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# Rational Agents Might be Overweight, Underweight, or the Physiologically Optimal Weight\*

Michael R. Caputo<sup>†</sup>  
University of Central Florida

Davide Dragone<sup>‡</sup>  
Università di Bologna

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## Abstract

Levy (2002) and Dragone (2009) showed that rational agents tend to become overweight. Their result is shown to be equivalent to an assumption placed on the instantaneous utility function, and their model is shown to admit multiple steady states, including being underweight.

**Keywords:** Intertemporal Consumer Choice; Obesity; Optimal Control;

**JEL codes:** D15; I12

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<sup>†</sup>Corresponding author. Department of Economics, University of Central Florida, P.O. Box 161400, Orlando, FL 32816-1400, email: mcaputo@ucf.edu

<sup>‡</sup>Dipartimento di Scienze Economiche, University of Bologna, Piazza Scaravilli 2, 40126, Bologna, Italy; e-mail: davide.dragone@unibo.it

# 1 Introduction

Worldwide, people are found to be underweight, an ideal body weight (hereafter weight), as well as overweight, and all three states of weight coexist in the population. A reasonable economic model of eating should therefore be able to rationalize the existence of people who are in any such state of weight and show that those states of weight can coexist. But nearly two decades ago Levy (2002) developed an optimal control model of rational eating and showed that it implied that an agent is overweight in a steady state, where overweight refers to a weight that exceeds the physiologically optimal weight. Seven years later Dragone (2009) generalized Levy's (2002) model and showed the same, further cementing the idea that rational agents eventually become overweight. Neither paper contemplated the coexistence of various steady states, and consequently were unable to explain the coexistence of different weights in the population.

In what follows, it is shown that the aforesaid conclusion of Levy (2002) and Dragone (2009) is not true in general and is instead equivalent to a seemingly innocuous assumption placed on the steady state value of instantaneous utility. This is achieved by deducing three general inequalities that hold in a steady state, from which it follows that a rational agent might be overweight, underweight, or even the physiologically optimal weight in a steady state. The inequalities yield an equivalence among four steady state expressions, and a fifth if instantaneous preferences can be represented by a popular subclass of constant relative risk aversion utility functions. The aforesaid equivalence helps build an intuitive understanding of the various steady state solutions. Furthermore, it is shown that underweight and overweight agents can coexist in a steady state, and that such coexisting states are local saddle points.

Before moving to the formal analysis, it is important to point out that the present work is not a criticism of the Levy (2002) and Dragone (2009) models. In particular, it does not point out shortcomings of their models nor does it propose a more general framework for the analysis of rational eating. Instead, it takes their models as given and identifies a seemingly innocuous assumption that curtails their reach. In doing so, and by uncovering an equivalence among four steady state expressions, it is shown that the Levy (2002) and Dragone (2009) models have more explanatory power than previously understood.

## 2 A Rational Eating Model and General Deductions

As Dragone's (2009) model contains Levy's (2002) as a special case, the former is the focus. Accordingly, the notation and assumptions that follow mimic those of Dragone (2009). To begin, let  $c(t)$  be the rate of consumption at time  $t$ , and  $\dot{c}(t)$  be the time rate of change of consumption at time  $t$ , that is,  $\dot{c}(t) = x(t)$ , with initial condition  $c(0) = c_0 > 0$ . Instantaneous preferences are represented by  $U(c(t)) - \frac{a}{2} [x(t)]^2$ , where  $U'(c(t)) > 0$ ,  $U''(c(t)) \leq 0$ , and  $a > 0$  is a parameter reflecting consumption habits. The equation of motion for weight  $w(t)$  is  $\dot{w}(t) = c(t) - \delta w(t)$ , where  $\delta > 0$  is a depreciation rate, or more accurately, a basal metabolic rate, and  $w(0) = w_0$  is an initial condition.

The lifetime of an agent is a random variable. As such, Levy (2002, p. 889) and Dragone (2009, p. 800) assumed that any deviation in the weight  $w(t)$  of an agent from the physiologically optimal weight  $w^* > 0$  affects an agent's probability of living beyond time  $t$ , i.e., an agent's survival function  $\Phi(z(t))$ , where  $\Phi'(z(t)) < 0$ ,  $z \triangleq (w(t) - w^*)^2$  and where its partial derivative with respect to weight, to wit,  $\Phi_w(\cdot)$ , is given by the expression  $\Phi_w((w - w^*)^2) = 2\Phi'((w - w^*)^2)(w - w^*)$ .

Pulling the above together, the optimal control problem of Dragone (2009) is given by

$$\begin{aligned} \max_{x(t)} \quad & \int_0^{+\infty} \Phi((w(t) - w^*)^2) \left[ U(c(t)) - \frac{a}{2} [x(t)]^2 \right] e^{-\rho t} dt \\ \text{s.t.} \quad & \dot{w}(t) = c(t) - \delta w(t), \quad w(0) = w_0, \\ & \dot{c}(t) = x(t), \quad c(0) = c_0, \end{aligned} \quad (1)$$

where  $\rho$  is a discount rate. It is noteworthy that the control problem defined by Eq. (1) is isomorphic to one in which  $\Phi(\cdot)$  is replaced by  $e^{-\beta t}\Phi(\cdot)$ ,  $\beta > 0$ . The current-value Hamiltonian associated with problem (1) is defined as

$$H(c, w, x, \lambda, \mu) \triangleq \Phi((w - w^*)^2) \left[ U(c) - \frac{a}{2} x^2 \right] + \lambda(c - \delta w) + \mu x$$

where  $\lambda$  and  $\mu$  are current-value costate variables. Assuming that  $H(\cdot)$  is concave in  $(c, w, x)$ , the usual necessary conditions for an interior solution, namely,

$$H_x(c, w, x, \lambda, \mu) = -\Phi((w - w^*)^2) ax + \mu = 0 \quad (2)$$

and

$$\dot{\lambda} = \lambda\rho - H_w(c, w, x, \lambda, \mu) = (\rho + \delta)\lambda - \Phi_w((w - w^*)^2) \left[ U(c) - \frac{a}{2} x^2 \right] \quad (3)$$

$$\dot{\mu} = \mu\rho - H_c(c, w, x, \lambda, \mu) = \rho\mu - \Phi((w - w^*)^2) U'(c) - \lambda \quad (4)$$

$$\dot{w} = H_\lambda(c, w, x, \lambda, \mu) = c - \delta w, \quad w(0) = w_0 \quad (5)$$

$$\dot{c} = H_\mu(c, w, x, \lambda, \mu) = x, \quad c(0) = c_0 \quad (6)$$

are also sufficient if the transversality condition of Theorem 14.4 of Caputo (2005) holds. By definition, a steady state solution, say  $(c^{ss}, w^{ss}, x^{ss}, \lambda^{ss}, \mu^{ss})$ , is a solution of Eqs. (2)–(6) when  $\dot{\lambda} = \dot{\mu} = \dot{w} = \dot{c} = 0$ , that is to say,  $(c^{ss}, w^{ss}, x^{ss}, \lambda^{ss}, \mu^{ss})$  is a solution to

$$-\Phi((w - w^*)^2) ax + \mu = 0, \quad (7)$$

$$(\rho + \delta)\lambda - \Phi_w((w - w^*)^2) \left[ U(c) - \frac{a}{2} x^2 \right] = 0, \quad (8)$$

$$\rho\mu - \Phi((w - w^*)^2) U'(c) - \lambda = 0, \quad (9)$$

$$c(t) - \delta w = 0, \quad (10)$$

$$x = 0. \quad (11)$$

It is now shown that three general inequalities hold in a steady state, from which it follows that  $w^{ss} \leq w^*$  and  $w^{ss} \geq w^*$  can be steady state solutions for weight.

By Eq. (11),  $x^{ss} = 0$ , which implies that  $\mu^{ss} = 0$  from Eq. (7). Substituting the latter in Eq. (9) gives  $\lambda^{ss} = -\Phi\left((w^{ss} - w^*)^2\right) U'(c^{ss})$ , the inequality following from the fact that  $\Phi\left((w^{ss} - w^*)^2\right) \in (0, 1]$  and the assumption  $U'(c) > 0$ . Using  $\lambda^{ss} \leq 0$  and  $x^{ss} = 0$  in Eq. (8) gives  $\Phi_w\left((w^{ss} - w^*)^2\right) U(c^{ss}) \leq 0$ , which is the first steady state inequality. But in view of the fact that  $\Phi_w\left((w^{ss} - w^*)^2\right) = 2\Phi'\left((w^{ss} - w^*)^2\right)(w^{ss} - w^*)$  and  $\Phi'\left((w^{ss} - w^*)^2\right) < 0$ , it then follows that  $(w^{ss} - w^*) U(c^{ss}) \geq 0$ , which is the second steady state inequality. For the third inequality, recall that  $H(\cdot)$  is concave in  $(c, w, x)$ , and therefore that  $\Phi_{ww}\left((w^{ss} - w^*)^2\right) U(c^{ss}) \leq 0$ .

In sum, it has been shown that

$$\Phi_w\left((w^{ss} - w^*)^2\right) U(c^{ss}) \leq 0, \quad \Phi_{ww}\left((w^{ss} - w^*)^2\right) U(c^{ss}) \leq 0, \quad (w^{ss} - w^*) U(c^{ss}) \geq 0 \quad (12)$$

Inspection of Eq. (12) reveals that there exists an *equivalence* between the signs of four terms, to wit,  $U(c^{ss})$ ,  $\Phi_w\left((w^{ss} - w^*)^2\right)$ ,  $\Phi_{ww}\left((w^{ss} - w^*)^2\right)$  and  $(w^{ss} - w^*)$ . That is, Eq. (12) shows that knowing the sign of any one of the four terms  $U(c^{ss})$ ,  $\Phi_w\left((w^{ss} - w^*)^2\right)$ ,  $\Phi_{ww}\left((w^{ss} - w^*)^2\right)$  and  $(w^{ss} - w^*)$ , is equivalent to knowing the sign of the other three. In particular, a nonpositive value of steady state utility is equivalent to, i.e., is a necessary and sufficient condition for, an agent being at or below the physiologically optimal weight and the survival function being a nondecreasing and convex function of weight at a steady state. Hence being underweight or at the physiologically optimal weight is a possible steady state of the model. This follows because  $U(c^{ss})$  can be any real number, as all that is required for the purpose at hand is that  $U(c)$  is strictly increasing and concave.

Levy (2002, p. 892) assumed that  $U(c) = c^\beta$  and  $\beta \in (0, 1)$ , implying that  $U(c) \geq 0$  for all  $c \geq 0$ , whereas Dragone (2009, p. 801) assumed that  $U(c^{ss})$  is positive. Given  $U(c^{ss}) \geq 0$ , it follows from  $(w^{ss} - w^*) U(c^{ss}) \geq 0$  that  $w^{ss} \geq w^*$ , and the converse. Thus, the seemingly innocuous assumption that the steady state value of utility is nonnegative is fully equivalent to the assumption that an agent is at or above the physiologically optimal weight in a steady state, and that the survival function is nonincreasing and concave at a steady state.

For further understanding of underweight and overweight steady states, set  $a \equiv 0$  for simplicity, define expected (instantaneous) utility by  $V(c, w) \triangleq \Phi\left((w - w^*)^2\right) U(c)$ , and employ Eq. (12) to arrive at  $V_w(c^{ss}, w^{ss}) = \Phi_w\left((w^{ss} - w^*)^2\right) U(c^{ss}) \leq 0$ . The inequality asserts that in a steady state, expected utility is a nonincreasing function of weight. Hence, according to expected utility, weight is not a good for overweight or underweight agents at a steady state, whereas consumption is, seeing as  $V_c(c^{ss}, w^{ss}) = \Phi\left((w^{ss} - w^*)^2\right) U'(c^{ss}) \geq 0$ . Moreover, because  $V_w(c^{ss}, w^{ss}) = \Phi_w\left((w^{ss} - w^*)^2\right) U(c^{ss}) \leq 0$ , if an agent is at or above the physiologically optimal weight, or equivalently if  $U(c^{ss}) \geq 0$ , then  $\Phi_w\left((w^{ss} - w^*)^2\right) \leq 0$ . In this case an agent lowers their

expected utility and their probability of survival when they gain weight. As these two costly effects of weight gain reinforce each other, an agent chooses a consumption rate that prevents them from getting too overweight in a steady state. If, instead, an agent is at or below the physiologically optimal weight, or equivalently if  $U(c^{ss}) \leq 0$ , then  $\Phi_w((w^{ss} - w^*)^2) \geq 0$ , in which case expected utility decreases and the probability of survival increases when weight increases, resulting in a tradeoff from higher weight that is not present in the overweight case. In other words, a higher weight increases an underweight agent's probability of survival, but it also makes them worse off in expectation.

Note too that there is another feature at work here. Consider the expected marginal rate of substitution of weight for consumption at a steady state, viz.,

$$MRS_{wc} = -\frac{V_w(c^{ss}, w^{ss})}{V_c(c^{ss}, w^{ss})} = -\frac{\Phi_w((w^{ss} - w^*)^2) U(c^{ss})}{\Phi((w^{ss} - w^*)^2) U'(c^{ss})} \geq 0, \quad \Phi((w - w^*)^2) \in (0, 1] \quad (13)$$

Equation (13) holds for all steady state values of utility. It shows that rational agents are indifferent between low consumption rates and their correspondingly low weights, and high consumption rates and their correspondingly high weights. But this is precisely what one would expect given that consumption is a good and weight is a bad in a steady state.

### 3 Further Intuition by Way of Examples

To further sharpen the intuition of the main result, it is useful to contemplate the constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) classes of instantaneous utility functions—hereafter utility functions—that are ubiquitous in problems involving risk. To begin, a utility function has CRRA equal to unity if and only if  $U(c) = \alpha \ln c + \gamma$ , where  $\alpha > 0$  and  $\gamma \geq 0$ , and is strictly increasing and strongly concave for all  $c > 0$ , with  $\lim_{c \rightarrow 0} U(c) = -\infty$ . Moreover, the value of  $c^{ss}$  relative to  $e^{-\gamma/\alpha} > 0$  fully determines the sign of  $U(c^{ss})$  and therefore whether  $w^{ss} > w^*$  or  $w^{ss} < w^*$ . In particular,  $c^{ss} \in (0, e^{-\gamma/\alpha}]$  is equivalent to  $U(c^{ss}) \leq 0$ , which is equivalent to an underweight steady state  $w^{ss} \leq w^*$ , while  $c^{ss} \in [e^{-\gamma/\alpha}, \infty)$  is equivalent to  $U(c^{ss}) \geq 0$ , which is equivalent to an overweight steady state  $w^{ss} \geq w^*$ . Thus preferences that can be represented by  $U(c) = \alpha \ln c + \gamma$  can rationalize overweight and underweight agents in a steady state, as also shown in the numerical example discussed at the end of the section.

More generally, a utility function has CRRA equal to  $s > 0$  and  $s \neq 1$ , if and only if  $U(c) = \frac{\alpha}{1-s} c^{1-s} + \gamma$ , where  $\alpha > 0$  and  $\gamma \geq 0$ , and is strictly increasing and strongly concave for all  $c > 0$ . In order to develop further intuition about steady state weight, make the common assumption that  $\gamma = 0$ . In this case,  $s \in (0, 1)$  is equivalent to  $U(c) = \frac{\alpha}{1-s} c^{1-s} > 0$  for all  $c > 0$ , with  $U(0) = 0$ , while  $s \in (1, \infty)$  is equivalent to  $U(c) = \frac{\alpha}{1-s} c^{1-s} < 0$  for all  $c > 0$ , with  $\lim_{c \rightarrow 0} U(c) = -\infty$ . Thus, for this subclass of CRRA utility functions, the size of an agent's coefficient of relative risk aversion relative to unity fully determines, and is fully determined by, the steady state weight of an agent relative to the physiologically optimal weight. That is, the intuition behind the steady state weight of an agent lies in their relative tolerance to risk. For example, being a relatively less risk averse

agent, i.e.,  $s \in (0, 1)$  is equivalent to the weight of an agent being greater than or equal to the physiologically optimal weight in a steady state. Similarly, being a relatively more risk averse agent, i.e.,  $s \in (1, \infty)$ , is equivalent to the weight of an agent being less than or equal to the physiologically optimal weight in a steady state.

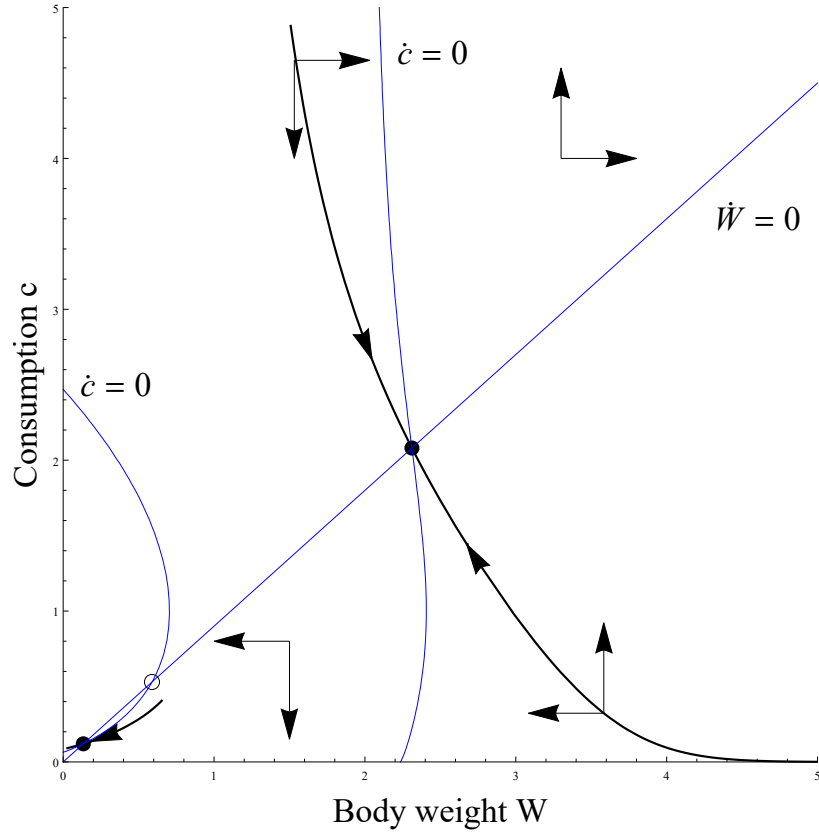
Finally, it is well known that a utility function has CARA equal to  $\theta > 0$  if and only if  $U(c) = -\frac{\alpha}{\theta}e^{-\theta c} + \gamma$ , where  $\alpha > 0$  and  $\gamma \geq 0$ , and is strictly increasing and strongly concave for all  $c \geq 0$ , with  $U(0) = \gamma - \frac{\alpha}{\theta} \geq 0$ . Thus preferences that can be represented by the popular subclass of CARA utility functions in which  $\gamma = 0$  are equivalent to agents being at or below the physiologically optimal weight in a steady state.

For some additional insight on the possible configurations of steady state solutions, consider a survival function  $\Phi(\cdot)$  that is nonincreasing and concave for some values of weight and nondecreasing and convex for others—the symmetric bell-shaped function given by  $\Phi\left((w - w^*)^2\right) = e^{-\frac{b}{2}(w - w^*)^2}$ ,  $b > 0$ , has these properties. In this instance Eq. (12) demonstrates that an agent who is underweight in a steady state is relatively further from the physiologically optimal weight and faces a lower probability of survival than an agent who is overweight.

The following example permits further appreciation of the above deductions. A logarithmic utility function is assumed, as it admits positive and negative values of utility, which in turn permit overweight and underweight steady states and allow the expected utility of weight to be positive or negative outside of a steady state. The assumed survival function admits convex and concave parts, but allows the Mangasarian sufficiency conditions to be locally satisfied. The numerical example reveals an additional result about the rational eating problem, namely, the possible emergence of multiple steady states of weight in which the same agent can alternatively end up being underweight or overweight. Notably, this result does not depend on the factors typically addressed in the literature, such as the availability of calorie-dense food, insufficient physical exercise, or self-control failures, but instead reflects the rational choice of a forward-looking agent that trades off the benefits and costs of eating and weight.

Consider  $U(c) = \ln c$  and  $\Phi\left((w - w^*)^2\right) = e^{-(w - w^*)^2}$ , and let  $\delta = 0.9$ ,  $\rho = 0.05$ ,  $w^* = 2$ , and  $a = 0$ . In this case three admissible steady states coexist, as shown in the phase diagram in Figure 1. The first is  $w_1^{ss} = 0.13$  and  $c_1^{ss} = 0.12$ , with utility  $U(c_1^{ss}) = -2.11$  and survival probability  $\Phi\left((w_1^{ss} - w^*)^2\right) = 0.03$ . The eigenvalues of the Jacobian matrix of the approximating linear dynamical system are  $(-0.6, 0.65)$ , implying that the steady state is a local saddle point. The second steady state is  $w_2^{ss} = 0.59$  and  $c_2^{ss} = 0.53$ , with utility  $U(c_2^{ss}) = -0.64$ , survival probability  $\Phi\left((w_2^{ss} - w^*)^2\right) = 0.14$ , and eigenvalues  $0.03 \pm 0.92i$ . It is a locally unstable spiral and thus cannot be reached from any initial weight that does not equal the steady state value of weight. And the third is  $w_3^{ss} = 2.31$  and  $c_3^{ss} = 2.08$ , with utility  $U(c_3^{ss}) = 0.73$ , survival probability  $\Phi\left((w_3^{ss} - w^*)^2\right) = 0.91$ , and eigenvalues  $(-2.87, 2.92)$ . It qualitatively mimics the first in that it too is a local saddle point and lies in a region where the conditions of the Mangasarian sufficiency theorem hold. Accordingly, if the initial value of weight is such that the corresponding value of consumption lies on the saddle path, then the trajectory passing through the initial point asymptotically converges to the steady state and is a solution of problem (1).





**Figure 1:** A phase diagram of the rational eating model with three steady states. Saddle points are depicted by full circles and the locally unstable steady state is depicted by a full circle. The saddle paths are represented by thick black lines and the nullclines by blue lines. The right-most steady state is the overweight steady state and the left-most is the underweight steady state. The overweight steady state is closer to the physiologically optimal weight than the underweight steady state and has a larger basin of attraction. The assumptions employed in generating the phase diagram are:  $U(c) = \ln c$ ,  $\Phi((w - w^*)^2) = e^{-(w - w^*)^2}$ ,  $\delta = 0.9$ ,  $\rho = 0.05$ ,  $w^* = 2$ , and  $a = 0$ .

Which of the saddle point steady states is reached depends on an agent's initial weight. Accordingly, if an agent's initial weight is sufficiently close to the underweight steady state, then the underweight steady state will be approached. On the other hand, if an agent's initial weight is sufficiently close to the overweight steady state, then the overweight steady state will be approached. Figure 1 shows that the overweight steady state has a larger basin of attraction than does the underweight steady state, and that is possible to be "locally trapped" by either saddle point steady state, making it hard for an agent to break free from them.

## 4 Summary and Conclusion

It has been shown that an agent in the models of Levy (2002) and Dragone (2009) might be overweight, underweight, or the physiologically optimal weight in a steady state. This deduction and the associated intuition about the possible steady state solutions were shown to follow from three general steady state inequalities and the resulting equivalence between four expressions. In the case of agents whose instantaneous preferences can be represented by a common subclass of CRRA utility functions, the equivalence was extended to the coefficient of relative risk aversion. In addition, it was shown that multiple steady states can coexist. This is an important deduction, seeing as underweight and overweight agents do indeed coexist in the population.

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