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Submanifolds of fixed degree in graded manifolds for perceptual completion*

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Abstract. We extend to a Engel type structure a cortically inspired model of perceptual completion initially proposed in the Lie group of positions and orientations with a sub-Riemannian metric. According to this model, a given image is lifted in the group and completed by a minimal surface. The main obstacle in extending the model to a higher dimensional group, which can code also curvatures, is the lack of a good definition of codimension 2 minimal surface. We present here this notion, and describe an application to image completion.

Keywords: perceptual completion· graded structures· fixed degree surfaces· area formula· degree preservig variations .

1 Introduction

Mathematical models of the visual cortex expressed in terms of differential geometry were proposed for the first time by Hoffman in [17], Mumford in [23], August Zucker in [2] to quote only a few. Petitot and Tondut in 1999 in [26] described the functional architecture of area V1 by a contact structure, and described the propagation in the cortex by a constrained Lagrangian operator. Only in 2003 Sarti and Citti in [6] and J. Petitot in [24] recognized that the geometry of the cortex is indeed sub-Riemannian. In [6], the functional architecture of V1 is described as a Lie group with a sub-Riemannian geometry: if the visual stimulus is corrupted, it is completed via a sub-Riemannian minimal surface. A large literature has been provided on sub-Riemannian models both for image processing or cortical modelling (we refer to the monograph [25] for a list of references). In section 2 we will present the model [6], and its extension in the Engel group provided in [25][1].

The notion of minimal surface in a sub-Riemannian setting as critical points of the first variation of the area functional is well known for co-dimension 1 surfaces (see [12], [11] for the area formula, and [9], [7], [16], [18] for the first variation). For higher codimension very few results are available: the notion of area has been introduced in [10], [20], [19], but the first variation, well-known

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for curves (see [21]), was studied for surfaces only very recently in [4], [5], [13]. We will devote section 3 to the description of these results.

We conclude this short presentation with an application of this result to the completion model in the Engel group, contained in section 4.

2 A subriemannian model of the visual cortex

The primary visual cortex is the first part of the brain processing the visual signal coming from the retina. The receptive profile (RP) $\psi(\xi)$ is the function that models the activation of a cortical neuron when a stimulus is applied to a point $\xi = (\xi_1, \xi_2)$ of the retinal plane. The hypercolumnar structure organizes the cortical cells of V1 in columns corresponding to different features. As a results we will identify cells in the cortex by means of two parameters (x, f), where $x = (x_1, x_2)$ is the position of the point, and f a vector of extracted features. We will denote F the set of features, and consequently the cortical space will be identified with $\mathbb{R}^2 \times F$. In the presence of a visual stimulus $I = I(\xi)$ the whole hypercolumn fires, giving rise to an output

$$O_F(x, f) = \int I(\xi)\psi_{(x,f)}(\xi)d\xi. \tag{1}$$

It is clear that the same image, filtered with a different family of cells, produces a different output.

For every cortical point, the cortical activity, suitably normalized, can be considered a probability density. Hence its maximum over the fibre F can be considered the most probable value of f, and can be considered the feature identified by the system (principle of non maxima suppression):

$$|O_F(x, f_I(x))| = \max_f |O_F(x, f)|. \tag{2}$$

The output of a family of cells is propagated in the cortical space $\mathbb{R}^2 \times F$ via the lateral connectivity.

2.1 Orientation and curvature selectivity

In [6] the authors considered only simple cells sensible to a direction $\theta \in \mathbb{S}^1$. Hence the set F becomes in this case S^1 and the underlying manifold reduces to $N := \mathbb{R}^2 \times \mathbb{S}^1$ with a subriemannian metric. The image I is lifted by the procedure (2) to a graph is this structure. If it is corrupted, it is completed via a subriemannian minimal surface.

The geometric description of the cortex was extended by Citti-Petitot-Sarti to a model of orientation and curvature selection, appeared in [25]. In this case the non maxima suppression process (2) selects a function

$$\Psi : \mathbb{R}^2 \to N_2 = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}, \quad \Psi(x) = (x, f_I(x)) = (x, \theta(x), k(x)).$$

Level lines of the input I are lifted to integral curves in N of the vector fields

$$X_1 = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} + k\frac{\partial}{\partial \theta}, \quad X_2 = \frac{\partial}{\partial k} \tag{3}$$

where $x = (x_1, x_2)$. The vector fields define an Hörmander type manifold since the whole space is spanned at every point by the vectors X_1, X_2 and their commutators

$$X_3 := [X_1, X_2] = -\frac{\partial}{\partial \theta}, \quad X_4 := [X_1, X_3] = -\sin\theta \frac{\partial}{\partial x_1} + \cos\theta \frac{\partial}{\partial x_2}. \tag{4}$$

These commutation conditions identify the structure of and Engel-type algebra. Integral curves of this model allow better completion of more complex images. In the image below we represent a grouping performed in [1] with an eigenvalue method in the sub-Riemannian setting. Precisely for every point x_i of the curves, we compute the orientation θ_i and the curvature k_i, and obtain a point $p_i = (x_i, \theta_i, k_i)$ in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$. Calling d the subriemannian distance induced by the choice of vector fields, we can compute the affinity matrix A, with entries $a_{ij} = d(p_i, p_j)$. The eigenvalues of this matrix, reprojected on $\mathbb{R}^2$ can be identified with the perceptual units present in the image. In figure 1 (from [1] the same curve is segmented in the geometry dependent only on orientation, and in the geometry of orientation and curvature: the second method correctly recovers the logarithmic spirals.

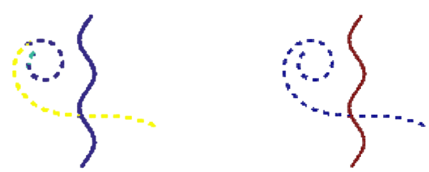

Fig. 1: grouping using orientation (left), or orientation and curvature (right).

It could be nice to see if it is possible to extend in this setting also the minimal surface algorithm for image completion [6]. The main obstacle in doing this was the fact that the notion of area and curvature was not well defined for codimension two surfaces in a sub-Riemannian metric, and that characterizing admissible variations presents intrinsic difficulties.

3 Graded structures

In the next section we will present the results obtained in [4], [5], [13] to define the notion of area of high codimension surfaces and its first variation in the setting of a graded structure. A graded structure is defined as following

Definition 31. *Let N be a smooth manifold and let $\mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s$ be an increasing filtration of sub-bundles of the tangent bundle TN s.t.*

$$X \in \mathcal{H}^i, Y \in \mathcal{H}^j \Rightarrow [X,Y] \in \mathcal{H}^{i+j} \text{ and } \mathcal{H}^s_p = T_pN \text{ for all } p$$

We will say that the fibration $\mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s$ is equiregular if $dim(\mathcal{H}^j)$ is constant in N. For such a manifold, we can define an homogeneous dimension $Q = \sum_{i=1}^s i\Big(dim(\mathcal{H}^i) - dim(\mathcal{H}^{i-1})\Big)$. We will say that a basis $(X_1, \ldots, X_n)$ of the tangent plane at every point is an adapted basis if $X_1, \ldots, X_{n_1}$ generate $\mathcal{H}^1$, $X_1, \ldots, X_{n_1}, X_{n_1+1}, \ldots, X_{n_2}$ generate $\mathcal{H}^2$, and so on. The presence of a filtration naturally allows to define the degree of a vector field and of an m-vector. In particular, we say that a vector $v \in T_p(M)$ has degree l, and we denote it $deg(v) = l$, if $v \in \mathcal{H}^l \setminus \mathcal{H}^{l-1}$. Given $m < n$, a multi-index $J = (j_1, \cdots, j_m)$, with $1 \leq j_1 < \ldots < j_m \leq n$, and an m-vector field $X_J = X_{j_1} \wedge \ldots \wedge X_{j_m}$ we define $\deg(X_J) = \deg(X_{j_1}) + \ldots + \deg(X_{j_m})$.

If a Riemannian metric g is defined on the graded manifold N we can introduce an orthogonal decomposition of the tangent space, which respects the grading, as follows: $\mathcal{K}^1_p := \mathcal{H}^1_p, \quad \mathcal{K}^{i+1}_p := (\mathcal{H}^i_p)^\perp \cap \mathcal{H}^{i+1}_p, \quad 1 \leq i \leq (s-1)$ So that $T_pN = \mathcal{K}^1_p \oplus \mathcal{K}^2_p \oplus \cdots \oplus \mathcal{K}^s_p$.

Example 1. *A Carnot manifold with a bracket generating distribution is a graded manifold.*

The interest of graded manifolds, is that a submanifold of a sub-Riemannian manifold, is not in general a sub-Riemannian manifold, but submanifolds of graded manifolds are graded.

3.1 Regular submanifolds

Given an immersion $\Phi : \bar{M} \to N, \quad M = \Phi(\bar{M})$. the manifold M inherits the graded structure $\tilde{\mathcal{H}}^i_p = T_pM \cap \mathcal{H}^i_p \subset \cdots \subset$. The pointwise degree (introduced in [14]) or local homogenous dimension is defined by

$$\deg_M(p) = \sum_{i=1}^s i \dim(\tilde{\mathcal{H}}_i(p) - \tilde{\mathcal{H}}_{i-1}(p))$$

For submanifolds, it is not possible in general to assume that the local homogeneous dimension is constant so that we will define the degree of M

$$d := \deg(M) = \max_{p \in M} \deg_M(p).$$

Given a graded manifold N with a Riemannian metric g, we are able to introduce a notion of area, as a limit of the corresponding riemannian areas. To begin with we define a family of Riemannian metrics g_r, adapted to the grading of the manifold as follows:

$$g_r|_{\mathcal{K}^i} = \frac{1}{r^{i-1}} g|_{\mathcal{K}^i}, \qquad i = 1, \ldots, s, \quad \text{for any} \quad r > 0$$

Now we assume that M is a submanifold of degree $d = deg(M)$, defined by a parametrization $\Phi : (\bar{M}, \mu) \to N$. Also assume that μ is a Riemannian metric on $\bar{M}$. For each $M' \subset \bar{M}$ we consider the Riemannian area, weighted by its degree

$$r^{\frac{d-m}{2}} \int_{M'} |E_1 \wedge \ldots \wedge E_m|_{g_r} d\mu(p),$$

where $E_1, \ldots, E_m$ a μ-orthonormal basis of T_pM. If the limit as $r \to 0$ exists, we call it the d-area measure. It can be explicitly expressed as

$$A_d(M') = \int_{M'} |\,(E_1 \wedge \ldots \wedge E_m)_d\,|_g \; d\mu(p).$$

where $(\cdot)_d$ the projection onto the space of m-vectors of degree d:

$$(E_1 \wedge \ldots \wedge E_m)_d = \sum_{X_J, \deg(X_J)=d} \langle E_1 \wedge \ldots \wedge E_m, X_J \rangle X_J$$

The same formula had been already established by [19].

3.2 Admissible variations

It would be natural to define the curvature as the first variation of the area. However in this case the area functional depends on the degree of the manifold. Hence we need to ensure that the degree does not change during variation, and we define degree preserving variations

Definition 32. *A smooth map $\Gamma : \bar{M} \times (-\epsilon, \epsilon) \to N$ is said to be an admissible variation of Φ if $\Gamma_s : \bar{M} \to N$, defined by $\Gamma_s(\bar{p}) := \Gamma(\bar{p}, s)$, satisfies*

(i) $\Gamma_0 = \Phi$,
(ii) $\Gamma_s(\bar{M})$ *is an immersion of the same degree as* $\Phi(\bar{M})$ *for small enough* s,
(iii) $\Gamma_s(\bar{p}) = \Phi(\bar{p})$ *for* $\bar{p}$ *outside a given compact subset of* $\bar{M}$.

We can always choose an adapted frame to a submanifold manifold M . First we choose a tangent basis $(E_1, \cdots, E_m)$, then we complete it to a basis of the space $X_{m+1}, \cdots, X_n$, where $X_{m+1}, \cdots, X_{m+k}$ have degree less or equal to $deg(E_1)$ while $X_{m+k+1}, \cdots, X_n$ have degree bigger than $deg(E_1)$.

Definition 33. *With the previous notation we define the variational vector field W as*

$$W(\bar{p}) = \frac{\partial \Gamma(\bar{p}, 0)}{\partial s}$$

It is always possible to assume that W has no tangential components, so that it will be represented as

$$W(\bar{p}) = \sum_{i=m+1}^{m+k} h_i X_i + \sum_{r=m+k+1}^{n} v_r X_r = H + V,$$

where $H = \sum_{i=m+1}^{m+k} h_i X_i$ and $V = \sum_{r=m+k+1}^{n} v_r X_r$.

Using the fact that if Γ is an admissible variation, which means that the degree of $\Gamma_s(\bar{M})$ is constant with respect to s, it is possible to prove the following

Proposition 1. *(see [5]) If W is an admissible vector field, then there exist matrices A, B such that*

$$E(V) + BV + AH = 0, \text{ where } E \text{ is the tangent basis.} \quad (5)$$

This property suggests the following definition

Definition 34. *We say that a compactly supported vector field W is admissible when it satisfies the admissibility system $E(V) + BV + AH = 0$.*

3.3 Variation for submanifolds

The phenomenon of minima which are isolated and do not satisfy any geodesic equation n (abnormal geodesic) was first discovered by Montgomery [22] for geodesic curves. In 1992 Hsu [15] proved a characterization of integrable vector fields along curves. We obtained in [4] a partial analogous of the previous result for manifolds of dimension bigger than one. Precisely we defined

Definition 35. *$\Phi : \bar{M} \to N$ is strongly regular at $\bar{p} \in \bar{M}$ if $A(\bar{p})$ has full rank, where A is defined in Proposition 1.*

Theorem 1. *[4] $\Phi : \bar{M} \to (N, \mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s)$ with a Riemannian metric g. Assume that Φ of degree d is strongly regular at $\bar{p}$. Then there exists an open neighborhood $U_{\bar{p}}$ of $\bar{p}$ such every admissible vector field W with compact support on $U_{\bar{p}}$ is integrable.*

The following properties are satisfied

Remark 1. *The admissibility of a vector field is independent of the Riemannian metric g.*

Remark 2. *All hypersurfaces in a sub-Riemannian manifold are deformable.*

4 Application to visual perception

Let us go back to the model of orientation and curvature introduced in Section 2.1. The underlying manifold is then $N = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$. Let us call g the metric which makes the vector fields X_1, X_2, X_3, X_4 in (3) and (4) an orthonormal basis. Let us consider a submanifold M defined by the parametrization

$$\Phi : \mathbb{R}^2 \supset \bar{M} \to \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}, \; M = \Phi(\bar{M})$$

In particular the surfaces obtained by non maxima suppression are expressed in the form

$$\Phi(x) = (x, \theta(x), \kappa(x))$$

If we impose the constraint $\kappa = X_1(\theta)$ the tangent vectors to M become $X_1 + X_1(\kappa)X_2$, $X_4 - X_4(\theta)X_3 + X_4(\kappa)X_2$, and the area functional reduces to

$$A_4(M) = \int_{\bar{M}} \sqrt{1 + X_1(\kappa)^2} \, dx. \quad (6)$$

We will denote $\bar{X}_1$ and $\bar{X}_4$ the projection of the vector fields X_1, X_4 onto $\bar{M}$. Then the admissibility system for a variational vector field $W = h_2(X_2 - X_1(\kappa)X_1) + v_3 X_3$ is given by

$$\bar{X}_1(v_3) = -\bar{X}_4(\theta)v_3 - Ah_2, \quad \text{where } A = (1 + (\bar{X}_1^2(\theta))^2)$$

Since $rank(A) = 1$ we deduce by Theorem 1 that for this type of surfaces each admissible vector is integrable, which implies that the minimal surfaces can be obtained via variational methods.

This property is quite important, since in [5] the authors proved that the manifold $\{(0, 0, \theta, k)\}$ do not admit degree preserving variation, so that it is isolated. This provides a generalization of the notion of abnormal geodesic.

4.1 Implementation and results

We directly implement the Euler Lagrangian of the functional (6). Since it is non linear, we compute a step 0 image with Euclidean Laplacian. After that, we compute at each step, orientation and curvature of level lines of the image at the previous step, update curvature and orientation via the linearized Euler Lagrangian equation and complete the 2D image diffusing along the vector field $\bar{X}_1$. We provide here a result in the simplified case with non corrupted points in the occluded region, and a preliminary result for the impainting problem. We plan to study the convergence of the algorithm and compare with existing literature in a forthcoming paper.

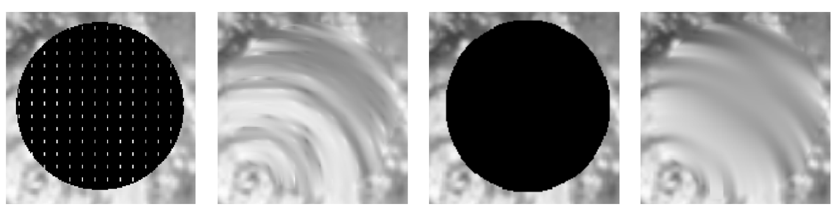

Fig. 2: Two examples of completion using the proposed algorithm.

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