Stable integral simplicial volume of 3-manifolds

Published Version:
Marco Moraschini (2020). Stable integral simplicial volume of 3-manifolds. OBERWOLFACH REPORTS, 17(1), 452-454 [10.4171/OWR/2020/7].

Availability:
This version is available at: https://hdl.handle.net/11585/888591 since: 2022-06-12

Published:
DOI: http://doi.org/10.4171/OWR/2020/7

Marco Moraschini, Stable integral simplicial volume of 3-manifolds. Oberwolfach Rep. 17 (2020), 452-454

Abstracts

Stable integral simplicial volume of 3-manifolds

Marco Moraschini
(joint work with Daniel Fauser, Clara Löh and José Pedro Quintanilha)

Simplicial volume is a homotopy invariant of compact manifolds introduced by Gromov [7] and it measures the complexity of a manifold in terms of its real singular chains.

Given an oriented compact connected n-manifold M (possibly with non-empty boundary) the *simplicial volume* is defined by

$$\|M, \partial M\| := \inf\Big\{ \sum_{j=1}^{m} |a_j| \;\Big|\; \sum_{j=1}^{m} a_j \cdot \sigma_j \in C_n(M;\mathbb{R}) \text{ is a relative fundamental cycle of } (M, \partial M) \Big\} .$$

One of the main original aims of the investigation of simplicial volume [7] was the understanding of the relation between the topology of a manifold and its (minimal) volume. In particular, in the case of hyperbolic manifolds, one can show that the simplicial volume is proportional to the Riemannian volume [7, 11].

I report in this talk a recent work in collaboration with Fauser, Löh and Quintanilha [4], in which we investigate an approximation problem of simplicial volume. A still open problem proposed by Gromov [8, p. 232] is the following:

Question 0.1. *Let M be an oriented closed connected aspherical manifold. Does $\|M\| = 0$ imply the vanishing of the Euler characteristic?*

One way for studying the previous open problem is the following: If M admits enough finite coverings (i.e. if $\pi_1(M)$ is residually finite), we introduce the *stable integral simplicial volume*

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} := \inf\Big\{ \frac{\|W, \partial W\|_{\mathbb{Z}}}{d} \;\Big|\; d \in \mathbb{N},\, W \text{ a } d\text{-sheeted covering of } M \Big\} ,$$

where the integral simplicial volume $\|W, \partial W\|_{\mathbb{Z}}$ is defined as the classical one but via $\mathbb{Z}$-singular chains instead of the real ones. One can easily check that the stable integral simplicial volume is always larger than or equal to the standard one:

$$\|M, \partial M\| \leq \|M, \partial M\|_{\mathbb{Z}}^{\infty} .$$

As for Betti numbers, ranks of fundamental groups, or logarithmic torsion of homology, one can ask which aspherical manifolds M satisfy *integral approximation* for simplicial volume, i.e. when the previous inequality is in fact an equality.

Since the stable integral simplicial volume of closed manifolds provides an upper bound (up to a uniform multiplicative factor depending only on the dimension of the manifold) of the Euler characteristic [7, 5], Gromov's Question 0.1 can be reformulated in the following stronger way:

Question 0.2. *Let M be an oriented closed connected aspherical manifold with $\|M\| = 0$. Does M satisfy integral approximation for simplicial volume?*

Of course, a positive answer to Question 0.2 would also imply an affirmative answer to Question 0.1

The following classes of manifolds are already known to satisfy integral approximation for simplicial volume: closed surfaces of positive genus [7, p. 9], closed hyperbolic 3-manifolds [6, Theorem 1.7] and graph manifolds with infinite fundamental group [3] (see also the work by Fauser [1] and Frigerio, Löh, Pagliantini and Sauer [6] for other examples).

In contrast, approximation fails uniformly for higher-dimensional hyperbolic manifolds [5, Theorem 2.1] and it fails for closed manifolds with non-abelian free fundamental group [6, Remark 3.9].

Our main result is the following:

Theorem 0.3. *Let M be an oriented closed connected aspherical 3-manifold, then*

$$\|M\| = \|M\|_{\mathbb{Z}}^{\infty} = \frac{\mathrm{hypvol}(M)}{v_3}.$$

Here, v_3 is the volume of any regular ideal tetrahedron in $\mathbb{H}^3$, and $\mathrm{hypvol}(M)$ denotes the sum of the volumes of the hyperbolic pieces in the JSJ decomposition of M. The equality $\|M\| = \mathrm{hypvol}(M)/v_3$ follows from the work of Soma [10].

Moreover, with regards to Question 0.2, we can provide the following complete picture of the 3-dimensional case:

Proposition 0.4. *Let M be an oriented closed connected 3-manifold with $\|M\| = 0$. Then the following are equivalent:*

(1) The simplicial volume of M satisfies integral approximation.
(2) The manifold M is aspherical or M is homeomorphic to either $\mathbb{S}^2 \times \mathbb{S}^1$ or the connected sum of two copies of $\mathbb{R}P^3$.

In this talk, we present the strategy for proving Theorem 0.3.

We only have to show that

$$\|M\|_{\mathbb{Z}}^{\infty} \leq \frac{\mathrm{hypvol}(M)}{v_3} .$$

The main difficulties arising from the JSJ decomposition of M are the following:

- to deal with the hyperbolic pieces with toroidal boundary and
- the subadditivity with respect to glueings along tori.

We explain here the strategy to overcome the previous issues: We work with a parametrized version of the simplicial volume instead of the stable integral simplicial volume. Since in this setting we can make use of the uniform boundary condition on tori studied by Fauser and Löh [2], this allows us to avoid involved bookkeeping for restrictions and compatibility of finite coverings to the glueing tori. This leads to a nice subadditivity formula with respect to glueings along tori in terms of parametrized simplicial volume.

One fundamental ingredient in the proof is that in some cases the most efficient parameter space is the profinite completion of the fundamental group. Hence, it is convenient to rewrite the stable integral simplicial volume as follows:

$$\|W, \partial W\|_{\mathbb{Z}}^{\infty} = \|W, \partial W\|_{\mathcal{F}}^{\widehat{\pi_1(W)}} ,$$

where $\widehat{\pi_1(W)}$ denotes the profinite completion of the fundamental group of a piece W appearing in the JSJ decomposition. Then, we also need to exploit some profinite properties of the JSJ decomposition and to keep control over the size of the boundary of the cycles appearing as representatives of the parametrised fundamental classes of its pieces.

Finally, we briefly discuss how to deal with the hyperbolic pieces. To this end, we have to prove a proportionality result between the parametrised simplicial volume of the hyperbolic pieces and their Riemannian volume:

Theorem 0.5. *Let W be an oriented compact connected hyperbolic 3-manifold with empty or toroidal boundary and let $M := W^\circ$. Then*

$$\|W, \partial W\|_{\partial}^{\widehat{\pi_1(W)}} = \frac{\mathrm{vol}(M)}{v_3} ,$$

where the subscript ∂ denotes the boundary control mentioned before.

References

[1] D. Fauser, *Integral foliated simplicial volume and S^1-actions*, arXiv:1704.08538 (2017).

[2] D. Fauser, C. Löh, *Variations on the theme of the uniform boundary condition*, To appear in J. Topol. Anal. DOI 10.1142/S1793525320500090,

[3] D. Fauser, S. Friedl, C. Löh, *Integral approximation of simplicial volume of graph manifolds*, Bull. Lond. Math. Soc. **51** (2019), 715–731.

[4] D. Fauser, C. Löh, M. Moraschini, J. P. Quintanilha, *Stable integral simplicial volume of 3-manifolds*, arXiv:1910.06120.

[5] S. Francaviglia, R. Frigerio, B. Martelli, *Stable complexity and simplicial volume of manifolds*, J. Topology **5** (2012), 977–1010.

[6] R. Frigerio, C. Löh, C. Pagliantini, R. Sauer, *Integral foliated simplicial volume of aspherical manifolds*, Israel J. Math. **216** (2016), 707–751.

[7] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **5** (1982), 5–99.

[8] M. Gromov, *Asymptotic invariants of infinite groups, Geometric group theory, Vol. 2*, London Math. Soc. Lectures Notes Ser. **182**, Cambridge Univ. Press (1993).

[9] C. Löh, C. Pagliantini, *Integral foliated simplicial volume of hyperbolic 3-manifolds*, Groups Geom. Dyn. **10** (2016), 825–865.

[10] T. Soma, *The Gromov invariant of links*, Invent. Math. **64** (1981), 445–454.

[11] W. P. Thurston, *The geometry and topology of 3-manifolds*, Princeton (1979).

Reporter: Zhicheng Han