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Disturbance Decoupling and Model Matching Problems for Discrete-time Systems with Time-varying Delays

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Abstract. In this paper, the disturbance decoupling problem and the model matching problem for discrete-time linear systems with time-varying delays are considered. Solvability of the above problems is characterized by means of structural necessary and sufficient conditions that can be checked by algorithmic procedures. The basic method used to analyze the considered problems consists in representing the discrete-time linear systems with time-varying delays as switching linear systems, whose properties can be studied by a powerful structural approach. In this way, the considered control problems can be reduced to the corresponding problems for switched linear systems, whose solvability has been recently characterized.

1 Introduction

Time-varying delays arise in several practical situations, for instance when modeling transfer lines or communication lines where modifications of the operating conditions (due to external events or internal causes) may force goods or information to change the velocity at which they are traveling. In relation with these and similar phenomena, several authors have considered, in particular, discrete-time linear systems with time-varying delays either from a theoretical or an applied point of view (see e.g. [4], [5], [6], [9], [12], [16], [25], [44]).
In the discrete-time framework, constant delays can be practically eliminated by augmenting the system internal variable by its finite history. The situation in which delays are not constant, but vary with time, can be dealt with in a similar way by resorting to switching linear systems. An early appearance in the literature of the idea to model a system with time-varying delays as a switching system without delays can be traced back to [15]. There, in analyzing the structure of integrated communication and control systems subject to time-varying delays, the authors considered discrete time models and, by augmenting the internal system variable, they constructed an equivalent model without delays, that was viewed as a time-varying system ([15, Equation (4.9)]) and that can, as well, be viewed as a discrete-time switching system. Thank to the correspondence so established between discrete-time linear systems with time-varying delays and switching linear systems, it is possible to apply in the framework of the first ones all the methods developed to study the second ones, for which a large literature is available (see e.g. [13], [14], [20], [21], [23], [34]).
After [15], other authors have exploited the same idea in a similar applied framework. In [24], discrete-time systems with time-varying delays were used to model networked control systems. Stability and performances problems involving these ones were dealt with simply by reducing them to the corresponding problems for switching systems, which have a known solution. The same idea was then applied in [35] and [28] to analyze the stability properties of control schemes for networked systems in presence of time-varying delays. Taking inspiration from this approach to networked systems, in [18], the correspondence between discrete-time linear systems with time-varying delays and switching linear systems was analysed in the general case and it was used to derive necessary and sufficient LMI conditions for the existence of delay dependent Lyapunov–Krasovskii functionals. The method based on those functionals to analyze stability for the considered class of time-delay systems was shown to be fully equivalent to the analogous one based on switching Lyapunov functions for switching systems. Much more recently, in [10], the authors resume the approach of [18] to give LMI sufficient conditions for the stabilizability of a class of discrete-time systems with time-varying delays and, in [31], the author uses the switching

system representation to characterize global asymptotic stability of discrete-time fully nonlinear systems with delays digraph in terms of time-dependent Lyapunov functions.

Here, we leverage the same approach for dealing with classical control problems, such as the disturbance decoupling problem and the asymptotic model matching problem, for discrete-time linear systems with time-varying delays. Referring to the augmented state space and to the dynamics of the switching representation of a given system with time varying delays, it is possible to introduce suitable notions of invariance and of controlled invariance that are useful in characterizing structural properties of the delayed dynamics. In this way, the geometric approach to linear systems of [2] and [37] can be extended to discrete-time linear systems with time-varying delays. Using recent results from the framework of switching systems [32], [42], we obtain, in particular, a complete characterization of the solvability of the disturbance decoupling problem with stability, either with unaccessible disturbance or with accessible disturbance, and of the asymptotic model matching problem. Solvability conditions can be algorithmically checked and viable procedures for constructing solutions, if any exists, are provided.

The paper is organized as follows. In Section 2, we introduce the class of discrete-time systems with time-varying delays we consider and we recall their equivalent representation as switching linear systems. As in the case of constant delays, the state variable of the switching representation is obtained by suitably augmenting the internal variable of the system with delay by its finite history, and the dynamics switches as the delays vary with time. In Section 3, we report some results on global asymptotic stability that are useful in the sequel. The notion of invariance and of controlled invariance for subspaces of the state space of a switching system are used in Section 4 to define the corresponding concepts with respect to a system with time-varying delays. Controlled invariance is characterized in terms of dynamic feedback in the time-varying delay framework and the relation with dynamic feedback stabilizability is illlustrated. Internal stabilizability for controlled invariant subspaces of the space of the augmented internal variables is the key property for dealing with dynamic feedback control problems in the time-delay framework if stability of the compensated system is required. In Section 5, we consider in particular the disturbance decoupling problem with stability and we characterize its solvability in the time-varying delay framework by specializing the results given in [42] and [41]. Analogously, in Section 6, we consider the asymptotic model matching problem and we characterize its solvability in the time-varying delay framework by specializing the results given in [32]. Section 7 contains the conclusions.

Notation: The symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}^+$ are used to denote the sets of real numbers, non negative real numbers and non negative integer numbers, respectively. Real vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. Linear maps between vector spaces and the associated matrices are denoted by the same slanted capital letters, like A. Therefore, the statements $A \in \mathbb{R}^{p\times q}$ and $A : \mathbb{R}^q \to \mathbb{R}^p$ are consistent. The image and the kernel of A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. Given a linear map $A : \mathcal{X} \to \mathcal{Y}$ and a subspace $\mathcal{V} \subseteq \mathcal{Y}$, the inverse image of $\mathcal{V}$ with respect to A is denoted by $A^{-1}(\mathcal{V})$.

2 Switching model representation of discrete-time systems with time-varying delays

It is known that in the discrete-time framework one can formally eliminate time delays from the equations of a dynamical system Σ by augmenting the dimension of the space of the internal variables (see e.g. [12]). This fact marks the substantial difference between discrete-time dynamical systems with delays and continuous-time ones, as the latter cannot be described by finite dimensional models.

In case of time-varying delays, a conceptually similar methodology can be applied to get a representation of the time-delayed system by means of a switching linear system in which the delays are no longer present. As recalled in the Introduction, this idea has been presented and exploited in a number of papers mainly to analyze the stability of systems with time-varying delays [15], [24], [28], [35], [18], [10], [31].

To describe formally this procedure, let us consider the unitary delay operator Δ whose action on a discrete-time function $f : \mathbb{Z} \to \mathbb{R}^q$ is defined recursively by

$$\begin{aligned} &\Delta f(t) = f(t-1) \\ &\Delta^{n+1} f(t) = \Delta(\Delta^n f(t)). \end{aligned} \tag{1}$$

A discrete-time linear system Σ with time-varying delays is an object defined by equations of the form

$$\Sigma \equiv \left\{ \begin{array}{rcl} x(t+1) & = & A_{\sigma(t)}(\Delta)x(t) + B_{\sigma(t)}(\Delta)u(t) \\ y(t) & = & C_{\sigma(t)}(\Delta)x(t) \end{array} \right. \tag{2}$$

where $t \in \mathbb{Z}^+$ denotes the time variable, $x \in \mathbb{R}^n$ is the internal variable; $u \in \mathbb{R}^m$ is the input; $y \in \mathbb{R}^p$ is the output; $\sigma : \mathbb{Z}^+ \to I$, with $I = \{1, ..., N\}$, is a time signal; $A_i(\Delta)$, $B_i(\Delta)$, $C_i(\Delta)$, with $i \in I$, are polynomial matrices in Δ of the form $A_i = \sum_{j=0}^{d} A_{ji}\Delta^j$, $B_i = \sum_{j=0}^{d'} B_{ji}\Delta^j$, $C_i = \sum_{j=0}^{d} C_{ji}\Delta^j$, with A_{ji}, B_{ji}, C_{ji} real matrices of dimensions $n \times n$, $n \times m$, $p \times n$ respectively. We say that the triple of matrices (A_i, B_i, C_i) defines, in particular, the (dynamical) structure of the delays Σ at time t if $\sigma(t) = i$ and that σ governs the variations of the structure of the delays. By d we denote the largest value of the delay that may affect any component of $x(t)$ in the equation of the dynamics and of the output of Σ and by d' we denote the largest value of the delay that may affect any component of the input $u(t)$. Note that the systems represented by (2) can be viewed as switching linear systems with delays and that their class include those considered, e.g., in [33].

Definition 1 *Given a discrete-time system Σ with time-varying delays of the form (2), consider the augmented variable X defined by*

$$X = (x^\top \, x_1^\top \, ... \, x_d^\top \, u_1^\top \, ... \, u_{d'}^\top)^\top \in \mathbb{R}^{n(d+1)} \times \mathbb{R}^{md'} \tag{3}$$

and define the augmented matrices $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ for $i \in I$ by

$$\bar{A}_i = \left(\begin{array}{c|c|c|c} A_{0i} \; ...... \; A_{(d-1)i} & A_{di} & B_{1i} \; ... \; B_{(d'-1)i} & B_{d'i} \\ \hline I_{nd} & 0_{nd \times n} & \multicolumn{2}{c}{0_{nd \times md'}} \\ \hline \multicolumn{4}{c}{0_{m \times (n(d+1)+md')}} \\ \hline \multicolumn{2}{c|}{0_{m(d'-1) \times n(d+1)}} & I_{m(d'-1)} & 0_{m(d'-1) \times m} \end{array} \right)$$
$$\bar{B}_i = \begin{pmatrix} B_{0i} \\ 0_{(nd) \times m} \\ I_m \\ 0_{m(d'-1) \times m} \end{pmatrix}; \; \bar{C}_i = (C_{0i} \; ... \; C_{di} \; 0_{p \times md'}). \tag{4}$$

The linear switching system Σ_σ defined by

$$\Sigma_\sigma \equiv \left\{ \begin{array}{rcl} X(t+1) & = & \bar{A}_{\sigma(t)}X(t) + \bar{B}_{\sigma(t)}u(t) \\ y(t) & = & \bar{C}_{\sigma(t)}X(t) \end{array} \right. \tag{5}$$

with modes

$$\Sigma_i \equiv \left\{ \begin{array}{rcl} X(t+1) & = & \bar{A}_i X(t) + \bar{B}_i u(t) \\ y(t) & = & \bar{C}_i X(t) \end{array} \right. \tag{6}$$

for $i \in I$ is called the switching representation of the discrete-time delayed system Σ.
We say that the augmented variable X defined by (3) is the state of Σ (and also of Σ_σ) and that $\mathcal{X}_{aug} = \mathbb{R}^{n(d+1)} \times \mathbb{R}^{md'}$ is the state space of Σ (and also of Σ_σ).

The representation given by (4) has been described in [24], [35], [18], [10], [31] in relation with the systems considered there.

Example 1 *A simple example is provided by the discrete-time system Σ with time-varying delays described by the equations*

$$\Sigma \equiv \left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + A_\delta x(t - \delta(t)) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{7}$$

where $t \in \mathbb{Z}^+$ denotes the time variable, $x \in \mathbb{R}^n$ is the internal variable; $u \in \mathbb{R}^m$ is the input; $y \in \mathbb{R}^p$ is the output; $\delta : \mathbb{Z}^+ \to \{1, ..., d\}$ is a function whose value defines a time-varying delay; A, A_δ, B and C are real matrices with suitable dimensions. The equations of Σ can be written in the form (2) by taking $\sigma(t) = \delta(t)$, $A_{\sigma(t)}(\Delta) = (A + A_\delta \Delta^{\sigma(t)}) = \sum_{j=0}^{d} A_{\sigma(t)j}\Delta^j$ (that is: $A_{i0} = A$ for all $i \in \{1, ..., d\}$, $A_{ij} = A_\delta$ for $j = i$ and $A_{ij} = 0_n$ for $0 \neq j \neq i$), $B_{\sigma(t)} = B$, $C_{\sigma(t)} = C$. Letting $X = (x^\top, x_1^\top, ..., x_d^\top)^\top$ denote the augmented internal *variable, the switching representation* Σ_σ *of* Σ *has the form (5) with the matrices* $\bar{A}_i$, $\bar{B}$ *and* $\bar{C}$*, respectively of dimensions* $(n(d+1)) \times (n(d+1))$, $(n(d+1)) \times m$ *and* $p \times (n(d+1))$*, given by*

$$\bar{A}_i = \left(\begin{array}{cccc|c} A_{i0} & A_{i1} & \dots & A_{i(d-1)} & A_{id} \\ \hline & & I_{nd} & & 0_{nd\times n} \end{array} \right); \ \bar{B} = \left(\begin{array}{c} B \\ 0_{nd\times m} \end{array} \right); \ \bar{C}^\top = \left(\begin{array}{c} C^\top \\ 0_{nd\times p} \end{array} \right).$$

Note that A *occupies the first position and* A_d *occupies the* $(i+1)$*-th position in the first row of the block matrix* $\bar{A}_i$*, while all other positions are occupied by* 0_n *(compare with [18, Eq. (6)], [10, Eq. (10)]).*

Example 2 *Let us consider a multi-agent system in which two agents with a discrete-time first order dynamics interact through unidirectional communication channels that are affected by independent time-varying delays. Assume that the delay on each channel may take the value* 1 *or* 2 *and that the internal dynamics of each agent and the weights* w_i *on each channel are constant. Then, the system can be described by equations of the form (2) as*

$$\left(\begin{array}{c} x_1(t+1) \\ x_2(t+1) \end{array} \right) = A_{\sigma(t)}(\Delta) \left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right) \tag{8}$$

where $\sigma : \mathbb{Z}^+ \to \{1,2,3,4\}$ *describes the variation of the delay structure and* $A_1(\Delta) = \left(\begin{array}{cc} a_1 & w_2\Delta \\ w_1\Delta & a_2 \end{array} \right)$, $A_2(\Delta) = \left(\begin{array}{cc} a_1 & w_2\Delta \\ w_1\Delta^2 & a_2 \end{array} \right)$, $A_3(\Delta) = \left(\begin{array}{cc} a_1 & w_2\Delta^2 \\ w_1\Delta^2 & a_2 \end{array} \right)$, $A_4(\Delta) = \left(\begin{array}{cc} a_1 & w_2\Delta^2 \\ w_1\Delta & a_2 \end{array} \right)$*. The associated switching representation* $\Sigma_{\sigma(t)}$ *has state space* $\mathbb{R}^6$ *and the dynamics matrices of its modes are given by*

$$\bar{A}_1 = \left(\begin{array}{cccc|cc} a_1 & 0 & 0 & w_2 & 0 & 0 \\ 0 & a_2 & w_1 & 0 & 0 & 0 \\ \hline & & I_4 & & & 0_{4\times 2} \end{array} \right); \ \bar{A}_2 = \left(\begin{array}{cccc|cc} a_1 & 0 & 0 & w_2 & 0 & 0 \\ 0 & a_2 & 0 & 0 & w_1 & 0 \\ \hline & & I_4 & & & 0_{4\times 2} \end{array} \right);$$

$$\bar{A}_3 = \left(\begin{array}{cccc|cc} a_1 & 0 & 0 & 0 & 0 & w_2 \\ 0 & a_2 & 0 & 0 & w_1 & 0 \\ \hline & & I_4 & & & 0_{4\times 2} \end{array} \right); \ \bar{A}_4 = \left(\begin{array}{cccc|cc} a_1 & 0 & 0 & 0 & 0 & w_2 \\ 0 & a_2 & w_1 & 0 & 0 & 0 \\ \hline & & I_4 & & & 0_{4\times 2} \end{array} \right).$$

Remark 1 *Although in the current literature (see e.g. [11, 26, 30, 36, 38]) the internal variable* x *is usually referred to as the state of the system* Σ *described by (7) or (2), it is more appropriate to assign the role of state of the delayed system to the augmented variable* X *defined in (3). Actually, it is well known that in order to compute the time evolution of the output* $y(t)$ *of* Σ *on a time interval* $[t_0, t_1]$*, the knowledge of* $x(t_0)$*, together with that of the input* $u(t)$ *on* $[t_0 - d', t_1 - 1]$*, is not enough, but it should be augmented by the information about the past finite history of* $x(t)$*, that is generally referred to as a "function of the initial state". More explicitly, in [12] the author states that* $x(t)$ *cannot be regarded as the state of the delayed system and in [3] and [36] the augmented state is employed in practice in some proofs. On the other hand, the augmented state* X *has the axiomatic property that characterizes the notion of state of a dynamical system as illustrated in [19, Cap. 10] and this motivates last part of Definition 1. Note, however, that it is not necessary to augment the variable* x *as much as indicated in Definition (1) to obtain an equivalent switching system representation of* Σ *in which the delay is no longer present. Actually, instead of* X*, it is enough to consider a state variable* X' *with* $\dim X' = (n + \sum_{i=1,\dots n} d_i + \sum j = 1, \dots m d'_j) \leq (n + nd + md') = \dim X$*, where* $d_i \leq d$ *and* $d'_j \leq d'$ *denote the maximum delay that affects, respectively, the component* x_i *of* x *and the component* u_j *of* u *in (2). The knowledge of* $X'(t_0)$*, together with that of the input* $u(t)$ *on* $[t_0, t_1 - 1]$*, is generally necessary and sufficient for computing the time evolution of the output* $y(t)$ *on* $[t_0, t_1]$*, provided the value of the switching functions* $\sigma(t)$ *is known at each time* t*.*

3 Stability

The qualitative properties of a discrete-time linear system Σ with time-varying delays of the form (7) are obviously the same as those of its switching representation Σ_σ and they depend, in particular, on the time signal σ that governs the variation of the structure of the delays in Σ and the switching in Σ_σ. In particular, global asymptotic stability of Σ for all $\sigma \in \mathscr{S}_1$, where $\mathscr{S}_1$ denote the set of all time signals $\sigma : \mathbb{Z}^+ \to I$, is equivalent to global asymptotic stability of Σ_σ for arbitrary switching, which is known to be equivalent to exponential stability. Analogously, the equivalence holds if variations of the structure of the delays may occur arbitrarily in time but according to sequences that are described by the paths of a given digraph, as in the so-called delay systems with delays digraphs, and in the corresponding so-called switching systems over digraphs. Since the literature contains several results on the stability of switching systems for arbitrary switching and for switching systems over digraphs, this fact has been exploited, as already recalled, for deriving result on the stability of discrete-time linear systems with time-varying delays in [24], [28] [35], [18] [10], [31]. A characterization of global asymptotic stability for all $\sigma \in \mathscr{S}_1$ is given in terms of LMIs in [18, Theorem 1] and in terms of matrix norm by the following proposition.

Proposition 1 *Let Σ be a discrete-time linear system with time-varying delays of the form (2) and, recalling (4), write the dynamic matrix $\bar{A}_{\sigma(t)}$ of the associated switching system Σ_σ as $\bar{A}_{\sigma(t)} = \begin{pmatrix} A^{11}_{\sigma(t)} & A^{12}_{\sigma(t)} \\ 0_{md' \times n(d+1)} & A^{22} \end{pmatrix}$, where $A^{22} = \begin{pmatrix} 0_{m(d'-1)\times m} & 0_m \\ I_{m(d'-1)} & 0_{m(d'-1)\times m} \end{pmatrix}$ is the component of the dynamics that models the presence of time-varying delays in the input of Σ. Then, Σ is globally asymptotically stable for all $\sigma \in \mathscr{S}_1$ if and only if there exists a finite integer k such that $||A_1\, A_2 \dots A_k||_\infty < 1$ for all k-tuple $(A_1, A_2, ..., A_k)$ with $A_i \in \{A^{11}_1, A^{11}_2, ..., A^{11}_d\}$.*

Proof: The stability of the switching system Σ_σ is not affected by A^{22}, since that component has a dead-beat behavior that does not depend on the rest of the system and on σ. Therefore, the component $(\, u_1^\top(t) \dots u_{d'}^\top(t))^\top$ of $X(t) = (x^\top(t)\, x_1^\top(t) \dots x_d^\top(t)\, u_1^\top(t) \dots u_{d'}^\top(t))^\top$ goes to 0 in finite time independently of all other conditions. The conclusion, then, follows from [22, Proposition 1]. □

Exploiting the structure of $\bar{A}_{\sigma(t)}$, we can derive from Proposition 1 the following result.

Proposition 2 *Let Σ be a discrete-time system with time-varying delays of the form (2). If the block submatrix $(A_{0i} \quad \dots \quad A_{(d-1)i} \quad A_{di})$ of the block matrix $\bar{A}_i$ satisfies the condition*

$$||(A_{0i} \quad \dots\dots \quad A_{(d-1)i} \quad A_{di})||_\infty < 1 \tag{9}$$

for all $i \in I$, then Σ is globally asymptotically stable for all $\sigma \in \mathscr{S}_1$.

Proof: As remarked above, we can neglect the component of the dynamics of Σ_σ that models the presence of time-varying delays in the input and concentrate on analyzing the submatrix of $\bar{A}_i$ given by

$$A^{11}_i = \left(\begin{array}{ccc|c} A_{0i} & \dots\dots & A_{(d-1)i} & A_{di} \\ \hline & I_{nd} & & 0_{nd\times n} \end{array}\right).$$

Choosing $\gamma \in \mathbb{R}$ such that $max_i||(A_{0i} \quad \dots \quad A_{(d-1)i} \quad A_{di})||_\infty < \gamma^d < 1$ and considering the nonsingular matrix $T \in \mathbb{R}^{n(d+1)\times n(d+1)}$ defined by

$$T = \left(\begin{array}{c|cccc} I_n & & 0_{n\times nd} & & \\ \hline & \gamma I_n & 0 & \dots & 0 \\ & 0 & \gamma^2 I_n & \dots & 0 \\ 0_{nd\times n} & & & \ddots & \\ & 0 & 0 & & 0 \\ & 0 & 0 & \dots & \gamma^d I_n \end{array}\right),$$

we get

$$\hat{A}_i = T A^{11}_i T^{-1} = \left(\begin{array}{cccc|c} A_{0i} & \gamma^{-1}A_{1i} & \dots\dots & \gamma^{-d+1}A_{(d-1)i} & \gamma^{-d}A_{di} \\ \hline & & \gamma \cdot I_{nd} & & 0_{nd\times n} \end{array}\right).$$

Since the inequality

$$\begin{aligned} &||(A_{0i} \quad \gamma^{-1}A_{1i} \quad \dots\dots \quad \gamma^{-d+1}A_{(d-1)i} \quad \gamma^{-d}A_{di})||_\infty \le ||\gamma^{-d}(A_{0i} \quad \dots\dots \quad A_{(d-1)i} \quad A_{di})||_\infty \le \\ &\le \gamma^{-d}||(A_{0i} \quad \dots\dots \quad A_{(d-1)i} \quad A_{di})||_\infty \le \gamma^{-d}(max_i||(A_{0i} \quad \dots \quad A_{(d-1)i} \quad A_{di})||_\infty) < 1 \end{aligned}$$

holds for all $i \in I$, we have that also $||\hat{A}_i||_\infty < 1$ holds for all $i \in I$. Then, applying the change of basis defined by $\begin{pmatrix} T^{-1} & 0 \\ 0 & I_{md'} \end{pmatrix}$, Σ_σ is shown to be globally asymptotically stable for arbitrary switching by Proposition 1. □

Example 3 *Consider the discrete-time linear system* Σ *with time-varying delays described by the equations*

$$\Sigma \equiv \begin{cases} x(t+1) &= -1/2x(t) + 1/4x(t-\delta_1(t)) + u(t) - 1/2u(t-\delta_2(t)) \\ y(t) &= Cx(t) \end{cases} \tag{10}$$

where $\delta_1 : \mathbb{Z}^+ \to \{1,2\}$ *and* $\delta_2 : \mathbb{Z}^+ \to \{1,2\}$ *represent two independent time-varying delays. In order to apply the above proposition, we consider, with the above notations, the matrices* $(A_{01} \quad A_{11} \quad A_{21}) = (-1/2 \quad 1/4 \quad 0)$ *and* $(A_{02} \quad A_{12} \quad A_{22}) = (-1/2 \quad 0 \quad 1/4)$. *Since the* ∞*-norm of both matrices is smaller than* 1, *the system* Σ *is globally asymptotically stable for all* $\sigma \in \mathscr{S}_1$, *that is for arbitrary variations of the delays. Note that this result can be obtained also by applying [18, Theorem 1] and computing a solution of the resulting LMI.*

Remark 2 *The stability condition (9) is conservative, but it may be interesting since it is very simple to check by elementary computations. In addition, if it holds for a given system* Σ, *it holds also for the system obtained by increasing the delays by substituting* Δ^j *with* Δ^{j+k_j} *in (2) with* $k_j \in \mathbb{Z}^+$, *provided that* $j + k_j \neq j' + k_{j'}$ *for all* $j \neq j'$.

Together with stability for arbitrary switching, it is interesting to consider also stability under the so-called restricted switching. This is the case, in particular, if the interval of time between two consecutive variations of the delays can be assumed to be greater than a given threshold. A situation of this kind is considered, for instance, in [24], where stability and performances of networked control systems subject to slow variations, in a suitable sense, of the delay structure is investigated and, from a more general point of view, in [39] (see also the references therein). To be more precise, given a time signal $\sigma : \mathbb{Z}^+ \to I$, with $I = \{1, ..., N\}$, consider the set $T = \{t, t \in \mathbb{Z}^+ \,\text{such that}\, \sigma(t+1) \neq \sigma(t)\} \cup \{0\}\}$. The dwell time τ_σ of σ is defined as $\tau_\sigma = min\{t_1 - t_2,\, with\, t_1, t_2 \in T\, and\, t_1 > t_2\}$. The dwell time is equal to the number of points contained in the smallest interval of $\mathbb{Z}^+$ on which σ is constant and it is greater than or equal to 1 for all $\sigma \in \mathscr{S}_1$. We denote by $\mathscr{S}_\alpha \subseteq \mathscr{S}_1$, with $\alpha \geq 1$, the subset of all time signals $\sigma : \mathbb{Z}^+ \to I$ such that $\tau_\sigma \geq \alpha$. A similar concept is expressed by the notion of average dwell time of σ that was introduced in dealing with switching systems in [17,43]. Letting $N_\sigma(k) = card\{T \cap [0,k]\}$, the average dwell time $\bar{\tau}_\sigma$ of σ is defined as $\bar{\tau}_\sigma = max\{\beta,\, such\, that\, N_\sigma(k) \leq N_0 + \frac{k}{\beta}\, for\, some\, N_0 \in \mathbb{Z}^*\}$. The value N_0 is called the chatter bound of σ. We denote by $\bar{\mathscr{S}}_\beta$, with $\beta \geq 1$, the subset of $\mathscr{S}_1$ consisting of all time signal $\sigma : \mathbb{Z}^+ \to I$ such that $\bar{\tau}_\sigma \geq \beta$. Intuitively, a time signal belongs to $\bar{\mathscr{S}}_\beta$ if the number of variations occurring in a generic interval $[0,k]$ is limited by a linear function whose slope is the inverse of the average dwell time. We speak of restricted switching, as opposed to arbitrary switching, if σ belongs to $\mathscr{S}_\alpha$ for some $\alpha > 1$ or to $\bar{\mathscr{S}}_\beta$ for some $\beta > 1$.
All stability results for switching linear systems with restricted switching apply to discrete-time linear systems with time-varying delays. In particular, by [27, Lemma 2] and by [43, Corollary 1], we have the following proposition.

Proposition 3 *Let* Σ *be a discrete-time system with time-varying delays of the form (2).*

1. *There exists* $\alpha \geq 1$ *such that* Σ *is globally asymptotically stable for all* $\sigma \in \mathscr{S}_\alpha$ *if and only if each mode* Σ_i *of the associated switching system* Σ_σ *is asymptotically stable.*

2. *There exists* $\beta \geq 1$ *such that* Σ *is globally asymptotically stable for all* $\sigma \in \bar{\mathscr{S}}_\beta$ *if and only if each mode* Σ_i *of the associated switching system* Σ_σ *is asymptotically stable.*

Example 4 *Let us consider the discrete-time linear system* Σ *with time-varying delays described by*

$$\Sigma \equiv \begin{cases} x(t+1) &= a_{\sigma(t)}(\Delta)x(t) + bu(t) \\ y(t) &= cx(t) \end{cases} \tag{11}$$

where $x \in \mathbb{R}$, $\sigma : \mathbb{Z}^+ \to \{1,2\}$ *and* $a_1 = 1 - \frac{1}{2}\Delta$, $a_2 = -1 + \frac{1}{2}\Delta^2$. *The associated switching system* Σ_σ, *with* $X \in \mathbb{R}^3$, *has two modes, whose dynamics is described, respectively, by the matrices* $A_1 = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *These matrices are Schur stable, and therefore, by Proposition 3, the system* Σ *is globally asymptotically stable for* $\sigma \in \mathscr{S}_\alpha$ *or for* $\sigma \in \bar{\mathscr{S}}_\beta$ *with* α *or* β *sufficiently large, or, in other terms, if the time instants at which the structure of the delays varies are sufficiently far away from each other or sufficiently far away from each other in the average. In particular, global asymptotic stability holds in this case if* $\alpha \geq 11$, *since* $||A_i^{11}||_\infty < 1$ *for* $i = 1, 2$. *The system* Σ, *however, is not globally asymptotically stable for arbitrary variations of the delay. To show this, let us consider the time signal* $\sigma(t)$ *defined by* $\sigma(t) = \begin{cases} 1 \text{ if } t \in \{0,2,4,6,...\} \\ 2 \text{ if } t \in \{1,3,5,7,...\} \end{cases}$ *that forces* Σ_σ *to switch between its two modes at all times instants. Since* $X(2t) = A^t X(0)$ *with* $A = A_2 A_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1/2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* A *has an eigenvalue with module greater than 1, the free evolution of the state is divergent and* Σ *is unstable.*

4 Geometric structure

The switching representation Σ_σ of a discrete-time system Σ with time-varying delays of the form (2) can be used to display the structural properties of Σ in order to analyze its dynamics. In doing this, one has to refer to the switching model representation Σ_σ and to the state space $\mathcal{X}_{aug}$ of Σ as they are described in Definition 1 and to apply the notions and methods developed for switching linear systems. We start by specializing to the class of systems at issue the notions of invariance and of controlled invariance.

Definition 2 *Given a linear discrete-time system* Σ *with time-varying delays of the form (2), a subspace* $\mathcal{V} \subseteq \mathcal{X}_{aug}$ *is said to be:*

1. *an invariant subspace for* Σ *if it is an invariant subspace for the switching model representation* Σ_σ*: that is if*
$$\bar{A}_i \mathcal{V} \subseteq \mathcal{V} \text{ for all } i \in I \tag{12}$$
2. *a controlled invariant subspace for* Σ *if it is a controlled invariant subspace for the switching model representation* Σ_σ*: that is if*
$$\bar{A}_i \mathcal{V} \subseteq \mathcal{V} + \operatorname{Im} \bar{B}_i \text{ for all } i \in I \tag{13}$$

where $\bar{A}_i$ *and* $\bar{B}_i$ *are, respectively, the dynamic matrix and the input distribution matrix of the* i*-th mode of* Σ_σ.

Remark 3 *Controlled invariance was introduced in the framework of linear switching system in [29] in order to deal with the problem of decoupling the output from a disturbance input by means of a state feedback. The notion of controlled invariant subspace for* Σ_σ *coincides with that of robust* (A_i, B_i)*-invariant subspace for the family of linear systems that define the modes of* Σ_σ *introduced earlier in [1]. A controlled invariant subspace for* Σ_σ *is, in particular, controlled invariant, or* (A_i, B_i)*-invariant, in the sense of [2] and [37] for each mode* Σ_i.

It is useful to remark that, denoting by V a matrix whose columns are a basis of $\mathcal{V}$, (13) is equivalent to the existence of a family $\{(L_i, M_i)\}_{i \in I}$ of pairs of matrices of suitable dimensions such that $A_i V = V L_i + B_i M_i$ for $i \in I$.

The set of controlled invariant subspaces for Σ that are contained in a given subspace $\mathcal{W} \subseteq \mathcal{X}_{aug}$ is a semi-lattice with respect to inclusion and sum of subspaces and, therefore, it has a maximum element. The maximum controlled invariant subspace for Σ contained in $\mathcal{W}$ is denoted by $\mathcal{V}^*(\mathcal{W})$ and it can be obtained as the limit of the sequence of subspaces $\mathcal{V}_k$ defined by

$$\mathcal{V}_0 = \mathcal{W}$$
$$\mathcal{V}_k = \mathcal{V}_{k-1} \cap \big(\bigcap_{i=1,\dots,N} \bar{A}_i^{-1}(\mathcal{V}_{k-1} + \mathrm{Im}\,\bar{B}_i)\big).$$

The above sequence converges in a number of steps that is smaller than or equal to $(\dim \mathcal{W}) + 1$, making it possible to construct $\mathcal{V}^*(\mathcal{W})$ by means of a finite algorithmic procedure (see [7], [29]). In the case where $\mathcal{W} = \bigcap_{i=1\dots N} \mathrm{Ker}\,\bar{C}_i$, we will denote $\mathcal{V}^*(\mathcal{W})$ simply by $\mathcal{V}^*$.
In addition to the geometric characterization given by (13), a dynamic characterization of controlled invariance can be given. To this aim, let us remark that, given a linear discrete-time system Σ with time-varying delays of the form (2), we can choose its input $u(t)$ in such a way to satisfy a relation of the form

$$u(t) = F_{\sigma(t)}(\Delta)x(t) + F'_{\sigma(t)}(\Delta)u(t) + v(t) \tag{14}$$

for $t \in \mathbb{Z}^+$, where $F_i(\Delta)$ and $F'_i(\Delta)$, for $i = \{1, ..., N\}$, are polynomial matrices of the form $F_i = \sum_{j=0}^{d} F_{ji}\Delta^j$ and $F'_i = \sum_{j=1}^{d'} F'_{ji}\Delta^j$, with F_{ij} and F'_{ij} real matrices of suitable dimensions, and where $v \in \mathbb{R}^m$ is a new input variable. If $F'_\sigma(\Delta)$ is different from the zero matrix, (14) defines, for each $i \in I$, a dynamic relation between $u(t)$ and $\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ whose transfer matrix is given by $\left(\frac{z^{\bar{d}}}{z^{\bar{d}}}(I_m - F'_i(z^{-1})^{-1}F_i(z^{-1}) \;\; I_m\right)$, where $\bar{d} = \max\{d, d'\}$ (note that $(I_m - F'_i(z^{-1}))$ is a nonsingular polynomial matrix in z^{-1}). It is quite natural to interpret (14) as a variable-delay dynamic feedback which, applied to Σ, gives rise to the compensated discrete-time variable delay system

$$\Sigma^{(F,F')} \equiv \begin{cases} x(t+1) &= (A_{\sigma(t)}(\Delta) + B_{\sigma(t)}(\Delta)F_{\sigma(t)}(\Delta))x(t) + B_{\sigma(t)}(\Delta){F'}_{\sigma(t)}(\Delta)u(t) + B_{\sigma(t)}(\Delta)v(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{cases} \tag{15}$$

We can associate to the dynamic feedback (14) a switching static state feedback acting on Σ_σ which is given by

$$u(t) = \bar{F}_{\sigma(t)}X(t) + v(t) \tag{16}$$

where $\bar{F}_i = (F_{0i}\; F_{1i}\; ...F_{di}\; F'_{1i}\; F'_{2i}\; ...F'_{d'i})$ and the state variable X is defined by (3). By applying the switching state feedback $u(t) = \bar{F}_{\sigma(t)}X(t) + v(t)$ to Σ, we get the compensated system

$$\Sigma_\sigma^{\bar{F}} \equiv \begin{cases} X(t+1) &= (\bar{A}_{\sigma(t)} + \bar{B}_{\sigma(t)}\bar{F}_{\sigma(t)})X(t) + \bar{B}_{\sigma(t)}v(t) \\ y(t) &= \bar{C}_{\sigma(t)}X(t) \end{cases} \tag{17}$$

which is associated to $\Sigma^{(F,F')}$ and we can state the following proposition.

Proposition 4 *Given a linear discrete-time system Σ with time-varying delays of the form (2), a subspace $\mathcal{V} \subseteq \mathcal{X}_{aug}$ is controlled invariant for Σ if and only if there exists a switching state feedback $u(t) = \bar{F}_{\sigma(t)}X(t) + v(t)$ of the form (16) such that $\mathcal{V}$ is an invariant subspace for $\Sigma_\sigma^{\bar{F}}$ or, equivalently, if and only if there exists a variable delay dynamic feedback $u(t) = F_{\sigma(t)}(\Delta)x(t) + F'_{\sigma(t)}(\Delta)u(t) + v(t)$ of the form (14) such that $\mathcal{V}$ is an invariant subspace for the compensated discrete-time variable delay system $\Sigma^{(F,F')}$.*

Proof: The proof given in [29] in the continuous-time framework holds □

Any feedback of the form (14), or equivalently of the form (16), that makes invariant in the compensated system a given controlled invariant subspace $\mathcal{V}$ is said to be a friend of $\mathcal{V}$. It is worthwhile to remark that employing the switching model representation of Σ the above dynamic characterization of controlled invariance is given in terms of static state feedback, as in the classic geometric approach developed for linear systems in [2] and [37].

Example 5 *Let us consider a discrete-time system Σ with time-varying delays of the form (7) and let $\mathcal{V} \subseteq \mathcal{X}$ be a subspace that is both (A,B)-controlled invariant (i.e. such that $A\mathcal{V} \subseteq \mathcal{V} + Im\, B$) and ($A_\delta$,B)-controlled invariant (i.e. such that $A_\delta \mathcal{V} \subseteq \mathcal{V} + Im\, B$). Let v be the dimension of $\mathcal{V}$ and, denoting by V a matrix whose columns are a basis of $\mathcal{V}$, consider the subspace $\mathcal{V}_{aug} \subseteq \mathcal{X}_{aug}$ spanned by the columns of the $(n \times (d+1)) \times (v \times (d+1))$ matrix* $V_{aug} = \begin{pmatrix} V & 0 & 0 & \cdots & 0 \\ 0 & V & 0 & \cdots & 0 \\ 0 & 0 & V & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & V \end{pmatrix}$. *It is possible to show that $\mathcal{V}_{aug}$ is a controlled invariant subspace for Σ. For this, let (L, M) and (L_δ, M_δ) be two pairs of matrices such that $AV = VL + BM$ and $A_\delta V = VL_\delta + BM_\delta$. Then, we have $\bar{A}_i V_{aug} = V_{aug}\bar{L}_i + \bar{B}\bar{M}_i$ for all $i = 1, ..., d$, where $\bar{L}_i = \left(\begin{array}{cccc|c} L & L_1 & \ldots & L_{d-1} & L_d \\ \hline & & I_{v(d-1)} & & 0_{v(d-1)\times v} \end{array} \right)$, with $L_j = 0_n$ for $j \neq i$ and $L_j = L_\delta$ for $j = i$ (i.e L_δ occupies the $(i+1)$-th position in the first block-line of $\bar{L}_i$), and where $\bar{M}_i = \begin{pmatrix} M & M_1 & \ldots & M_{d-1} & M_d \end{pmatrix}$, with $M_j = 0_n$ for $j \neq i$ and $M_j = M_\delta$ for $j = i$ (i.e. M_δ occupies the $(i+1)$-th position in the block representation of $\bar{M}_i$). If $\mathcal{V}$ is contained in $Ker\, C$, then $\mathcal{V}_{aug}$ is contained in $Ker\, \bar{C}$, but in general $\mathcal{V}_{aug}$ does not coincide with the maximum controlled invariant subspace for Σ contained in $Ker\, \bar{C}$, namely $\mathcal{V}^*$, even if $\mathcal{V}$ is the largest subspace of $Ker\, C$ that is both (A,B)-controlled invariant and (A_δ,B)-controlled invariant. This can be shown by taking, for instance, the simple discrete-time linear system Σ, where the delay is not variable, defined by*

$$\begin{cases} x(t+1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t-1) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_d(t) \\ y(t) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(t) \end{cases} \tag{18}$$

Referring to the system matrices of Σ by the notations of used in equation (7), we have that the largest subspace of $Ker\, C$ that is both (A,B)-controlled invariant and (A_δ,B)-controlled invariant is the null subspace of $\mathbb{R}^3$, but $\mathcal{V}^* = \mathrm{Im} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Further structural notions and properties can be defined for discrete-time systems with time-varying delays by means of the corresponding notions and properties of the associated switching systems. This is the case, in particular, of the notion of observability and those of unobservable subspace and output nulling subspace. More precisely, given a linear discrete-time system Σ with time-varying delays of the form (2), we can define the unobservable subspace of Σ as the unobservable subspace $\mathcal{X}_{NO} \subseteq \mathcal{X}_{aug}$ of Σ_σ: that is the subspace consisting of all the states $X_0 \in \mathcal{X}_{aug}$ whose free evolution $X(t)$ is such that $y(t) = \bar{C}(X(t)) = 0$ for $t \geq 0$. Analogously, we can define the output nulling subspace of Σ as the output nulling subspace $\mathcal{X}_{null} \subseteq \mathcal{X}_{aug}$ of Σ_σ: that is the subspace consisting of all the states $X_0 \in \mathcal{X}_{aug}$ for which there exists an input sequence $u(t)$, with $t \geq 0$, such that the corresponding forced evolution $X(t)$ gives $y(t) = \bar{C}(X(t)) = 0$ for $t \geq 0$. The following proposition illustrate the relations between these notions and $\mathcal{V}^*$.

Proposition 5 *Given a linear discrete-time system Σ with time-varying delays of the form (2), the maximum controlled invariant subspace $\mathcal{V}^*$ contained in $\bigcap_{i=1...N} \mathrm{Ker}\, \bar{C}_i$ coincides with the output nulling subspace $\mathcal{X}_{null}$ of Σ and also with the unobservable subspace $\mathcal{X}_{NO}$ of $\Sigma^{(F,F')}$, where $u(t) = F_{\sigma(t)}(\Delta)x(t) + F'_{\sigma(t)}(\Delta)u(t) + v(t)$ is a friend of $\mathcal{V}^*$.*

The propositions above indicate that controlled invariance is the key notion in the analysis and solution of a number of noninteracting control problems, as the disturbance decoupling problem that will be dealt with in

Section 5. Moreover, the switching dynamics induced by the dynamics of $\Sigma_\sigma^{\bar{F}}$ on the subspace $\mathcal{V}^* \subseteq \mathcal{X}_{aug}$ plays a fundamental role in the solution of the same problems when stability is an additional requirement. This motivates the introduction of the following notion by specializing to the class of systems at issue the corresponding one introduced for switching systems in [41].

Definition 3 *A controlled invariant subspace $\mathcal{V}$ for the linear discrete-time system Σ with time-varying delays of the form (2) is said internally stabilizable if for some friend $\bar{F}_{\sigma(t)}$ of $\mathcal{V}$ there exists α such that the switching dynamics induced on $\mathcal{V}$ by that of $\Sigma_\sigma^{\bar{F}}$ is globally asymptotically stable for all $\sigma \in \mathscr{S}_\alpha$.*

The set of all internally stabilizable controlled invariant subspaces that are contained in a given subspace $\mathcal{W} \subseteq \mathcal{X}_{aug}$ has a maximum element. The maximum internally stabilizable controlled invariant subspace for Σ contained in $\mathcal{W} \subseteq \mathcal{X}_{aug}$ is denoted by $\mathcal{V}_g^*(\mathcal{W})$, or simply by $\mathcal{V}_g^*$ if $\mathcal{W} = \bigcap_{i=1\dots N} \operatorname{Ker} \bar{C}_i$ and it is called the good controlled invariant subspace for Σ. The notion of good controlled invariant subspace was introduced in the framework of switching systems in [42], where also a procedure to construct $\mathcal{V}_g^*(\mathcal{W})$ was given. To recall such procedure, let us denote by $\mathcal{V}_g(\bar{A}_i, \bar{B}_i, \mathcal{W})$ the maximum controlled invariant subspace for the i-th mode Σ_i contained in $\mathcal{W}$ that is internally stabilizable for Σ_i. The subspace $\mathcal{V}_g(\bar{A}_i, \bar{B}_i, \mathcal{W})$ was first considered in [37] and a procedure to construct it is given in [37, Section 5.6]. Then, we have that $\mathcal{V}_g^*(\mathcal{W})$ is the last term of the sequence $\mathcal{V}^j$, with $j = 0, 1, ..., l$, generated by the recursive algorithm

$$\begin{cases} \mathcal{W}^0 = \mathcal{W} \\ \mathcal{K}^0 = \bigcap_{i \in \mathcal{I}} \max \mathcal{V}_g(\bar{A}_i, \bar{B}_i, \mathcal{W}^0) \\ \mathcal{V}^0 = \max \mathcal{V}(\bar{A}_i, \bar{B}_i, \mathcal{K}^0) \end{cases} \qquad \begin{cases} \mathcal{W}^j = \mathcal{V}^{j-1} \\ \mathcal{K}^j = \bigcap_{i \in \mathcal{I}} \max \mathcal{V}_g(\bar{A}_i, \bar{B}_i, \mathcal{W}^j) \\ \mathcal{V}^j = \max \mathcal{V}(\bar{A}_i, \bar{B}_i, \mathcal{K}^j) \end{cases} \qquad j = 1, 2, ..., l$$

where l, with $0 \leq l \leq \dim \mathcal{K}$, is the least integer such that $\mathcal{V}^l$ is internally stabilizable with respect to each mode Σ_i, for $i \in I$, of Σ_σ.

Remark 4 *The structural notions introduced in this section are of interest mainly in the case in which the value $\sigma(t)$ of the time signal that governs the variation of the structure of the delays in Σ is known at each time t, or, in other terms, in which σ is measurable (note that this does not implies that $\sigma(t)$ is known in advance and it allows random occurrences of the variations). From a general point of view, this assumption is restrictive, but it can reasonably be made in a number of interesting situations: for instance, in modelling transfer lines in which the modifications of the structure that causes the variations of the delays are controlled by a supervisor, who takes decisions according to the specific information that is available at time t. More generally, this may happens in a multi-agent system where delays are due to the necessity of sharing limited resources, like e.g. communication lines, by allocating them according to variable needs and to some rules.*

5 Disturbance Decoupling Problem

The switching model representation and the geometric notions illustrated above can be used to deal with the problem of decoupling, by means of a feedback, the output of a linear discrete-time system Σ with time-varying delays from a disturbance input, either in case this is unaccesible and in case this is accessible. Formally, the problems we want to deal with are defined as follows.

Problem 1 *Given a discrete-time linear system Σ with time-varying delays of the form (2), assume that the input variable u is partitioned in two components as $u = (u_c^\top \; u_d^\top)^\top$, with $u_c \in \mathbb{R}^{m_c}$, $u_d \in \mathbb{R}^{m_d}$ and $m_c + m_d = m$, and that the matrix $B_{\sigma(t)}(\Delta)$ is correspondingly partitioned as $B_{\sigma(t)}(\Delta) = (B^c_{\sigma(t)}(\Delta) \; B^d_{\sigma(t)}(\Delta))$. Interpreting u_c as the control input variable and u_d as the unaccessible disturbance input variable, the Disturbance Decoupling Problem with Stability (DDPS) for Σ consists in finding a variable delay dynamic feedback $u_c(t) = F_{\sigma(t)}(\Delta)x(t) + F'_{\sigma(t)}(\Delta)u_c(t) + v(t)$ of the form (14), such that the compensated system $\Sigma^{(F,F')}$ is globally asymptotically stable for all $\sigma \in \mathscr{S}_\alpha$ for some $\alpha > 1$ and its output $y(t)$ is not affected by the disturbance input $u_d(t)$.*

Problem 2 *In the same hypotheses as above, but assuming that the disturbance u_d is accessible, the Accessible Disturbance Decoupling Problem with Stability (ADDPS) for Σ consists in finding a variable delay dynamic compensator of the form*

$$u_c(t) = F_{\sigma(t)}(\Delta)x(t) + F'_{\sigma(t)}(\Delta)u_c(t) + G_{\sigma(t)}(\Delta)u_d(t) + v(t) \tag{19}$$

such that the compensated system $\Sigma^{(F,F',G)}$ is globally asymptotically stable for all $\sigma \in \mathscr{S}_\alpha$ for some $\alpha > 1$ and its output $y(t)$ is not affected by the disturbance input $u_d(t)$.

In order to characterize the solvability of the above problems, let us consider the switching system Σ_σ given by (5) and associated to the disturbed system Σ. Since $B_i = (B_i^c \; B_i^d) = \sum_{j=0}^{d'} (B_{ji}^c \; B_{ji}^d)\Delta^j$ for $i \in I$, we write the equations of Σ_σ in the form

$$\Sigma_\sigma \equiv \left\{ \begin{array}{rcl} X(t+1) & = & \bar{A}_{\sigma(t)}X(t) + \bar{B}^c_{\sigma(t)}u_c(t) + \bar{B}^d_{\sigma(t)}u_d(t) \\ y(t) & = & \bar{C}_{\sigma(t)}X(t) \end{array} \right. \tag{20}$$

with

$$\bar{B}_i^c = \begin{pmatrix} B_{0i}^c \\ 0_{(nd)\times m_1} \\ I_{m_1} \\ 0_{m_2\times m_1} \\ 0_{m(d'-1)\times m_1} \end{pmatrix}; \;\; \bar{B}_i^d = \begin{pmatrix} B_{0i}^d \\ 0_{(nd)\times m_2} \\ 0_{m_1\times m_2} \\ I_{m_2} \\ 0_{m(d'-1)\times m_2} \end{pmatrix}. \tag{21}$$

Denoting by $\mathcal{V}_g^*$ the good invariant subspace of the undisturbed switching system obtained by disregarding the input distribution matrix $\bar{B}^d_{\sigma(t)}$ in (20), we can state the following result.

Theorem 1 *Given a disturbed discrete-time linear system Σ with time-varying delays of the form (2) with $u = (u_c^\top \; u_d^\top)^\top$, assume that each mode of the associated switching system Σ_σ of the form (20) is asymptotically stable. Then*

1. *the corresponding DDPS is solvable if and only if the condition*

$$\operatorname{Im} \bar{B}_i^d \subseteq \mathcal{V}_g^* \tag{22}$$

   holds for all $i \in I$;

2. *the corresponding ADDPS is solvable if and only if the condition*

$$\operatorname{Im} \bar{B}_i^d \subseteq \mathcal{V}_g^* + \operatorname{Im} \bar{B}_i^c \tag{23}$$

   holds for all $i \in I$.

Proof: To solve the problem for Σ is equivalent to solve the corresponding problem for the associated switching system Σ_σ. Hence, since there are no conceptual differences between the discrete-time case and the continuous-time one with regard to the decoupling problems, the proof given in [42, Theorem 1] for the first statement and inside the proof of [32, Theorem 1] (see also [40]) for the second statement hold. □

It is important to remark that the proofs given in [42] and in [32] are constructive, in the sense that they provide viable algorithmic procedures to check the solvability conditions and to construct solutions to the considered problems, if any exists, in the framework of the linear switching system. Exploiting those procedures and the representation by means of switching models we can practically do the same in the framework of discrete-time linear systems with time-varying delays.

6 Model Matching Problem

The solution of the ADDPS can be exploited to solve the problem of compensating a given discrete-time linear system Σ with time-varying delays in such a way to force its output to follow that of a given model of the same kind as done, in particular in the framework of switching systems, in [32] and in [8]. In order to state precisely the problem we want to deal with, let us consider two discrete-time linear systems with time-varying delays Σ_P and Σ_M, called respectively the plant and the model, defined by the following equations of the form (2)

$$\Sigma_P \equiv \left\{ \begin{array}{rcl} x_P(t+1) & = & A_{P\sigma(t)}(\Delta)x_P(t) \\ & & +B_{P\sigma(t)}(\Delta)w(t) \\ y_P(t) & = & C_{P\sigma(t)}(\Delta)x_P(t) \end{array} \right. \qquad \Sigma_M \equiv \left\{ \begin{array}{rcl} x_M(t+1) & = & A_{M\sigma(t)}(\Delta)x_M(t) \\ & & +B_{M\sigma(t)}(\Delta)u(t) \\ y_M(t) & = & C_{M\sigma(t)}(\Delta)x_M(t) \end{array} \right. \tag{24}$$

with $y_P, y_m \in \mathbb{R}^P$ and $\sigma : \mathbb{Z}^+ \to I$. In order to compare the output of the plant and that of the model, we employ the output difference system

$$\Sigma_D \equiv \left\{ \begin{array}{rcl} x_P(t+1) & = & A_{P\sigma(t)}(\Delta)x_P(t) + B_{P\sigma(t)}(\Delta)w(t) \\ x_M(t+1) & = & A_{M\sigma(t)}(\Delta)x_M(t) + B_{M\sigma(t)}(\Delta)u(t) \\ y(t) & = & C_{P\sigma(t)}(\Delta)x_P(t) - C_{M\sigma(t)}(\Delta)x_M(t) \end{array} \right. \tag{25}$$

and, we state the matching problem as follows.

Problem 3 *Given a plant Σ_P and a model Σ_M of the form (24), assume that the internal variables of both the plant and the model are accessible and let d_1, d_2, d_3, d_4 denote respectively the maximum degree of the polynomial matrices $\begin{pmatrix} A_{Pi}(\Delta) \\ C_{Pi}(\Delta) \end{pmatrix} \begin{pmatrix} A_{Mi}(\Delta) \\ C_{Pi}(\Delta) \end{pmatrix}$, $B_{Pi}(\Delta)$, $B_{Mi}(\Delta)$ for $i \in I$ (i.e. the maximum delay that affects respectively the internal variables x_P and x_M and the input variables w and u in (24)). The Asymptotic Model Matching Problem (AMMP) consists in finding a variable delay dynamic compensator of the form*

$$w(t) = F_{P\sigma(t)}(\Delta)x_P(t) + F_{M\sigma(t)}(\Delta)x_M(t) + F'_{\sigma(t)}(\Delta)w(t) + G_{\sigma(t)}(\Delta)u(t) \tag{26}$$

that forces the output $y_P(t)$ of the compensated plant $\Sigma^{(F_P, F_M, F', G)}$ to match asymptotically the output $y_M(t)$ of the models, i.e. such that $lim_{t\to\infty}||y_P(t) - y_P(t)|| = 0$, for all $\sigma \in \mathscr{S}_\alpha$ for some $\alpha \geq 1$ and for any value of $x_P(t)$, $x_M(t)$, $w(t)$ and $u(t)$ over, respectively, the intervals $[-d_1, 0]$, $[-d_2, 0]$, $[-d_3, 0]$, $[-d_4, +\infty)$.

Solvability of the AMMP is characterized by the following theorem.

Theorem 2 *Given a plant Σ_P and a model Σ_M of the form (24), whose internal variables are assumed to be accessible, the related AMMP is solvable if and only if, considering $u(t)$ as a disturbance input in Σ_D, the related ADDPS is solvable.*

Proof: It follows from remarking that the AMMP is solved by a variable delay feedback compensator of the form (26) if and only if the forced response of the compensated systems $\Sigma_D^{(F_P, F_M, F', G)}$ is annihilated and its free response (i.e. the response for $u(t) = 0$ for $t \in [-d_4, +\infty)$) goes to 0 as t goes to $+\infty$ for all $\sigma \in \mathscr{S}_\alpha$ for some $\alpha \geq 1$. This is equivalent to decoupling the output $y(t)$ from the disturbance input $u(t)$ and making, at the same time, the compensated system globally asymptotically stable for all $\sigma \in \mathscr{S}_\alpha$ (see also [32] and [8]). □

By the above theorem, the solvability of the AMMP can be practically checked using the condition (23) (see also [32]).

7 Conclusions

By modelling discrete-time linear systems with time-varying delays as switching linear systems one has the possibility to import methods and results from the switching framework to the time-delay one. In particular, the structural geometric approach that has already been extended from the classical linear framework to the linear switching one can be applied to investigate control problems that involve discrete-time linear systems with time-varying delays. Two of them, namely the disturbance decoupling problem with stability and the asymptotic model matching problem, have been considered in this paper. Solutions have been given in the case corresponding to restricted switching, that in which the variations of the delay structure Regulation problems and observation problems can be dealt with in a similar way.

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