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Absence of covariant singularities in pure gravity

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Abstract

The assumptions of the Hawking-Penrose singularity theorem are not covariant under field redefinitions. Following the works on the covariant formulation of quantum field theory initiated by Vilkovisky and DeWitt in the 80's, we propose to study singularities in field space, where the spacetime metric is treated as a coordinate along with the other fields in the theory. From this viewpoint, a spacetime singularity might be just a singularity in the field-space coordinates, analogously to the standard coordinate singularities in General Relativity. Objects invariant under field-space coordinate transformations can then reveal whether certain spacetime singularity is indeed singular. We recall that observables in quantum field theory are scalar functionals in field space. Therefore, in principle, spacetime singularities corresponding to regular field-space curvature invariants would not affect physical observables. In this paper, we show that the field-space Kretschmann scalar for a certain choice of the DeWitt field-space metric is everywhere finite. This fact could be interpreted as an indication that no singularities actually exist in pure gravity for any gravitational action. In particular, all vacuum singularities of General Relativity result from an unhappy choice of field variables. The extension to the case in which matter fields are present, as required by singularity theorems, is left for future development.

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1 Introduction

Spacetime singularities are a trademark of General Relativity [1]. In static and stationary spacetimes, these singular regions must be excised from the classical spacetime manifold [2], thus affecting the spacetime topology. What makes them more interesting is their possible occurrence at the end of the gravitational collapse or at the beginning of the cosmological expansion, when the energy density of matter would classically grow unbounded. In fact, mathematical theorems predict the general occurrence of geodesic incompleteness in the classical theory of gravity when matter satisfies suitable energy conditions [1]. However, there are indications that this unlimited growth does not survive the quantization of any toy model which is simple enough to be analysed exactly [3], or even approximately [4]. As the Heisenberg uncertainty principle naively suggests, the gravitational pull is overbalanced by the quantum pressure which prevents matter to be localised too narrowly, albeit the scale of this "bounce" remains hard to estimate reliably.

Another important result applying to high energy density scenarios is obtained by treating the Einstein-Hilbert action as a field theory on a fixed background. In this instance, loop corrections switch on new interaction terms, which can be viewed as a quantum completion of the initial theory [5]. Conversely and more appropriately, one can regard the Einstein-Hilbert action as the effective low-energy theory which holds when the extra terms are small with respect to the Planck scale and study loop effects on the singularity theorems [6–8].

At the same time, attempts at "reconciling" the presence of singularities with the classical theory of General Relativity have also been undertaken. In particular, by reconciliation we mean that points corresponding to singularities are not excluded from the spacetime manifold but, instead, mechanisms for crossing these points are elaborated. With regard to such mechanisms, it is useful to recall that, besides "strong" singularities, such as the Big Bang–Big Crunch in cosmology or the Schwarzschild central singularity, there exist also the so called "soft" or sudden singularities [9–17]. The particularity of soft singularities is that, while curvature invariants diverge, the Christoffel symbols remain finite (or even vanish, like for the Big Brake [12]), hence geodesics are well defined and can be extended through the singularity itself. A complete spacetime is so reconstructed that the singularity crossing takes place in it. The lesson that one can extract from these cases is that the presence of curvature singularities does not always represent too grave of a menace.

The idea of resolving or passing through a strong singularity of the Big Bang–Big Crunch type or of the Schwarzschild type, looks much more counterintuitive with respect to the crossing of soft singularities. Nevertheless, a number of works devoted to this topic have been published recently [18–33]. From our point of view, it seems that all of these approaches resort to one of two ideas, or a combination thereof. One of these ideas is to employ a reparameterization of the field variables which makes the singular geometrical invariant nonsingular. Another idea is to find such a parameterization of the fields, including, naturally, the metric, that gives enough information to describe consistently the crossing of the singularity even if some of the curvature invariants diverge. It must be said that there is no consensus about approaches describing the crossing of singularities of the Big Bang–Big Crunch type, and sometimes the debate escalates. In Refs. [26, 27] the procedure for the crossing of the Big Bang–Big Crunch singularity based on the use of Weyl symmetry, was elaborated. Using a Weyl-invariant theory, with two scalar fields conformally coupled to gravity, the authors obtained the geodesic completeness of the corresponding spacetime. The consequence of this geodesic completeness is the crossing of the Big Bang singularity and the emergence of antigravity regions in the Einstein frame. The use of Weyl symmetry to describe the passage through the Big Crunch–Big Bang singularity accompanied by a change of sign for the effective Newton constant, has led to some discussion. In Ref. [34, 35], it was noticed that for such a passage through the singularity some of the curvature invariants still become infinite. In Ref. [27] a counter-argument was put forward according to which, if one has enough conditions so as to match the nonsingular quantities before and after crossing the singularities, then the singularities can be traversed.

Given the complexity of the singularity problem in General Relativity, and in modifications or generalisations thereof, we think that it makes sense to try and take the complementary point of view of analysing the structure of singularities in the functional space of all field configurations. This is indeed lined up with the covariant approach to quantum field theory [1] [36–38], which builds up on the field-space formulation of canonical quantum gravity as proposed in DeWitt's seminal paper [39]. We shall, however, work in the classical limit and leave the generalization to the quantum regime for another work. It is particularly tempting to investigate whether the absence of spacetime singularities could be a property of the space of fields of the complete theory of quantum gravity. This requires understanding the connection between spacetime singularities and singular points in field space.

Field-space singularities are linked to spacetime singularities once a choice for the parameterisation of fields is made. This choice, however, is not unique. One can in fact perform field redefinitions, as customary in field theory, without affecting the physical observables of the theory. The freedom of choosing the field parameterisation allows for the removal of spacetime singularities in some cases. This naturally leads to distinct types of singularities, classified in accordance with the possibility of removing them via field redefinitions. Our perspective is that removable spacetime singularities under field redefinitions reflect an initial bad choice of field parameterisation, like removable spacetime singularities reflect a bad choice of coordinates.

It is not always easy, however, to find a convenient field redefinition (if it exists) able to remove spacetime singularities without introducing other defects. Clearly, one cannot conclude that such singularities are non-removable before exhausting all infinite possibilities of field parameterisation. A covariant approach under field redefinitions is thus of utmost importance and a practical necessity. In this paper, we shall investigate singularities in field space by adopting a field-covariant approach. The main result of this paper is the calculation

[1] That physics should remain the same under field reparameterizations has also been called "field relativity" by Wetterich [18]. This idea indeed goes back to the works of Vilkovisky and DeWitt in the early 80's, where they introduced a metric and a connection in field space to enforce reparameterization invariance at the quantum level. Wetterich states "field relativity" as a principle, without showing how one can obtain an invariant effective action. We also take it as a principle, but within Vilkovisky-DeWitt's formalism for the effective action. That being said, our work here concerns the classical regime, thus one needs not dwell on this more complicated subject.

of the field-space Kretschmann scalar in pure gravity, which turns out to be constant and free of singularities.

This paper is organised as follows. In Section 2, we review coordinate singularities in spacetime and use the same reasoning to classify singularities in field space. In Section 3, we review one known example, namely the Hawking-Turok instanton, where singularities can be removed by field redefinitions. These removable field-space singularities motivate the search of a more general formalism, which we start to develop in the rest of the paper. We show in Section 4 that the assumptions in the Hawking-Penrose theorem are not invariant under field redefinitions, urging the investigation of singularities in the field space. Section 5 is devoted to the geometry of field space for pure gravity, which is then used to study singularities in a field-covariant way. We discuss our results in Section 6.

Before continuing, we must remark a potential source of confusion due to the differences in nomenclature adopted in the literature when it comes to the space of all fields. Field space, superspace, space of histories, configuration space and functional space usually refer to some notion of "set of all fields". Sometimes these terms are used interchangeably and other times they are used with different meanings, *e.g.* superspace might refer to the space of three-dimensional fields in the ADM formalism [40] or the space of fields defined on the entire four-dimensional spacetime. To avoid misunderstandings, throughout this paper we shall adopt the term "field space" to denote the set of all possible field configurations in the entire spacetime manifold. This set is infinite dimensional and is not restricted to solutions of the classical equations of motion. [2] The adjectives "functional" and "field-space", as in "functional tensor" or "field-space tensor", shall both be used interchangeably to refer to objects or concepts in the field space.

2 Field space and singularities

Like the principle of General Relativity says that physics does not depend on the spacetime coordinates, it is widely accepted that observables (like cross sections) should not depend on the fields parametrisations we use to describe physical processes [3]. The field space $\mathcal{S}$ was developed in order to treat on the same (geometrical) footing both changes of coordinates in the spacetime $\mathcal{M}$ and field redefinitions in the functional approach to quantum field theory (for a review, see *e.g.* Refs. [5, 41]). In fact, the action can be written as a functional on $\mathcal{S}$ in a way that makes it explicitly invariant under spacetime diffeomorphisms and small field

[2] As we shall see below, field-space singularities will correspond to points in field space that are solutions of some equations of motion. The formalism is nonetheless general and independent of the particular theory that one has in mind.

[3] Expectation values, and more general correlation functions, are also observables in a quantum theory (at least formally). However, we should again stress that we focus on the classical level here, for which examples of observables include the field-space curvature invariants, such as the field-space Kretschmann scalar (see Section 5). These objects are rather abstract and it is difficult to conceive a way to measure them directly. One thus needs to relate known quantities measured in a laboratory to field-space curvature invariants. At the quantum level, on the other hand, the Vilkovisky-DeWitt formalism guarantees that all correlation functions remain covariant under field redefinitions.

variations. [4] For this purpose, all field components are denoted by indices which represent both the spacetime point and the field and tensor components (the so-called DeWitt condensed notation [38]). We shall thus reserve capital Latin letters for discrete indices (*e.g.*, tensorial, gauge and spinor indices), Greek letters shall correspond to usual spacetime indices and lower-case Latin letters will denote the union of all indices (discrete and continuum, the latter being denoted by spacetime coordinates such as x^μ or y^μ). For example, if the theory contains a scalar field ϕ and the metric $\boldsymbol{g}$, one collectively writes both fields together as

$$\varphi^a = \varphi^{(A,\mu)} = \varphi^A(x^\mu) = \big(\phi(x^\mu), g^{\alpha\beta}(x^\mu)\big) \ . \tag{2.1}$$

Note that field reparameterizations can be viewed as a particular case of field-space transformations acting on the index A (leaving coordinates unaffected), whereas changes in the spacetime coordinates act both on the argument x^μ and the tensor indices included in A.

All results regarding coordinate invariance in a given spacetime manifold and field parameterisation should be automatically accounted for in this formalism. However, this is partly hindered by the convention which assumes spacetime integration for repeated indices. For example,

$$\varphi^a \, \varphi_a = \sum_{B\in\mathcal{A}} \int_B \varphi^A(x) \, \varphi_A(x) \, \mathrm{d}\mu(x) \ , \tag{2.2}$$

where (B, x^μ) are suitable charts of the atlas $\mathcal{A}$ for the spacetime $\mathcal{M}$ and $\mathrm{d}\mu$ a suitable spacetime measure. Including the above integration is convenient for the purpose of writing functionals in $\mathcal{S}$, but makes it more problematic to analyse the nature of local singularities in spacetime. Moreover, one typically has $\mathrm{d}\mu = \sqrt{-g}\, \mathrm{d}^4x$, where $g = \det(\boldsymbol{g})$ so that the metric $\boldsymbol{g}$ would always play a special role (even if it were not dynamical).

Finally, it is important to remark that the topology of $\mathcal{M}$ will affect the possible metrics that belong to the gravitational sector of the field space

$$\mathcal{G}(\mathcal{M}) = \frac{\mathrm{Lor}(\mathcal{M})}{\mathrm{Diff}(\mathcal{M})} \ , \tag{2.3}$$

where $\mathrm{Lor}(\mathcal{M})$ denotes Lorentzian metrics and $\mathrm{Diff}(\mathcal{M})$ the diffeomorphisms on $\mathcal{M}$. The existence of singularities of the kind predicted by the singularity theorems should therefore reflect in the topology of $\mathcal{M}$ and, consequently, the global structure of $\mathcal{S} \supseteq \mathcal{G}$. Moreover, the field content of the theory is also an ingredient that can be set independently but which will affect the local and global features of both $\mathcal{M}$ and $\mathcal{S}$. In the following we shall therefore consider the topology of $\mathcal{M}$ and the specific fields as parts of what needs to be determined compatibly with local scalar quantities in $\mathcal{S}$.

[4] Note the technical and substantial fact that coordinate transformations and field redefinitions are not (and should not - more later) be restricted to the "small" ones which can be connected with the identity when singularities are present.

2.1 Spacetime singularities: known facts

We know pretty much everything about this kind of singularities from textbooks in General Relativity, where singularities are analysed for specific solutions of field equations. The general definitions can however be given without assuming the metric tensor and other fields satisfy equations of motion. In particular, we can consider the following two types of singularities [1].

2.1.1 Removable (coordinate) singularities

Given a spacetime chart (B, x^μ), there might be field components (of the metric or otherwise) which behave badly (diverge or vanish) on the border ∂B, although all local spacetime scalar quantities remain smooth there. In this case, we expect that the singularities in field components can be removed by a suitable change of coordinates $x^\mu \to y^\mu(x^\nu)$ about the border of B, so that the new chart $(\bar{B}, y^\mu)$ covers ∂B and extends past the singularities.

Note that the fact a singularity is removable does not necessarily mean it is physically irrelevant. Typical example of a purely accidental bad behaviour is given by the origin $r = 0$ in spherical coordinates, where the angular components of the flat Euclidean metric vanish. This singularity can of course be removed by changing to Cartesian coordinates, which cover the point $r = 0$. A different example is given by the Schwarzschild radius in the usual Schwarzschild coordinates, where the metric $\boldsymbol{g}$ reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2\,M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2\,M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\ . \tag{2.4}$$

The time component of the above covariant metric vanishes and the radial component diverges for $r \to 2\,M$. This singularity can also be removed by a change of coordinates, but its deep physical meaning as an event horizon can only be assessed by studying the behaviour of geodesics [1]. After all, the role of the metric is precisely to define geodesic motion, that is to describe gravity according to the equivalence principle. Finally, note that any change of coordinates which removes the singular behaviour in the components of the metric (2.4) around $r = 2\,M$ must depend on the mass M,

$$x^\mu \to y^\mu(x^\nu, M)\ , \tag{2.5}$$

and cannot be a diffeomorphism, in that there exists no parameter which smoothly connects the above change of coordinates to the identity. [5] (One usually refers to such changes of coordinates as "large".) This example is a nice reminder that the symmetry group of General Relativity (meaning general coordinate invariance, not the dynamics of the particular Einstein-Hilbert action) is the Bergmann-Komar group [42]

$$x^\mu \to y^\mu(x^\nu, \boldsymbol{g}) \tag{2.6}$$

which includes, but is not exhausted by, spacetime diffeomorphisms.

[5] One should not consider $M \to 0$ as the way to connect to the identity, since changing M means to change the spacetime metric.

2.1.2 Non-removable singularities

Changes of coordinates, either small or large, cannot affect spacetime scalar quantities. Therefore, if there exists scalar quantities which behave badly at the border ∂B of the spacetime chart (B, x^μ), such singularities cannot be eliminated by defining a new chart $(\bar{B}, y^\mu)$ and one must conclude that $\partial B \subseteq \partial\mathcal{M}$. We remark once more that local (scalar) quantities are used to probe and determine the global structure of $\mathcal{M}$.

The typical examples are given by the origin $r = 0$ in the Schwarzschild spacetime, where the Kretschmann scalar diverges, and the big bang singularity in cosmology, where the scale factor of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric vanishes. In both cases, those singular points do not belong to $\mathcal{M}$, which is therefore said to be geodesically incomplete. These non-removable singularities are the subject of Penrose-Hawking theorems which prove their general existence in General Relativity, albeit under suitable auxiliary conditions about the matter fields. Note that curvature singularities cannot generally be seen as a consequence of the Hawking-Penrose theorem. In fact, nowhere in the theorem's proof appears divergent energy densities. On physical grounds, however, it is reasonable to conjecture that both concepts are somehow related. In practice, one can adopt such a conjecture by puncturing the points in spacetime where curvature invariants diverge. When this is performed, geodesics terminate at such missing points for a finite value of the affine parameter/proper time, thus the spacetime becomes geodetically incomplete. In this sense, geodesic incompleteness would capture both divergent curvatures and conical singularities (in which geodesic incompleteness is present but the curvature remains regular). In this paper, we shall adopt this point of view.

2.2 Field-space singularities: a conjecture

Since the field-space formalism includes the spacetime tensor structure, the above classification remains valid, although somewhat hindered by the convention (2.2) if the singularity is localised (like in the Schwarzschild manifold). [6] There is however hope that some of the spacetime singularities which cannot be removed by changes of coordinates could still be removed by field redefinitions. Note that removing such singularities, strictly speaking, changes the topology of spacetime and is therefore tantamount to extend the singular spacetime manifold $\mathcal{M}_1$ to a different $\mathcal{M}_2$ which includes the originally singular points. It now seems natural to consider local quantities behaving like scalars under field reparameterisations to likewise define removable and non-removable field-space singularities.

This approach requires introducing a local metric $\boldsymbol{G}$ in field space $\mathcal{S}$ and computing the associated geometric scalars by defining a covariant derivative which is compatible with $\boldsymbol{G}$. Note that $\boldsymbol{G}$ is actually determined by the kinetic part of the action and its dimension depends on the field content of the latter. There is therefore a huge freedom in defining $\boldsymbol{G}$ and one might hope that adding enough fields always allows one to remove all divergences. However, classification of such high dimension metric manifolds with respect to singularities

[6] Cosmology is again simpler because one assumes very strong symmetries like homogeneity, which makes the spacetime integration somewhat less harmful.

is not complete and only specific examples can be worked out.

A few more remarks are also in order. Firstly, like for spacetime singularities, it is clear that "large" field redefinitions will be needed to remove singularities. Moreover, it is also clear that such field redefinitions will have to depend on the specific configuration we wish to cure, so that the relevant transformations generalise the Bergmann-Komar ones to include a dependence on all fields in the specific configuration. Secondly, note that none of the above assumes or needs the field configurations to satisfy any equations of motion. The fact that $\mathbf{G}$ is related to the kinetic term in the action simply reflects the requirement that a canonical choice for the fields should exist locally.

3 Hawking-Turok instanton

Several examples in the literature suggest that some non-removable spacetime singularities can still be removed by field redefinitions [17–19, 21–27]. In this section, we briefly review one such example in cosmology, namely the Hawking-Turok instanton describing an open universe obtained by analytical continuation of a singular Euclidean spacetime [43]. Hawking and Turok have pointed out that a singular solution might be acceptable should the on-shell action evaluated at such a solution be regular [43]. At the level of classical General Relativity, this implies that only the spacetime Ricci scalar and matter fields are required to be finite. This is indeed the case for the aforementioned instanton, as we shall now see.

The Euclidean action in this model reads

$$S_E = \int \mathrm{d}^4x \sqrt{g} \left[-\frac{1}{2} R + \frac{1}{2} \nabla_\mu \phi \, \nabla^\mu \phi + V(\phi) \right] \ , \tag{3.1}$$

where R is the Ricci scalar for the metric $\boldsymbol{g}$, ϕ is a canonical scalar field and $V(\phi)$ its potential. The equations of motion corresponding to Eq. (3.1) contains the solution $\phi = \phi(\sigma)$ and

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + b^2(\sigma) \left[\mathrm{d}\chi^2 + \sin^2(\chi) \, \mathrm{d}\Omega^2 \right] \ , \tag{3.2}$$

where the factor

$$b(\sigma) \approx \begin{cases} \sigma \ , & \text{for} \quad \sigma \sim 0 \ , \\ (\sigma_f - \sigma)^{1/3} \ , & \text{for} \quad \sigma \sim \sigma_f \ , \end{cases} \tag{3.3}$$

and the scalar field

$$\phi(\sigma) \approx \begin{cases} \frac{1}{2} \sigma^2 \ , & \text{for} \quad \sigma \sim 0 \ , \\ -\sqrt{\frac{2}{3}} \ln(\sigma_f - \sigma), & \text{for} \quad \sigma \sim \sigma_f \ . \end{cases} \tag{3.4}$$

After Wick rotating χ, one then obtains an open universe. The Ricci scalar for the solution (3.2) reads

$$R \sim \frac{1}{(\sigma_f - \sigma)^2} \ , \tag{3.5}$$

thus there is a non-removable spacetime singularity at $\sigma = \sigma_f$. The on-shell action is nonetheless regular should the potential $V(\phi)$ not grow faster than b^3 for $\sigma \to \sigma_f$:

$$S_E|_{\text{on-shell}} = - \int \mathrm{d}^4x \, b^3(\sigma) \, V(\phi) \ , \tag{3.6}$$

where $S_E|_{\text{on-shell}}$ denotes the action (3.1) evaluated at the solution (3.2). Because the action is a scalar functional in the field space, the finiteness of $S_E|_{\text{on-shell}}$ suggests that the singularity observed in Eq. (3.5) should instead be removable by field redefinitions. In fact, a simple rescaling of the metric is able to remove this singularity as we now explain.

Changing spacetime coordinates to $\mathrm{d}\bar{\sigma} = b^{-1}\,\mathrm{d}\sigma$, followed by a Weyl transformation $\bar{g}_{\mu\nu} = b^{-2}\,g_{\mu\nu}$, leads to

$$\mathrm{d}s^2 = b^2(\bar{\sigma})\,\mathrm{d}\bar{s}^2 \;, \tag{3.7}$$

where we defined

$$\mathrm{d}\bar{s}^2 = \mathrm{d}\bar{\sigma}^2 + \mathrm{d}\chi^2 + \sin^2(\chi)\,\mathrm{d}\Omega^2 \;. \tag{3.8}$$

Near the singularity, that is for $\sigma \sim \sigma_f$, one has

$$\bar{\sigma} = \frac{3}{2}\,(\sigma_f - \sigma)^{2/3} \;, \tag{3.9}$$

and the new geometry $\bar{\boldsymbol{g}}$ is flat and obviously regular. However, the scalar field

$$\phi = -\sqrt{6}\,\log b(\bar{\sigma}) \tag{3.10}$$

becomes singular as $b(\bar{\sigma}) \to 0$ (for $\bar{\sigma} \to 0$). We have thus shifted the singularity from the geometry to the scalar field.

In order to remove the singularity from the scalar field as well, one can perform another Weyl transformation [22]

$$\bar{g}_{\mu\nu} = \Omega^2\,\tilde{g}_{\mu\nu} \;, \tag{3.11}$$

with

$$\Omega = 1 + \beta\,e^{-\alpha\sqrt{2/3}\,\phi} \;. \tag{3.12}$$

where α and β are free parameters. Note that the absence of singularity in the geometry $\bar{\boldsymbol{g}}$ is not spoiled by this additional transformation as it goes to unity at the singularity ($\phi \to \infty$). Near the singularity at $\bar{\sigma} = 0$, the new Ricci scalar takes the form

$$\tilde{R} \sim \bar{\sigma}^{\alpha-2} \;, \tag{3.13}$$

thus it is regular for $\alpha > 2$. The action in the field frame $\tilde{\boldsymbol{g}}$ becomes

$$\begin{aligned} S_E \;=\; \int \mathrm{d}^4x\,\sqrt{\tilde{g}}\,\Bigg[&-\frac{1}{2}\,e^{-\sqrt{2/3}\,\phi}\,\Omega^2\,\tilde{R} + 3\,e^{-\sqrt{2/3}\,\phi}\,\Omega^2\,\frac{\partial\ln\Omega}{\partial\phi}\left(\sqrt{\frac{2}{3}} - \frac{\partial\ln\Omega}{\partial\phi}\right)\tilde{\nabla}_\mu\phi\,\tilde{\nabla}^\mu\phi \\ &+ \Omega^4\,\bar{V}(\phi)\Bigg] \;. \end{aligned} \tag{3.14}$$

Now we can see that the canonically-normalised scalar field

$$\mathrm{d}\tilde{\phi}^2 = 6\,e^{-\sqrt{2/3}\,\phi}\,\Omega^2\,\frac{\partial\ln\Omega}{\partial\phi}\left(\sqrt{\frac{2}{3}} - \frac{\partial\ln\Omega}{\partial\phi}\right)\mathrm{d}\phi^2 \tag{3.15}$$

is also regular in the spacetime with metric $\tilde{\boldsymbol{g}}$. In fact, near the singularity at $\bar{\sigma} = 0$, one finds

$$\tilde{\phi} \sim \bar{\sigma}^{(1+\alpha)/2} \;, \tag{3.16}$$

where we set $\beta = -1/4\alpha$. Therefore, in conformity with the finitude of $S_E|_{\text{on-shell}}$, we have found a field-space frame where both the geometry and the scalar field are singularity-free. More precisely, the above sequence of field transformations for the metric and scalar field allows one to extend the spacetime manifold to include the singularity $\sigma = \sigma_f$, which is therefore removable in field space.

It is easy to see that the sequence of field transformations mapping $\varphi^a = (\phi, \boldsymbol{g})$ into $\tilde{\varphi}^a = (\tilde{\phi}, \tilde{\boldsymbol{g}})$ does not map all spacetime geodesics of the original metric $\boldsymbol{g}$ in Eq. (3.2) into geodesics (straight lines) of the flat metric $\tilde{\boldsymbol{g}}$ in Eq. (3.8). Therefore, one runs into the problem of having to decide *a priori* whether test particles (*e.g.*, the classical and non-relativistic electron) fall freely in the metric $\boldsymbol{g}$ or in the metric $\tilde{\boldsymbol{g}}$. This ambiguity is removed in a fundamental theory where test particles are fully replaced by matter fields (such as spinors in the example of the electron), in which case the matter content of the Hawking-Turok model is solely given by the scalar field ϕ. In that case, the entire dynamics is unique and one does not need to study spacetime geodesics at all, although the distinction between matter and gravity is lost under the field redefinitions. The notable exception is given by trajectories of constant χ, which are geodesics in both field frames. However, the expansion of congruences of such geodesic, namely

$$\theta = u^\mu{}_{;\mu} \;, \tag{3.17}$$

clearly vanishes in $\tilde{\boldsymbol{g}}$ and takes the well-known FLRW form [44]

$$\theta = 3\,\frac{b'(\sigma)}{b(\sigma)} \tag{3.18}$$

in the initial metric $\boldsymbol{g}$. There is no geodesic focusing in $\boldsymbol{g}$, whereas the Hawking-Penrose theorem applies in $\tilde{\boldsymbol{g}}$. This exemplifies the issue of field frame dependence of the singularity theorems we are going to discuss next.

4 Functional frame dependence of Hawking-Penrose theorem

The causal structure of a given spacetime can be inspected by studying congruences of geodesics, whose behaviour is determined by the Raychaudhuri equation [44]. In particular, the positivity of the discriminant

$$\Delta = R_{\mu\nu}\, u^\mu\, u^\nu \;, \tag{4.1}$$

where $\boldsymbol{u}$ is the geodesic four-velocity field and $\boldsymbol{R}$ the Ricci tensor, ensures the focusing property of gravity, that is, the fact that the expansion (3.17) of time- and light-like congruences diverges at focal points. The Hawking-Penrose theorem [1] then relies on the positivity of

Δ (along with other assumptions) for light-like geodesics in order to further establish the geodesic-incompleteness of spacetime. Nonetheless, Δ is not covariant under field redefinitions and the focusing property can therefore be affected by changing the field frame. This fact can already be seen by considering a pure metric redefinition $\boldsymbol{g} \to \tilde{\boldsymbol{g}}$ (of course, not corresponding to a change of spacetime coordinates, but not involving other fields either), under which the Ricci tensor transforms as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + A_{\mu\nu} \ , \tag{4.2}$$

with

$$A_{\mu\nu} = 2\, C^{\sigma}{}_{\nu[\mu}\, C^{\rho}{}_{\rho]\sigma} - 2\, \nabla_{[\mu} C^{\rho}{}_{\rho]\nu} \ , \tag{4.3}$$

where

$$C^{\alpha}{}_{\mu\nu} = \frac{1}{2}\, \tilde{g}^{\alpha\beta} \left(\nabla_{\mu} \tilde{g}_{\beta\nu} + \nabla_{\nu} \tilde{g}_{\beta\mu} - \nabla_{\beta} \tilde{g}_{\mu\nu} \right) \ , \tag{4.4}$$

and covariant derivatives are taken with respect to the original metric $\boldsymbol{g}$. It is precisely $A_{\mu\nu}$ that makes the Ricci tensor a non-tensorial quantity under these restricted field redefinitions. [7].

The focusing condition $\Delta > 0$, as much as the singularity theorems in their current form, are thus not covariant under field redefinitions. Therefore, the physical meaning of singularities in field space should be put on the same footing as (removable or non-removable) singularities of spacetime. In fact, we should also recall that ways of evading those theorems are known (see, *e.g.* Ref. [45]). Strictly speaking, singularities implied by the Hawking-Penrose theorem are most likely field-space coordinate singularities, unless one also has $A_{\mu\nu}\, u^{\mu}\, u^{\nu} > 0$ or

$$2\, C^{\sigma}{}_{\nu[\mu}\, C^{\rho}{}_{\rho]\sigma}\, u^{\mu}\, u^{\nu} > 2\, \nabla_{[\mu}\, C^{\rho}{}_{\rho]\nu}\, u^{\mu}\, u^{\nu} \ . \tag{4.5}$$

This condition will depend on the metric redefinition one makes and it seems rather unlikely that it will hold for any frame in field space without being too restrictive. Resorting to energy conditions will not change this scenario because energy conditions also depend on the frame in field space [46, 47].

We should also point out that the proper time

$$\Delta\tau = \int \sqrt{g_{\mu\nu}\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu}} \tag{4.6}$$

is obviously dependent on the functional frame, thus the rate of change of the expansion θ for time-like geodesics has yet another instance of dependence on the field-space frame, and so consequently do the focusing theorems in this case. Affine parameters generally depend on the field-space frame due to their definition via the geodesic equation, which contains an implicit dependence on the metric. Finally, the very definition of the expansion θ, given in Eq. (3.17) for any tangent vector u^{μ}, also spoils the covariance under field redefinitions because of the metric dependence in the covariant derivative. Therefore, there are three different instances of functional non-covariance in the Hawking-Penrose theorems [8] originating from (1) affine parameters, (2) the geodesic expansion and (3) the discriminant Δ.

[7] Note, however, that $A_{\mu\nu}$ is itself a tensor in field space.

[8] Similar conclusions can be found in Ref. [48].

Note that, at first sight, it is somewhat expected that the singularity theorem would not hold under field redefinitions, as Einstein's equations certainly take a different form in different field-space coordinates. It is important to remark, however, that there are other sources of non-covariance in addition to the equations of motion, as we outlined above. Although the form of the equations of motion change, it could be the case that all sources of non-covariance could cancel each other out, leaving the Hawking-Penrose theorem invariant. This turns out not to be the case.

Although the Hawking-Penrose theorem sets the conditions for the formation of spacetime singularities, the above analysis shows that it cannot discriminate between non-removable and field-coordinate singularities. We thus need an object that is invariant under both coordinate transformations and field redefinitions. This invariably requires the study of the geometry of field space.

5 Geometry of field space for pure gravity

The results of the preceding Sections 3 and 4 suggest that fully non-removable singularities, in addition to being coordinate-independent, must be independent of the choice we make to describe the dynamical fields. In this Section, we propose to look at local invariants in the field space $\mathcal{S} = \mathcal{S}(\mathcal{M})$, such as the functional Kretschmann scalar, to decide whether singularities are removable by field redefinitions.

We shall now revert to the DeWitt condensed notation briefly recalled in Section 2, with the field space parameterized by φ^a, which collectively denotes all fields of arbitrary spin present in the theory. These fields are supposed to be interpreted as mere coordinates in the field space $\mathcal{S}$. Notice that the field space, as the set of all field configurations, comprises points that are solutions and points that are not solutions of some field equations. We nonetheless define a singularity $\varphi^a = \varphi^a_{\rm s}$ in field space as a solution of some field equations for which the field-space curvature invariants are infinite at $\varphi^a_{\rm s}$. Without specifying the theory, one can analyse the existence of possible field-space singularities by calculating the curvature invariants with some choice of the field-space metric.

We should also stress that under field redefinitions $\varphi^a \to \tilde{\varphi}^a$ the classical equations of motion and their solutions transform in such a way that the redefined solutions will be solutions of the redefined equations of motion. This indeed reflects the fact that the classical action transforms as a scalar under field redefinitions $\tilde{S}[\tilde{\varphi}^a] = S[\varphi^a]$, thus the classical equations of motion transform covariantly, [9]

$$\frac{\delta \tilde{S}[\tilde{\varphi}]}{\delta \tilde{\varphi}^a} = \frac{\delta \varphi^b}{\delta \tilde{\varphi}^a}\,\frac{\delta S[\phi]}{\delta \varphi^b}\ . \tag{5.1}$$

This implies

$$\frac{\delta \tilde{S}[\tilde{\varphi}]}{\delta \tilde{\varphi}^a} = 0 \quad \Leftrightarrow \quad \frac{\delta S[\varphi]}{\delta \varphi^b} = 0\ , \tag{5.2}$$

[9] As we mentioned in footnote 3, this does not automatically hold at the quantum level, since the standard effective action acquires off-shell corrections under field redefinitions. The Vilkovisky-DeWitt effective action is precisely designed in order to keep the covariance of the effective equations of motion.

which means that solutions in one field-space coordinates will be taken into solutions in another field-space chart. Covariant singularities can thus be studied via the field-space invariants.

With the purpose of being as general as possible, we shall leave the classical action unspecified. The metric $\boldsymbol{G}$ of the field space is supposed to be seen as part of the definition of the theory. The line element in $\mathcal{S}$ is naturally defined by

$$\mathrm{d}s^2 = G_{ab}\,\mathrm{d}\varphi^a\,\mathrm{d}\varphi^b = \int \mathrm{d}^4x \int \mathrm{d}^4x'\, G_{AB}(x,x')\,\mathrm{d}\varphi^A(x)\,\mathrm{d}\varphi^B(x')\ . \tag{5.3}$$

For pure gravity theories, $\mathcal{S} = \mathcal{G}$ and one identifies $\varphi^a = g^{\mu\nu}(x)$. In this case, assuming the field-space metric to be ultralocal (*i.e.* proportional to the spacetime Dirac delta and independent of derivatives of the spacetime metric), there is a unique (up to a global factor) one-parameter familiy of field-space metrics

$$G_{ab} = G_{AB}\,\delta(x,x')\ , \tag{5.4}$$

where

$$G_{AB} = \frac{1}{2}\left(g_{\mu\rho}\,g_{\sigma\nu} + g_{\mu\sigma}\,g_{\rho\nu} + c\,g_{\mu\nu}\,g_{\rho\sigma}\right) \tag{5.5}$$

is called the DeWitt field-space metric [39, 49] and involves only a dimensionless parameter c. Its inverse is found by solving $G_{AB}\,G^{BC} = \delta_A^C$, which gives

$$G^{AB} = \frac{1}{2}\left(g^{\mu\rho}\,g^{\sigma\nu} + g^{\mu\sigma}\,g^{\rho\nu} - \frac{2\,c}{2+n\,c}\,g^{\mu\nu}\,g^{\rho\sigma}\right)\ , \tag{5.6}$$

where n is the dimension of the spacetime $\mathcal{M}$. The DeWitt functional metric is thus invertible only for $c \neq -2/n$. The parameter c cannot be determined without some additional assumption. For example, Vilkovisky [36, 50] suggested that $\boldsymbol{G}$ should be identified from the highest derivative term in the classical action. In that case, $c = -1$ for the Einstein-Hilbert action, but it would be different for higher-derivative gravity. We shall leave c unspecified.

The connection on $\mathcal{S}$ is of the Levi-Civita type:

$$\Gamma^a{}_{bc} = \Gamma^A{}_{BC}\,\delta(x_A, x_B)\,\delta(x_A, x_C)\ , \tag{5.7}$$

with

$$\Gamma^A{}_{BC} = \frac{1}{2}\,G^{AD}\left(\partial_B G_{DC} + \partial_C G_{BD} - \partial_D G_{BC}\right)\ . \tag{5.8}$$

In particular, the Levi-Civita connection for the DeWitt functional metric (5.4) reads

$$\begin{aligned}\Gamma^A{}_{BC} = &-\delta^{(\lambda}{}_{(\mu}\,g_{\nu)(\rho}\,\delta^{\tau)}{}_{\sigma)} + \frac{1}{4}\,g_{\mu\nu}\,\delta^{(\lambda}_{\rho}\,\delta^{\tau)}_{\sigma} + \frac{1}{4}\,g_{\rho\sigma}\,\delta^{(\lambda}_{\mu}\,\delta^{\tau)}_{\nu} \\ &- \frac{1}{2\,(2+n\,c)}\,g^{\lambda\tau}\,g_{\mu(\rho}\,g_{\sigma)\nu} - \frac{c}{4\,(2+n\,c)}\,g^{\lambda\tau}\,g_{\mu\nu}\,g_{\rho\sigma}\ .\end{aligned} \tag{5.9}$$

The functional Riemann curvature is then defined in the usual way as

$$\mathcal{R}^a{}_{bcd} = \partial_c\Gamma^a{}_{db} - \partial_d\Gamma^a{}_{cb} + \Gamma^a{}_{ce}\,\Gamma^e{}_{db} + \Gamma^a{}_{de}\,\Gamma^e{}_{cb}\ , \tag{5.10}$$

with $\mathcal{R}_{bd} = \mathcal{R}^a{}_{bad}$ and $\mathcal{R} = \mathcal{R}^a{}_a$ being the the functional Ricci tensor and functional Ricci scalar, respectively. For the DeWitt functional metric, the Ricci tensor is given by

$$\mathcal{R}_{AB} = \frac{1}{4}\left(g_{\mu\nu}\, g_{\rho\sigma} - n\, g_{\mu(\rho}\, g_{\sigma)\nu}\right) \tag{5.11}$$

and the Ricci scalar by

$$\mathcal{R} = \frac{n}{4} - \frac{n^2}{8} - \frac{n^3}{8}\ . \tag{5.12}$$

We remark that both quantities are local in $\mathcal{M}$ because of Eq. (5.4), and can therefore be used to inspect the field space $\mathcal{S}$, like spacetime scalars are used to probe $\mathcal{M}$.

The standard practice in General Relativity to decide whether a singularity is removable or just a coordinate singularity is to seek singularities in the curvature invariants, such as the Kretschmann scalar. Since diffeomorphism invariants are the same in all coordinate systems, only "true" [10] singularities, if any, would be manifested. Analogously, we must seek a scalar functional defined in the field space in order to investigate the appearance of singularities. We shall define the functional Kretschmann scalar of the underlying field space $\mathcal{S}$ as

$$\mathcal{K} = \mathcal{R}_{ABCD}\, \mathcal{R}^{ABCD}\ . \tag{5.13}$$

The functional Kretschmann scalar is naturally invariant under field redefinitions, it thus exhibits singularities that can neither be removed by field redefinitions nor spacetime coordinate transformations. To assess whether a singularity is real or only a consequence of a bad choice of field variables, one must therefore calculate $\mathcal{K}$. Note that the functional Kretschmann scalar depends only on the field-space metric $\boldsymbol{G}$. The dependence on a particular action would become manifest once the components G_{AB} are identified through the highest derivative term of the classical action as explained before. For the DeWitt functional metric (5.5), the dependence on the theory is thus encoded by the parameter c.

The difficulty now is to compute $\mathcal{K}$ explicitly. For a spacetime $\mathcal{M}$ of dimension n, each capital index corresponds to $(n^2+n)/2$ degrees of freedom. The functional Riemann tensor alone thus contains $(n^2+n)^4/16$ components. Futhermore, one must contract it with itself to obtain $\mathcal{K}$. The use of a good computer algebra system is clearly very convenient and we used the software `Cadabra2` [51, 52] to obtain the fairly simple result

$$\mathcal{K} = \frac{n}{8}\left(\frac{n^3}{4} + \frac{3n^2}{4} - 1\right). \tag{5.14}$$

This clearly shows that $\mathcal{K}$ is smooth for any spacetime metric $\boldsymbol{g}$ in any spacetime dimension n. More importantly, $\mathcal{K}$ turns out to not depend on the DeWitt parameter c. Therefore, *every theory of pure gravity* is absent of curvature singularities in the field space $\mathcal{G}$. Note that, as we explained before, curvature invariants generally depend on the spacetime metric, thus spacetime singularities are indeed related to field-space singularities. Because field-space curvature scalars are invariant under field redefinitions, any singularity manifested

[10] As we advocate in this paper, true singularities must be invariant not only under diffeomorphisms but also under field redefinitions. That is the reason we wrote true between quotes.

in $\mathcal{K}$ would represent singularities that cannot be removed by field redefinitions. This is analogous to the fact that singularities in the spacetime Kretschmann $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ cannot be removed by coordinate transformations.

As an example, let us make a slight modification of the DeWitt metric (5.5) by including a global factor $g = \det(\boldsymbol{g})$ for the determinant of the spacetime metric, that is

$$G_{AB} = \frac{1}{2} (-g)^{\epsilon} \left(g_{\mu\rho}\, g_{\sigma\nu} + g_{\mu\sigma}\, g_{\rho\nu} + c\, g_{\mu\nu}\, g_{\rho\sigma} \right) \ , \tag{5.15}$$

where ϵ is an arbitrary exponent. In this case, the Kretschmann scalar becomes

$$\mathcal{K} = \frac{n}{8} (-g)^{-2\epsilon} \left(\frac{n^3}{4} + \frac{3n^2}{4} - 1 \right) \ . \tag{5.16}$$

Such a result signals the potential existence of field-space singularities at $g = 0$ for $\epsilon > 0$ or at $g \to \infty$ for $\epsilon < 0$. It is only for $\epsilon = 0$ that one obtains an everywhere-finite result, in agreement with Eqs. (5.5) and (5.14). These potential singularities will, of course, only become actual singularities if, given a theory (be it General Relativity or else), there exist solutions of the corresponding equations of motion which satisfy $g = 0$ for the field-space metric with $\epsilon > 0$ or $g \to \infty$ for $\epsilon < 0$. Spacetime singularities for which g is either finite (for $\epsilon < 0$) or non-zero (for $\epsilon > 0$) can in principle be removed by field redefinitions. A simple example that can be used to understand the dependence of the field-space metric (here parameterized by ϵ) on the presence of covariant singularities is the Schwarzschild black hole,

$$\mathrm{d}s^2 = -\left(1 - \frac{2MG}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2MG}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin\theta\, \mathrm{d}\phi^2\right) \ , \tag{5.17}$$

whose determinant reads

$$g = -r^4 \sin^2\theta \ . \tag{5.18}$$

Assuming that the dynamics is described by General Relativity, of which Eq. (5.17) is a well-known solution, one can assess the presence of covariant singularities by combining Eqs. (5.16) and (5.18). Thus, if we take $\epsilon > 0$, a covariant singularity exists at the origin $r = 0$, whereas a covariant singularity is present at $r \to \infty$ for the $\epsilon < 0$. No covariant singularity seems however to exist in the Schwarzschild black hole for $\epsilon = 0$. Indeed, the fact that the Kretschmann scalar $\mathcal{K}$ is constant and finite for $\epsilon = 0$ suggests that no covariant singularity would be present, regardless of the underlying theory, in this case. Therefore, the absence of covariant singularities largely depends on the choice of the field-space metric.

It so happens that the Kretschmann scalar is everywhere finite for the DeWitt metric with $\epsilon = 0$, which suggests that all spacetime singularities could be removed by field redefinitions in this case. This means that, in pure gravity, spacetimes $\mathcal{M}_1$ with curvature singularities signalled by the diverging Kretschmann scalar can always be extended to regular spacetimes $\mathcal{M}_2$ which do not contain such singularities by a redefinition of the metric $\boldsymbol{g}$. It thus seems that a field-space metric with $\epsilon = 0$ is the optimal choice to avoid covariant singularities in pure gravity.

Note that the above discussion is restricted to curvature singularities. Other types of singularities in field space, such as the conical ones, could very well be present and be reflected in the physical observables. The study of general singularities is left for future developments (*e.g,* see Ref. [53]).

6 Conclusions

In this paper, we have proposed to investigate singularities in the field space rather than in spacetime. This is particularly important in order to perform an analysis that is covariant under field redefinitions. Such a generalised form of the principle of covariance is indeed required in the study of quantum effects in cosmology and astrophysics, where observables are in-in correlation functions rather than S-matrix elements. Although the equivalence theorem guarantees that scattering amplitudes are invariant under the set of field redefinitions that keep the asymptotic states fixed, the same cannot be said about in-in correlations functions, whose calculation requires the geometrical approach introduced by Vilkovisky and DeWitt [36,37]. As a result of Vilkovisky and DeWitt's works, one obtains a formalism where fields are mere coordinates in the field space and observables are thus scalars defined on the same space. Spacetime singularities are field-dependent, thus their real significance is not clear until one proves their presence for every choice of field variables. Existing examples in the literature indeed show that certain singularities in spacetime can be removed by field redefinitions albeit being non-removable under change of coordinates. We corroborated such examples by remarking that the Hawking-Penrose theorem is not a covariant result under field redefinitions.

Finding field redefinitions that can eliminate singularities is obviously not always feasible in practice as there are infinitely many choices of parameterisation for a field theory. The most promising approach is to calculate curvature invariants in field space. We showed that the Kretschmann scalar of the DeWitt functional metric turns out to be free of singularities for $\epsilon = 0$. Surprisingly, this result does not depend on the choice of the gravitational action, which is encoded in the parameter c in Eq. (5.5). Any pure gravity theory is therefore devoid of curvature singularities should one take $\epsilon = 0$.

We should note that removing a singularity from a field configuration certainly makes its description more complete, but the fact that a singularity is removable in field space does not imply that there is no interesting physics occurring around it. Horizons as removable spacetime singularities clearly teach us that investigating the physics likely requires a case by case study. In fact, studying specific models of self-gravitating systems is one of the natural developments of the present work, as it is the formal analysis of field-space invariants for more general theories than pure gravity. We will pursue both directions in future works.

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