where $a_1, \ldots, a_n$ are the abscissae of points $A_1, \ldots, A_n$.

If we take $a_1 = 1, a_2 = 2, \ldots, a_n = n$, then the abscissa of the centre of gravity will be:

$$\frac{P_1 \cdot 1 + P_2 \cdot 2 + \cdots + P_n \cdot n}{P_1 + P_2 + \cdots + P_n}. \tag{3}$$

By choosing the values of the weights such that $P_1 = 1, P_2 = 2, \ldots, P_n = n$, we will have the following equality:

$$\frac{P_1 \cdot 1 + P_2 \cdot 2 + \cdots + P_n \cdot n}{P_1 + P_2 + \cdots + P_n} = \frac{2n+1}{3}. \tag{4}$$

Finally, we evaluate the number $(2n+1)/3$ (which gives the abscissa of the centre of gravity). If $n = 3k+1$, then the number $(2n+1)/3 = 2k+1$ is a whole number and the characteristic consists of one term. If $n = 3k$, then $(2n+1)/3 = 2k+1/3$; if $n = 3k+2$, then $(2n+1)/3 = (2k+1)+2/3$. In both the latter cases, $(2n+1)/3$ is not a whole number and therefore the characteristic consists of two terms. This concludes our mechanical proof of the arithmetical theorem.

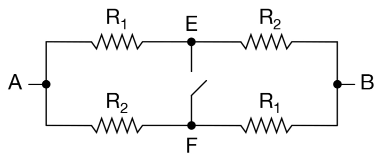

Figure 5. Circuit with four resistors of resistances R_1 and R_2.

The fourth and last example presented here illustrates an application of physics in number theory that, unlike the previous one, does not involve mechanical considerations.[15]

Consider the circuit shown in Figure 5, in which we have four resistors: two resistors of resistance R_1 and the other two of resistance R_2. When the switch between Nodes E and F is open, we have two paths in parallel, each of resistance R_1+R_2, and therefore the total resistance between Nodes A and B is $(R_1+R_2)/2$. If we close the switch, we have two resistors in sequence, each of resistance $R_1R_2/(R_1+R_2)$, and the total resistance is $2R_1R_2/(R_1+R_2)$.[16]

Shorting E with F we get a circuit with a total resistance that is the same as or less than the total resistance of the open circuit.[17] This condition, which is empirical, can be given using the formulas for the total resistance found in the previous paragraph:

$$\frac{R_1+R_2}{2} \geq \frac{2R_1R_2}{R_1+R_2}. \tag{5}$$

Inequality (5) expresses the following arithmetical proposition: the arithmetic mean is greater than or equal to the harmonic mean. This can be seen in a more explicit way if we replace R_1 with a and R_2 with b. We get the inequality:

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}N. \tag{6}$$

[15] The example is discussed in Levi ([2009], pp. 76–78).

[16] The resistance R of $n \geq 2$ resistors in parallel $(R_1, R_2, \ldots, R_n)$ is given by the following formula: $1/R = 1/R_1 + 1/R_2 + \cdots + 1/R_n$. The resistance R of $n \geq 2$ resistors in series $(R_1, R_2, \ldots, R_n)$ is given by the sum of the individual resistances: $R = R_1 + R_2 + \cdots + R_n$.

[17] The reason for this is that current and resistance are inversely proportional according to Ohm's law. If the switch is closed, we have higher current and less resistance.

3 Why Successful?

In the introduction, I claimed that the four cases illustrated above are examples of successful applications (of physics to mathematics). Before giving an analysis of how physics is successfully applied in these examples, which will be the goal of the next two sections, I want to address the question of why these applications are successful.

The claim that a discipline A (for instance, mathematics or physics) is successfully applied to another discipline B (for instance, physics or mathematics) depends on the fact that the result obtained in B through the use of A is successful. Nevertheless, to say that we are confronted with a successful result in B, we should adopt a criterion, and this criterion should be somehow independent from the application we are considering (otherwise, the reasoning would strike us as circular).

Let's first consider the case of direct applications. Although in many studies on the applicability of mathematics a criterion to evaluate the success of the application is not explicitly offered, it is easy to see how in such studies 'successful application of mathematics' is tantamount to 'mathematics is used to obtain a successful prediction' or 'the result obtained using mathematics is confirmed by empirical means'. For instance, in one of his works on the applicability of mathematics, Pincock ([2004]) considers the case of a set of equations that are used, together with the appropriate constants and initial conditions, to find the velocity and the altitude of a rocket at burnout. After presenting the case, he observes: 'As philosophers, we want to know how and why this appeal to mathematics works' (Pincock [2004], p. 136). In this context, the expression 'mathematics works' clearly means 'the results obtained through our set of equations receive successful empirical confirmation'. We can therefore state, for cases of direct applications, the following criterion: the success of mathematics in the empirical sciences is given by the fact that the (empirical) results obtained through mathematics receive successful empirical confirmations or they allow to make successful empirical predictions.

But what about converse applications? If we consider cases of converse applications the criterion cannot be the same and, more importantly, it cannot appeal to empirical considerations. The reason is that successful applications of non-mathematical methods to mathematics are successful in mathematics. And therefore the criterion to evaluate the success of such applications should be a wholly mathematical one. This is why I see only one relevant possibility, namely, the success of non-mathematical methods in mathematics is given from the existence of a purely mathematical proof of the result.[18]

Now that we have a criterion to evaluate the effectiveness of physics in mathematics, we can see whether the four cases of application reported in the previous section are successful.

First, consider Archimedes' case. According to the criterion proposed a few lines above, the application of physics to geometry made by Archimedes is successful if the same result obtained in the *Method* is also obtained through a purely mathematical proof. Do we have such a proof?

The answer is positive and it is given by Archimedes himself. At the end of his discussion of Proposition 1, Archimedes remarks that the treatment he has just presented cannot be said to be a proof of the result. Rather, he observes, it gives 'the impression' that the conclusion is true. And for a rigorous mathematical proof the reader is referred to another treatise that has been already published. Here is the passage:

> This as not therefore been proved by the above, but a certain impression has been created that the conclusion is true. Since we thus see that the conclusion has not been proved, but we suppose it is true, we shall mention the previously published geometrical proof, which we ourselves have found for it, in its appointed place. (Dijksterhuis [1987], p. 318)

This quotation informs us of the existence of a purely geometrical demonstration of Proposition 1. Such demonstration is given by Archimedes in his treatise *Quadrature of the Parabola*, where he obtains the same result without recurring to non-mathematical considerations.[19] We have therefore that the result obtained in Proposition 1 through the application of the law of the lever is confirmed on independent grounds, within pure mathematics, and the example should be considered as a successful case of application.[20]

Before moving on to see if the criterion for a (converse) application being successful is also met by the other three examples, let me briefly consider another aspect of the previous quotation that, although not directly connected to the content of this section, is relevant to

[18] The proof should be formally valid and contain no mistakes. Otherwise it would not count as proof.

[19] The proof is based on Eudoxus' method of exhaustion and it is completed through a double *reductio ad absurdum* (see Katz [2009], pp. 108–9).

[20] The mathematical result is also confirmed by modern calculus. Nevertheless, it is not necessary to resort to modern mathematics, since the geometrical proof given in the *Quadrature of the Parabola* is still regarded by mathematicians as a flawless demonstration.

analysis of applicability.

The passage above sheds light on Archimedes' conception of legitimate mathematical proof. In particular, it reveals that Archimedes does not consider the mathematical treatment given in the *Method* as a proof. Why? To answer this question, and discover why Archimedes is considering that the treatment given does not satisfy all requirements of exactness, we have to turn to two methods that he employs to reach Proposition 1: the barycentric method and the method of indivisibles. The first, the barycentric method, amounts to conceiving geometrical figures as attached to a lever that remains in equilibrium, on which the law of the lever can be applied. The second, the method of indivisibles, consists in seeing a plane figure as the sum of innumerably many parallel segments of straight lines. As observed by Dijksterhuis ([1987], pp. 318–19), the mathematical deficiency Archimedes is pointing to in the passage above is exclusively a consequence of the use of indivisibles, since he has no mathematical objection against the use of barycentric considerations and he furthermore uses them in other mathematical treatises. Thus, contrary to what we may think from a modern perspective, the lack in mathematical exactness is not attributed by Archimedes to the use of mechanical considerations but to the use of indivisibles, which we accept today on the basis of infinitesimal calculus.[21] Such observation is interesting because it sheds light on the specific way in which mathematics was practiced during the third century BCE and brings back the issue of accounting for the legitimate use of non-mathematical considerations in mathematics.[22]

Let's now check the effectiveness of the second, third and fourth examples. Concerning the Pythagorean theorem, it is easy to find many purely mathematical proofs of the result. For instance, in his book *The Pythagorean Proposition*, the mathematician Elisha Scott Loomis ([1968]) reports 370 different proofs of the theorem, and the majority of them are purely geometric or algebraic demonstrations. As for the third example, in which we saw how to find the

[21] Archimedes' concern about the method of indivisibles stems from the difficulties posed by the use, in mathematics, of techniques involving the concept of infinity. For a modern reader, however, the application of indivisibles is not problematic because infinitesimal calculus provides a mathematical justification for the application of such technique.

[22] The issue of accounting for the introduction and assimilation of new methods in mathematical practice has received growing attention in philosophy of mathematics (see, for example, Grosholz and Breger [2000]; Ferreirós [2015]). Even if I cannot address this discussion here, I take the investigation of how the realm of pure mathematics is influenced by new methods, and especially by those methods that are non-mathematical and that come from the empirical sciences (see Urquhart [2008]), as another crucial component of the general problem of applicability.

number of terms (one or two) that compose the characteristic of a row representing a segment of the natural series $(1, 2, 3, \dots, n)$, a purely mathematical proof is given in (Uspenskii [1961]).[23] Finally, concerning the fourth example, where we used physical considerations to state that the arithmetic mean is greater than or equal to the harmonic mean, we have that the inequality can be proved in pure mathematics via different proof methods (see Bullen [2003]).

We have therefore seen how, according to the criterion given, the four examples are cases of successful applications of physics to mathematics. In the next section I will address the following question: what kind of physical considerations are involved in the examples of converse applications presented here? Thus, while in this section I focused on the effectiveness of some converse applications, in the following one I will examine the physical content that makes these applications—as well as other cases that we can identify in scientific practice—successful.

4 The Role of Physical Principles in Converse Applications

Let's see what kind of physical considerations are involved in the examples of Section 2. *Prima facie*, it may thought that in these examples the physical content of the application (that is, what is applied in mathematics) is given by (some) concrete entities. For instance, in Archimedes' case, we may think that what is applied in mathematics are weights and levers. Nevertheless, it is easy to see that this view is misleading. First of all, the concrete entities that appear in the mathematical treatment are idealizations (in Archimedes' case the weights are homogeneous and the lever is weightless and homogeneous). And therefore there is nothing concrete involved in the application, since idealizations do not correspond to concrete entities. Second, and more importantly for the point I want to make in the present section, the (idealized) entities that figure in the application are the content of (and are subject to) specific physical laws that are used. It seems then that in converse applications, we do not apply concrete entities such as levers and tables to mathematics. Rather, we apply physical laws that are about these concrete objects (or, better, idealizations of these concrete objects).[24]

Take Archimedes' case. In his geometrical treatment of the parabolic segment, Archimedes

[23] Uspenskii shows how to calculate the terms of the characteristic for a determinate row. In this calculation, he employs only arithmetical formulas.

[24] There are, of course, many idealizations in place when we apply a physical law. For instance, in Archimedes' case the gravitational field is uniform and the system is closed and nothing interferes with it.

applies the law of the lever, which states that bodies placed on opposite sides of a fulcrum are in equilibrium at distances reciprocally proportional to their weights. Obviously, he does not posses the concept of torque (the intensity with which a force tries to rotate an object it is applied to around a pivot).[25] Furthermore, and more crucially, he does not posses the concept of energy. And we know that Archimedes' law of the lever is a direct consequence of energy conservation. Such law, in fact, can be obtained by applying the principle of conservation of energy on a (closed and isolated) lever-system.[26]

Concerning the relation between energy conservation and other physical laws, Feynman ([1967], pp. 71–72) notes that

> [...] knowing the law of conservation of energy and the formulae for calculating energy, we can understand other laws. In other words many other laws are not independent, but are simply secret ways of talking about the conservation of energy. The simplest is the law of the lever.

Therefore the law of the lever essentially depends on a conservation principle, conservation of energy, which is among the most fundamental principles of physics. It is this principle that Archimedes is implicitly applying in his geometrical treatment. And thus, in Archimedes' case, the converse applicability problem, namely, the problem of accounting for the success of physics in mathematics, can be restated in terms of the successful application of a physical principle in mathematics.[27]

In the second and third examples there is a similar involvement of physics. In the second example we saw that what guarantees the vanishing of the combined torque around point P is the principle of conservation of energy (since the prism is initially at rest and the system

[25] We know today that a lever is in balance if the total left side torque is equal to the total right side torque. Such zero torque condition can be used to get the law of the lever: for the simple lever considered by Archimedes, if m_1 and m_2 are the masses of the two bodies placed on the arms and d_1 and d_2 are the distances of the bodies' centres of mass from the fulcrum, the zero torque condition reads $m_1 g \cdot d_1 = m_2 g \cdot d_2$. It is easy to recognize in this expression Archimedes' law of the lever: $m_1/d_2 = m_2/d_1$.

[26] For the derivation, see (Feynman [1967], pp. 72–73). The key idea is to evaluate the potential energy of an horizontal lever that has been tilted and, then, to apply the principle of conservation of energy.

[27] Here I am specifically focusing on the import of principles, and not laws, in converse applications. The distinction between principles and laws in physics is the object of several debates in philosophy of science, but for the point at stake here it is sufficient to add that I regard conservation laws as 'principles' because I take them as more fundamental than other laws. For instance, Archimedes' law of the lever can be obtained from energy conservation. And I consider the 'principle' of conservation of energy as more fundamental than the law of the lever. This view is usually shared by physicists (for example, by Feynman [1967]), who take conservation laws as more fundamental than other physical laws. I do not exclude, however, that other physical laws besides conservation laws may have such fundamental status. For instance, Newton's laws are sometimes regarded as fundamental principles of physics (see Dilworth [1994]).

is considered as an isolated one, the net torque on the prism must be zero). It is therefore this principle that is successfully applied and permits to get the Pythagorean proposition. In the third example, we studied our mathematical problem in terms of a system of loads. In particular, we saw how considerations about the centre of gravity of this system played an important role in the proof: when moving from the original system of loads to the derived systems of loads, the centre of gravity of the mechanical system did not change (each derived system of loads had the same centre of gravity of the original system of loads). By proceeding in this way, in our mechanical proof we tacitly used a conservation principle. Indeed, the key insight to reach the mathematical result was that the centre of gravity of the mechanical system always remains the same. But this is possible only if we suppose that when we move from the original system of loads to the derived system(s) of loads, the total mass of the system remains exactly the same at each step in the process. And what guarantees that this condition is fulfilled is the principle of conservation of mass. Thus, similarly to what we saw in the first two cases, also in our second example the application of physics is guaranteed by the application of a conservation principle.

What about our fourth example? In the circuit example, we used the formulas for the resistance to calculate the total resistance of a circuit in two different configurations (switch open and switch closed, respectively). More precisely, we used the following results for a number of resistors in parallel and in series: the resistance R of $n \geq 2$ resistors in parallel is given by the formula $1/R = 1/R_1 + 1/R_2 + \cdots + 1/R_n$; the resistance R of $n \geq 2$ resistors in series is the sum of the individual resistances, that is, $R = R_1 + R_2 + \cdots + R_n$. These formulas can be obtained from Kirchhoff's laws plus Ohm's law, which gives a definition of the resistance of an electrical conductor. Kirchhoff's first law (K_1) tells us that the sum of currents entering a node equals the sum of currents exiting a node; Kirchhoff's second law (K_2) states that the algebraic sum of all voltages within any closed path of a circuit must be equal to zero. Ohm's law expresses the linear relationship between the applied voltage V between two points and the current I through a conductor between the same points: $V/I = constant = R$ (the constant of proportionality R is called 'resistance'). Now, these three laws are applied to any circuit, which consists of resistors, inductances, batteries, capacitors and so on. Nevertheless, there is an important difference between Kirchhoff's laws and Ohm's. While Ohm's law is empirical, since it is a generalization that comes from experiments, Kirchhoff's laws are a restatement of two basic principles of physics: K_1 is a restatement of the conservation of electric charge; K_2 is a restatement of the principle of conservation of energy.[28] Therefore, also in this case, the successful application of physics in mathematics amounts to the (successful) application of conservation laws in mathematics.

Is there a similar involvement of physics in other examples that can be found in scientific practice? Although the converse applicability problem has not yet been addressed by philosophers interested in issues of applicability, at the beginning of Section 2 I observed that we do have many textbooks that illustrate how physics is applied with success in mathematics. For the most part, the applications reported in these works draw on conservation principles, even if a reference to such principles is not always explicit. This is, for instance, the case of Levi's ([2009], p. 76) discussion of the application of Kirchhoff laws in mathematics. Although not mentioned by Levi, we know that Kirchhoff laws express charge conservation and conservation of energy. And therefore the application of these laws can be rephrased in terms of the application of charge conservation and conservation of energy.

Nevertheless, it is worth noting that in other examples taken from textbooks, we also find the application of laws that do not involve conserved quantities, as for instance when the laws of optics are used in mathematics (see Uspenskii [1961], p. 10). Moreover, in contemporary mathematical practice there also cases in which considerations coming from physics are successfully applied in pure mathematics, but the exact import of conservation laws in these applications is difficult to be assessed and deserves closer examination.[29] One example can be found in the work of Edward Witten, the sole physicist ever to win the Fields Medal. Witten used path integrals—which are generally regarded as tools of physics because they lack mathematical rigour—to solve geometry problems, and this application is an essential ingredient of the work for which he was awarded the Fields Medal by the International Mathematical Union in 1990.[30] Another interesting example can be found in the application of mirror

[28] Since voltage is defined as electric potential energy per unit charge, K_2 states that the total amount of energy gained per unit charge must be equal to the amount of energy lost per unit charge.

[29] A major difficulty is that the physics involved is such applications is highly mathematized and it is therefore hard to trace the boundary between physics and mathematics. I am grateful to an anonymous referee for bringing to my attention the two examples that follow.

[30] There are many aspects of Witten's work that can be read in terms of the successful interaction between

symmetry, which was originally discovered by string theorists in association with particular complex manifolds called Calabi–Yau manifolds, in pure mathematics. Mirror symmetry is an example of a duality, namely, an equivalence between two systems.[31] It is extremely useful in theoretical physics because it gives an equivalence between the physics of string theory on two different Calabi–Yau manifolds, thus permitting to translate a difficult computation on the original manifold into a relatively easy one on an equivalent ('mirror') manifold (see Quigley [2015]). In 1990 physicist Philip Candelas and his collaborators Xenia de la Ossa, Paul Green, and Linda Parkes ([1991]) showed that mirror symmetry could also be used to solve problems in enumerative geometry, which is a branch of algebraic geometry.[32]

Thus it seems that there is a cluster of cases of converse applicability that involve the application of conservation principles, as for instance conservation of energy and conservation of mass. These principles are neither mathematical, since they are about physical entities and phenomena, nor strictly empirical, as they are about idealized concrete entities and state of affairs. They are abstract principles of physics.[33]

After having established what kind of physical considerations are applied in some converse applications, it is time to face the tricky question of how these considerations (physical principles) are applied with success to mathematics. The goal of the next section is to outline some strategies of analysis that can be used to address this question.

mathematics and physics. A general exposition of these aspects is offered in a article written by the influential mathematician Michael Francis Atiyah and presented at the International Congress of Mathematicians in 1990. In his description of the main achievements of Witten, Atiyah ([1991], p. 31) remarks how 'his [Witten's] ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems'.

[31] Duality 'gives two different points of view of looking at the same object' (Atiyah [unpublished], p. 69).

[32] On the history of mirror symmetry in string theory and its impact on algebraic geometry, see (Galison [2004]). In his article, Galison also shows how scientists working in string theory are operating in a 'trading zone' that lies right at the interface of physics and mathematics, in which they are uncertain of whether they are doing physics or mathematics. The activities of the scientists working in this interzone between the two disciplines well reflect the uncertain status of strings—neither fully mathematical objects nor fully physical ones. Moreover, they result in different interpretations of the theory and new styles of argumentation, and also prompt the introduction of new concepts and categories of objects in physics and in mathematics.

[33] The characterization of conservation principles proposed here should be taken with a grain of salt, as there exist different philosophical stances on the status of conservation principles. On the other hand, it is endorsed by various philosophers of science. For instance, Ellis ([2002], p. 94) observes how 'even some of the most fundamental laws of nature are abstract. The conservation laws, for example, all refer to idealized systems. These laws are supposed to hold exactly only for systems that are closed and isolated. They are not thought to hold precisely for the kinds of open and interacting systems we find in nature'.

5 The Reasonable Effectiveness of Physical Principles in Mathematics

According to the view proposed above, some converse applications essentially depend on the use of conservation principles in the application process. This suggests that it is the successful interaction between mathematics and physical principles that guarantees the success of (some) converse applications. But how can we account for this successful interaction?

A first, a potential trigger for such analysis can be found in a proposal that has recently been put forward by Lange ([2017]) in his studies on mathematical explanation. Lange considers that in some non-causal explanations, mathematical facts act as constraints because they bring into the explanation a particular degree of necessity (mathematical necessity) that is stronger than the necessity of ordinary laws of nature. Although Lange does not connect his proposal to issues of applicability, such perspective suggests a way to analyse the effectiveness of the converse applications examined here. How? The idea it to focus on the degrees of necessity of the mathematical facts and physical principles that are involved in converse applications. For instance, by comparing grades of necessity of conservation principles and mathematical truths involved in the application, it may be argued that conservation principles and mathematical truths successfully interact exactly because they have the same degree of necessity. And therefore, according to this view, the successful application of physical principles in mathematics would be no longer 'unreasonable' (since it can be accounted from this metaphysical stance).[34]

Another strategy would be to adopt the view that conservation laws are metaphysically necessary and analyse the successful interplay between conservation principles and mathematical truths from this standpoint. Unlike the approach outlined in the previous paragraph, this proposal would be centred on the species of necessity involved in converse applications (rather than on the grades of necessity of conservation principles and mathematical truths). Laws of nature—and thus conservation principles—are usually regarded by philosophers as universal and necessary truths, where the necessity in question is taken to be distinct from metaphysical

[34] Let me note that although the proposal suggested here exploits an idea put forward by Lange (that of comparing the grades of necessities of mathematical truths and physical laws involved in the interaction between physics and mathematics), it differs from Lange's conception of conservation principles in several ways. The most important is perhaps the following: according to Lange ([2007], [2017], Chapter 2) conservation principles are naturally necessary, and they are sometimes modally stronger than force laws, however they are not as necessary as mathematical truths.

necessity (Frisch [2014]). Nevertheless, the view that conservation principles are metaphysically necessary has been defended by various philosophers (see, for example, Swoyer [1982]; Leeds [2007]; Wolff [2013]).[35] If we endorse this view, we may consider that the successful interplay between conservation principles and mathematics holds in virtue of the fact that conservation principles and mathematical truths share the very same status of being metaphysically necessary. Indeed, mathematical truths are usually seen as 'paradigmatic metaphysical necessities' (Clarke-Doane [2019], p. 5; see also Sider [2011], Chapter 12).

A third strategy sketched here departs from the previous ones insofar as it draws on the idea that although physically significant, conservation laws hold in virtue of mathematics alone.[36] Such suggestion is clearly compatible with the proposal put forward in the previous paragraph. In fact, if conservation laws can be derived as a matter of mathematics, then they can be seen as metaphysically necessary.[37] On the other hand, by considering that conservation laws hold in virtue of mathematics alone, we are adopting a very different stance on the way in which applications as those considered here work. As a matter of fact, in this scenario we regard the application of conservation principles in mathematics not as an application of physics to mathematics, but rather as an application of mathematics to mathematics.[38] And therefore it is easy to see how, more than a road towards a resolution of the issue discussed in this section, this strategy would point towards a dissolution of it (since there is no converse application to begin with).

Can we implement these strategies and provide a robust account of the successful interaction of conservation principles and mathematics? Obviously, the sketchy discussion just given does not provide a conclusive answer to this question. Furthermore, it is not exhaustive of the many and multifaceted philosophical debates that concern the status of conservation principles (since it has been limited to some positions presented in these debates) and it cannot be taken as fully representative of the converse applicability problem (since it only deals with a

[35] Conservation principles are also considered metaphysically necessary by scientific essentialists, for example, Ellis ([2002]) and Bird ([2007]), who take laws of nature as metaphysically necessary.

[36] This idea, which hinges on the argument that conservation laws are merely mathematical because they result from mathematical identities that can be derived from Noether theorems, is presented and discussed in (Brading and Brown [2003]; Martin [2003]; Wolff [2013]).

[37] This line of reasoning is pursued in (Wolff [2013]).

[38] That is, an 'internal' application of the kind examined in (Steiner [1998], [2005]).

subset of converse applications—those in which physical principles are successfully applied in mathematics).

Thus, what exactly is to be gained by pursuing the three strategies above is yet to be assessed. And surely the content of this section needs further elaboration. Nevertheless, the discussion offered here prompts new questions about applicability and it acquires particular significance in the light of the whole architecture of the article: it connects the analysis presented in the previous sections to various philosophical stances presented in the literature and it traces some research-paths that once suitably refined, may give unexpected twists to the study of the successful interactions between physics and mathematics.

6 Conclusion

Discussions of the unreasonable effectiveness of mathematics in physics are an important component of the study of the successful interplay between these two disciplines. Nevertheless, they miss a crucial aspect of it, namely, the successful use of physics in mathematics. In this article, I have focused on this aspect. I have shown how there exist cases of converse applications in scientific practice and how in some of them the successful application of physics essentially depends on the use of conservation principles. This is why I referred to such cases as a cluster, or subset, of converse applications. Finally, in Section 5, I have outlined some approaches that can be adopted to target the issue of how conservation principles and mathematics successfully interact.

In the present article I did not give an account of application, nor did I address the broad question of how mathematics and the empirical sciences successfully interact. Rather, I concentrated on the successful interplay between mathematics and physics and I shed light on an aspect of the applicability issue that has remained unexplored until now. Surely, it may be objected that my considerations about the role of conservation principles in mathematics only stem from the analysis of some case studies, and that I did not generalize to any case of successful application. That is certainly true and, indeed, it was not my aim to offer such generalization. By specifically focusing on some cases of converse applications, my goal was to highlight how there is a specific cluster of converse applications, and how in these cases the success, or effectiveness, of the application is granted by the use of conservation principles.

The analysis presented here provides (this is my hope) a first step towards the understanding of the unreasonable effectiveness of a natural science—physics—in mathematics. And, as typically happens with investigations that target a new research topic, it leaves many open questions. Nevertheless, it may be developed further and also have strong repercussions on some aspects of applicability that have not been directly addressed in this article. For instance, it may prompt investigations of whether and how conservation principles are applied in other empirical sciences, as for instance in biology or in economics. Furthermore, it can be used to implement current analysis of direct applications and come to better grips with the interconnections between direct and converse applications. By examining and comparing cases of direct and converse applications, in fact, we may elucidate the question of whether the direct and the converse applicability problems are really distinct issues (or whether they are connected in some way), that of whether there is a potential 'unificationist' approach to them, or even the question of whether conservation principles are mathematical (in the sense of the third strategy outlined in Section 5) and therefore they should be treated as such in analysis of direct applications. Clearly, these questions cannot be pursued here. But they hint at the connections that the present article has with issues of concern to philosophers interested in the applicability problem. Moreover, they suggest that the philosophical pay-off to be expected by deepening our understanding of converse applications is high.

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