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# DEFORMATIONS OF LOG CALABI–YAU PAIRS CAN BE OBSTRUCTED

SIMON FELTEN, ANDREA PETRACCI, AND SHARON ROBINS

**ABSTRACT.** We exhibit examples of pairs  $(X, D)$  where  $X$  is a smooth projective variety and  $D$  is an anticanonical reduced simple normal crossing divisor such that the deformations of  $(X, D)$  are obstructed. These examples are constructed via toric geometry.

## 1. INTRODUCTION

In the introduction we assume that the ground field, denoted by  $\mathbb{C}$ , is algebraically closed and has characteristic 0.

A *log Calabi–Yau pair* is a pair  $(X, D)$  where  $X$  is a smooth projective variety and  $D$  is a reduced normal crossing divisor on  $X$  such that  $K_X + D$  is linearly trivial. Most authors, e.g. [HX15, KX16, CK16, Kal20, Mau20], consider more general definitions and allow singularities on  $X$ , but we do not do that.

By comparing with the celebrated Bogomolov–Tian–Todorov theorem [Bog78, Tia87, Tod89] (see also [Ran92, Kaw92, FM99, IM10, Iac17, CLM, FFR21]), which asserts that deformations of smooth proper varieties over  $\mathbb{C}$  with trivial canonical bundle are unobstructed, one could ask:

**Question 1.1.** *Are deformations of log Calabi–Yau pairs  $(X, D)$  over  $\mathbb{C}$  unobstructed?*

By definition, deformations of a pair  $(X, D)$  are (not necessarily locally trivial) deformations of the closed embedding  $D \hookrightarrow X$ ; in particular, the singularities of  $D$  are allowed to be smoothed.

The answer to Question 1.1 is positive if at least one of the following three additional assumptions is satisfied.

- (i)  $D$  is smooth. This is due to Iacono [Iac15] and to Sano [San14, Remark 2.5] independently (see also [Kon, KKP08, LRW19, Wan, FP]). Note that in this case the deformations of  $(X, D)$  are locally trivial and coincide with the log smooth deformations of the log scheme given by  $X$  equipped with the divisorial log structure associated to  $D$ .
- (ii)  $X$  is weak Fano, i.e.  $-K_X$  is big and nef. This follows from the unobstructedness of deformations of weak Fano manifolds by Sano [San14, Theorem 1.1] and the vanishing of  $H^1(N_{D/X})$  by Kawamata–Viehweg vanishing.
- (iii)  $X$  is a surface. If  $X$  is rational, this is due to Friedman [Fri, Proposition 3.5]. If  $X$  is irrational, then  $D$  is smooth by a slight generalisation of the proof of [GHK15, Proposition 1.3] (see also [KX16, Proposition 19]) and so one concludes thanks to (i).

Examples of log Calabi–Yau pairs are *toric pairs*, i.e. pairs  $(X, D)$  where  $X$  is a smooth projective toric variety and  $D$  is the *toric boundary* of  $X$ , i.e. the reduced sum of the torus-invariant prime divisors of  $X$ . Note that the complement of the toric boundary of  $X$  is the big torus of  $X$ , i.e. the open torus-orbit.

If  $(X, D)$  is a toric pair, then  $X$  is rational,  $-K_X$  is big,  $D$  has simple normal crossings (i.e. the irreducible components of  $D$  are smooth), and the pair  $(X, D)$  has maximal intersection in the sense that the snc divisor  $D$  has 0-dimensional strata (see [KX16]).

The purpose of this paper is to provide a negative answer to Question 1.1 by exhibiting particular toric pairs  $(X, D)$ :

**Theorem 1.2.** *For every integer  $n \geq 3$ , there exists a smooth projective toric  $n$ -fold  $X$  such that the deformations of the pair  $(X, D)$ , where  $D$  denotes the toric boundary of  $X$ , are obstructed.*

The idea of the proof is as follows. With the methods of [IT20] one produces examples of smooth projective toric varieties  $X$  which are obstructed. More precisely, one finds a first-order deformation  $\xi$  of  $X$  which cannot be extended to the second order. Let  $D$  denote the toric boundary of  $X$ , and consider the forgetful map

$$(1.1) \quad \mathrm{Def}_{(X,D)} \longrightarrow \mathrm{Def}_X.$$

By using the torus action and the consequent grading on all relevant cohomology groups (see §4), we show that, in some examples, the deformation  $\xi$  lies in the image of (1.1). Obviously, no preimage of  $\xi$  can be extended to the second order, therefore we have constructed a first-order deformation of  $(X, D)$  which is obstructed.

Actually the varieties  $X$  we consider have a description also outside toric geometry: they are products of  $\mathbb{P}^{n-3}$  with the projectivisation of a certain split rank-2 vector bundle on the second Hirzebruch surface.

Note that the obstructed first-order deformation  $\xi$  of  $X$  is not homogeneous with respect to the grading on  $H^1(X, T_X)$  induced by the Euler sequence; indeed, homogeneous first-order deformations of smooth projective toric varieties are unobstructed [IV12] (see also [Mav, HI13, Pet21]).

We conclude the introduction with a question in Hodge theory. Assume now that the ground field  $\mathbb{C}$  is the field of complex numbers. The unobstructedness of deformations of  $n$ -dimensional smooth projective complex varieties  $X$  with trivial canonical bundle can be proved by using the fact that  $H^1(T_X) \simeq H^1(\Omega_X^{n-1})$  is topological, i.e. a Hodge piece of  $H^n(X, \mathbb{C})$  (see the  $T^1$ -lifting criterion in [Ran92]). Similarly, if  $(X, D)$  is an  $n$ -dimensional snc log Calabi–Yau pair, then the *locally trivial* deformations of  $(X, D)$  are unobstructed because  $H^1(T_X(-\log D)) \simeq H^1(\Omega_X^{n-1}(\log D))$  is topological, indeed a part of  $H^n(X \setminus D, \mathbb{C})$ . It is an interesting open question, suggested to us by Richard Thomas, to investigate whether there is a topological/Hodge-theoretic explanation of the obstructedness of the (not necessarily locally trivial) deformations of our examples.

**Notation and conventions.** In the rest of the paper the ground field, denoted by  $\mathbb{k}$ , is an arbitrary field of characteristic different from 2. However, §3 and Proposition 4.1 are valid over a field of arbitrary characteristic.

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## 2. CUP PRODUCT

Let  $X$  be a smooth variety over  $\mathbb{k}$ . Let  $\text{Def}_X$  denote the deformation functor of  $X$ . It is well known that  $H^1(T_X)$  is the tangent space of  $\text{Def}_X$  and  $H^2(T_X)$  is an obstruction space of  $\text{Def}_X$ , where  $T_X$  is the tangent sheaf of  $X$ , i.e. the sheaf of  $\mathbb{k}$ -derivations of  $\mathcal{O}_X$ .

By composing the cohomology product  $H^1(T_X) \otimes_{\mathbb{k}} H^1(T_X) \rightarrow H^2(T_X \otimes_{\mathbb{k}} T_X)$  with the morphism induced on  $H^2$  by the Lie bracket  $[\cdot, \cdot]: T_X \otimes_{\mathbb{k}} T_X \rightarrow T_X$ , one gets a symmetric bilinear form

$$(2.1) \quad b: H^1(T_X) \otimes_{\mathbb{k}} H^1(T_X) \longrightarrow H^2(T_X)$$

which is called the *cup product*. If one chooses an affine open cover of  $X$  and uses Čech cohomology to describe  $H^i(T_X)$ , then the cup product is given by

$$b(\xi, \xi') = \{[\xi_{ij}, \xi'_{jk}]\}_{i,j,k}$$

for  $\xi = \{\xi_{ij}\}_{i,j}$  and  $\xi' = \{\xi'_{ij}\}_{i,j}$ . If one uses alternating Čech cocycles, then

$$b(\xi, \xi') = \left\{ \frac{[\xi_{ij}, \xi'_{jk}] - [\xi_{jk}, \xi'_{ij}]}{2} \right\}_{i,j,k}.$$

We refer the reader to [IT20, §2] for a thorough account.

Let

$$q: H^1(T_X) \longrightarrow H^2(T_X)$$

be the quadratic form associated to  $b$ , i.e.  $q(\xi) = b(\xi, \xi)$ . In terms of Čech cocycles we have

$$q\left(\{\xi_{ij}\}_{i,j}\right) = \{[\xi_{ij}, \xi_{jk}]\}_{i,j,k}.$$

The quadratic form  $q$  is very important in deformation theory and is called the *first obstruction*: if  $\xi \in H^1(T_X)$  is a first-order deformation of  $X$  (i.e. a deformation of  $X$  over  $\text{Spec } \mathbb{k}[t]/(t^2)$ ), then  $\frac{1}{2}q(\xi) \in H^2(T_X)$  is the obstruction to lift  $\xi$  to a deformation over  $\text{Spec } \mathbb{k}[t]/(t^3)$ .

This implies that, up to a multiplicative constant, the quadratic form  $q$  coincides with the degree 2 terms of the equations which define the base of the miniversal deformation of  $X$  as a closed subspace of  $H^1(T_X)$ . In particular, if  $q$  is non-zero, then  $\text{Def}_X$  is not smooth, i.e.  $X$  is obstructed.

## 3. DEFORMATIONS OF PAIRS

Let  $X$  be a smooth variety over  $\mathbb{k}$ , and let  $D$  be an effective (Cartier) divisor on  $X$ . Let  $\text{Def}_{(X,D)}$  be the deformation functor of the pair  $(X, D)$ , i.e. of the closed embedding  $D \hookrightarrow X$ . There is an obvious natural transformation (1.1) which forgets the divisor  $D$ .

Let  $N_{D/X}$  be the normal bundle of  $D$  in  $X$ : this is  $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-D), \mathcal{O}_D)$ . Consider the map

$$(3.1) \quad d: T_X \longrightarrow N_{D/X}$$

which maps every  $\mathbb{k}$ -derivation  $\partial: \mathcal{O}_X \rightarrow \mathcal{O}_X$  to the composition  $\pi \circ \partial|_{\mathcal{O}_X(-D)}$ , where  $\partial|_{\mathcal{O}_X(-D)}$  is the restriction of  $\partial$  to the ideal  $\mathcal{O}_X(-D)$  of  $D$  in  $X$  and  $\pi: \mathcal{O}_X \twoheadrightarrow \mathcal{O}_D$  is the surjection induced by the closed embedding  $D \hookrightarrow X$ . It is easy to show that  $d$  is a well-defined homomorphism of  $\mathcal{O}_X$ -modules.

Let  $A^\bullet$  be the 2-term cohomological complex given by  $d$ , i.e.  $T_X$  is the term in degree 0,  $N_{D/X}$  is the term in degree 1, all the other terms are zero, and the differential from degree 0 to degree 1 is  $d$ . There is an obvious exact triangle

$$A^\bullet \longrightarrow T_X \xrightarrow{d} N_{D/X} \longrightarrow$$

which induces the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^0(A^\bullet) & \longrightarrow & H^0(T_X) & \longrightarrow & H^0(N_{D/X}) \\ & & \longrightarrow & & H^1(T_X) & \longrightarrow & H^1(N_{D/X}) \\ & & \longrightarrow & & H^2(T_X) & \longrightarrow & H^2(N_{D/X}). \end{array}$$

By [SV90, Proposition 8]  $\mathbb{H}^1(A^\bullet)$  is the tangent space of  $\text{Def}_{(X,D)}$  and the map induced by (1.1) on the tangent spaces coincides with the map  $\mathbb{H}^1(A^\bullet) \rightarrow H^1(T_X)$  in the long exact sequence above.

**Remark 3.1.** One could prove that  $\mathbb{H}^2(A^\bullet)$  is an obstruction space for  $\text{Def}_{(X,D)}$ , but we will not need this result below. We just remark that the identification of  $\mathbb{H}^2(A^\bullet)$  as an obstruction space for  $\text{Def}_{(X,D)}$  allows one to recover the well-known criterion that says that the forgetful map (1.1) is smooth if  $H^1(N_{D/X}) = 0$ ; indeed, under the assumption  $H^1(N_{D/X}) = 0$ , the map (1.1) induces a surjection on tangent spaces and an injection on obstruction spaces.

**Remark 3.2.** All the discussion above works for pairs  $(X, D)$  where  $X$  is a smooth variety and  $D$  is an effective Cartier divisor on  $X$ . Note that  $D$  might be non-reduced. If, in addition, one assumes that  $D$  is smooth, then one gets the residue sequence

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X(\log D) \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

and by dualising it one gets the short exact sequence

$$0 \longrightarrow T_X(-\log D) \longrightarrow T_X \longrightarrow N_{D/X} \longrightarrow 0,$$

which shows that  $T_X(-\log D)$  is quasi-isomorphic to  $A^\bullet$ . If  $D$  is only assumed to be snc, then  $A^\bullet$  might not be quasi-isomorphic to  $T_X(-\log D)$ : indeed, whereas  $A^\bullet$  controls all deformations of the pair  $(X, D)$ ,  $T_X(-\log D)$  controls only locally trivial deformations of  $(X, D)$ .

#### 4. TORIC GEOMETRY AND $M$ -GRADINGS

Let  $N$  be a lattice of rank  $n$ , let  $M$  be its dual, and let  $\langle \cdot, \cdot \rangle: M \times N \rightarrow \mathbb{Z}$  be the duality pairing. Consider the torus  $T_N = N \otimes_{\mathbb{Z}} \mathbb{G}_m = \text{Spec } \mathbb{k}[M]$ , where  $\mathbb{G}_m = \text{Spec } \mathbb{k}[x, x^{-1}]$  is the 1-dimensional algebraic torus. If one has a fan  $\Sigma$  in  $N$ , one gets a toric variety equipped with an action of  $T_N$ .

The set of the 1-dimensional cones (aka rays) of  $\Sigma$  is denoted by  $\Sigma(1)$ , and the divisor corresponding to  $\rho \in \Sigma(1)$  is denoted by  $D_\rho$ . With slight abuse of notation, the primitive lattice generator of a ray  $\rho \in \Sigma(1)$  is denoted again by  $\rho$ . For more details about toric varieties we refer the reader to [CLS11].

If  $X$  is a toric variety and  $U \subseteq X$  is an *affine toric* open subscheme (i.e. the affine toric variety associated to a cone in the fan  $\Sigma$  defining  $X$ ), then the torus

action on  $U$  induces a natural  $M$ -grading on  $\Gamma(U, \mathcal{O}_X(D))$  for every torus-invariant  $\mathbb{Z}$ -divisor  $D$ . If  $U' \subseteq U$  is a smaller affine toric subscheme, then the restriction maps preserve the  $M$ -grading. In particular, the Čech complex with respect to the open affine covering given by the maximal cones of the fan  $\Sigma$  is naturally  $M$ -graded; therefore, one has  $M$ -gradings on the cohomology groups  $H^i(X, \mathcal{O}_X(D))$  for every torus-invariant  $\mathbb{Z}$ -divisor  $D$ . Note that the isomorphism class of the sheaf  $\mathcal{O}_X(D)$  depends only on the linear equivalence class of  $D$ , whereas the  $M$ -gradings on  $H^i(X, \mathcal{O}_X(D))$  depend on the divisor  $D$ . The homogeneous part of  $H^i(X, \mathcal{O}_X(D))$  of degree  $u \in M$  is denoted by  $H^i(X, \mathcal{O}_X(D))_u$ .

Now let  $X$  be a smooth toric variety. Using the  $M$ -grading on  $\mathcal{O}_X(D)$ , we induce an  $M$ -grading on  $T_X$  via the (dual) Euler sequence. For  $D$  the toric boundary, we get also an  $M$ -grading on the normal bundle  $N_{D/X}$  such that  $d: T_X \rightarrow N_{D/X}$  preserves the gradings. In order to construct the gradings and show that they are preserved by  $d$ , we employ the following proposition, which holds more generally for every effective divisor  $D$  on  $X$ .

**Proposition 4.1.** *Let  $X$  be a smooth complete toric variety, let  $D$  be the effective divisor on  $X$  defined by the zero-locus of a homogeneous polynomial  $F$  in the Cox ring of  $X$ , let  $\beta \in \text{Pic}(X)$  be the degree of  $F$ .*

*Then there is a commutative diagram of coherent sheaves on  $X$  with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_1(X) \otimes_{\mathbb{Z}} \mathcal{O}_X & \xrightarrow{(\beta_\rho x_\rho)_\rho} & \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho) & \longrightarrow & T_X \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \left( \frac{\partial F}{\partial x_\rho} \right)_\rho & & \downarrow d \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{F} & \mathcal{O}_X(D) & \longrightarrow & N_{D/X} \longrightarrow 0 \end{array}$$

where  $x_\rho$  denotes the Cox coordinate associated to the ray  $\rho$  and  $\beta_\rho \in \text{Pic}(X)$  denotes its degree, the group  $N_1(X)$  is  $\text{Hom}_{\mathbb{Z}}(\text{Pic}(X), \mathbb{Z})$ , the top exact sequence is the dual of the Euler sequence [CLS11, Theorem 8.1.6], and the right vertical map is the homomorphism (3.1).

By [IT20, Equation (5)] the homomorphism  $\bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho) \rightarrow T_X$  is defined as follows: the image of a local section  $\chi^w$  of  $\mathcal{O}_X(D_\rho)$  over an affine toric subscheme  $U \subseteq X$  is the derivation  $\partial(\rho, w): \mathcal{O}_U \rightarrow \mathcal{O}_U$  defined by

$$(4.1) \quad \partial(\rho, w)(\chi^u) = \langle u, \rho \rangle \chi^{u+w}.$$

**Example 4.2.** For  $X = \mathbb{P}^n$  the diagram in Proposition 4.1 is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{(x_0, \dots, x_n)} & \xrightarrow{\quad} & \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} & \longrightarrow & T_{\mathbb{P}^n} \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \left( \frac{\partial F}{\partial x_i} \right)_i & & \downarrow d \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \xrightarrow{F} & \mathcal{O}_{\mathbb{P}^n}(\beta) & \longrightarrow & N_{D/\mathbb{P}^n} \longrightarrow 0 \end{array}$$

where  $F \in \mathbb{k}[x_0, \dots, x_n]$  is a homogeneous polynomial of degree  $\beta$ . The commutativity of the left square is the Euler identity

$$\beta F = \sum_{i=0}^n x_i \frac{\partial F}{\partial x_i}.$$

Let us now analyse the commutativity of the right square. Restrict to the affine chart  $U = \{x_0 \neq 0\}$  which is isomorphic to  $\mathbb{A}^n$  with affine coordinates  $y_k = x_k/x_0$

for  $k = 1, \dots, n$ . Consider the dehomogenisation

$$f(y_1, \dots, y_n) = F(x_0, \dots, x_n)/x_0^\beta = F(1, y_1, \dots, y_n).$$

We have the equalities

$$(4.2) \quad \frac{\partial F}{\partial x_k}(1, y_1, \dots, y_n) = \frac{\partial f}{\partial y_k} \quad \text{for } k = 1, \dots, n$$

and

$$(4.3) \quad \beta f = \sum_{k=1}^n y_k \frac{\partial f}{\partial y_k} + \frac{\partial F}{\partial x_0}(1, y_1, \dots, y_n).$$

By using the trivialisations

$$\begin{aligned} \mathcal{O}_U &\simeq \mathcal{O}_{\mathbb{P}^n}(1)|_U & g &\mapsto x_0 g, \\ \mathcal{O}_U &\simeq \mathcal{O}_{\mathbb{P}^n}(\beta)|_U & g &\mapsto x_0^\beta g, \\ \mathcal{O}_U^{\oplus n} &\simeq T_{\mathbb{P}^n}|_U & (g_1, \dots, g_n) &\mapsto \sum_{i=1}^n g_i \frac{\partial}{\partial y_i} \quad \text{and} \quad (\partial(y_1), \dots, \partial(y_n)) \leftarrow \partial, \end{aligned}$$

the right square in the diagram becomes

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_U)^{\oplus(n+1)} & \xrightarrow{\begin{pmatrix} -y_1 & 1 & & \\ & \vdots & & \\ & & \ddots & \\ -y_n & & & 1 \end{pmatrix}} & \Gamma(U, \mathcal{O}_U)^{\oplus n} \\ \downarrow \left( \frac{\partial F}{\partial x_i}(1, y_1, \dots, y_n) \right)_{i=0, \dots, n} & & \downarrow \left( \frac{\partial f}{\partial y_i} \right)_{i=1, \dots, n} \\ \Gamma(U, \mathcal{O}_U) & \xrightarrow{\quad \bar{\cdot} \quad} & \Gamma(U, \mathcal{O}_{D \cap U}) \end{array}$$

where  $\bar{\cdot}: \Gamma(U, \mathcal{O}_U) = \mathbb{k}[y_1, \dots, y_n] \twoheadrightarrow \Gamma(U, \mathcal{O}_{D \cap U}) = \mathbb{k}[y_1, \dots, y_n]/(f)$  denotes the projection modulo  $f$ . This square commutes because of (4.2) and (4.3).

*Proof of Proposition 4.1.* The existence of the two short exact sequences is clear. The commutativity of the left square is the Euler relation [CLS11, Exercise 8.1.8]. It remains to prove the commutativity of the right square. We proceed in a way analogous to Example 4.2.

We restrict to the affine toric subscheme  $U \subseteq X$  associated to an  $n$ -dimensional cone  $\sigma \in \Sigma$ . Assume that the rays of  $\sigma$  are  $\rho_1, \dots, \rho_n$ , and  $\rho_{n+1}, \dots, \rho_{n+r}$  are the rays of  $\Sigma$  not in  $\sigma$ . Here  $r$  is the Picard rank of  $X$ . For brevity, set  $x_i := x_{\rho_i}$ ,  $D_i := D_{\rho_i}$ ,  $\beta_i := \beta_{\rho_i}$ .

Since  $X$  is smooth,  $\rho_1, \dots, \rho_n$  form a basis of the lattice  $N$ . Hence we can write  $\rho_j = -\sum_{k=1}^n a_{kj} \rho_k$  for every  $j = n+1, \dots, n+r$ . We have that  $\beta_{n+1}, \dots, \beta_{n+r}$  is a basis of  $\text{Pic}(X)$  and  $\beta_k = \sum_{j=n+1}^{n+r} a_{kj} \beta_j$  for every  $k = 1, \dots, n$ . We also write  $\beta = \sum_{j=n+1}^{n+r} b_j \beta_j$ . Let  $w_1, \dots, w_n \in M$  be the dual basis of  $\rho_1, \dots, \rho_n$ .

Set  $y_k := \chi^{w_k}$  for every  $k = 1, \dots, n$ . It is clear that  $U$  is isomorphic to  $\mathbb{A}^n$  with coordinates  $y_1, \dots, y_n$ . Under the Cox isomorphism [CLS11, Equation (5.3.1)]

$y_k = x_k \prod_{j=n+1}^{n+r} x_j^{-a_{kj}}$ . The dehomogenisation of  $F$  is

$$f(y_1, \dots, y_n) = \frac{F}{\prod_{j=n+1}^{n+r} x_j^{b_j}} = F(y_1, \dots, y_n, 1, \dots, 1).$$

There are the following Euler relations:

$$b_j F = \sum_{k=1}^n a_{kj} x_k \frac{\partial F}{\partial x_k} + x_j \frac{\partial F}{\partial x_j} \quad \text{for } j = n+1, \dots, n+r.$$

We have

$$(4.4) \quad \frac{\partial f}{\partial y_k} = \frac{\partial F}{\partial x_k}(y_1, \dots, y_n, 1, \dots, 1) \quad \text{for } k = 1, \dots, n$$

and

$$(4.5) \quad b_j f = \sum_{k=1}^n a_{kj} y_k \frac{\partial f}{\partial y_k} + \frac{\partial F}{\partial x_j}(y_1, \dots, y_n, 1, \dots, 1) \quad \text{for } j = n+1, \dots, n+r.$$

We want to show that the diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^{n+r} \Gamma(U, \mathcal{O}_X(D_i)) & \longrightarrow & \Gamma(U, T_X) \\ \downarrow \frac{\partial F}{\partial x_i} & & \downarrow d \\ \Gamma(U, \mathcal{O}_X(D)) & \longrightarrow & \Gamma(U, N_{D/X}) \end{array}$$

commutes. By using the trivialisations

$$\begin{aligned} \Gamma(U, T_X) &\simeq \Gamma(U, \mathcal{O}_U)^{\oplus n} & \partial &\mapsto (\partial(y_1), \dots, \partial(y_n)) \\ \Gamma(U, \mathcal{O}_U) &\simeq \Gamma(U, \mathcal{O}_X(D_k)) & g &\mapsto g y_k^{-1} \quad \text{for } k = 1, \dots, n \\ \Gamma(U, \mathcal{O}_U) &= \Gamma(U, \mathcal{O}_X(D_j)) & & \text{for } j = n+1, \dots, n+r \end{aligned}$$

and the two following consequences of (4.1)

$$\begin{aligned} \partial(\rho_i, -w_i)(\chi^{w_k}) &= \langle w_k, \rho_i \rangle \chi^{w_k - w_i} = \delta_{ik} & i = 1, \dots, n, \\ \partial(\rho_j, 0)(\chi^{w_k}) &= \langle w_k, \rho_j \rangle \chi^{w_k} = -a_{kj} y_k & j = n+1, \dots, n+r, \end{aligned}$$

this diagram becomes

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_U)^{\oplus(n+r)} & \xrightarrow{\left( \begin{array}{c|c} 1 & \\ \vdots & \\ 1 & -a_{kj} y_k \end{array} \right)} & \Gamma(U, \mathcal{O}_U)^{\oplus n} \\ \downarrow \left( \frac{\partial F}{\partial x_i}(y_1, \dots, y_n, 1, \dots, 1) \right)_{i=1, \dots, n+r} & & \downarrow \left( \frac{\partial f}{\partial y_i} \right)_{i=1, \dots, n} \\ \Gamma(U, \mathcal{O}_U) & \xrightarrow{\bar{\cdot}} & \Gamma(U, \mathcal{O}_{D \cap U}) \end{array}$$

where  $\bar{\cdot}: \Gamma(U, \mathcal{O}_U) = \mathbb{k}[y_1, \dots, y_n] \twoheadrightarrow \Gamma(U, \mathcal{O}_{D \cap U}) = \mathbb{k}[y_1, \dots, y_n]/(f)$  denotes the projection modulo  $f$ . This square commutes because of (4.4) and (4.5).  $\square$



The restriction to every affine toric open subscheme  $U \subseteq X$  of the sheaf homomorphism  $N_1(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \rightarrow \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho)$  appearing in the top short exact sequence in Proposition 4.1 is homogeneous with respect to the  $M$ -grading. This implies that the cohomology groups  $H^i(X, T_X)$  inherit an  $M$ -grading. We denote by  $H^i(X, T_X)_u$  the part of  $H^i(X, T_X)$  of degree  $u \in M$ .

**Remark 4.3.** One can prove that the cup product (2.1) preserves the  $M$ -grading; in [IT20] Ilten and Turo give a combinatorial description of its homogeneous parts with respect to the  $M$ -grading.

## 5. THE EXAMPLE

We construct an explicit smooth projective toric variety of each dimension  $n \geq 3$  that turns out to give a counterexample to Question 1.1.

**Proposition 5.1.** *For every  $n \geq 3$ , there exist a smooth projective toric  $n$ -fold  $X$  and elements  $u', u'' \in M$ , where  $M$  denotes the character lattice of the big torus of  $X$ , such that:*

- (1) *the cup product  $H^1(X, T_X)_{u'} \otimes_{\mathbb{K}} H^1(X, T_X)_{u''} \rightarrow H^2(X, T_X)_{u'+u''}$  is non-zero,*
- (2)  *$H^1(X, \mathcal{O}_X(D))_{u'} = 0$  and  $H^1(X, \mathcal{O}_X(D))_{u''} = 0$ , where  $D$  is the toric boundary of  $X$ .*

The rest of this section is devoted to the proof of Proposition 5.1. Firstly, we consider the case  $n = 3$ ; the 3-fold we construct is the projectivisation of a split rank-2 vector bundle on the second Hirzebruch surface. Secondly, in dimension  $n \geq 4$  we consider the product of this 3-fold with  $\mathbb{P}^{n-3}$ .

**Example 5.2.** Consider the lattice  $N = \mathbb{Z}^3$  and its dual  $M$ . Let  $\rho_1, \dots, \rho_6 \in N$  be the columns of the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 3 & 1 & -1 \end{pmatrix}.$$

Let  $\Sigma$  be the fan in  $N$  with rays given by  $\rho_1, \dots, \rho_6$  and with the following 3-dimensional cones:  $\sigma_{125}, \sigma_{126}, \sigma_{145}, \sigma_{146}, \sigma_{235}, \sigma_{236}, \sigma_{345}, \sigma_{346}$ , where  $\sigma_{ijk}$  denotes the cone with rays  $\rho_i, \rho_j, \rho_k$ . The fan  $\Sigma$  can be visualised by looking at Figure 1: by considering  $\rho_6$  as a vertex at infinity, we can describe the maximal cones of  $\Sigma$  by the 2-simplices of the simplicial complex in Figure 1.

Let  $X$  be the toric variety associated to the fan  $\Sigma$ ; it is a smooth projective 3-fold with Picard rank 3. One can prove that  $X$  is the projectivisation of a split rank-2 vector bundle over the second Hirzebruch surface – see [Rob, Theorem 3.6].

For a torus invariant divisor  $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$  and  $u \in M$ , we consider the simplicial complex

$$V_{D,u} = \bigcup_{\sigma \in \Sigma} \text{conv}(\rho \in \sigma(1) \mid \langle u, \rho \rangle < -a_\rho).$$

When  $D = D_\rho$ , we simply denote it by  $V_{\rho,u}$ . Various  $V_{D,u}$ 's are depicted in Figure 2.

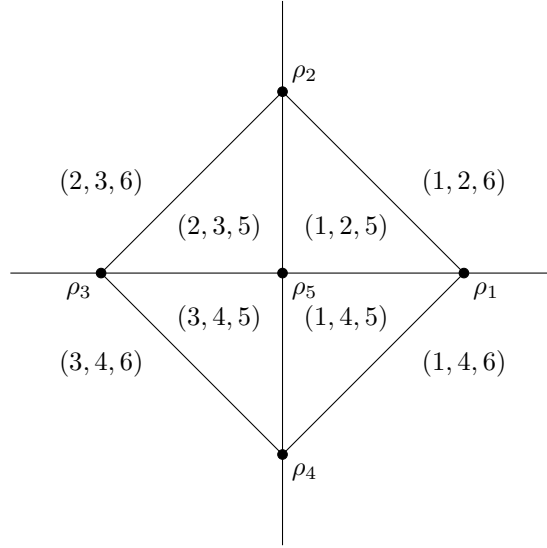


FIGURE 1. A schematic picture of the fan  $\Sigma$  in Example 5.2. The four triangles correspond to the four 3-dimensional cones in  $\Sigma$  which contain  $\rho_5$ , and the four 2-dimensional unbounded regions correspond to the four 3-dimensional cones in  $\Sigma$  which contain  $\rho_6$ .

We have the following isomorphisms from [CLS11, Theorem 9.1.3] and [IT20, Proposition 3.1]:

$$(5.1) \quad H^i(X, \mathcal{O}_X(D))_u \simeq \tilde{H}^{i-1}(V_{D,u}, \mathbb{k}),$$

$$(5.2) \quad H^i(X, T_X)_u \simeq \bigoplus_{\rho \in \Sigma(1)} \tilde{H}^{i-1}(V_{\rho,u}, \mathbb{k}).$$

Now fix  $u' = (-1, -1, 0)$  and  $u'' = (0, -1, -1)$  in  $M$ . By looking at (A), (B), (C) in Figure 2 and by using (5.2), we get

$$H^1(X, T_X)_{u'} = \mathbb{k}, \quad H^1(X, T_X)_{u''} = \mathbb{k}, \quad H^2(X, T_X)_{u'+u''} = \mathbb{k}.$$

Now we have all the necessary conditions in [IT20, Theorem 4.3], but it is not immediately clear whether the cup product is non-zero. To see this, we proceed as follows.

We use the notation and the constructions from [IT20, §5]. Consider  $Z = \rho_1$ ,  $Z' = \rho_2$  and the simple 1-cycle  $\alpha = E_1 + E_2 + E_3 + E_4$  in  $V_{D_5, u'+u''}$  as in Figure 2C. We have then  $\alpha(Z) = \{E_1, E_4\}$ ,  $\alpha(Z') = \{E_1, E_2\}$ ,  $b_1 = b_4 = 1$ ,  $b_2 = b_3 = 0$ . It is immediate to show  $Z *_{\alpha} Z' \neq 0$ . Now [IT20, Theorem 5.3] implies that the cup product is non-zero. This proves (1) in Proposition 5.1.

Now we consider the toric boundary  $D = \sum_{i=1}^6 D_i$  of  $X$ . From Figure 2D and Figure 2E we can see that  $V_{D, u'}$  is empty and  $V_{D, u''}$  is a point. Therefore by (5.1) we get  $H^1(X, \mathcal{O}_X(D))_{u'} = 0$  and  $H^1(X, \mathcal{O}_X(D))_{u''} = 0$ . This proves (2) in Proposition 5.1.

**Remark 5.3.** Let  $X$  be the smooth projective toric 3-fold in Example 5.2. One can prove that  $H^1(X, T_X)$  has dimension 3 and the following degrees:  $u' = (-1, -1, 0)$ ,

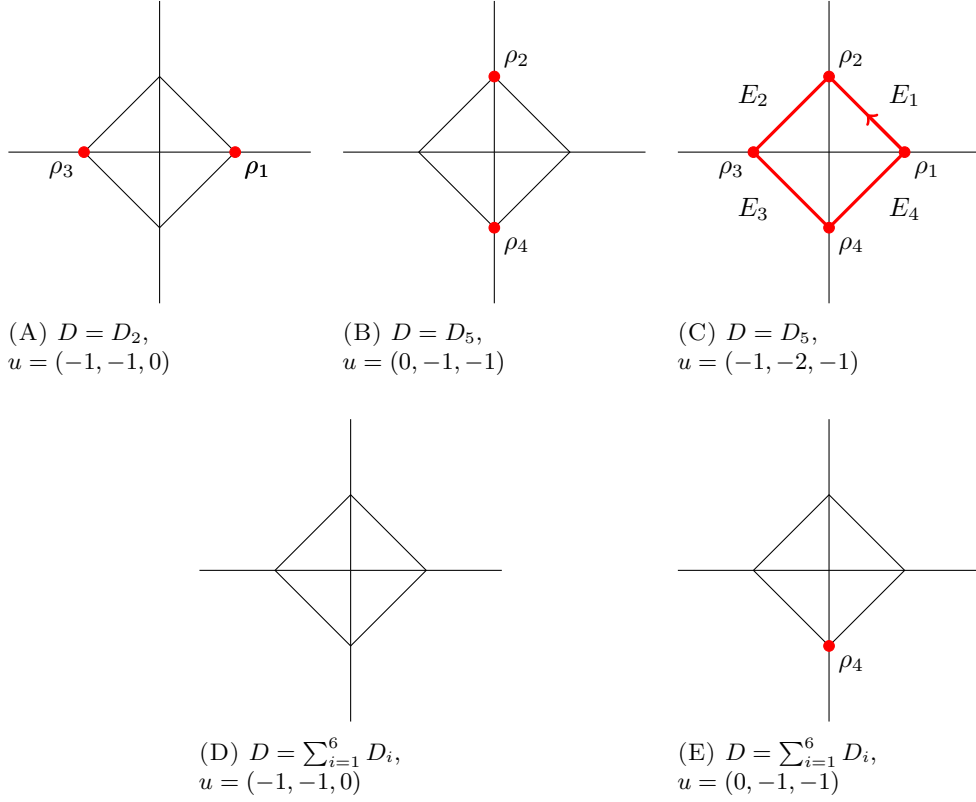


FIGURE 2. The simplicial complex  $V_{D,u}$  in red for different values of  $D$  and  $u$ , in Example 5.2

$u'' = (0, -1, -1), (-1, 0, 1)$ . Moreover,  $H^2(X, T_X)$  has dimension 1 and degree  $u' + u'' = (-1, -2, -1)$ . Using the  $M$ -grading and the fact that the cup product is non-zero on  $H^1(X, T_X)_{u'} \otimes_{\mathbb{k}} H^1(X, T_X)_{u''}$ , one can show that the hull of  $\text{Def}_X$  is  $\mathbb{k}[[t_1, t_2, t_3]]/(t_1 t_2)$ .

**Remark 5.4.** One can check that the toric variety in [IT20, §6] is a smooth projective toric 3-fold which has Picard rank 6 and satisfies the conditions of Proposition 5.1. The variety described in Example 5.2 is much simpler and has minimal Picard rank among obstructed smooth projective toric varieties.

*Proof of Proposition 5.1.* Let  $X$  be the smooth projective toric 3-fold in Example 5.2. Set  $\tilde{X} = X \times \mathbb{P}^{n-3}$ ; the character lattice of the big torus of  $\tilde{X}$  is  $\tilde{M} = M \oplus \mathbb{Z}^{n-3}$ . We will show that  $\tilde{X}$  satisfies the conditions (1) and (2) for  $v' = (u', 0) \in \tilde{M}$  and  $v'' = (u'', 0) \in \tilde{M}$ .

If  $\Sigma$  is the fan of  $X$  and  $\Sigma_0$  is the fan of  $\mathbb{P}^{n-3}$ , then  $\tilde{X}$  can be described by the fan  $\tilde{\Sigma} = \Sigma \times \Sigma_0$ . The image of  $\rho_i \in \Sigma(1)$  in  $\tilde{\Sigma}(1)$  is denoted by  $\tilde{\rho}_i$ , and the image of  $\tau_i \in \Sigma_0(1)$  in  $\tilde{\Sigma}(1)$  is denoted by  $\tilde{\tau}_i$ .

The first-order deformation of  $X$  corresponding to the degree  $u'$  (resp.  $u''$ ) induces a first-order deformation of  $\tilde{X} = X \times \mathbb{P}^{n-3}$  corresponding to the degree  $v'$

(resp.  $v''$ ) by deforming only the first factor. From a combinatorial point of view this can be seen from (5.2) and by observing that  $V_{\rho_2, u'} = V_{\widetilde{\rho}_2, v'}$  and  $V_{\rho_5, u''} = V_{\widetilde{\rho}_5, v''}$ .

For computing the cup product, first observe that  $\langle \widetilde{\tau}_i, v' \rangle = 0 = \langle \widetilde{\tau}_i, v'' \rangle$  for  $\tau_i \in \Sigma_0(1)$ . Hence by restricting the attention to the cones generated by the rays  $\widetilde{\rho}_i$ , following the recipe in [IT20, §5] is exactly the same as in Example 5.2. Thus, we can see that the cup product is non-zero.

Finally, observe that  $V_{D, u'} = V_{\widetilde{D}, v'}$  and  $V_{D, u''} = V_{\widetilde{D}, v''}$ , where  $D$  (resp.  $\widetilde{D}$ ) is the toric boundary of  $X$  (resp.  $\widetilde{X}$ ). Hence the condition (2) is immediate from (5.1).  $\square$

## 6. PROOF OF THEOREM 1.2

Let  $X$  be one of the smooth projective toric varieties constructed in Proposition 5.1, let  $D$  be the toric boundary of  $X$ , and let  $u', u'' \in M$  be the degrees satisfying the conditions (1) and (2) in Proposition 5.1.

Let  $\xi' \in H^1(T_X)_{u'}$  and  $\xi'' \in H^1(T_X)_{u''}$  be such that the cup product  $b(\xi', \xi'') \in H^2(T_X)_{u'+u''}$  is non-zero – they exist by (1). By the polarisation identity for quadratic forms we have that either  $q(\xi' + \xi'') \neq 0$  or  $q(\xi' - \xi'') \neq 0$ . Hence there exists a linear combination  $\xi$  of  $\xi'$  and  $\xi''$  such that  $q(\xi) \neq 0$ . This implies that the first-order deformation of  $X$  associated to  $\xi$  cannot be extended to the second order.

Let  $\Sigma$  be the fan defining the toric variety  $X$ . Consider the monomial  $F = \prod_{\rho \in \Sigma(1)} x_\rho$  in the Cox ring of  $X$ . The zero-locus of  $F$  is exactly the toric boundary  $D$  of  $X$ . Since  $F$  is a monomial, the homomorphism  $\Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X(D))$  given by the multiplication by  $F$  preserves the natural  $M$ -gradings for every open affine toric subscheme  $U \subseteq X$ . By the bottom exact sequence in the diagram in Proposition 4.1, the cohomology groups  $H^i(N_{D/X})$  carry natural  $M$ -gradings. Since  $X$  is rational,  $\mathcal{O}_X$  does not have higher cohomology, therefore we have an isomorphism of  $M$ -graded vector spaces

$$H^1(\mathcal{O}_X(D)) \simeq H^1(N_{D/X}).$$

By (2) in Proposition 5.1 we have  $H^1(N_{D/X})_{u'} = 0$  and  $H^1(N_{D/X})_{u''} = 0$ .

Since the derivatives of  $F$  are monomials in Cox coordinates, the multiplication by  $\frac{\partial F}{\partial x_\rho}$  preserves the  $M$ -grading when restricted to every open affine toric subscheme of  $X$ . The commutativity of the diagram in Proposition 4.1 implies that, for every  $i$ , the homomorphism

$$H^i(T_X) \longrightarrow H^i(N_{D/X})$$

induced by (3.1) preserves the  $M$ -grading. From the above result on the  $M$ -grading of  $H^1(N_{D/X})$ , we deduce that the two elements  $\xi', \xi'' \in H^1(T_X)$  maps to zero in  $H^1(N_{D/X})$ . Therefore  $\xi$  maps to zero in  $H^1(N_{D/X})$ .

By the long exact sequence in §3 we have that  $\xi$  lies in the image of  $\mathbb{H}^1(A^\bullet) \rightarrow H^1(T_X)$ . This implies that the first-order deformation  $\xi$  of  $X$  can be lifted to a first-order deformation  $\eta$  of the pair  $(X, D)$ . Since  $\xi$  cannot be extended to the second order, also the first-order deformation  $\eta$  of  $(X, D)$  cannot be extended to the second order. In particular,  $\text{Def}_{(X, D)}$  is not smooth.

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