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# SIMPLICIAL VOLUME VIA NORMALISED CYCLES

CLARA LÖH AND MARCO MORASCHINI

ABSTRACT. We show that the Connes-Consani semi-norm on singular homology with real coefficients, defined via s-modules, coincides with the ordinary  $\ell^1$ -semi-norm on singular homology in all dimensions.

## 1. INTRODUCTION

Connes and Consani introduced a semi-norm on singular homology via s-modules and established that this semi-norm is equivalent to the  $\ell^1$ -semi-norm defined by Gromov [CC20]. Moreover, they proved that their semi-norm is equal to the  $\ell^1$ -semi-norm in the case of surfaces [CC20, Theorem 1.4], using a delicate construction specific to surfaces. In this note, we show that the two semi-norms agree in *all* dimensions, thereby confirming and extending a conjecture of Connes and Consani [CC20, p. 4]:

**Theorem 1.1.** *Let  $X$  be a topological space, let  $n \in \mathbb{N}$ , let  $\alpha \in H_n(X; \mathbb{R})$ , and let  $\lambda \in \mathbb{R}_{>0}$ . Then  $\|\alpha\|_1 < \lambda$  if and only if  $\alpha$  lies in the image of the canonical map  $H_n(X; \|H\mathbb{R}\|_\lambda) \rightarrow H_n(X; \mathbb{R})$ .*

In particular, the simplicial volume of closed manifolds can also be expressed in terms of homology of s-modules.

As explained by Connes and Consani, in order to show Theorem 1.1 it suffices to prove that the  $\ell^1$ -semi-norm on singular homology can be computed via *normalised* singular cycles (see Section 2.3):

**Proposition 1.2.** *Let  $X$  be a topological space and let  $n \in \mathbb{N}$ . Then, for all  $\alpha \in H_n(X; \mathbb{R})$ , we have*

$$\|\alpha\|_1 = \|\alpha\|_1^{\text{norm}}.$$

In Section 2, we recall basic definitions and notation. The proof of Proposition 1.2 is given in Section 3, based on a symmetrisation construction.

## 2. THE (NORMALISED) $\ell^1$ -SEMI-NORM

**2.1. The singular chain complex.** Let  $n \in \mathbb{N}$  and let  $\Delta^n$  be the standard  $n$ -simplex. For  $j \in \{0, \dots, n\}$ , we denote by  $\iota_j^n: \Delta^{n-1} \rightarrow \Delta^n$  the affine inclusion of the  $j$ -th facet of  $\Delta^n$ .

Given a topological space  $X$ , we consider the singular simplicial set  $S(X)$ : For  $n \in \mathbb{N}$ , we have  $S_n(X) := \text{map}(\Delta^n, X)$  and for  $j \in \{0, \dots, n\}$ , the face

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maps  $\partial_j: S_n(X) \rightarrow S_{n-1}(X)$  are given by

$$\partial_j(\sigma) := \sigma \circ \iota_j^n$$

for all  $\sigma \in \text{map}(\Delta^n, X)$ . The singular chain complex  $C_\bullet(X; \mathbb{R})$  with real coefficients is the free  $\mathbb{R}$ -chain complex associated with  $S(X)$ .

Furthermore, we have the *Moore normalisation*  $NC_\bullet(X; \mathbb{R})$  of  $C_\bullet(X; \mathbb{R})$ , given by the submodules

$$NC_n(X; \mathbb{R}) := \bigcap_{j=0}^{n-1} \ker(\partial_j) \subseteq C_n(X; \mathbb{R})$$

and the boundary maps  $d := \partial_n: NC_n(X; \mathbb{R}) \rightarrow NC_{n-1}(X; \mathbb{R})$ .

**Definition 2.1** (normalised chain). A singular chain  $c \in C_n(X; \mathbb{R})$  is *normalised* if it lies in the submodule  $NC_n(X; \mathbb{R})$ .

**2.2. The  $\ell^1$ -semi-norms.** We briefly recall Gromov's  $\ell^1$ -semi-norm on singular homology [Gro82]: The  $\ell^1$ -norm  $\|\cdot\|_1$  on  $C_n(X; \mathbb{R})$  associated with the basis  $S_n(X)$  induces a semi-norm on  $H_n(X; \mathbb{R})$ , the  $\ell^1$ -*semi-norm*, which we will also denote by  $\|\cdot\|_1$ .

Following Connes and Consani [CC20], one can also endow  $H_n(X; \mathbb{R})$  with the semi-norm induced by the  $\ell^1$ -norm on the normalised complex  $NC_\bullet(X; \mathbb{R})$ : For  $\alpha \in H_n(X; \mathbb{R})$  one sets

$$\|\alpha\|_1^{\text{norm}} := \inf \{ \|c\|_1 \mid c \in C_n(X; \mathbb{R}) \text{ is a normalised cycle representing } \alpha \}.$$

Connes and Consani prove that the two semi-norms are equivalent [CC20, Lemma 3.4], namely, for every  $\alpha \in H_n(X; \mathbb{R})$ , we have

$$\|\alpha\|_1 \leq \|\alpha\|_1^{\text{norm}} \leq \max(1, 2^{n-1}) \cdot \|\alpha\|_1.$$

Proposition 1.2 states that they are in fact equal.

**2.3. Deriving Theorem 1.1 from Proposition 1.2.** Connes and Consani introduce a filtration of the s-module  $H\mathbb{R}$  by a family  $(\|H\mathbb{R}\|_\lambda)_{\lambda \in \mathbb{R}_{>0}}$  of sub-s-modules and, for topological spaces  $X$ , associated singular homology objects  $(H_n(X; \|H\mathbb{R}\|_\lambda)_{\lambda \in \mathbb{R}_{>0}})$  [CC20]. Moreover, these come with canonical maps

$$\varrho_{n,\lambda}: H_n(X; \|H\mathbb{R}\|_\lambda) \rightarrow H_n(X; \mathbb{R})$$

to the singular homology of  $X$  [CC20, Section 3.4]. This filtration defines a semi-norm on  $H_n(X; \mathbb{R})$  that is equivalent to  $\|\cdot\|_1$  [CC20, Corollary 3.6]. More precisely, the image of  $\varrho_{n,\lambda}$  coincides with the set of elements  $\alpha \in H_n(X; \mathbb{R})$  with  $\|\alpha\|_1^{\text{norm}} < \lambda$  [CC20, Theorem 3.5]. Therefore, Theorem 1.1 is a direct consequence of Proposition 1.2.

### 3. PROOF OF PROPOSITION 1.2

**3.1. Symmetrisation of chains.** We recall the *symmetrisation map* on singular chains, which is given by averaging singular simplices over all vertex-permutations of the standard simplex: In the following, let  $X$  be a topological space and  $n \in \mathbb{N}$ . Let  $\Sigma_{n+1}$  denote the symmetric group on  $\{0, \dots, n\}$  and  $\text{sgn}: \Sigma_{n+1} \rightarrow \{\pm 1\}$  the sign function. For a map  $\pi: \{0, \dots, k\} \rightarrow \{0, \dots, n\}$ , we write  $\Delta(\pi) := [\pi(0), \dots, \pi(k)]: \Delta^k \rightarrow \Delta^n$  for the affine map that extends the map  $\pi$  on the vertices.

**Definition 3.1** (symmetrisation map). The *symmetrisation map*

$$\text{symm}_n: C_n(X; \mathbb{R}) \rightarrow C_n(X; \mathbb{R})$$

is the  $\mathbb{R}$ -linear map defined on each singular  $n$ -simplex  $\sigma$  as

$$\text{symm}_n(\sigma) := \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi) \cdot \sigma \circ \Delta(\pi).$$

**Lemma 3.2** ([FM11, Lemma 2.6]). *The symmetrisation map  $\text{symm}_\bullet$  is a chain map  $C_\bullet(X; \mathbb{R}) \rightarrow C_\bullet(X; \mathbb{R})$  that is chain homotopic to the identity. Moreover, for all  $c \in C_n(X; \mathbb{R})$ , we have*

$$\|\text{symm}_n(c)\|_1 \leq \|c\|_1.$$

For us, the key observation is that symmetrisation enforces normalisation on cycles:

**Lemma 3.3** (normalisation via symmetrisation).

(1) For all  $j \in \{0, \dots, n\}$ , we have

$$\partial_j \circ \text{symm}_n = (-1)^j \cdot \partial_0 \circ \text{symm}_n.$$

(2) In particular: If  $c \in C_n(X; \mathbb{R})$  is a cycle, then  $\partial_j(\text{symm}_n(c)) = 0$  for all  $j \in \{0, \dots, n\}$ .

*Proof.* Ad 1. Using the cyclic permutation  $\tau_j := (j \ j-1 \ \dots \ 1 \ 0) \in \Sigma_{n+1}$ , we can re-write  $\partial_j \circ \text{symm}_n$  as follows: Each permutation  $\pi \in \Sigma_{n+1}$  satisfies

$$\begin{aligned} \Delta(\pi) \circ \iota_j^n &= [\pi(0), \dots, \pi(j-1), \pi(j+1), \dots, \pi(n)] \\ &= [\pi \circ \tau_j(1), \dots, \pi \circ \tau_j(j), \pi \circ \tau_j(j+1), \dots, \pi \circ \tau_j(n)] \\ &= \Delta(\pi \circ \tau_j) \circ \iota_0^n. \end{aligned}$$

Therefore, for all singular  $n$ -simplices  $\sigma$  on  $X$  we have

$$\begin{aligned} \partial_j \circ \text{symm}_n(\sigma) &= \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi) \cdot \sigma \circ \Delta(\pi) \circ \iota_j^n \\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi \circ \tau_j) \cdot \sigma \circ \Delta(\pi) \circ \iota_j^n \\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi \circ \tau_j) \cdot \sigma \circ \Delta(\pi \circ \tau_j) \circ \iota_0^n \\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\eta \in \Sigma_{n+1}} \text{sgn}(\eta) \cdot \sigma \circ \Delta(\eta) \circ \iota_0^n \\ &= (-1)^j \cdot \partial_0 \circ \text{symm}_n(\sigma). \end{aligned}$$

Ad 2. As  $\text{symm}_\bullet$  is a chain map (Lemma 3.2), if  $c \in C_n(X; \mathbb{R})$  is a cycle, then  $\text{symm}_n(c)$  is a cycle, and in combination with the first part we see that

$$\begin{aligned} 0 = \partial(\text{symm}_n(c)) &= \sum_{j=0}^n (-1)^j \cdot \partial_j(\text{symm}_n(c)) = \sum_{j=0}^n (-1)^{2j} \cdot \partial_0(\text{symm}_n(c)) \\ &= (n+1) \cdot \partial_0(\text{symm}_n(c)). \end{aligned}$$

Therefore,  $\partial_0(\text{symm}_n(c)) = 0$ . Applying the first part once more shows that  $\partial_j(\text{symm}_n(c)) = 0$  for all  $j \in \{0, \dots, n\}$ .  $\square$

**3.2. Proof of Proposition 1.2.** We already know that  $\|\alpha\|_1 \leq \|\alpha\|_1^{\text{norm}}$  for every  $\alpha \in H_n(X; \mathbb{R})$ . Let us prove the opposite inequality. Let  $c \in C_n(X; \mathbb{R})$  be a cycle representing  $\alpha \in H_n(X; \mathbb{R})$ . Then, we can consider  $\text{symm}_n(c) \in C_n(X; \mathbb{R})$ . By Lemma 3.2 we know that  $\text{symm}_n(c)$  is homologous to  $c$  and satisfies  $\|\text{symm}_n(c)\|_1 \leq \|c\|_1$ . Moreover, Lemma 3.3 implies that  $\text{symm}_n(c)$  is normalised. This shows that

$$\|\alpha\|_1^{\text{norm}} \leq \|\text{symm}_n(c)\|_1 \leq \|c\|_1.$$

Taking the infimum over all cycles representing  $\alpha$  completes the proof.

#### REFERENCES

- [CC20] A. Connes and C. Consani,  $\overline{\text{Spec } \mathbb{Z}}$  and the Gromov norm, *Theory Appl. Categories* **35** (2020), no. 6, 155–178.
- [FM11] K. Fujiwara and J. K. Manning, *Simplicial volume and fillings of hyperbolic manifolds*, *Algebr. Geom. Topol.* **11** (2011), 2237–2264.
- [Gro82] M. Gromov, *Volume and bounded cohomology*, *Publ. Math. Inst. Hautes Études Sci.* **56** (1982), 5–99.

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