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# QUANTUM BLACK HOLES AND RESOLUTION OF THE SINGULARITY

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Astrophysical objects and the surrounding geometry must be described by suitable states in the complete quantum theory of matter and gravity. We attempt at reconstructing such a state for a Schwarzschild black hole from the analysis of a collapsing ball of dust and describing the outer geometry by means of a coherent state.

*Keywords:* Quantum gravity; black holes; singularities.

## 1. Introduction

Astronomical observations are compatible with classical descriptions of astrophysical objects and of the Universe as a whole. This fact might lead us to believe that quantum physics only regards “small fluctuations” around classical configurations.<sup>a</sup> However, a consistent view of Nature demands that all is quantum<sup>b</sup> and therefore what the experimental observations really tell us is that the quantum states for those observed systems reproduce very closely classical solutions of the dynamical equations. In other words, the expectation values of quantum gravity observables taken on the actual states that are relevant for the description of our world must be very close to classical solutions of General Relativity where data are available.

A quantum theory can differ from its classical counterpart 1) by admitting states that have no classical analogue (for example, half integer spinors) and 2) by not admitting the existence of all the quantum states which can be approximated by classical solutions. A prototypical example of the latter is the atom: according to classical electrodynamics, an electron accelerated by the Coulomb field of the protons should radiate and spiral down into the nucleus in a very brief time. However, quantum mechanics predicts a discrete spectrum of energy states for the electron, with the ground state being several orders of magnitude broader than the size of

<sup>a</sup>This is the hard core version of the background field approach.

<sup>b</sup>Or nothing is, which we know is not the case from plenty of experiments carried out in our laboratories.

the nucleus. <sup>c</sup> This makes the probability for the electron to be captured by the nucleus negligible, thus explaining the existence of the world we see and ourselves.

Several solutions of the Einstein field equations show similarly pathological behaviours in the form of spacetime singularities. It is rather natural to conjecture that quantum gravity disposes of those like the ultraviolet catastrophe of electrodynamics is cured in quantum mechanics, the latter being an approximation of quantum field theory stemming from the smart choice of the position of the electron as the relevant observable. One kind of such singularities of General Relativity appears in the gravitational collapse of compact objects if a trapping surface appears [1]. It is obvious that the complexity of real astrophysical objects makes such a problem intractable without the “smart choice” of a suitable observable to describe the collapse, like it is done with the (much simpler case of the) electron in the atom. In fact, there are also obvious differences between the collapse of astrophysical objects and the electron capture by an atom. For instance, we cannot literally see the electron in its trajectory towards the nucleus, but we only detect the discrete emission when it transitions from an excited state to a lower energy one. On the other hand, we have astronomical data from supernovae explosions and accretion disks around black holes, although our present resolution does not allow us to see very close to the horizon and such processes involve huge amounts of matter.

The descriptions presented in this topical review should therefore be viewed as very simple toy models for what actually occurs in nature. In particular, we will first consider results from Ref. [2], in which the collapse of a ball of dust of mass  $M$  was studied to show that its ground state is much wider than the (naively expected) Planck length and is in fact characterised by a “principal quantum number” proportional to  $M^2$  (in Planck units). A realistic description of the gravitational collapse goes way beyond our scope here, but we will also provide some preliminary comments about how the classical collapse can be recovered in the quantum picture. The scaling of the ground state size with the mass  $M^2$  will then allow us to connect with results from Ref. [3], in which the existence of a proper quantum state reproducing the Schwarzschild geometry outside the collapsed body is shown to imply both the absence of a central singularity and again a scaling of the “graviton number” with  $M^2$ , like is required by the horizon area quantisation [4].

## 2. Discrete spectrum of dust ball

In General Relativity, the areal radius  $R = R(\tau)$  of a ball of dust with ADM mass  $M$  follows a radial geodesic in the Schwarzschild spacetime

$$ds^2 = -(1 + 2V_N) dt^2 + \frac{dr^2}{1 + 2V_N} + r^2 d\Omega^2, \quad (1)$$

<sup>c</sup>And much larger than the Compton length of the electron and well.

where <sup>d</sup>

$$V_N = -\frac{G_N M}{r} . \quad (2)$$

The evolution of  $R$  can then be determined from the effective Hamiltonian [2] <sup>e</sup>

$$H \equiv \frac{P^2}{2M} - \frac{G_N M^2}{R} = \frac{M}{2} \left( \frac{E^2}{M^2} - 1 \right) \equiv \mathcal{E} , \quad (3)$$

where the momentum  $P = M (dR/d\tau)$ . Eq. (3) is formally the same as the Newtonian conservation law for the energy  $\mathcal{E}$ , but it is in fact the fully General Relativistic Hamiltonian constraint. General Relativity differs from Newtonian physics in that it is non-linear and that difference in this simple toy model is given by the non-linear relation between  $\mathcal{E}$  and  $E$ .

Quantising the system is tantamount to assuming the uncertainty relation  $\Delta R \Delta P \gtrsim \hbar = \ell_p m_p$ , which is obtained by means of the usual canonical commutator <sup>f</sup>

$$[\hat{R}, \hat{P}] = i \hbar , \quad (4)$$

and expectation values are then taken on wavefunctions  $\Psi = \Psi(R)$  satisfying

$$\hat{H} \Psi = \mathcal{E} \Psi . \quad (5)$$

This is just the Schrödinger equation for a gravitational atom and the spectrum contains the eigenstates

$$\Psi_n = \sqrt{\frac{M^9}{\pi n^5 \ell_p^3 m_p^9}} e^{-\frac{M^3 r}{n m_p^3 \ell_p}} L_{n-1}^1 \left( \frac{2 M^3 r}{n m_p^3 \ell_p} \right) . \quad (6)$$

where  $L_{n-1}^1$  are generalised Laguerre polynomials with  $n \geq 1$  for zero angular momentum. The corresponding eigenvalues are given by

$$\frac{\mathcal{E}_n}{M} \simeq -\frac{G_N^2 M^4}{2 \hbar^2 n^2} = -\frac{1}{2 n^2} \left( \frac{M}{m_p} \right)^4 = \frac{1}{2} \left( \frac{E_n^2}{M^2} - 1 \right) , \quad (7)$$

and one also has

$$R_n \equiv \langle \Psi_n | R | \Psi_n \rangle \simeq \frac{\hbar^2 n^2}{G_N M^3} = n^2 \ell_p \left( \frac{m_p}{M} \right)^3 . \quad (8)$$

In particular,  $R_{n \sim 1} \sim \ell_p (m_p/M)^3 \ll \ell_p$ , and the spectrum contains states  $\Psi_n$  of infinitesimally small width for  $M \gg m_p$ . These lowest states would also have an energy density of the order of  $M/R_1^3 \sim (M^{10}/m_p^9) \ell_p^{-3}$ , which is hardly a satisfying alternative to the classical singularity of infinite energy density.

<sup>d</sup>We use units with  $c = 1$ ,  $G_N = \ell_p/m_p$  and  $\hbar = \ell_p m_p$ , where  $\ell_p$  is the Planck length and  $m_p$  the Planck mass.

<sup>e</sup>Numerical coefficients of order one will often be omitted or approximated for the sake of clarity.

<sup>f</sup> $\Delta O \equiv \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$  for  $\hat{O} = \hat{R}$  or  $\hat{P}$ .

However, the non-linear relation in Eq. (7) yields the constraint

$$0 \leq \frac{E_n^2}{M^2} \simeq 1 - \frac{1}{n^2} \left( \frac{M}{m_p} \right)^4, \quad (9)$$

and we thus find that

$$n \geq N_M \simeq \left( \frac{M}{m_p} \right)^2. \quad (10)$$

This means that, for  $M \gg m_p$ , the actual ground state for the collapsed ball in quantum General Relativity is given by  $\Psi_{N_M}$  with  $N_M \gg 1$  and

$$R_n \gtrsim R_{N_M} \sim G_N M. \quad (11)$$

Moreover, since  $R_{N_M} < R_H \equiv 2 G_N M$ , we could also argue that the ground state is hidden behind the Schwarzschild horizon of the metric (1). It is now important to remark that the above description is only useful when the dust ball is of macroscopic mass  $M \gg m_p$ , so that the radius  $R$  emerges as an effective degree of freedom from the collective behaviour of a many body system. For example, if we applied the above description to elementary particles with  $M \ll m_p$ , we would find that  $R_{N_M} = R_1$  exceeds the size of the visible Universe for an electron. Since  $R_1 \gg R_H$ , this simply means that the corresponding wavefunction  $\Psi_1 = \Psi_1(R)$  gives a negligible probability for the electron to be found inside its gravitational radius and be a black hole (see Refs. [7] for similar conclusions). However, the wavefunction  $\Psi = \Psi(R)$  only determines the probability (amplitude) for the radius of the object to take a given value  $R$ , whereas this conclusion would need us to reconstruct the outer geometry from a suitable quantum state. This will be attempted in the next Section.

It is remarkable that the principal quantum number  $N_M$  for the ground state shows the same dependence on the mass  $M$  as the one derived from the quantisation of the horizon area [4], which is at the heart of the corpuscular model of black holes [5], hence hinting to the *classicalization* of gravity [6]. In particular, we notice that, for  $n > N_M$ , the radial spacing between different states is given by

$$\delta R = R_n - R_{n-1} \lesssim \ell_p \frac{m_p}{M}, \quad (12)$$

so that the size of the dust ball effectively forms a continuum for  $M \gg m_p$ . Moreover, since the total energy  $M$  is the same for all states in the spectrum, nothing prevents the ball from “decaying” into the ground state. <sup>§</sup> This process would be the analogue of the classical Oppenheimer-Snyder collapse [8], albeit ending at a “bouncing” radius [9] of the order of  $R_{N_M}$ , and will be further investigated in future works. Finally, the same scaling in Eq. (10) will arise again in the next Section from the quantum description of the geometry.

<sup>§</sup>This is a remarkable difference with respect to the capture of the electron by an atom, which instead requires the emission of energy from the system.

### 3. Quantum Schwarzschild geometry

We next look for a quantum description of the geometry (1) outside the collapsed object by means of a scalar field  $\Phi = V_N/\sqrt{G_N}$  [3]. It is important to remark that this description is not fundamental and  $\Phi$  is just meant to represent the (non-perturbative) behaviour of the degrees of freedom of General Relativity. What is important is that we regard the vacuum state  $|0\rangle$  of  $\Phi$  as the quantum state of a truly empty spacetime, in which no modes of matter or gravity are excited. It is therefore natural to quantise  $\Phi$  as a massless field satisfying the free wave equation in Minkowski spacetime

$$\left[ -\frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \right] \Phi(t, r) = 0, \quad (13)$$

whose normal modes can be conveniently written as

$$u_k(t, r) = e^{-i k t} j_0(k r), \quad (14)$$

where  $j_0 = \sin(k r)/k r$  are spherical Bessel functions. We can now introduce the usual annihilation operators  $\hat{a}_k$  and creation operators  $\hat{a}_k^\dagger$  for these modes. The quantum Minkowski vacuum is then defined by  $\hat{a}_k |0\rangle = 0$  and the corresponding Fock space is built as usual.

Classical configurations of the scalar field that can be realised in the quantum theory must correspond to suitable states in this Fock space, and a natural choice is given by coherent states

$$|g\rangle = e^{-N_G/2} \exp \left\{ \int_0^\infty \frac{k^2 dk}{2\pi^2} g_k \hat{a}_k^\dagger \right\} |0\rangle \quad (15)$$

such that

$$\sqrt{\frac{\ell_p}{m_p}} \langle g | \hat{\Phi}(t, r) | g \rangle = V(r) = \int_0^\infty \frac{k^2 dk}{2\pi^2} \tilde{V}(k) j_0(k r). \quad (16)$$

The latter condition determines the occupation numbers for each mode  $k$  as

$$g_k = \sqrt{\frac{k}{2}} \frac{\tilde{V}(k)}{\ell_p}. \quad (17)$$

It is now crucial that the state (15) is well-defined only if it is normalisable, that is if the total occupation number

$$N_G = \int_0^\infty \frac{k^2 dk}{2\pi^2} g_k^2 \quad (18)$$

is finite.

For the potential  $V_N$  in the metric (1) we have

$$g_k = -\frac{4\pi M}{\sqrt{2} k^3 m_p}, \quad (19)$$

and the integral in Eq. (18) diverges both in the infrared (IR) and the ultraviolet (UV). This implies that no quantum state exists in our Fock space which can

reproduce  $V_N$  exactly. Any quantum realisation of the Schwarzschild geometry must therefore contain occupation numbers  $g_k$  which differ from those in Eq. (17) for  $k \rightarrow 0$  and  $k \rightarrow \infty$ , so as to make the quantum state normalisable. Since the explicit form of such proper occupation numbers will depend on the (unknown and presumably very complicated) state of matter inside the horizon, we will instead try to derive some general conclusions from qualitative arguments. In particular, the IR divergence is simply due to the assumption that the system is completely static and the potential  $V_N$  extends to infinite distance from the source centred at  $r = 0$ . To cure the IR divergence we can introduce a cut-off  $k_{\text{IR}} = 1/R_\infty$  to account for the necessarily finite life-time  $\tau \sim R_\infty$  of any realistic source. The UV divergence is instead due to the behaviour of  $V_N$  for  $r \rightarrow 0$  and would not be present if the source were extended. This allows us to connect the description of the outer geometry with the ground state in the previous Section by introducing a cut-off  $k_{\text{UV}} \sim 1/R_{N_M}$ .

The total occupation number finally reads

$$N_G = \frac{4 M^2}{m_p^2} \int_{k_{\text{IR}}}^{k_{\text{UV}}} \frac{dk}{k} = \frac{4 M^2}{m_p^2} \ln \left( \frac{R_\infty}{R_{N_M}} \right), \quad (20)$$

and we have again recovered a scaling of the mass compatible with the horizon area quantisation [4]. Moreover, the average radial momentum is given by

$$\langle k \rangle = \frac{4 M^2}{m_p^2} \int_{k_{\text{IR}}}^{k_{\text{UV}}} dk = \frac{4 M^2}{m_p^2} \left( \frac{1}{R_{N_M}} - \frac{1}{R_\infty} \right), \quad (21)$$

and the typical wavelength  $\lambda_G = N_G / \langle k \rangle \sim \ell_p M / m_p$  also reproduces the scaling found in the corpuscular picture of black holes [5].

We can next recompute the expectation value of the scalar field in the proper quantum state  $|g\rangle$  and find

$$\begin{aligned} V_{\text{QN}} &\simeq \int_{k_{\text{IR}}}^{k_{\text{UV}}} \frac{k^2 dk}{2\pi^2} \tilde{V}_N(k) j_0(kr) \\ &\simeq V_N \left\{ 1 - \left[ 1 - \frac{2}{\pi} \text{Si} \left( \frac{r}{R_{N_M}} \right) \right] \right\}, \end{aligned} \quad (22)$$

where Si denotes the sine integral function (see Fig. 1 for some examples). We remark that the oscillations occur around the expected classical behaviour  $V_N$  and become smaller and smaller for decreasing values of  $R_{N_M}$  in the region  $r > R_H$ .

On the other hand,  $V'_{\text{QN}}(r=0) = 0$  and tidal forces should vanish at the centre. In fact, the quantum corrected metric will have the form in Eq. (1) with  $V_N$  replaced by  $V_{\text{QN}}$ . The Ricci scalar  $\mathcal{R} \sim r^{-2}$  and the Kretschmann scalar  $\mathcal{R}_{\alpha\beta\mu\nu} \mathcal{R}^{\alpha\beta\mu\nu} \sim \mathcal{R}^2 \sim r^{-4}$ . Hence, tidal forces remain finite all the way into the centre of the system, which is technically an *integrable singularity*. This can be confronted with the standard Schwarzschild geometry, for which  $\mathcal{R}_{\alpha\beta\mu\nu} \mathcal{R}^{\alpha\beta\mu\nu} \sim \mathcal{R}^2 \sim r^{-6}$  and tidal forces acting on neighbouring geodesics diverge for  $r \rightarrow 0$ , causing the so-called “spaghettification” of infalling matter (for more details, see Ref. [3]). It is perhaps surprising that the avoidance of diverging tidal forces can just follow from the very



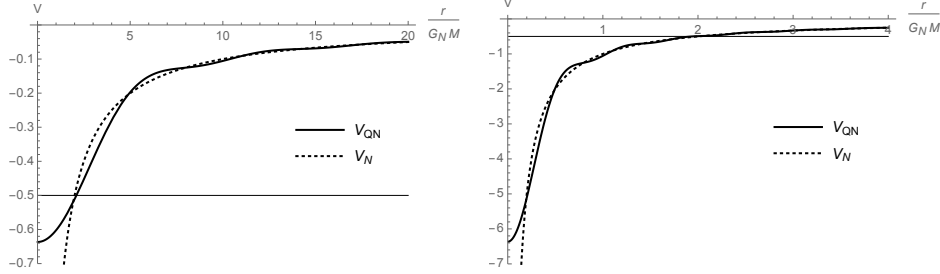


Fig. 1. Quantum potential  $V_{QN}$  in Eq. (22) (solid line) compared to  $V_N$  (dashed line) for  $R_{N_M} = G_N M = R_H/2$  (left panel) and  $R_{N_M} = R_H/20$  (right panel). The horizontal thin line marks the location of the horizon for  $V = -1/2$ .

existence of proper quantum states and does not seem to require any modification of the fundamental dynamics of gravity given by the General Relativistic Eq. (3).

For  $R_{N_M} \lesssim G_N M$ , it also follows that the quantum corrected metric always contains an event horizon (see thin horizontal line in Fig. 1) and the matter core in the ground state of the previous Section therefore gives rise to a black hole geometry. Another nice property of this metric is that it contains no inner horizon and none of the conceptual issues associated with Cauchy horizons.

#### 4. Conclusions

In this short topical review, we brought together the analyses from Refs. [2] and [3] about the quantum description of black holes. The main results are: 1) the matter which forms a black hole does not end into a singularity but maintains macroscopic size; 2) the effective metric is regular everywhere, including the centre; 3) the outer region contains information about (at least) the size of the material core, which constitutes a form of quantum hair [10], and 4) both the quantum ground state of the collapsed matter and the quantum state of the outer geometry are characterised by similar scalings of the mass  $M^2 \sim N_M \sim N_G$ , from Eqs. (10) and (20). The above picture is thus compatible with the quantisation of the horizon area.

We would also like to mention that a similar description of the Schwarzschild-de Sitter spacetime in terms of a coherent quantum state was given in Ref. [11] and that a contribution reproducing MOND was obtained from the finite size of the visible Universe. Finally, from the conceptual point of view, it would certainly be interesting to improve our understanding of the interplay between spacetime singularities (or the removal thereof) and the fundamental quantum dynamics of matter and gravity, as it is described, for instance, in Refs. [12] and [13].

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